PROBABILITY DENSITY FUNCTIONS

P.D.F.

CALCULATIONS

Question 1 (***)

The lifetime of a certain brand of battery, in tens of hours, is modelled by the continuous random variable X with probability density function $f(x)$ given by

$$f(x)=\begin{cases}\dfrac{2}{75}x & 0\le x\le 5\\ \dfrac{2}{15} & 5<x\le 10\\ 0 & \text{otherwise}\end{cases}$$

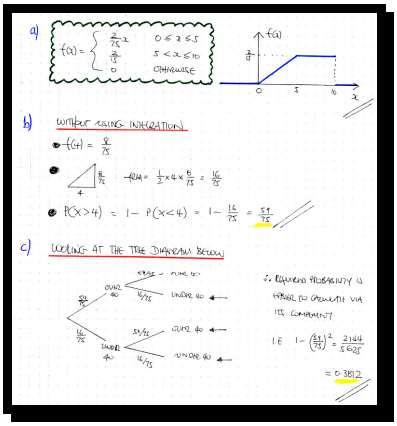
a) Sketch $f(x)$ for all x.

b) Determine $\mathrm{P}(X>4)$.

Two such batteries are needed by a piece of electronic equipment. This equipment will only operate if **both** batteries are still functional.

c) If two new batteries are fitted to this equipment, determine the probability that this equipment will stop working within the next 40 hours.

$\square$, $\boxed{\mathrm{P}(X>4)=\dfrac{59}{75}}$, $\boxed{\dfrac{2144}{5625}\approx 0.381}$



Question 2 (***)

The lengths of telephone conversations, in minutes, by sales reps of a certain company are modelled by the continuous random variable T.

The probability density function of T is denoted by $f(t)$, and is given by

$$f(t)=\begin{cases} kt & 0\le t\le 12 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=\dfrac{1}{72}$.

b) Determine $\mathrm{P}(T>5)$.

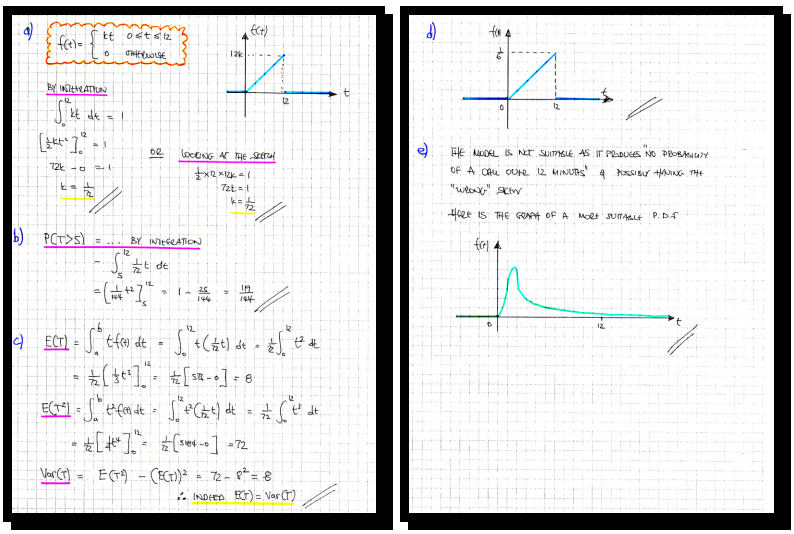
c) Show by calculation that $\mathrm{E}(T)=\mathrm{Var}(T)$.

d) Sketch $f(t)$ for all t.

A statistician suggests that the probability density function $f(t)$ as defined above, might not provide a good model for T.

e) Give a reason for his suggestion.

$\boxed{}$, $\boxed{\mathrm{P}(T>5)=\dfrac{119}{144}}$, $\boxed{\mathrm{E}(T)=\mathrm{Var}(T)=8}$



Question 3 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} 2x+k & 3\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=-6$.

b) Sketch $f(x)$ for all x.

c) State the mode of X.

d) Calculate, showing detailed workings, the value of …

i. … $\mathrm{E}(X)$.

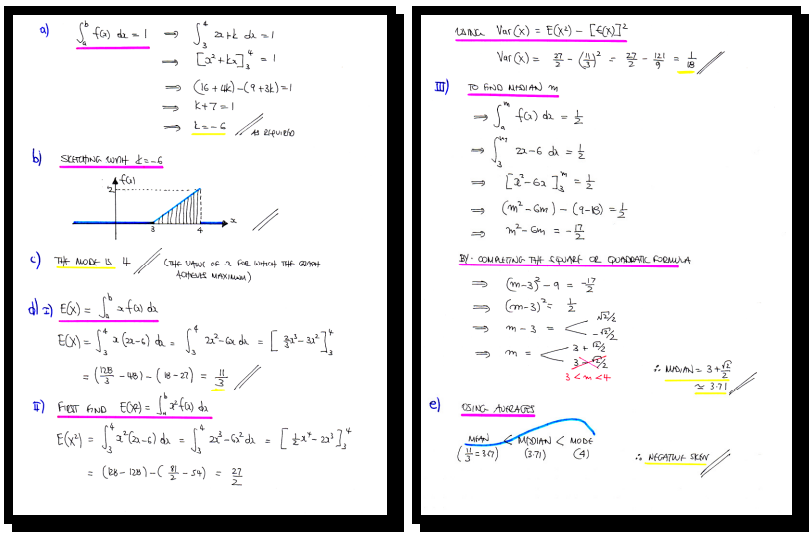
ii. … $\mathrm{Var}(X)$.

iii. … the median of X.

e) Determine with justification the skewness of the distribution.

$\boxed{}$, $\boxed{\text{mode}=4}$, $\boxed{\mathrm{E}(X)=\frac{11}{3}\approx 3.67}$, $\boxed{\mathrm{Var}(X)=\frac{1}{18}\approx 0.0556}$,

$\boxed{\text{median}=3+\frac{\sqrt{2}}{2}\approx 3.71}$, $\boxed{\text{mean} < \text{median} < \text{mode} \Rightarrow \text{negative skew}}$



Question 4 (****)

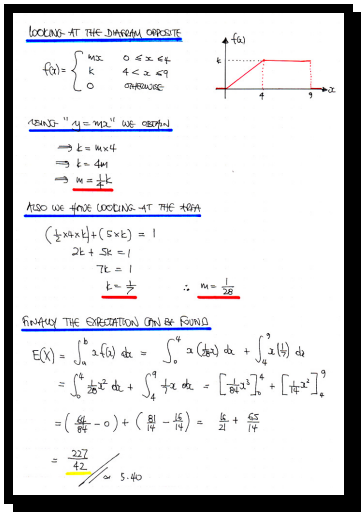
A continuous random variable X has probability density function $f(x)$ given by

$$f(x)\equiv\begin{cases} mx & 0\le x\le 4 \\ k & 4\le x\le 9 \\ 0 & \text{otherwise} \end{cases}$$

where m and k are positive constants.

Find as an exact simplified fraction the value of $\mathrm{E}(X)$.

$\boxed{}$, $\boxed{\mathrm{E}(X)=\dfrac{227}{42}}$



Question 5 (**+)**

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} kx\left(16-x^2\right) & 0\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=\dfrac{1}{64}$.

b) Calculate, showing detailed workings, the value of …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{Var}(X)$.

c) Show by calculation, that the median is 2.165, correct to 3 decimal places.

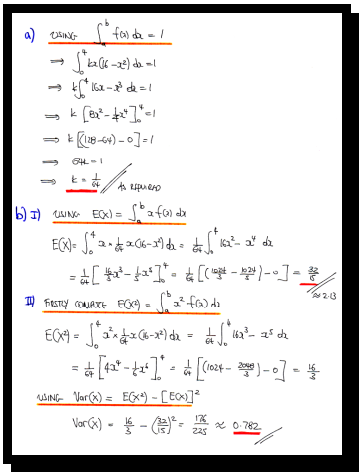
d) Use calculus to find the mode of X.

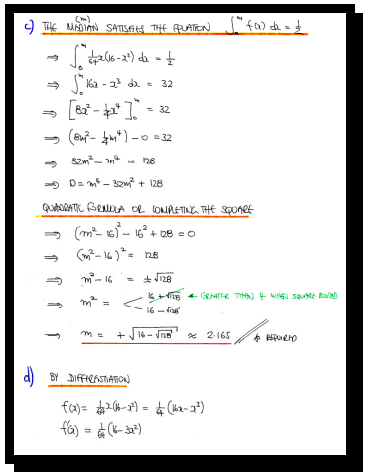
e) Sketch the graph of $f(x)$ for all x.

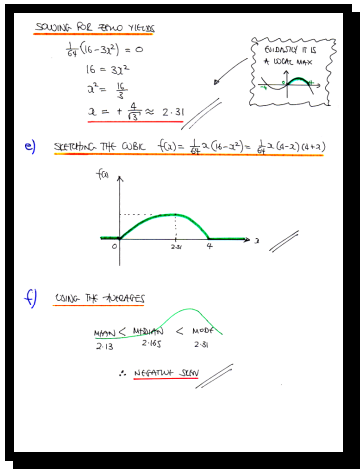
f) Determine with justification the skewness of the distribution.

$\square$, $\boxed{\mathrm{E}(X)=\frac{32}{15}\approx 2.13}$, $\boxed{\mathrm{Var}(X)=\frac{176}{225}\approx 0.782}$, $\boxed{\text{mode}\approx 2.31}$,

$\boxed{\text{mean} < \text{median} < \text{mode} \Rightarrow \text{negative skew}}$







Question 6 (**+)**

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} kx(a-x) & 0\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

where k and a are positive constants.

A statistician claims that $a\ge 4$.

a) Justify the statistician's claim.

b) Show clearly that

$$k=\frac{3}{8(3a-8)}.$$

It is further given that $\mathrm{E}(X)=2.4$.

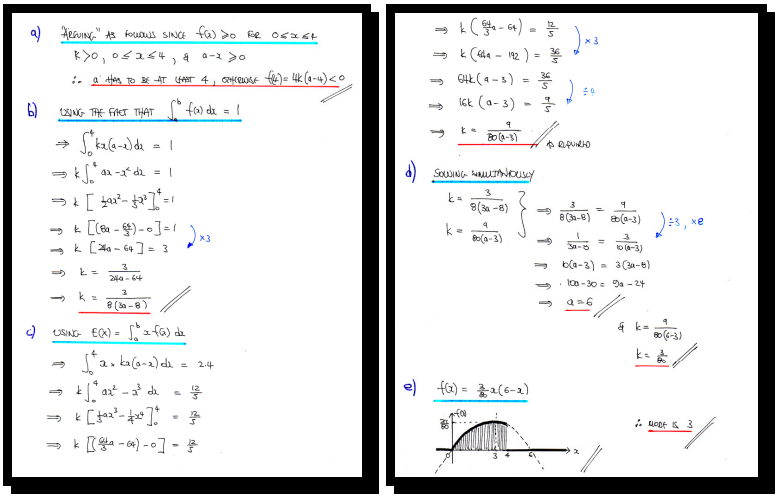
c) Show further that

$$k=\frac{9}{80(a-3)}.$$

d) Hence determine the value of a and the value of k.

e) Sketch the graph of $f(x)$ for all x and hence state the mode of X.

$\boxed{}$, $\boxed{a=6}$, $\boxed{k=\frac{3}{80}=0.0375}$, $\boxed{\text{mode}=3}$



Question 7 (**+)**

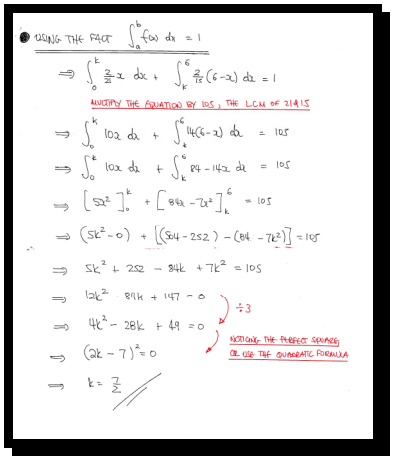
The continuous random variable X has probability density function $f(x)$, given by

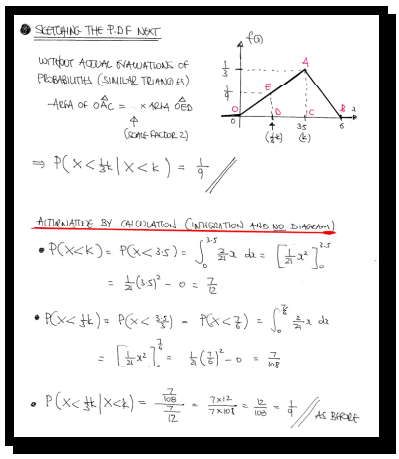
$$f(x)=\begin{cases}\frac{2}{21}x & 0\le x\le k\\ \frac{2}{15}(6-x) & k<x\le 6\\ 0 & \text{otherwise}\end{cases}$$

Determine the value of the positive constant k, and hence or otherwise find

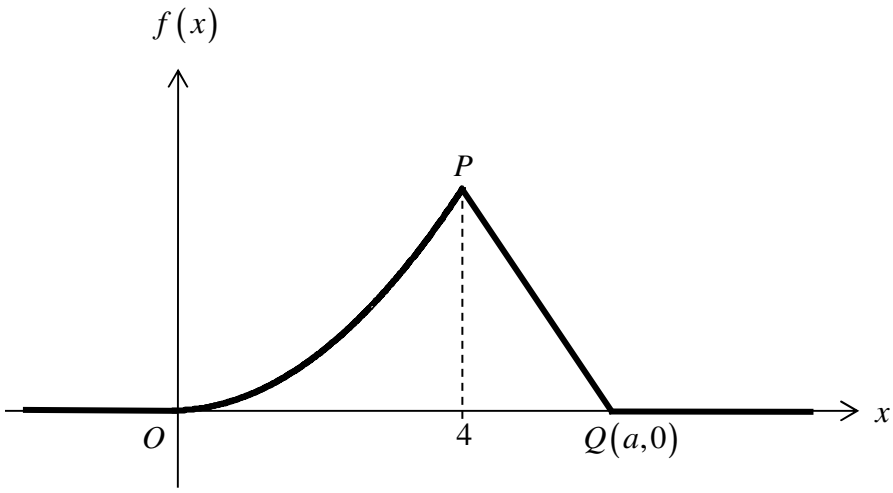
$$\mathrm{P}\left[X<\tfrac{1}{3}k\,\middle|\,X<k\right].$$

$\square$, $\boxed{k=3.5}$, $\boxed{\mathrm{P}\left[X<\tfrac{1}{3}k\,\middle|\,X<k\right]=\frac{1}{9}}$





Question 8 (**+)**



The figure above shows the graph of the probability density function $f(x)$ of a continuous random variable X.

The graph consists of the curved segment OP with equation

$$f(x)=kx^2,\ \ 0\le x\le 4,$$

where k is a positive constant.

The graph of $f(x)$ further consists ofm a straight line segment from P to $Q(a,0)$, for $4<x\le a$, where a is a positive constant.

For all other values of x, $f(x)=0$.

a) State the mode of X.

It is given that the mode of X is equal to the median of X.

b) Show that $k=\dfrac{3}{128}$, and find the value of a.

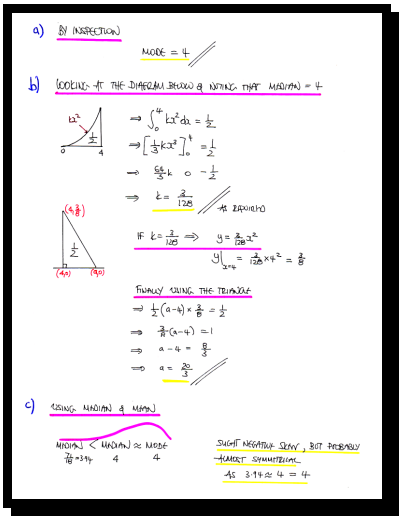
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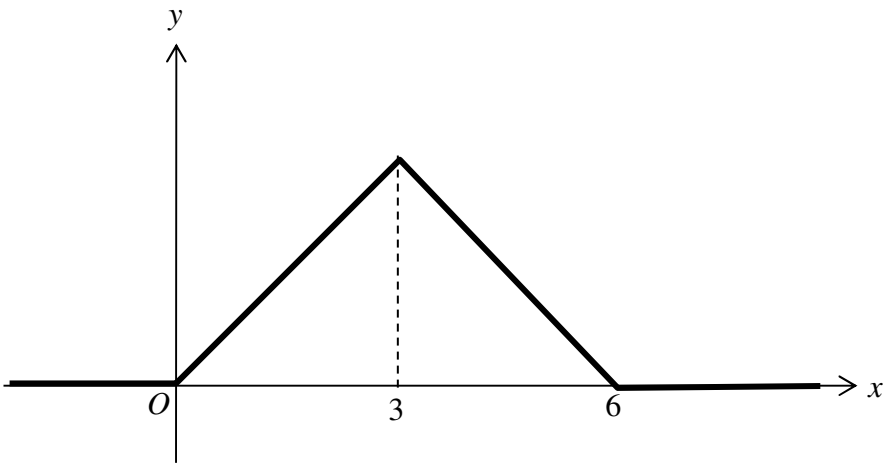
It is further given that $\mathrm{E}(X)=\dfrac{71}{18}$.

c) Determine with justification the skewness of the distribution.

$\boxed{}$, $\boxed{\text{mode}=4}$, $\boxed{a=\dfrac{20}{3}}$, $\boxed{\text{mean} < \text{median} = \text{mode} \Rightarrow \text{negative skew}}$



Question 9 (**+)**



The figure above shows the graph of a probability density function $f(x)$ of a continuous random variable X.

The graph consists of two straight line segments of **equal** length joined up at the point where $x=3$.

The probability density function $f(x)$ is fully specified as

$$f(x)=\begin{cases} ax & 0\le x\le 3 \\ b+cx & 3<x\le 6 \\ 0 & \text{otherwise} \end{cases}$$

where a, b and c are non zero constants.

a) Show that $b=\frac{2}{3}$, $c=-\frac{1}{9}$ and find the value of a.

b) State the value of $\mathrm{E}(X)$.

c) Show that $\mathrm{Var}(X)=1.5$.

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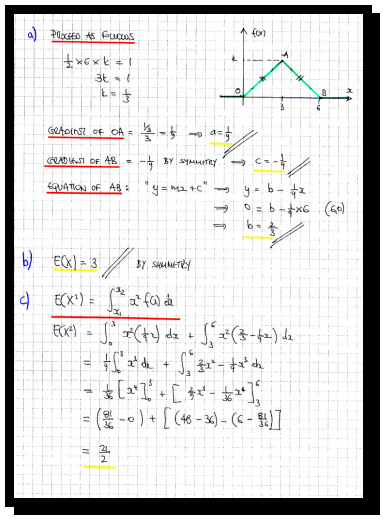
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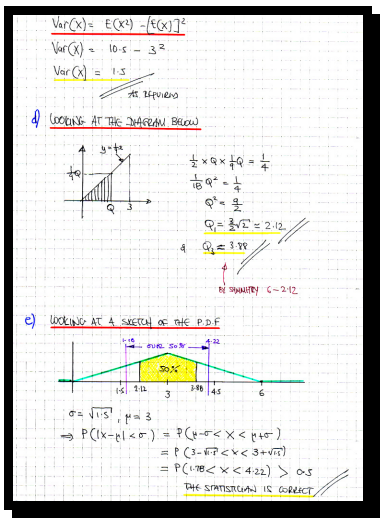
d) Determine the upper and lower quartile of X.

A statistician claims that $P\left(\left|X-\mu\right|<\sigma\right)>0.5$.

e) Show that the statistician's claim is correct.

$\boxed{}$, $\boxed{a=\frac{1}{9}}$, $\boxed{\mathrm{E}\left(X\right)=3}$, $\boxed{Q_1\approx 2.121}$, $\boxed{Q_3\approx 3.879}$





Question 10 (**+)**

The continuous random variable X has probability density function $f(x)$, given by

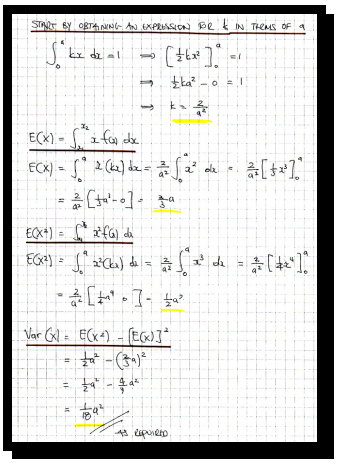
$$f(x)=\begin{cases} kx & 0\le x\le a \\ 0 & \text{otherwise} \end{cases}$$

where a and k are positive constants.

Show, by a detailed method, that

$$\text{Var}(X)=\frac{1}{18}a^2 .$$

[ ], [proof]



P.D.F. to C.D.F CALCULATIONS

Question 1 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases}\frac{2}{9}(5-x) & 2\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

The cumulative distribution function of X, is denoted by $F(x)$.

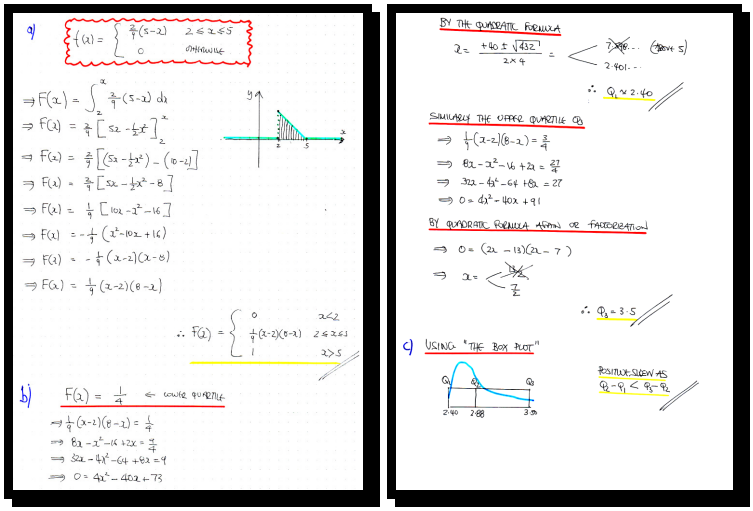
a) Find and specify fully $F(x)$.

b) Use $F(x)$, to show that the lower quartile of X is approximately 2.40, and find the value of the upper quartile.

c) Given that the median of X is 2.88, comment on the skewness of X.

$\square$, $\boxed{F(x)=\begin{cases}0 & x<2\\ -\frac{1}{9}(x-8)(x-2) & 2\le x\le 5\\ 1 & x>5\end{cases}}$, $\boxed{Q_3=3.5}$,

$\boxed{Q_2-Q_1<Q_3-Q_2\Rightarrow \text{+ve skew}}$



Question 2 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} kx(x-3) & 3\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Use integration to show that $k=\frac{6}{11}$.

b) Calculate the value of $\mathrm{E}(X)$.

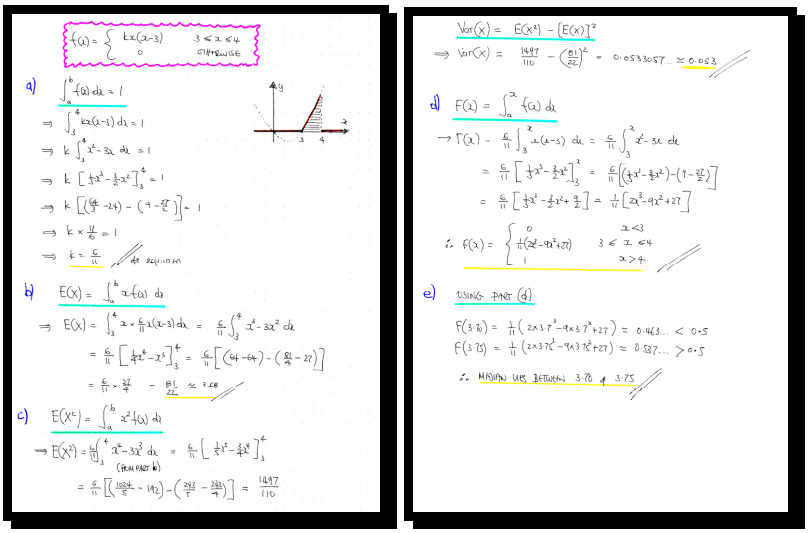
c) Show that $\mathrm{Var}(X)=0.053$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Show that the median of X lies between 3.70 and 3.75.

$\square$, $\boxed{\mathrm{E}(X)=\frac{81}{22}\approx 3.68}$, $\boxed{F(x)=\begin{cases} 0 & x<3 \\ \frac{1}{11}\left(2x^3-9x^2+27\right) & 3\le x\le 4 \\ 1 & x>4 \end{cases}}$



Question 3 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} k(x+2)^2 & -2\le x\le 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k=\frac{3}{8}$.

b) Find the value of $\mathrm{E}(X)$.

c) Show that $\mathrm{Var}(X)=0.15$.

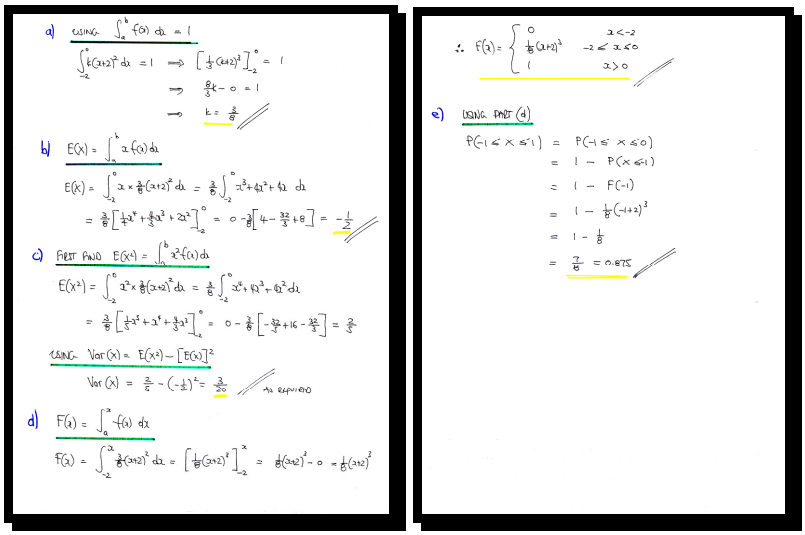
The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Determine $P(-1\le X\le 1)$.

$\square$, $\boxed{\mathrm{E}(X)=-0.5}$, $\boxed{F(x)=\begin{cases} 0 & x<-2 \\ \frac{1}{8}x^3+\frac{3}{4}x^2+\frac{3}{2}x+1 & -2\le x\le 0 \\ 1 & x>0 \end{cases}}$,

$\boxed{P(-1\le X\le 1)=0.875}$



Question 4 (****)

The continuous random variable X has probability density function $f(x)$, given by:

$$f(x)=\begin{cases}\frac{1}{60}x^3 & 2\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

a) Find the value of $\mathrm{E}(X)$.

b) Show that the standard deviation of X is 0.516, correct to 3 decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

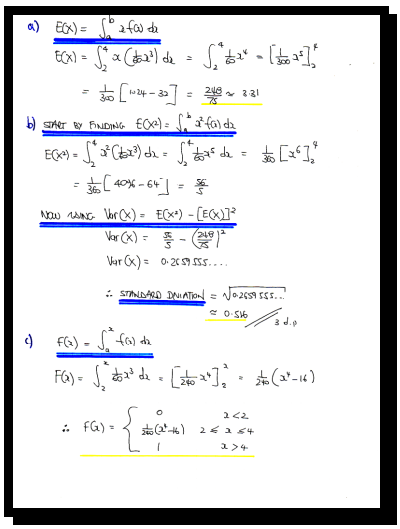
c) Find and specify fully $F(x)$.

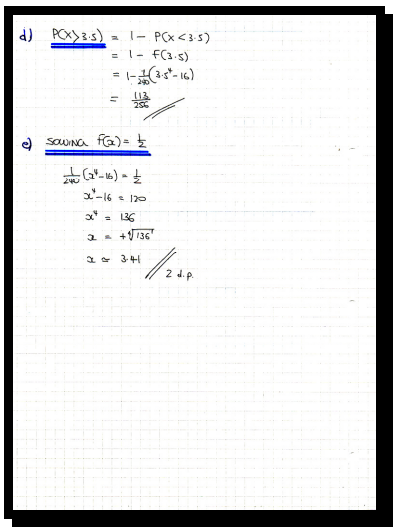
d) Determine $P(X>3.5)$.

e) Calculate the median of X, correct to two decimal places.

$\square$, $\boxed{\mathrm{E}(X)=\frac{248}{75}\approx 3.31}$, $\boxed{F(x)=\begin{cases}0 & x<2\\ \frac{1}{240}\left(x^4-16\right) & 2\le x\le 4\\ 1 & x>4\end{cases}}$,

$\boxed{P(X>3.5)=\frac{113}{256}}$, $\boxed{\text{median}\approx 3.41}$





Question 5 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} kx(6-x) & 0\le x\le 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k=\dfrac{3}{100}$.

b) Calculate the value of $\mathrm{E}(X)$.

c) Find the mode of X.

The cumulative distribution function of X, is denoted by $F(x)$.

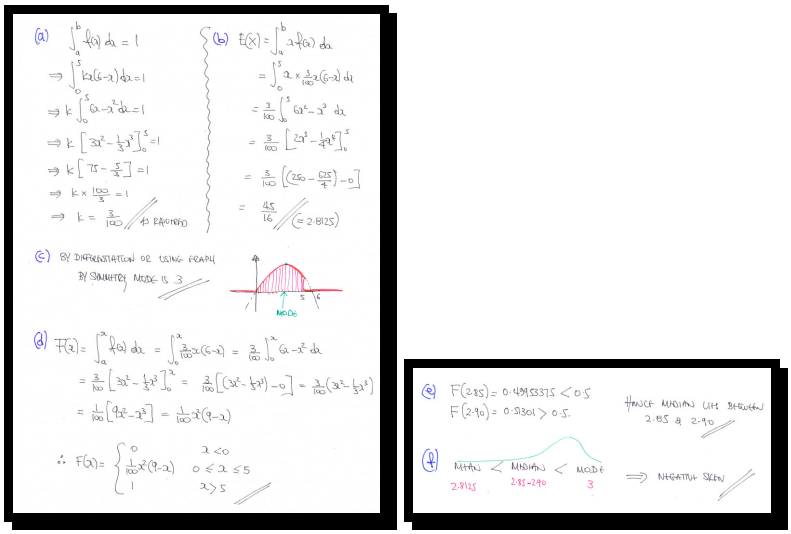
d) Find and specify fully $F(x)$.

e) Verify that the median of X lies between 2.85 and 2.9.

f) Determine with justification the skewness of the distribution.

$\boxed{\mathrm{E}(X)=\dfrac{45}{16}}$, $\boxed{\text{mode}=3}$, $\boxed{F(x)=\begin{cases} 0 & x<0 \\ \dfrac{1}{100}x^2(9-x) & 0\le x\le 5 \\ 1 & x>5 \end{cases}}$,

$\boxed{\text{mean}<\text{median}<\text{mode}\Rightarrow\text{negative skew}}$



Question 6 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} k(x-1)(x-4) & 1\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k=-\frac{2}{9}$

b) Sketch the graph of $f(x)$, for all x.

c) State the value of $\mathrm{E}(X)$.

d) Calculate the $\mathrm{Var}(X)$.

The cumulative distribution function of X, is denoted by $F(x)$.

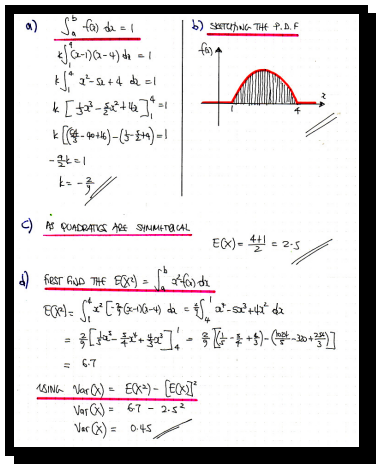
e) Find and specify fully $F(x)$.

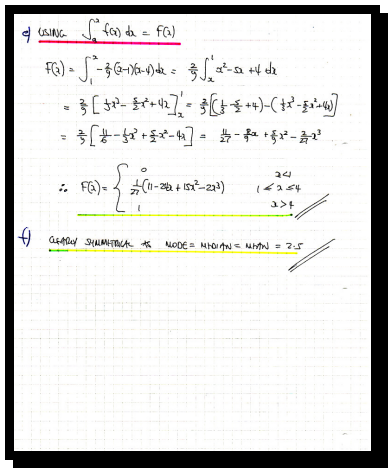
f) Determine with justification the skewness of the distribution.

$\square$, $\boxed{\mathrm{E}(X)=2.5}$, $\boxed{\mathrm{Var}(X)=0.45}$

$$\boxed{F(x)=\begin{cases} 0 & x<1 \\ \frac{1}{27}\left(11-24x+15x^2-2x^3\right) & 1\le x\le 4 \\ 1 & x>4 \end{cases}},$$

$\boxed{\text{mean}=\text{median}=\text{mode}=2.5\Rightarrow \text{zero skew}}$





Question 7 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases}\dfrac{2+x}{k} & 2\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

a) Show clearly that $k=\dfrac{33}{2}$.

b) Find the value of $\mathrm{E}(X)$.

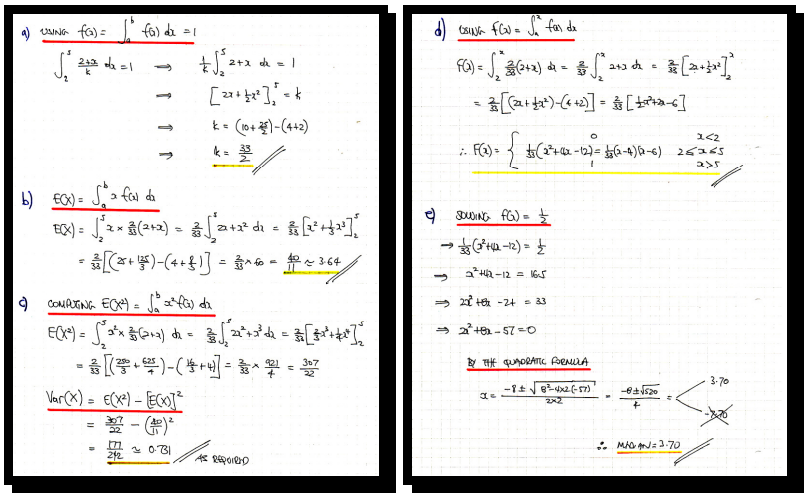
c) Show that $\mathrm{Var}(X)=0.731$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Determine the median of X.

$\square$, $\boxed{\mathrm{E}(X)=\dfrac{40}{11}}$, $\boxed{F(x)=\begin{cases}0 & x<2\\ \dfrac{1}{33}(x+6)(x-2) & 2\le x\le 5\\ 1 & x>5\end{cases}}$, $\boxed{\text{median}\approx 3.70}$



Question 8 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases}\frac{1}{3} & 0\le x\le 2\\ \frac{4}{195}x^3 & 2<x\le 3\\ 0 & \text{otherwise}\end{cases}$$

a) Show that $\mathrm{E}(X)=1.532$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

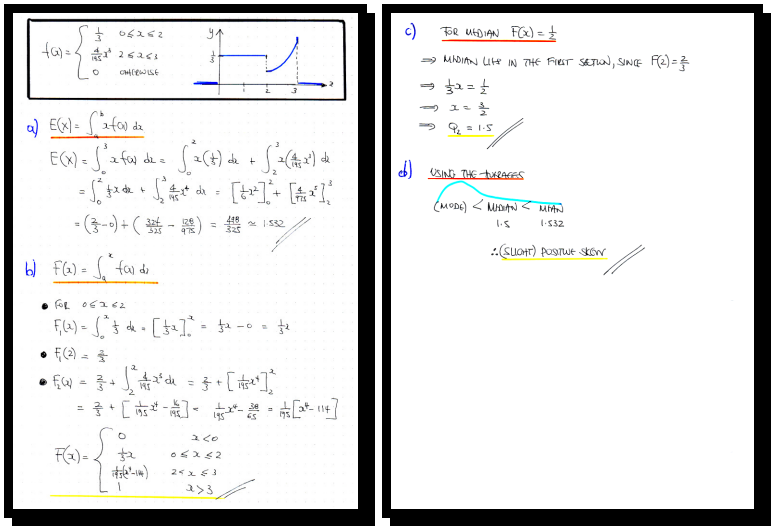
b) Find and specify fully $F(x)$.

c) Calculate the median of X.

d) Determine with justification the skewness of the distribution.

$\square$, $\boxed{F(x)=\begin{cases}0 & x<0\\ \frac{1}{3}x & 0\le x\le 2\\ \frac{1}{195}x^4+\frac{38}{65} & 2<x\le 3\\ 1 & x>3\end{cases}}$, $\boxed{\text{median}=1.5}$,

$\boxed{(\text{mode})<\text{median}<\text{mean}\Rightarrow\text{slight positive skew}}$



Question 9 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases}\frac{1}{4}x & 0\le x\le 2\\ kx^3 & 2<x\le 4\\ 0 & \text{otherwise}\end{cases}$$

a) Show clearly that $k=\frac{1}{120}$.

The cumulative distribution function of X, is denoted by $F(x)$.

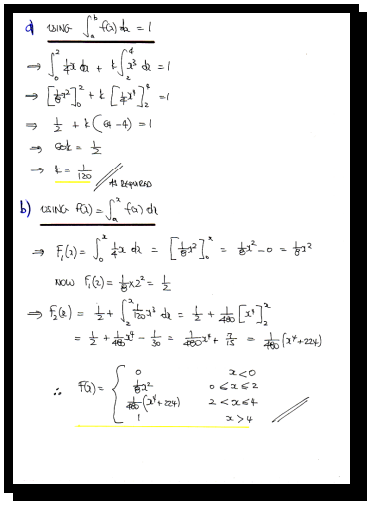
b) Find and specify fully $F(x)$.

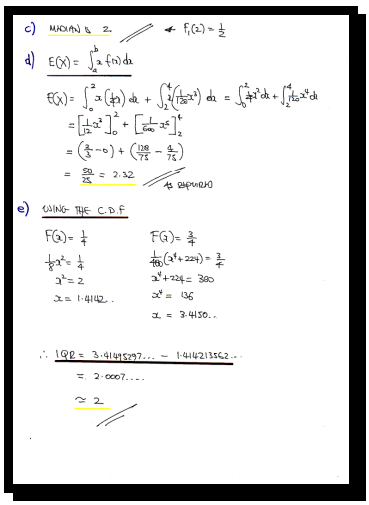
c) State the median of X.

d) Show that $\mathrm{E}(X)=\frac{58}{25}$.

e) Calculate the interquartile range of X.

$\square$, $\boxed{F(x)=\begin{cases}0 & x<0\\ \frac{1}{8}x^2 & 0\le x\le 2\\ \frac{x^4+224}{480} & 2<x\le 4\\ 1 & x>4\end{cases}}$, $\boxed{\text{median}=2}$, $\boxed{\text{IQR}\approx 2.00}$





Question 10 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} k\left(x^2-2x+3\right) & 0\le x<2 \\ \frac{1}{3}k & 2\le x\le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k=\frac{1}{5}$.

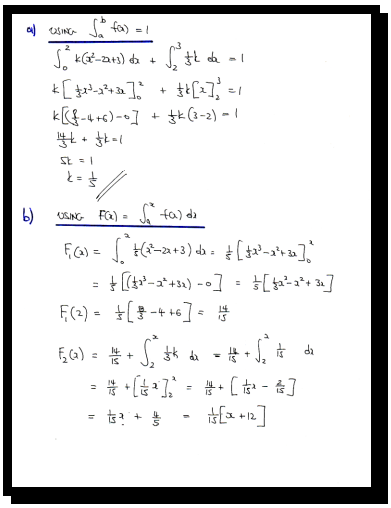
The cumulative distribution function of X, is denoted by $F(x)$.

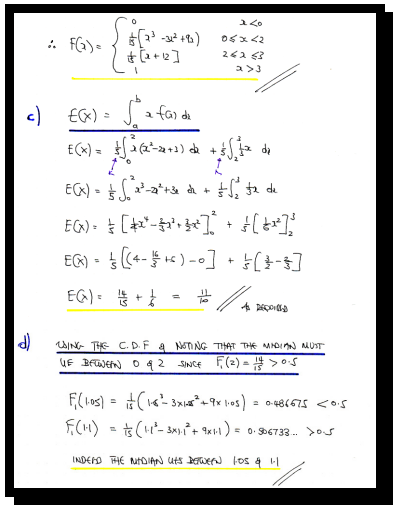
b) Find and specify fully $F(x)$.

c) Show that $\mathrm{E}(X)=\frac{11}{10}$.

d) Show that the median of X lies between 1.05 and 1.1.

$$\square,\quad F(x)=\begin{cases} 0 & x<0 \\ \frac{1}{15}x\left(x^2-3x+9\right) & 0\le x<2 \\ \frac{1}{15}(x+12) & 2\le x\le 3 \\ 1 & x>4 \end{cases}$$





Question 11 (**+)**

The continuous random variable X has probability density function $f(x)$, given by:

$$f(x)=\begin{cases}\frac{1}{10}x & 0\le x<4\\ 2-\frac{2}{5}x & 4\le x\le 5\\ 0 & \text{otherwise}\end{cases}$$

a) Sketch the graph of $f(x)$ for all x.

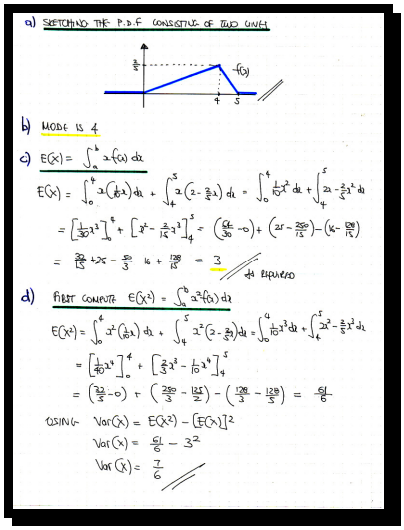
b) State the mode of X.

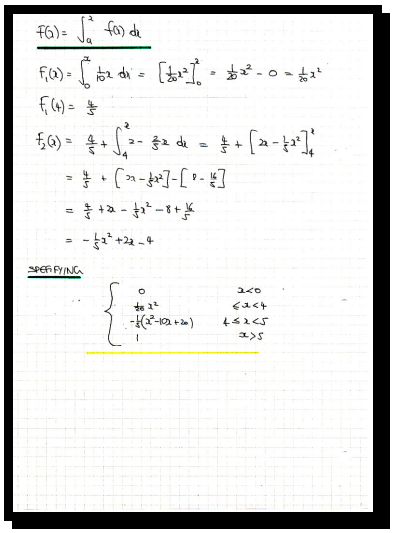
c) Show clearly that $\mathrm{E}(X)=3$.

d) Calculate the value of $\mathrm{Var}(X)$.

e) Find and specify fully the cumulative distribution function of X, $F(x)$.

$\boxed{}$, $\boxed{\text{mode}=4}$, $\boxed{\mathrm{Var}(X)=\frac{7}{6}}$, $\boxed{F(x)=\begin{cases}0 & x<0\\ \frac{1}{20}x^2 & 0\le x<4\\ -\frac{1}{5}\left(x^2-10x+20\right) & 4\le x\le 5\\ 1 & x>5\end{cases}}$





Question 12 (**+)**

The continuous random variable X has probability density function $f(x)$, defined by the piecewise continuous function

$$f(x)=\begin{cases}\frac{1}{12}(x-1) & 1\le x\le 3\\ \frac{1}{6} & 3<x\le 6\\ \frac{5}{12}-\frac{1}{24}x & 6<x\le 10\\ 0 & \text{otherwise}\end{cases}$$

a) Sketch the graph of $f(x)$, for all values of x.

The cumulative distribution function of X, is denoted by $F(x)$.

b) Show by detailed calculations that

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{24}x^2-\frac{1}{12}x+\frac{1}{24} & 1\le x\le 3\\ \frac{1}{6}x-\frac{1}{3} & 3<x\le 6\\ -\frac{1}{48}x^2+\frac{5}{12}x-\frac{13}{12} & 6<x\le 10\\ 1 & x>10\end{cases}$$

[continues overleaf]

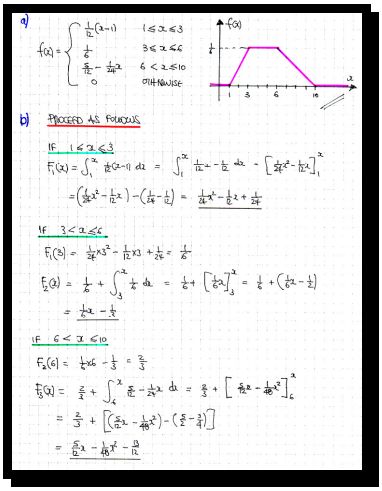
[continued from overleaf]

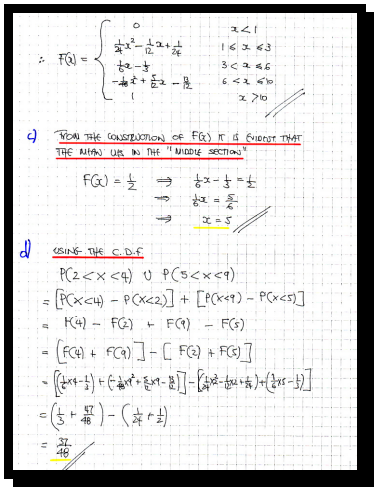
c) Calculate the median of X.

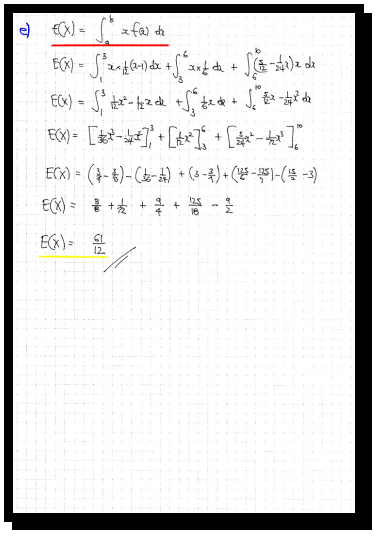
d) Determine the value of $P(2<X<4)\cup P(5<X<9)$.

e) Find the value for $\mathrm{E}(X)$.

$\boxed{}$, $\boxed{\text{median}=5}$, $\boxed{P(2<X<4)\cup P(5<X<9)=\frac{37}{48}}$, $\boxed{\mathrm{E}(X)=\frac{61}{12}}$







C.D.F. to P.D.F CALCULATIONS

Question 1 **(***)**

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x)=\begin{cases} 0 & x<1 \\ k\left(x^4+2x^2-3\right) & 1\le x\le 3 \\ 1 & x>3 \end{cases}$$

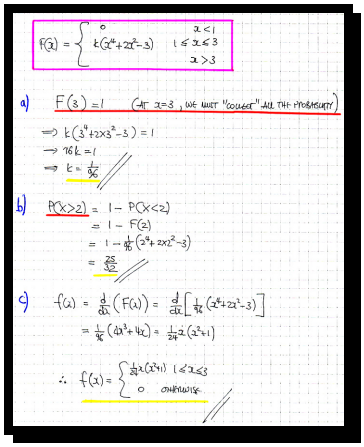
where k is a positive constant.

a) Show clearly that $k=\dfrac{1}{96}$.

b) Find $\mathrm{P}(X>2)$.

c) Determine the probability density function of X, for all values of x.

$\square$, $\boxed{\mathrm{P}(X>2)=\frac{25}{32}}$, $\boxed{f(x)=\begin{cases} \frac{1}{24}\left(x^3+x\right) & 1\le x\le 3 \\ 0 & \text{otherwise} \end{cases}}$



Question 2 (***)

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by

$$F(x)=\begin{cases}0 & x<-3\\ k(x+3) & -3\le x\le 8\\ 1 & x>8\end{cases}$$

where k is a positive constant.

a) Show clearly that $k=\frac{1}{11}$.

b) Calculate $\mathrm{P}(X<1)$.

c) Find $\mathrm{P}(X=1)$.

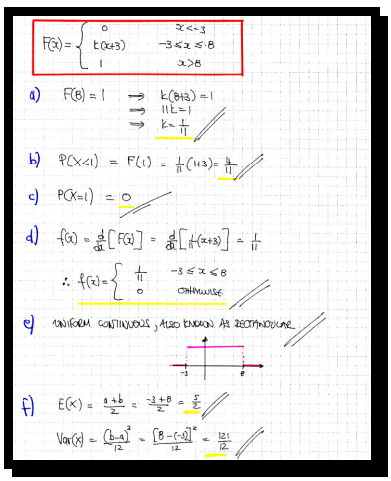
d) Define the probability density function of X, for all values of x.

e) State the name of this probability distribution.

f) Determine the value of $\mathrm{E}(X)$ and the value of $\mathrm{Var}(X)$.

$\boxed{}$, $\boxed{\mathrm{P}(X<1)=\frac{4}{11}}$, $\boxed{\mathrm{P}(X=1)=0}$, $\boxed{f(x)=\begin{cases}\frac{1}{11} & -3\le x\le 8\\ 0 & \text{otherwise}\end{cases}}$,

$\boxed{\text{uniform continuous}}$, $\boxed{\mathrm{E}(X)=\frac{5}{2}}$, $\boxed{\mathrm{Var}(X)=\frac{121}{12}}$



Question 3 (***)

The continuous random variable T represents the lifetime, in tens of hours, for a certain brand of battery.

The cumulative distribution function $F(t)$ of the variable T is given by

$$F(t)=\begin{cases} 0 & t<2 \\ \frac{1}{20}(t+6)(t-2) & 2\le t\le 4 \\ 1 & t>4 \end{cases}$$

a) Calculate the value of …

i. … $\mathrm{P}(T>3.5)$.

ii. … $\mathrm{P}(T=3)$.

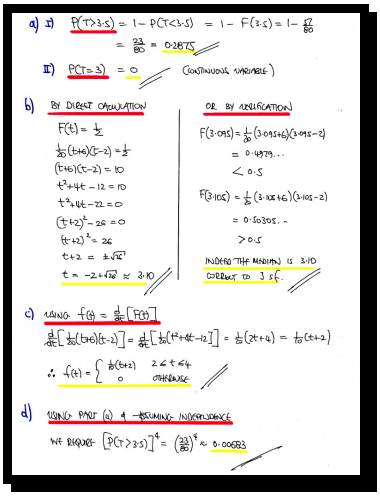
b) Show that the median of T is 3.10, correct to three significant figures.

c) Define the probability density function of T, for all values of t.

Four new batteries, whose lifetime is modelled by T, are fitted to a road hazard lantern. This lantern will only remain operational if all four batteries are working.

d) Determine the probability that the lantern will operate for more than 35 hours.

$\boxed{}$, $\boxed{\mathrm{P}(T>3.5)=\frac{23}{80}}$, $\boxed{\mathrm{P}(T=3)=0}$, $\boxed{f(t)=\begin{cases} \frac{1}{10}(t+2) & 2\le t\le 4 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{0.00683}$



Question 4 (***)

The cumulative distribution function, $F(y)$, of a continuous random variable Y is given by

$$F(y)=\begin{cases}0 & y<0\\ ky^3 & 0\le y\le b\\ 1 & y>b\end{cases}$$

It is further given that $\mathrm{P}(1<Y<3)=\frac{13}{32}$.

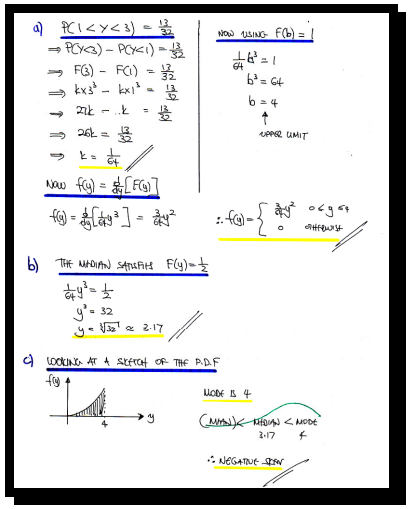
a) Find the value of k and hence determine the probability density function of Y for all y.

b) Calculate the median of Y.

c) **State** the mode of Y and hence comment on the skewness of the distribution.

$\boxed{}$, $\boxed{f(y)=\begin{cases}\frac{3}{64}y^2 & 0\le y\le 4\\ 0 & \text{otherwise}\end{cases}}$, $\boxed{\text{median}\approx 3.17}$, $\boxed{\text{mode}=4}$,

$\boxed{\text{median}<\text{mode}\Rightarrow\text{negative skew}}$



Question 5 (***)

The cumulative distribution function $F(x)$ of a continuous random variable X is given by the following expression

$$F(x)=\begin{cases}0 & x<0\\ kx(x+2) & 0\le x\le 2\\ 1 & x>2\end{cases}$$

where k is a positive constant.

a) Find the value of k.

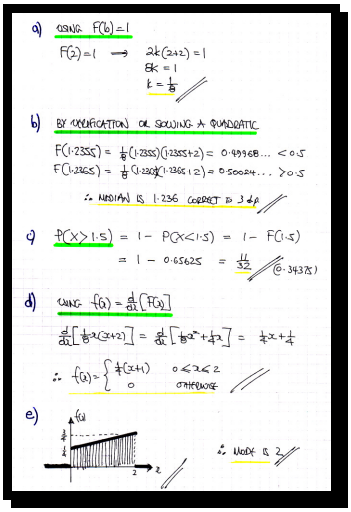
b) Show that the median of X is 1.236, correct to 3 decimal places.

c) Calculate $\mathrm{P}(X>1.5)$.

d) Define the probability density function of X, $f(x)$, for all x.

e) Sketch the graph of $f(x)$ for all x, and hence state the mode of X.

$\square$, $\boxed{k=\frac{1}{8}}$, $\boxed{\mathrm{P}(X>1.5)=\frac{11}{32}}$, $\boxed{f(x)=\begin{cases}\frac{1}{4}(x+1) & 0\le x\le 2\\ 0 & \text{otherwise}\end{cases}}$, $\boxed{\text{mode}=2}$



Question 6 (*+)**

The time, in weeks, a patient has to wait for an appointment to see a psychiatrist in a certain hospital, is modelled by the continuous random variable T.

The cumulative distribution function of T is given by

$$F(t)=\begin{cases}0 & t<0\\ \frac{1}{1728}t^3 & 0\le t\le 12\\ 1 & t>12\end{cases}$$

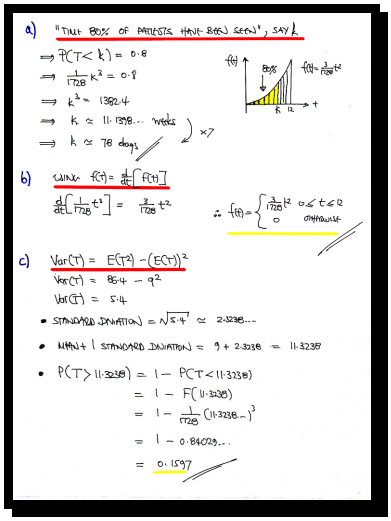
a) Find, to the nearest day, the time within which 80% of the patients have been given an appointment.

b) Define the probability density function of T, for all values of t.

It is given that $\mathrm{E}(T)=9$ and $\mathrm{E}\left(T^2\right)=86.4$.

c) Determine the probability that the time a patient has to wait for an appointment, is more than one standard deviation above the mean.

$$\boxed{},\ \boxed{\approx 78\text{ days}},\ \boxed{f(t)=\begin{cases}\frac{1}{576}t^2 & 0\le t\le 12\\ 0 & \text{otherwise}\end{cases}},\ \boxed{\approx 0.1597}$$



Question 7 (*+)**

The time, in hours, spent on a piece of homework by a group of students is modelled by the continuous random variable T.

The cumulative distribution function of T is given by

$$F(t)=\begin{cases} 0 & t<0 \\ k\left(2t^3-t^4\right) & 0\le t\le b \\ 1 & t>b \end{cases}$$

where k is a positive constant.

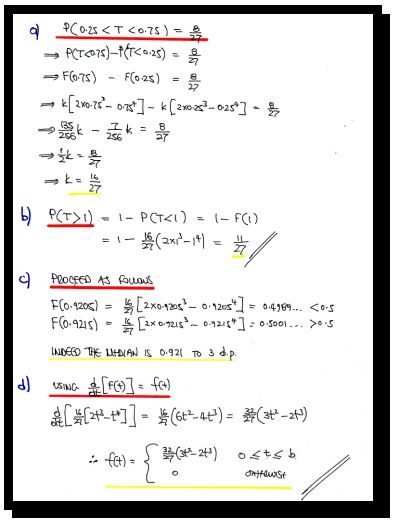
a) If the probability that a student from this group took between one quarter and three quarters of an hour to complete the homework is $\frac{8}{27}$, show that $k=\frac{16}{27}$.

b) Find the proportion of students who spent over an hour in their homework.

c) Verify that the median is 0.921, correct to 3 decimal places.

d) Define the probability density function of T, for all values of t.

$\square$, $\boxed{\mathrm{P}(T>1)=\frac{11}{27}}$, $\boxed{f(t)=\begin{cases} \frac{32}{27}\left(3t^2-2t^3\right) & 0\le t\le b \\ 0 & \text{otherwise} \end{cases}}$



Question 8 (*+)**

A continuous random variable X has the following cumulative distribution function $F(x)$, defined by

$$F(x)=\begin{cases}0 & x<0\\ \frac{1}{5}x^2\left(6-x^2\right) & 0\le x\le 1\\ 1 & x>1\end{cases}$$

a) Find $\mathrm{P}(X>0.5)$.

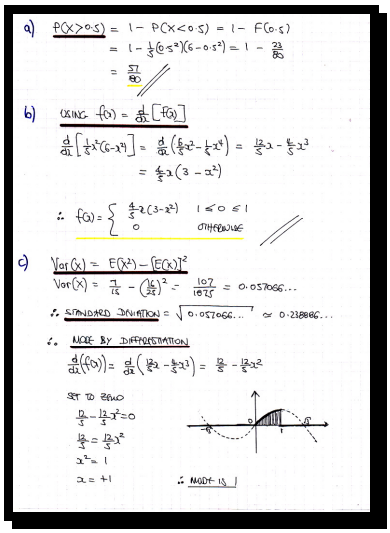
b) Define $f(x)$, the probability density function of X, for all values of x.

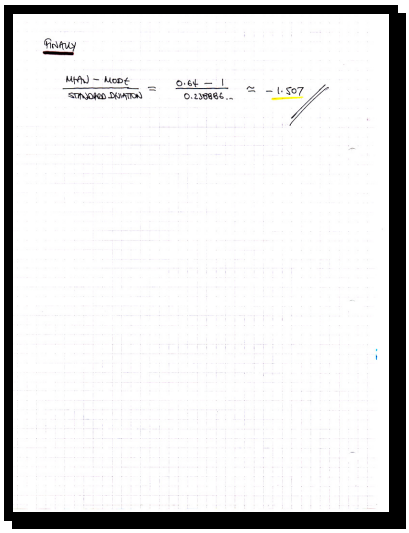
A skewness coefficient can be calculated by the formula

$$\frac{\text{mean}-\text{mode}}{\text{standard deviation}}.$$

c) Given that $\mathrm{E}(X)=\frac{16}{25}$ and $\mathrm{E}\left(X^2\right)=\frac{7}{15}$, evaluate the skewness coefficient for this distribution.

$\boxed{}$, $\boxed{\mathrm{P}(X>0.5)=\frac{57}{80}}$, $\boxed{f(x)=\begin{cases}\frac{4}{5}x\left(3-x^2\right) & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}}$, $\boxed{-1.507}$





Question 9 **(****)**

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x)=\begin{cases} 0 & x<1 \\ k\left(-x^3+6x^2-5\right) & 1\le x\le 4 \\ 1 & x>4 \end{cases}$$

where k is a positive constant.

a) Show clearly that $k=\frac{1}{27}$.

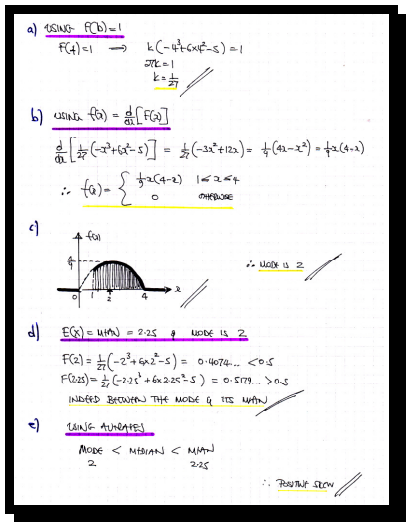
b) Define $f(x)$, the probability density function of X, for all values of x.

c) Sketch the graph of $f(x)$ for all x, and hence state the mode of X.

d) Given that $\mathrm{E}(X)=2.25$, show that the median of X lies between its mode and its mean.

e) State the skewness of the distribution.

$\boxed{}$, $\boxed{f(x)=\begin{cases} \frac{1}{9}x(4-x) & 1\le x\le 4 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\text{mode}=2}$, $\boxed{\text{positive skew}}$



Question 10 (****)

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x)=\begin{cases} 0 & x<1 \\ (3-x)(x-1)^2 & 1\le x\le 2 \\ 1 & x>2 \end{cases}$$

a) Find $\mathrm{P}(X>1.2)$.

b) Show that the median of X lies between 1.59 and 1.60.

c) Show that for $1\le x\le 2$, the probability density function of X, $f(x)$, is

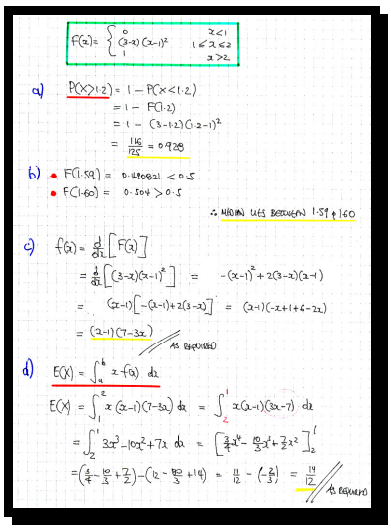
$$f(x)=(1-x)(3x-7).$$

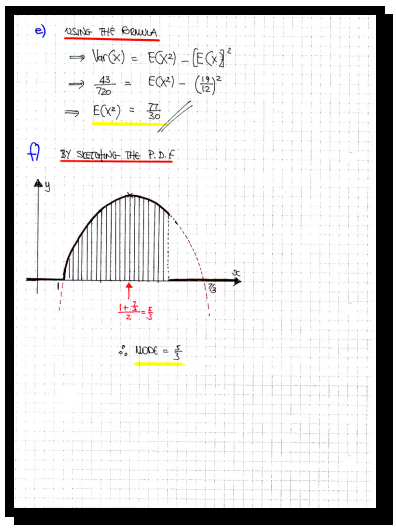
d) Show further that the mean of X is $\dfrac{19}{12}$.

e) Given that the variance of X is $\dfrac{43}{720}$, find the exact value of $\mathrm{E}\left(X^2\right)$.

f) Find the mode of X.

$\square$, $\boxed{\mathrm{P}(X>1.2)=0.928}$, $\boxed{\mathrm{E}\left(X^2\right)=\frac{77}{30}}$, $\boxed{\text{mode}=\frac{5}{3}}$





Question 11 (**+)**

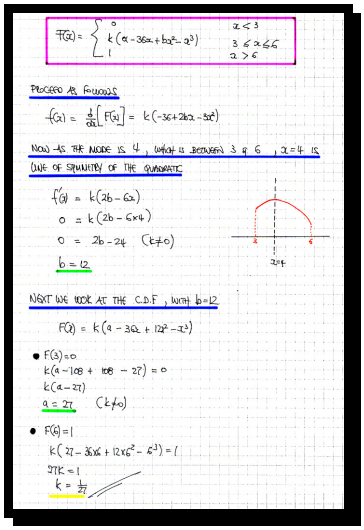
The cumulative distribution function, $F(x)$, of a continuous random variable X is given by

$$F(x)=\begin{cases} 0 & x<3 \\ k\left(a-36x+bx^2-x^3\right) & 3\le x\le 6 \\ 1 & x>6 \end{cases}$$

where k, a and b are non zero constants.

Determine the value of k, given that the mode of X is 4.

$\boxed{}$, $\boxed{k=\frac{1}{27}}$



Question 12 (*****)

The continuous random variable X has the following cumulative distribution function

$$F(x)=\begin{cases} 0 & x<0 \\ ax-bx^2 & 0\le x\le k \\ 1 & x>k \end{cases}$$

where vehicles a, b and k are positive constants.

The variable Y is related to X by

$$Y=3X-2.$$

Determine the value of a, b and k given further that $\mathrm{E}(Y)=2$ and $\mathrm{Var}(Y)=8$.

$\square$, $\boxed{a=\frac{1}{2}}$, $\boxed{b=\frac{1}{16}}$, $\boxed{k=4}$

