# TRIGONOMETRY

## THE PYTHAGOREAN IDENTITIES

**Question 1**

Prove the validity of each of the following trigonometric identities.

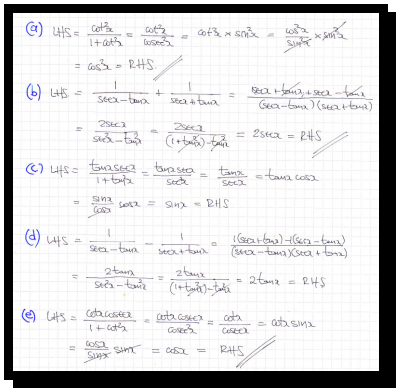
**a)** $\dfrac{\cot^2 x}{1+\cot^2 x} \equiv \cos^2 x$

**b)** $\dfrac{1}{\sec x - \tan x} + \dfrac{1}{\sec x + \tan x} \equiv 2\sec x$

**c)** $\dfrac{\tan x \sec x}{1+\tan^2 x} \equiv \sin x$

**d)** $\dfrac{1}{\sec x - \tan x} - \dfrac{1}{\sec x + \tan x} \equiv 2\tan x$

**e)** $\dfrac{\cot x \operatorname{cosec} x}{1+\cot^2 x} \equiv \cos x$



**Question 2**

Prove the validity of each of the following trigonometric identities.

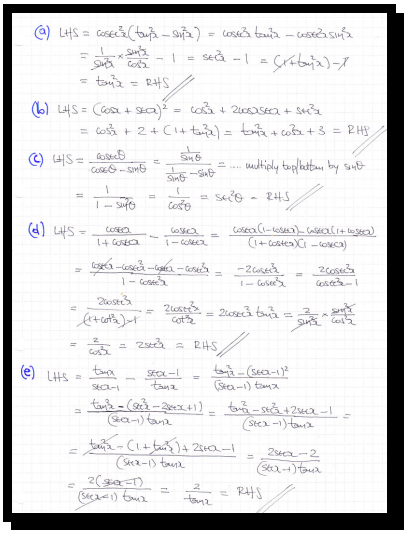
**a)** $\operatorname{cosec}^2 x\left(\tan^2 x-\sin^2 x\right)\equiv\tan^2 x$

**b)** $\left(\cos x+\sec x\right)^2\equiv\tan^2 x+\cos^2 x+3$

**c)** $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sin\theta}\equiv\sec^2\theta$

**d)** $\dfrac{\operatorname{cosec} x}{1+\operatorname{cosec} x}-\dfrac{\operatorname{cosec} x}{1-\operatorname{cosec} x}\equiv 2\sec^2 x$

**e)** $\dfrac{\tan x}{\sec x-1}-\dfrac{\sec x-1}{\tan x}\equiv 2\cot x$



(a) LHS $= \operatorname{cosec}^2\left(\tan^2 - \sin^2\right) = \operatorname{cosec}^2\tan^2 - \operatorname{cosec}^2\sin^2$
$= \frac{1}{\sin^2}\times\frac{\sin^2}{\cos^2} - 1 = \sec^2 - 1 = (\sec^2 - 1) - 1$
$= \tan^2 = \text{RHS}$

(b) LHS $= (\cos + \sec)^2 = \cos^2 + 2\cos\sec + \sec^2$
$= \cos^2 + 2 + (1 + \tan^2) = \tan^2 + \cos^2 + 3 = \text{RHS}$

(c) LHS $= \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} = \ldots$ multiply top/bottom by $\sin\theta$
$= \frac{1}{1 - \sin^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta = \text{RHS}$

(d) LHS $= \frac{\operatorname{cosec}}{1+\operatorname{cosec}} - \frac{\operatorname{cosec}}{1-\operatorname{cosec}} = \frac{\operatorname{cosec}(1-\operatorname{cosec}) - \operatorname{cosec}(1+\operatorname{cosec})}{(1+\operatorname{cosec})(1-\operatorname{cosec})}$
$= \frac{\operatorname{cosec} - \operatorname{cosec}^2 - \operatorname{cosec} - \operatorname{cosec}^2}{1 - \operatorname{cosec}^2} = \frac{-2\operatorname{cosec}^2}{1 - \operatorname{cosec}^2} = \frac{2\operatorname{cosec}^2}{\operatorname{cosec}^2 - 1}$
$= \frac{2\operatorname{cosec}^2}{(1+\cot^2)-1} = \frac{2\operatorname{cosec}^2}{\cot^2} = 2\operatorname{cosec}^2\tan^2 = \frac{2}{\sin^2}\times\frac{\sin^2}{\cos^2}$
$= \frac{2}{\cos^2} = 2\sec^2 = \text{RHS}$

(e) LHS $= \frac{\tan}{\sec - 1} - \frac{\sec - 1}{\tan} = \frac{\tan^2 - (\sec - 1)^2}{(\sec - 1)\tan}$
$= \frac{\tan^2 - (\sec^2 - 2\sec + 1)}{(\sec - 1)\tan} = \frac{\tan^2 - \sec^2 + 2\sec - 1}{(\sec - 1)\tan} =$
$= \frac{\tan^2 - (1 + \tan^2) + 2\sec - 1}{(\sec - 1)\tan} = \frac{2\sec - 2}{(\sec - 1)\tan}$
$= \frac{2(\sec - 1)}{(\sec - 1)\tan} = \frac{2}{\tan} = \text{RHS}$

**Question 3**

Prove the validity of each of the following trigonometric identities.

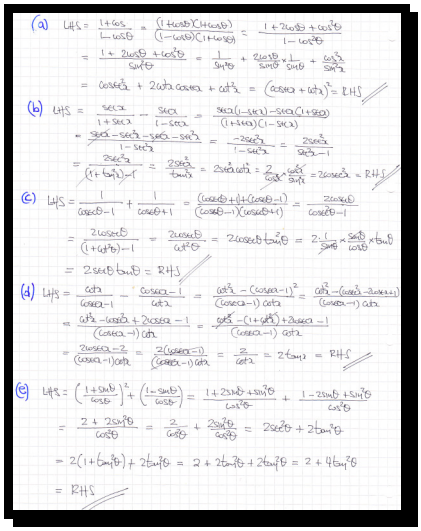
**a)** $\dfrac{1+\cos\theta}{1-\cos\theta} \equiv \left(\operatorname{cosec}\theta+\cot\theta\right)^2$

**b)** $\dfrac{\sec x}{1+\sec x}-\dfrac{\sec x}{1-\sec x} \equiv 2\operatorname{cosec}^2 x$

**c)** $\dfrac{1}{\operatorname{cosec}\theta-1}+\dfrac{1}{\operatorname{cosec}\theta+1} \equiv 2\sec\theta\tan\theta$

**d)** $\dfrac{\cot x}{\operatorname{cosec} x-1}-\dfrac{\operatorname{cosec} x-1}{\cot x} \equiv 2\tan x$

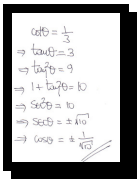
**e)** $\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2+\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2 \equiv 2+4\tan^2\theta$



**Question 4**

If $\cot\theta = \frac{1}{3}$, show clearly that $\cos\theta = \pm\frac{\sqrt{10}}{10}$.

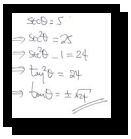
proof




**Question 5**

If $\sec\theta = 5$, show clearly that $\tan\theta = \pm\sqrt{24}$.

proof



**Question 6**

Solve each of the following equations.

**a)** $2\tan^2\theta = 11\sec\theta - 7, \qquad 0 \le \theta < 360°$

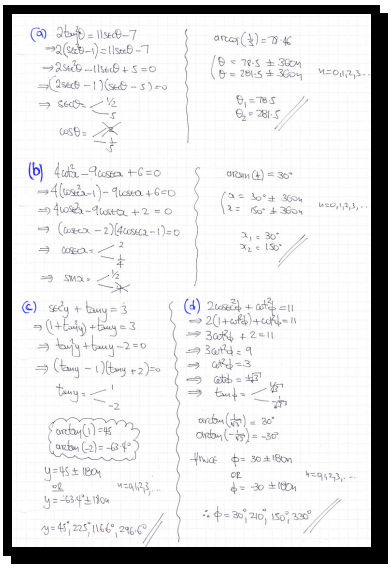
**b)** $4\cot^2 x - 9\operatorname{cosec} x + 6 = 0, \qquad 0 \le x < 360°$

**c)** $\sec^2 y + \tan y = 3, \qquad 0 \le y < 360°$

**d)** $2\operatorname{cosec}^2\varphi + \cot^2\varphi = 11, \quad 0 \le \varphi < 360°$

$\boxed{\theta = 78.5°,\ 281.5°}$, $\boxed{x = 30°,\ 150°}$, $\boxed{y = 45°,\ 225°\quad y \approx 116.6°,\ 296.6°}$,

$\boxed{\varphi = 30°,\ 150°,\ 210°,\ 330°}$



**Question 7**

Solve each of the following equations.

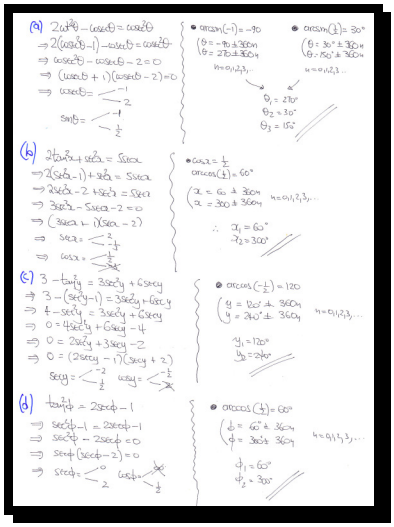
**a)** $2\cot^2\theta - \operatorname{cosec}\theta = \operatorname{cosec}^2\theta, \quad 0 \le \theta < 360°$

**b)** $2\tan^2 x + \sec^2 x = 5\sec x, \quad 0 \le x < 360°$

**c)** $3 - \tan^2 y = 3\sec^2 y + 6\sec y, \quad 0 \le y < 360°$

**d)** $\tan^2\varphi = 2\sec\varphi - 1, \quad 0 \le \varphi < 360°$

$\boxed{\theta = 30°,\ 150°,\ 270°}$, $\boxed{x = 60°,\ 300°}$, $\boxed{y = 120°,\ 240°}$, $\boxed{\varphi = 60°,\ 300°}$



**Question 8**

Solve each of the following equations.

**a)** $2\cot^2\theta+6=9\operatorname{cosec}\theta, \qquad 0\le\theta<360°$

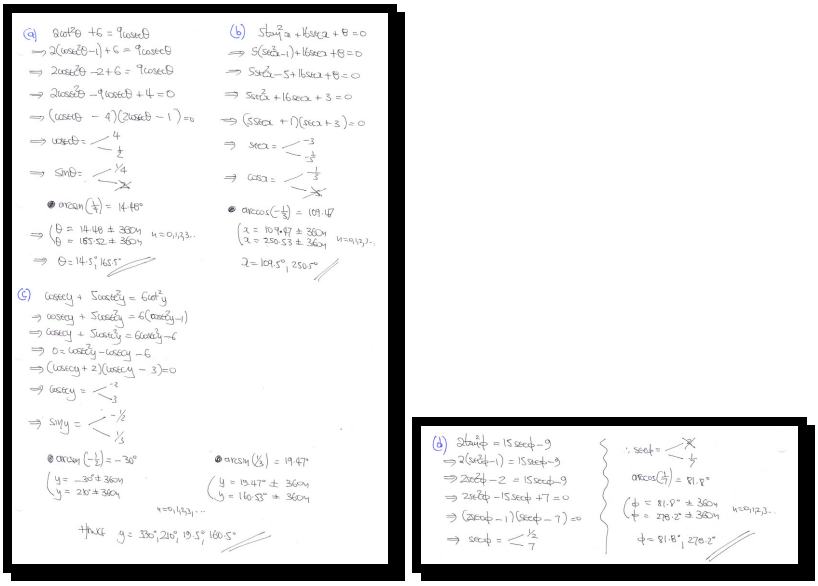
**b)** $5\tan^2 x+16\sec x+8=0, \qquad 0\le x<360°$

**c)** $\operatorname{cosec} y+5\operatorname{cosec}^2 y=6\cot^2 y, \qquad 0\le y<360°$

**d)** $2\tan^2\varphi=15\sec\varphi-9, \quad 0\le\varphi<360°$

$\boxed{\theta\approx 14.5°,\ 165.5°}$, $\boxed{x\approx 109.5°,\ 250.5°}$, $\boxed{y\approx 19.5°,\ 160.5°\quad y=210°,\ 330°}$,

$\boxed{\varphi\approx 81.8°,\ 278.2°}$



**Question 9**

Solve each of the following equations.

**a)** $4\cot^2\theta = 1 + \operatorname{cosec}\theta, \qquad 0 \le \theta < 360°$

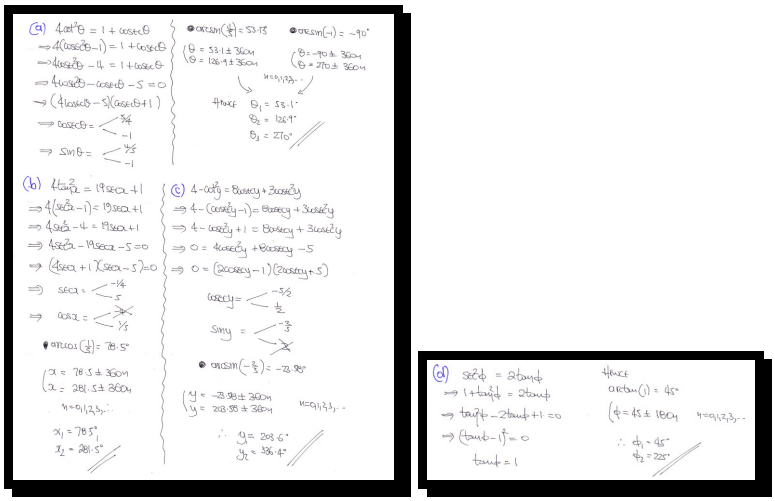
**b)** $4\tan^2 x = 19\sec x + 1, \qquad 0 \le x < 360°$

**c)** $4 - \cot^2 y = 8\operatorname{cosec} y + 3\operatorname{cosec}^2 y, \qquad 0 \le y < 360°$

**d)** $\sec^2\varphi = 2\tan\varphi, \quad 0 \le \varphi < 360°$

$\boxed{\theta \approx 53.1°,\ 126.9° \quad \theta = 270°}$, $\boxed{x \approx 78.5°,\ 281.5°}$, $\boxed{y \approx 203.6°,\ 336.4°}$,

$\boxed{\varphi = 45°,\ 225°}$



**Question 10**

Solve each of the following equations.

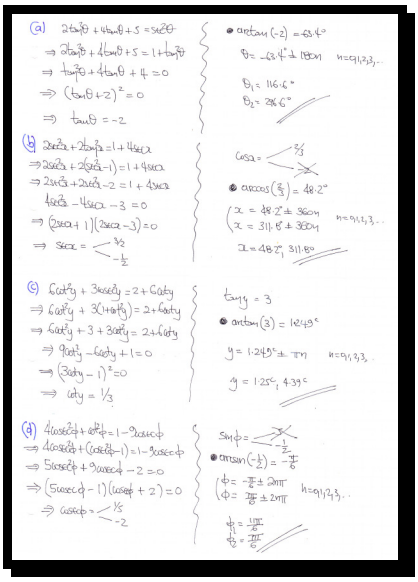
**a)** $2\tan^2\theta+4\tan\theta+5=\sec^2\theta\,,\qquad 0\le\theta<360°$

**b)** $2\sec^2 x+2\tan^2 x=1+4\sec x\,,\qquad 0\le x<360°$

**c)** $6\cot^2 y+3\operatorname{cosec}^2 y=2+6\cot y\,,\qquad 0\le y<2\pi$

**d)** $4\operatorname{cosec}^2\varphi+\cot^2\varphi=1-9\operatorname{cosec}\varphi\,,\quad 0\le\varphi<2\pi$

$\boxed{\theta\approx 116.6°,\ 296.6°}$, $\boxed{x\approx 48.2°,\ 311.8°}$, $\boxed{y\approx 1.25^{\text{c}},\ 4.39^{\text{c}}}$, $\boxed{\varphi=\frac{7\pi}{6},\ \frac{11\pi}{6}}$



**Question 11**

Solve each of the following equations.

**a)** $10\sec^2\theta = 11\tan\theta + 16, \qquad 0 \le \theta < 360°$

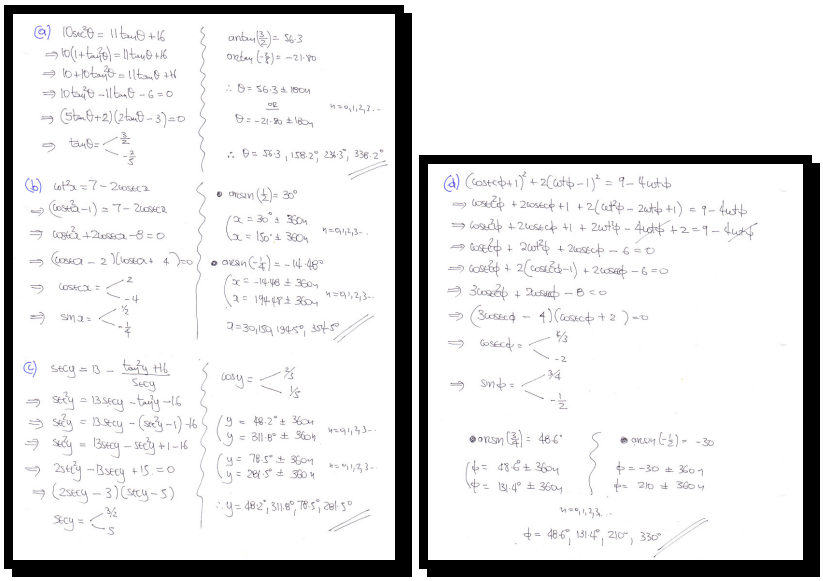
**b)** $\cot^2 x = 7 - 2\operatorname{cosec} x, \qquad 0 \le x < 360°$

**c)** $\sec y = 13 - \dfrac{\tan^2 y + 16}{\sec y}, \qquad 0 \le y < 360°$

**d)** $(\operatorname{cosec}\varphi + 1)^2 + 2(\cot\varphi - 1)^2 = 9 - 4\cot\varphi, \quad 0 \le \varphi < 360°$

$\boxed{\theta \approx 56.3°,\ 158.2°,\ 236.3°,\ 338.2°}$, $\boxed{x = 30°,\ 150°\quad x \approx 194.5°,\ 344.5°}$,

$\boxed{y \approx 48.2°,\ 78.5°,\ 281.5°,\ 311.8°}$, $\boxed{\varphi \approx 48.6°,\ 131.4°\quad \varphi = 210°,\ 330°}$



**Question 12**

Solve each of the following equations.

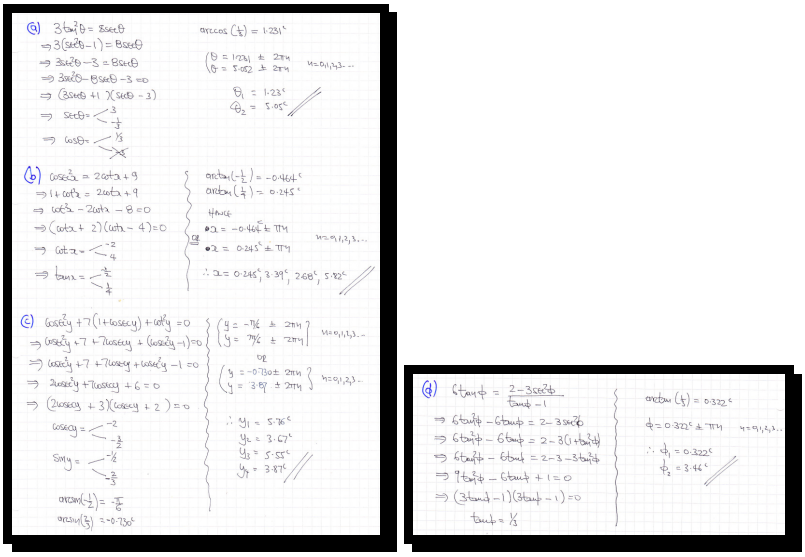
**a)** $3\tan^2\theta = 8\sec\theta, \qquad 0 \le \theta < 2\pi$

**b)** $\operatorname{cosec}^2 x = 2\cot x + 9, \qquad 0 \le x < 2\pi$

**c)** $\operatorname{cosec}^2 y + 7(1+\operatorname{cosec} y) + \cot^2 y = 0, \qquad 0 \le y < 2\pi$

**d)** $6\tan\varphi = \dfrac{2-3\sec^2\varphi}{\tan\varphi - 1}, \quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 1.23^{\text{c}},\ 5.05^{\text{c}}}$, $\boxed{x \approx 0.245^{\text{c}},\ 2.68^{\text{c}},\ 3.39^{\text{c}},\ 5.82^{\text{c}}}$,
$\boxed{y \approx 3.67^{\text{c}},\ 3.87^{\text{c}},\ 5.55^{\text{c}},\ 5.76^{\text{c}}}$, $\boxed{\varphi \approx 0.322^{\text{c}},\ 3.46^{\text{c}}}$



**Question 13**

Solve each of the following equations.

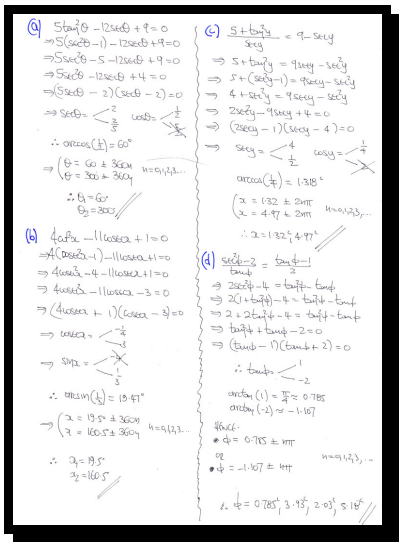
**a)** $5\tan^2\theta - 12\sec\theta + 9 = 0, \qquad 0 \le \theta < 360°$

**b)** $4\cot^2 x - 11\operatorname{cosec} x + 1 = 0, \qquad 0 \le x < 360°$

**c)** $\dfrac{5+\tan^2 y}{\sec y} = 9 - \sec y, \qquad 0 \le y < 2\pi$

**d)** $\dfrac{\sec^2\varphi - 2}{\tan\varphi} = \dfrac{\tan\varphi - 1}{2}, \qquad 0 \le \varphi < 2\pi$

$\boxed{\theta = 60°,\ 300°}$, $\boxed{x \approx 19.5°,\ 160.5°}$, $\boxed{y \approx 1.32^{\text{c}},\ 4.97^{\text{c}}}$,
$\boxed{\varphi \approx 0.785^{\text{c}},\ 2.03^{\text{c}},\ 3.93^{\text{c}},\ 5.18^{\text{c}}}$



**Question 14**

Solve each of the following equations.

**a)** $\sec\theta = \dfrac{1-\tan^2\theta}{4\sec\theta - 9}, \qquad 0 \le \theta < 2\pi$

**b)** $\dfrac{\sec^2 x + 8}{4 - \tan x} = 3\tan x, \qquad 0 \le x < 2\pi$

**c)** $\dfrac{1 - 2\operatorname{cosec}^2 y}{2\cot y} - 2 = \cot y, \qquad 0 \le y < 2\pi$

**d)** $\dfrac{2\cot^2\varphi + 5}{\operatorname{cosec}\varphi} + 2\operatorname{cosec}\varphi = 13, \quad 0 \le \varphi < 2\pi$

$\boxed{\theta = \dfrac{\pi}{3},\ \dfrac{5\pi}{3}}$, $\boxed{x \approx 0.983^{\text{c}},\ 4.12^{\text{c}}}$, $\boxed{y \approx 2.03^{\text{c}},\ 5.18^{\text{c}}}$, $\boxed{\varphi \approx 0.340^{\text{c}},\ 2.80^{\text{c}}}$

