TRIGONOMETRY

THE COMPOUND ANGLE IDENTITIES

Question 1

Prove the validity of each of the following trigonometric identities.

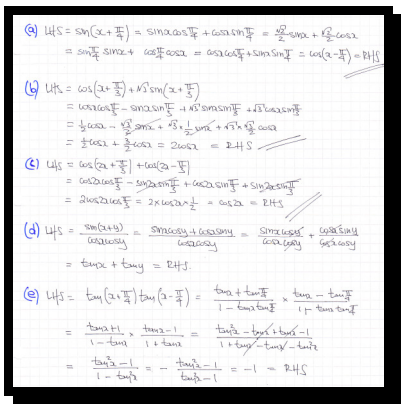
a) $\sin\left(x+\frac{\pi}{4}\right)\equiv\cos\left(x-\frac{\pi}{4}\right)$

b) $\cos\left(x+\frac{\pi}{3}\right)+\sqrt{3}\sin\left(x+\frac{\pi}{3}\right)\equiv 2\cos x$

c) $\cos\left(2x+\frac{\pi}{3}\right)+\cos\left(2x-\frac{\pi}{3}\right)\equiv\cos 2x$

d) $\frac{\sin(x+y)}{\cos x\cos y}\equiv\tan x+\tan y$

e) $\tan\left(x+\frac{\pi}{4}\right)\tan\left(x-\frac{\pi}{4}\right)\equiv -1$



Question 2

Prove the validity of each of the following trigonometric identities.

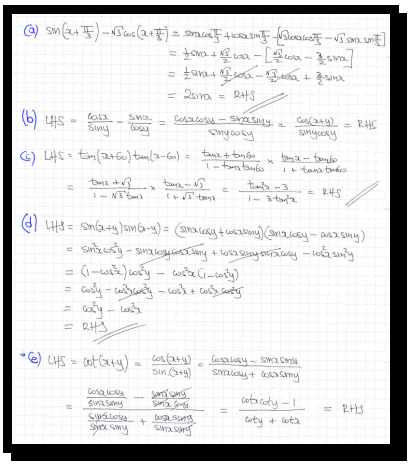
a) $\sin\left(x+\frac{\pi}{3}\right)-\sqrt{3}\cos\left(x+\frac{\pi}{3}\right)\equiv 2\sin x$

b) $\frac{\cos x}{\sin y}-\frac{\sin x}{\cos y}\equiv\frac{\cos\left(x+y\right)}{\sin y\cos y}$

c) $\tan\left(x+60°\right)\tan\left(x-60°\right)\equiv\frac{\tan^2 x-3}{1-3\tan^2 x}$

d) $\sin\left(x+y\right)\sin\left(x-y\right)\equiv\cos^2 y-\cos^2 x$

e) $\cot\left(x+y\right)\equiv\frac{\cot x\cot y-1}{\cot x+\cot y}$



Question 3

Prove the validity of each of the following trigonometric identities.

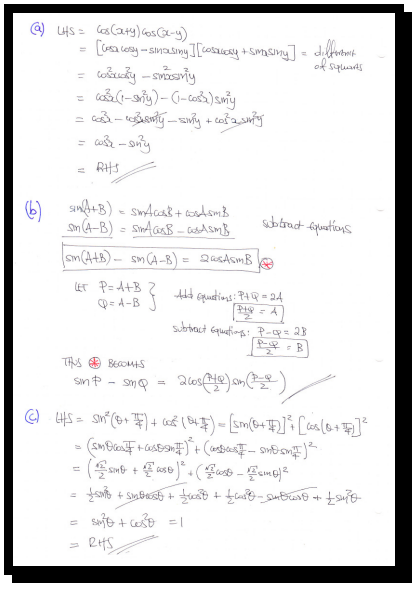
a) $\cos(x+y)\cos(x-y) \equiv \cos^2 x - \sin^2 y$

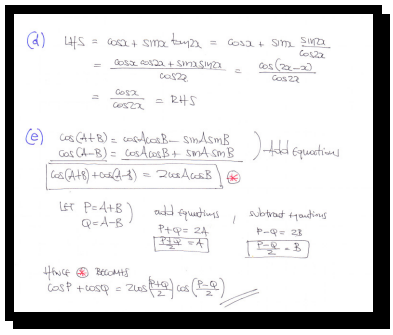
b) $\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

c) $\sin^2\left(\theta+\frac{\pi}{4}\right)+\sin^2\left(\theta-\frac{\pi}{4}\right) \equiv 1$

d) $\cos x + \sin x \tan 2x \equiv \frac{\cos x}{\cos 2x}$

e) $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$



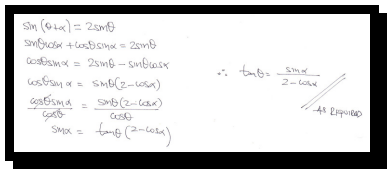


Question 4

If $\sin(\theta+\alpha)=2\sin\theta$, show clearly that

$$\tan\theta=\frac{\sin\alpha}{2-\cos\alpha}.$$

proof

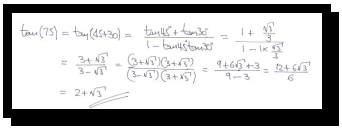


Question 5

By expanding $\tan(\theta+45°)$ with a suitable value for θ, show clearly that

$$\tan 75°=2+\sqrt{3}.$$

proof

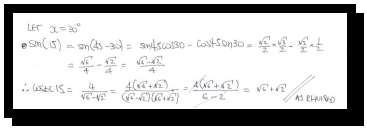


Question 6

By expanding $\sin(45°-x)$ with a suitable value for x, show clearly that

$$\operatorname{cosec}15°=\sqrt{2}+\sqrt{6}.$$

proof

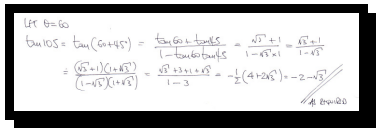


Question 7

By expanding $\tan(\theta+45°)$ with a suitable value for θ, show clearly that

$$\tan 105° = -2-\sqrt{3}\,.$$

proof

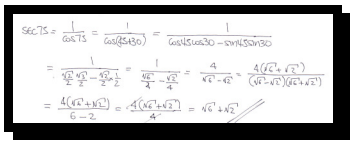


Question 8

By expanding $\cos(y+45°)$ with a suitable value for y, show clearly that

$$\sec 75° = \sqrt{2}+\sqrt{6}\,.$$

proof

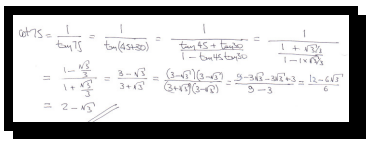


Question 9

By considering the expansion of $\tan(A+B)$ with suitable values for A and B, show clearly that

$$\cot 75° = 2-\sqrt{3}\,.$$

proof

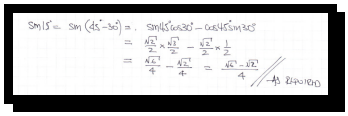


Question 10

Show clearly, by using the compound angle identities, that

$$\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

proof

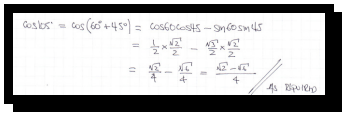


Question 11

Show clearly, by using the compound angle identities, that

$$\cos 105^\circ = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

proof

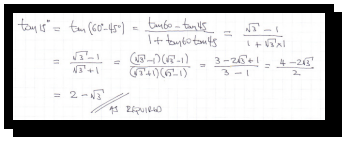


Question 12

Show clearly, by using the compound angle identities, that

$$\tan 15^\circ = 2-\sqrt{3}\,.$$

proof



Question 13

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B .$$

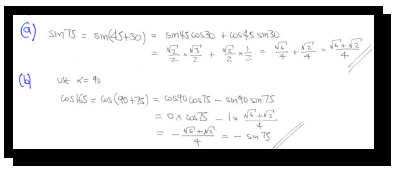
a) Use the above trigonometric identity with suitable values for A and B, to show that

$$\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} .$$

b) Hence by using the trigonometric expansion of $\cos(75^\circ + \alpha)$ with a suitable value for α, show clearly that

$$\cos 165^\circ = -\sin 75^\circ .$$

proof



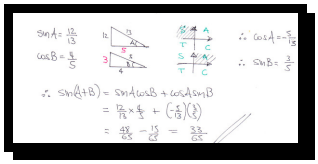
Question 14

$$\sin A = \frac{12}{13} \quad \text{and} \quad \cos B = \frac{4}{5} .$$

If A is obtuse and B is acute, show clearly that

$$\sin(A+B) = \frac{33}{65} .$$

proof



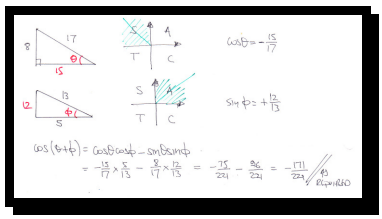
Question 15

$$\sin\theta=\frac{8}{17}\quad\text{and}\quad\cos\varphi=\frac{5}{13}.$$

If θ is obtuse and φ is acute, show clearly that

$$\cos(\theta+\varphi)=-\frac{171}{221}.$$

proof



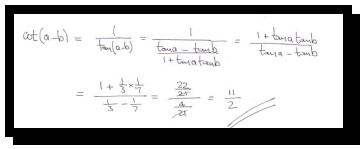
Question 16

The constants a and b are such so that

$$\tan a=\frac{1}{3}\quad\text{and}\quad\tan b=\frac{1}{7}.$$

Determine the exact value of $\cot(a-b)$, showing all the steps in the workings.

$$\boxed{\cot(a-b)=\frac{11}{2}}$$



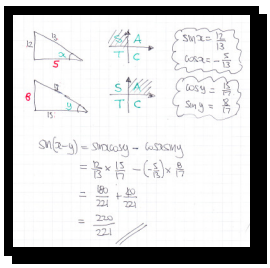
Question 17

$$\sin x = \frac{12}{13} \text{ and } \cos y = \frac{15}{17}.$$

If x is obtuse and y is acute, show clearly that

$$\sin(x-y) = \frac{220}{221}.$$

proof



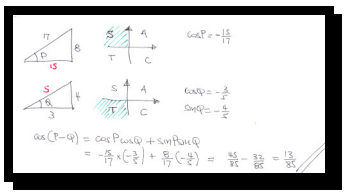
Question 18

$$\sin P = \frac{8}{17} \text{ and } \tan Q = \frac{4}{3}.$$

If P is obtuse and Q is reflex, show clearly that

$$\cos(P-Q) = \frac{13}{85}.$$

proof



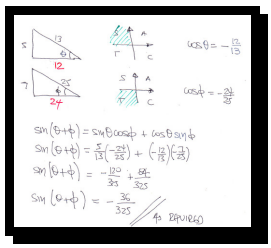
Question 19

$$\sin\theta=\frac{5}{13}\quad\text{and}\quad\sin\varphi=-\frac{7}{25}.$$

If θ is obtuse and φ is such so that $180°<\varphi<270°$, show clearly that

$$\sin\left(\theta+\varphi\right)=-\frac{36}{325}.$$

proof

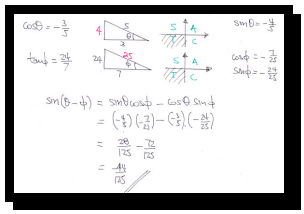


Question 20

$$\cos\theta=-\frac{3}{5}\quad\text{and}\quad\tan\varphi=\frac{24}{7}.$$

If θ is reflex, and φ is also reflex, show clearly that

$$\sin\left(\theta-\varphi\right)=-\frac{44}{125}.$$



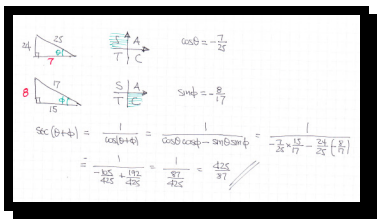
Question 21

$$\sin\theta = \frac{24}{25} \text{ and } \cos\varphi = \frac{15}{17}.$$

If θ is obtuse and φ is reflex, show clearly that

$$\sec(\theta+\varphi) = \frac{425}{87}.$$

proof



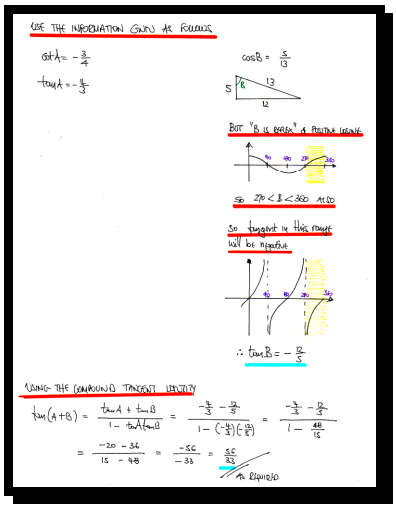
Question 22

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show clearly that

$$\tan\left(A+B\right) = \frac{56}{33}.$$

proof



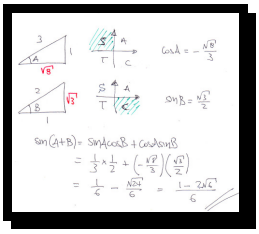
Question 23

$$\sin A = \frac{1}{3} \quad \text{and} \quad \cos B = \frac{1}{2}.$$

If A is obtuse and B is reflex, show clearly that

$$\sin\left(A+B\right) = \frac{1-2\sqrt{6}}{6}.$$

proof



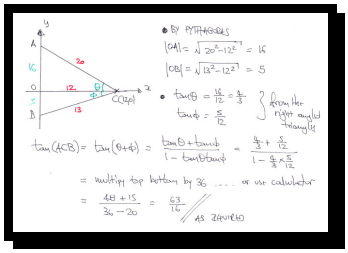
Question 24

The point A lies on the y axis above the origin O and the point B lies on the y axis below the origin O.

The point $C\left(12,0\right)$ is at a distance of 20 units from A and at a distance of 13 units from B.

By considering the tangent ratios of $\measuredangle OCA$ and $\measuredangle OCB$, show that the tangent of the angle ACB is exactly $\dfrac{63}{16}$.

proof



Question 25

Solve each of the following trigonometric equations.

a) $\cos(\theta+30^\circ)=\sin\theta\,, \qquad 0\le\theta<360^\circ$

b) $3\cos(x+30^\circ)=\sin(x-60^\circ)\,, \qquad 0\le x<360^\circ$

c) $\sin(y-30^\circ)=\sin(y+45^\circ)\,, \qquad 0\le y<360^\circ$

d) $\sin(\varphi+30^\circ)=\cos(\varphi-45^\circ)\,, \qquad 0\le\varphi<360^\circ$

e) $\cos(\alpha-60^\circ)=\cos(\alpha-45^\circ)\,, \qquad 0\le\alpha<360^\circ$

$\boxed{\theta=30^\circ,\ 210^\circ}$, $\boxed{x=60^\circ,\ 240^\circ}$, $\boxed{y=82.5^\circ,\ 262.5^\circ}$, $\boxed{\varphi=52.5^\circ,\ 232.5^\circ}$,
$\boxed{\alpha=52.5^\circ,\ 232.5^\circ}$



Question 26

Solve each of the following trigonometric equations.

a) $\sin(\theta-45^\circ)=\sin\theta, \qquad 0\le\theta<360^\circ$

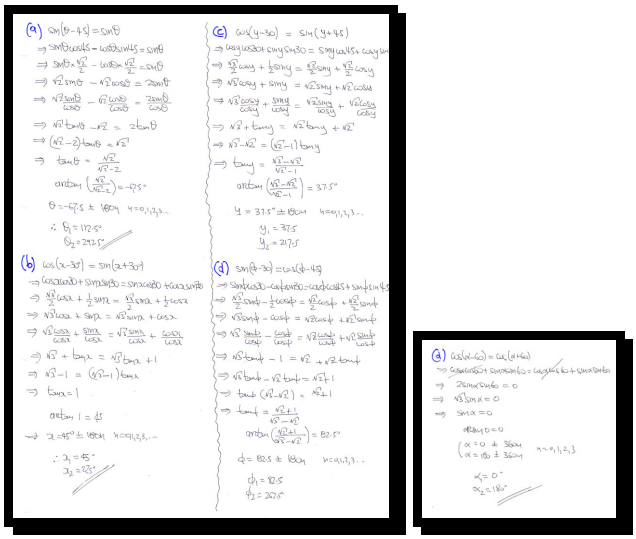
b) $\cos(x-30^\circ)=\sin(x+30^\circ), \qquad 0\le x<360^\circ$

c) $\cos(y-30^\circ)=\sin(y+45^\circ), \qquad 0\le y<360^\circ$

d) $\sin(\varphi-30^\circ)=\cos(\varphi-45^\circ), \qquad 0\le\varphi<360^\circ$

e) $\cos(\alpha-60^\circ)=\cos(\alpha+60^\circ), \qquad 0\le\alpha<360^\circ$

$\boxed{\theta=112.5^\circ,\ 292.5^\circ}$, $\boxed{x=45^\circ,\ 225^\circ}$, $\boxed{y=37.5^\circ,\ 217.5^\circ}$, $\boxed{\varphi=82.5^\circ,\ 262.5^\circ}$, $\boxed{\alpha=0^\circ,\ 180^\circ}$



Question 27

Solve each of the following trigonometric equations.

a) $\sin\left(\theta+\frac{\pi}{4}\right)=\sin\theta\,, \qquad 0\le\theta<2\pi$

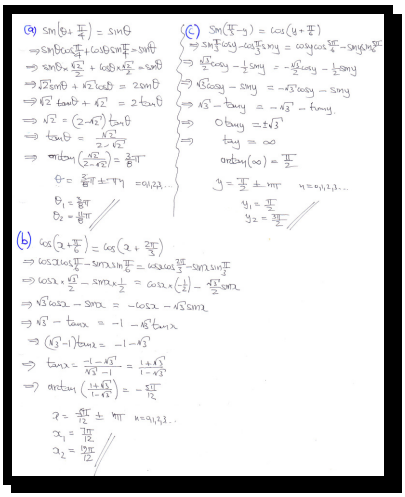
b) $\cos\left(x+\frac{\pi}{6}\right)=\cos\left(x+\frac{2\pi}{3}\right), \qquad 0\le\theta<2\pi$

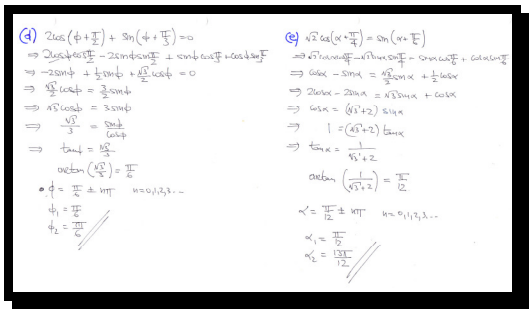
c) $\sin\left(\frac{\pi}{3}-y\right)=\cos\left(y+\frac{5\pi}{6}\right), \qquad 0\le y<2\pi$ (very hard)

d) $2\cos\left(\varphi+\frac{\pi}{2}\right)+\sin\left(\varphi+\frac{\pi}{3}\right)=0\,, \qquad 0\le\varphi<2\pi$

e) $\sqrt{2}\cos\left(\alpha+\frac{\pi}{4}\right)=\sin\left(\alpha+\frac{\pi}{6}\right), \qquad 0\le\alpha<2\pi$

$$\boxed{\theta=\frac{3\pi}{8},\ \frac{11\pi}{8}}\,,\quad \boxed{x=\frac{7\pi}{12},\ \frac{19\pi}{12}}\,,\quad \boxed{y=\frac{\pi}{2},\ \frac{3\pi}{2}}\,,\quad \boxed{\varphi=\frac{\pi}{6},\ \frac{7\pi}{6}}\,,\quad \boxed{\alpha=\frac{\pi}{12},\ \frac{13\pi}{12}}$$





Question 28

Solve each of the following trigonometric equations.

a) $\sin(\theta-20°)=\sin(\theta+60°), \qquad 0\le\theta<360°$

b) $\cos(x-35°)=\cos(x-55°), \qquad 0\le x<360°$

c) $\sin(y-48°)=\cos(y+12°), \qquad 0\le y<360°$

d) $\sin(\varphi+72°)=\cos(\varphi-38°), \qquad 0\le\varphi<360°$

e) $\cos(\alpha-36°)=\cos(\alpha-72°), \qquad 0\le\alpha<360°$

$\boxed{\theta=70°,\ 250°}$, $\boxed{x=45°,\ 225°}$, $\boxed{y=63°,\ 243°}$, $\boxed{\varphi=28°,\ 208°}$, $\boxed{\alpha=54°,\ 234°}$

