93 EXAM QUESTIONS ON INVERSE TRIGONOMETRIC FUNCTIONS

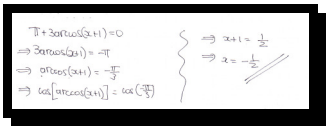
22 BASIC QUESTIONS

Question 1 (+)**

Solve the following trigonometric equation

$$\pi + 3\arccos\left(x+1\right) = 0 .$$

$$\boxed{x = -\frac{1}{2}}$$



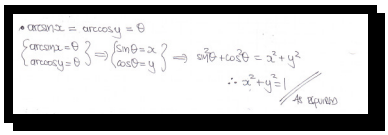
Question 2 (***)

It is given that $\arcsin x = \arccos y$.

Show, by a clear method, that

$$x^2 + y^2 = 1 .$$

$$\boxed{}\text{, } \boxed{\text{proof}}$$

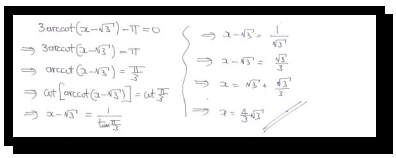


Question 3 (***)

Solve the following trigonometric equation

$$3\operatorname{arccot}\left(x-\sqrt{3}\right) - \pi = 0 .$$

$$\boxed{x = \frac{4}{3}\sqrt{3}}$$



Question 4 (*+)**

A curve C is defined by the equation

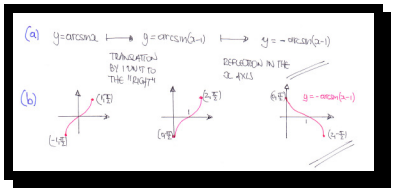
$$y=-\arcsin(x-1),\ 0\le x\le 2.$$

a) Describe the 2 geometric transformations that map the graph of $\arcsin x$ onto the graph of C.

b) Sketch the graph of C.

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes and the coordinates of the endpoints of C.

[ ], [translation by 1 unit to the right, followed by reflection in the x axis]



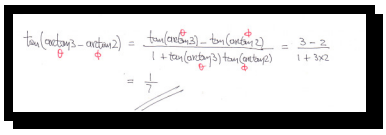
Question 5 (*+)**

Simplify, showing all steps in the calculation, the following expression

$$\tan(\arctan 3-\arctan 2),$$

giving the final answer as an exact fraction.

$$\frac{1}{7}$$

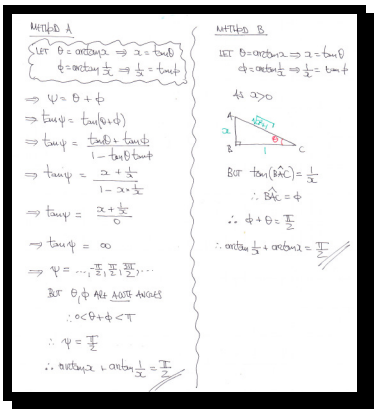


Question 6 (*+)**

Show clearly that if $x>0$

$$\arctan x+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}.$$

proof



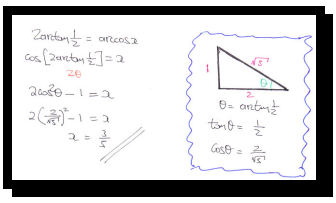
Question 7 (*+)**

Solve the equation

$$2\arctan\left(\frac{1}{2}\right)=\arccos x\,,$$

showing clearly all the workings.

$$x=\frac{3}{5}$$



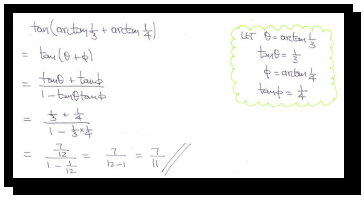
Question 8 (*+)**

Simplify, showing all steps in the calculation, the expression

$$\tan\left[\arctan\frac{1}{3}+\arctan\frac{1}{4}\right],$$

giving the final answer as an exact fraction.

$$\boxed{\frac{7}{11}}$$

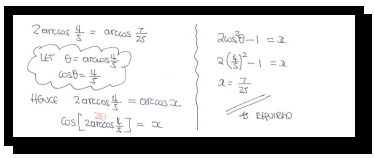


Question 9 (*+)**

Show clearly that

$$2\arccos\left(\frac{4}{5}\right)=\arccos\left(\frac{7}{25}\right).$$

$$\boxed{\text{proof}}$$

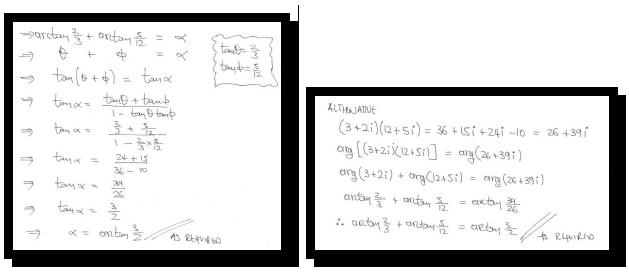


Question 10 (*+)**

Show clearly that

$$\arctan\frac{2}{3}+\arctan\frac{5}{12}=\arctan\frac{3}{2}.$$

proof

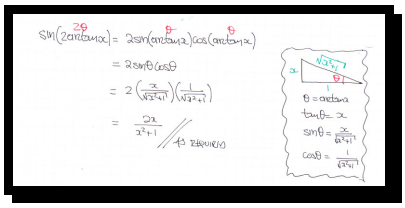


Question 11 (*+)**

Show clearly that

$$\sin\left(2\arctan x\right)=\frac{2x}{x^2+1}.$$

proof

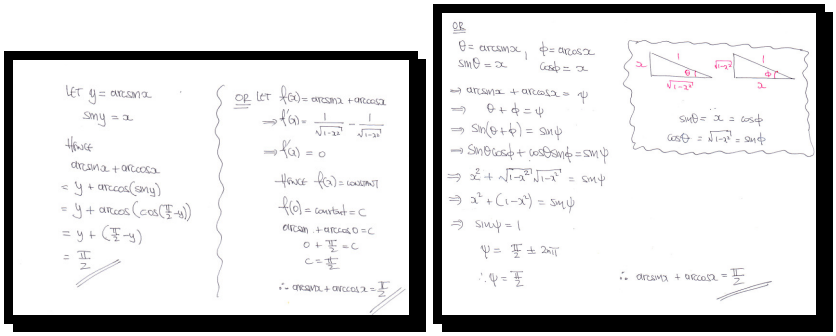


Question 12 (*+)**

Prove the trigonometric identity

$$\arcsin x+\arccos x=\frac{\pi}{2}.$$

proof

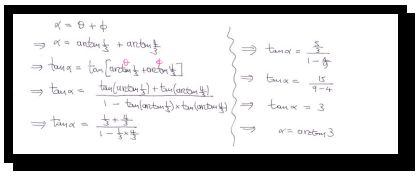


Question 13 (*+)**

Show clearly that

$$\arctan\frac{1}{3}+\arctan\frac{4}{3}=\arctan 3.$$

proof

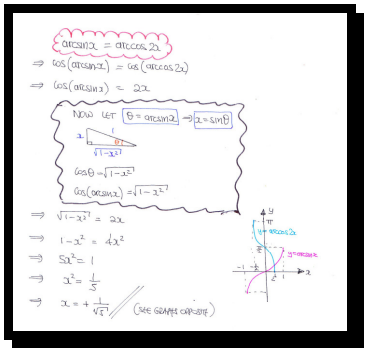


Question 14 (*+)**

Solve the trigonometric equation

$$\arcsin x = \arccos 2x .$$

$$\boxed{x = \frac{1}{\sqrt{5}}}$$

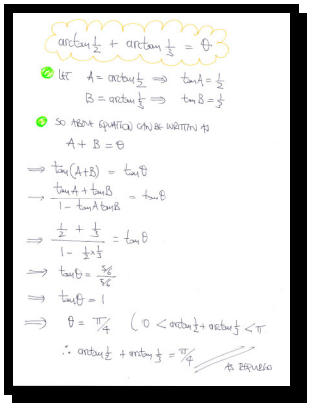


Question 15 (*+)**

Using a detailed method, show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{1}{4}\pi .$$

$$\boxed{\text{proof}}$$

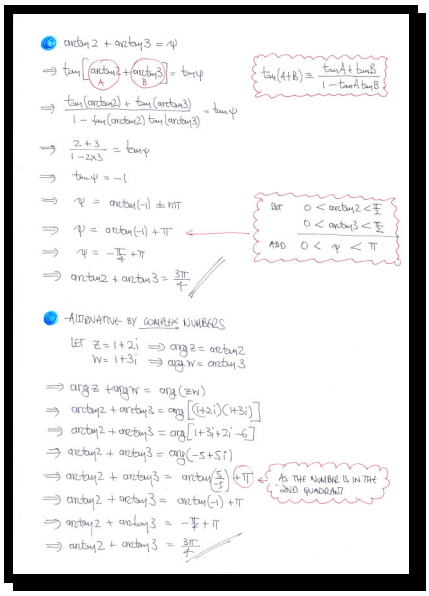


Question 16 (*+)**

Show, by detailed workings, that

$$\arctan 2+\arctan 3=\frac{3\pi}{4}.$$

proof

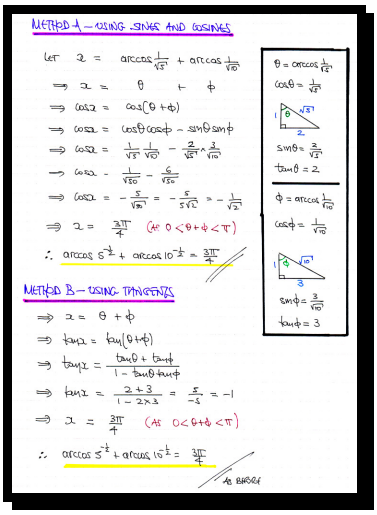


Question 17 **(***+)**

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right)+\arccos\left(10^{-\frac{1}{2}}\right)=\frac{3\pi}{4}.$$

$\boxed{}$, $\boxed{\text{proof}}$

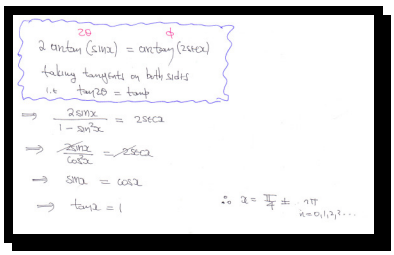


Question 18 **(***+)**

Find the general solution of the following trigonometric equation

$$2\arctan\left(\sin x\right)=\arctan\left(\sec x\right).$$

$\boxed{}$, $\boxed{x=\frac{\pi}{4}+k\pi,\quad k\in\mathbb{Z}}$

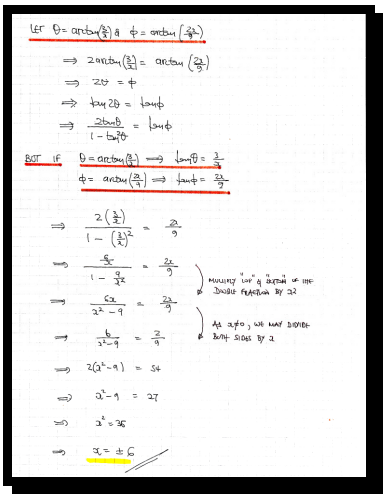


Question 19 (*+)**

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right)=\arctan\left(\frac{6x}{25}\right).$$

$\boxed{}$, $\boxed{x=\pm 6}$

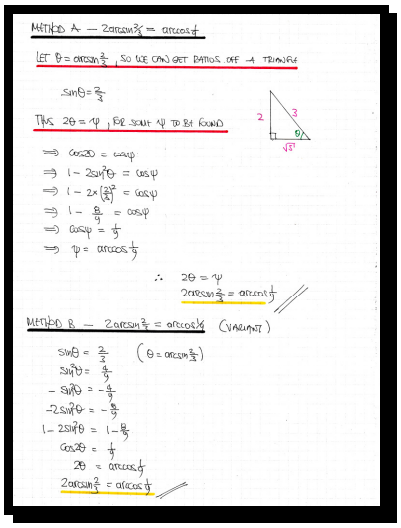


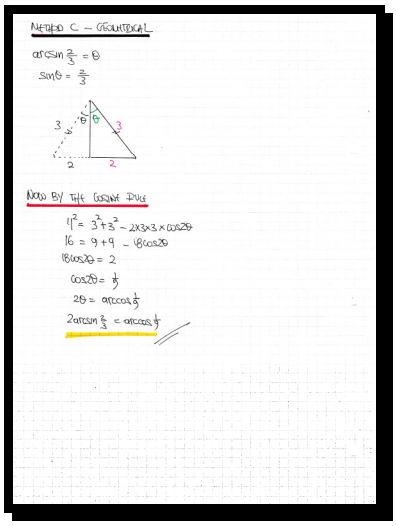
Question 20 (*+)**

Prove that

$$2\arcsin\left(\frac{2}{3}\right)=\arccos\left(\frac{1}{9}\right).$$

V, [ ], proof





Question 21 **(***+)**

Differentiate with respect to x

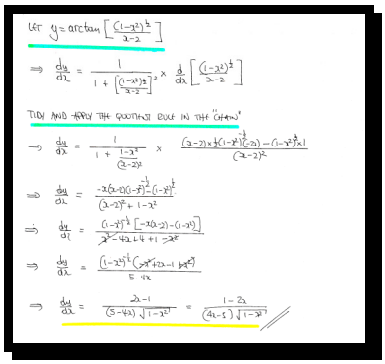
$$\arctan\left[\frac{\sqrt{1-x^2}}{x-2}\right].$$

Give a simplified answer in the form

$$\frac{A+Bx}{(Cx+D)\sqrt{1-x^2}},$$

where A, B, C and D are integers to be found.

$\boxed{}$, $\boxed{A=1}$, $\boxed{B=-2}$, $\boxed{C=4}$, $\boxed{D=-5}$



Question 22 **(***+)**

Differentiate with respect to x

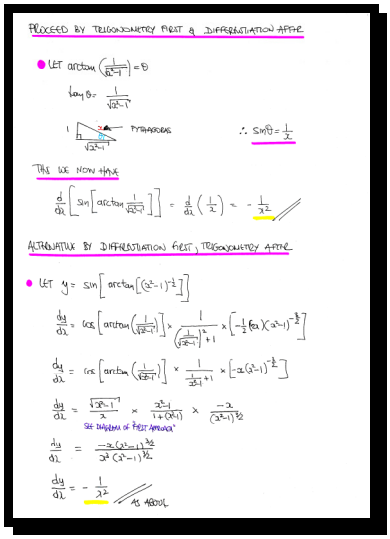
$$\sin\left[\arctan\left[\frac{1}{\sqrt{1-x^2}}\right]\right].$$

Give a simplified answer in the form

$$\frac{A}{x^n},$$

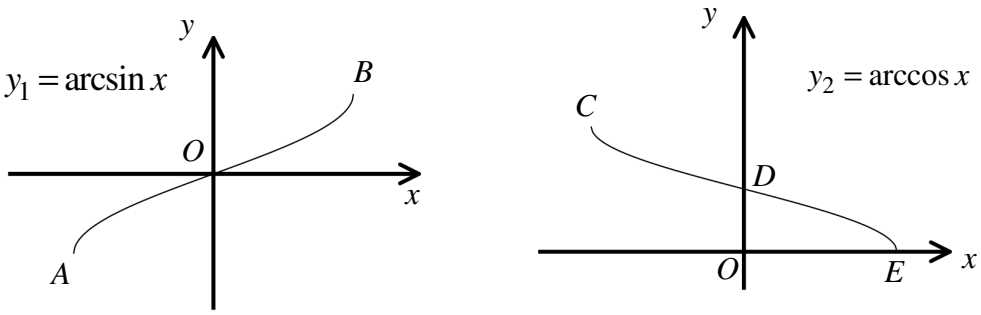
where A and n are integers to be found.

$\boxed{}$, $\boxed{A=-1}$, $\boxed{n=2}$



19 STANDARD QUESTIONS

Question 1 **(****)**



The diagrams above shows the graphs of $y_1 = \arcsin x$ and $y_2 = \arccos x$.

The graph of y_1 has endpoints at A and B.

The graph of y_2 has endpoints at C and E, and D is the point where the graph of y_2 crosses the y axis.

a) State the coordinates of A, B, C, D and E.

The graph of y_2 can be obtained from the graph of y_1 by a series of two geometric transformations which can be carried out in a specific order.

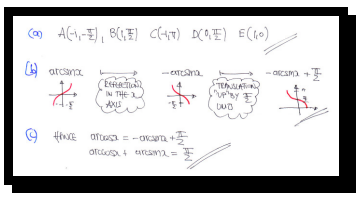
b) Describe the two geometric transformations.

c) Deduce using valid arguments that

$$\arcsin x + \arccos x = \text{constant},$$

stating the exact value of this constant.

$\boxed{A\left(-1,\frac{\pi}{2}\right)}$, $\boxed{B\left(1,\frac{\pi}{2}\right)}$, $\boxed{C(-1,\pi)}$, $\boxed{D\left(0,\frac{\pi}{2}\right)}$, $\boxed{E(1,0)}$, $\boxed{\text{constant} = \frac{\pi}{2}}$



Question 2 (****)

$$y = \arcsin x, \quad -1 \le x \le 1.$$

a) Show that

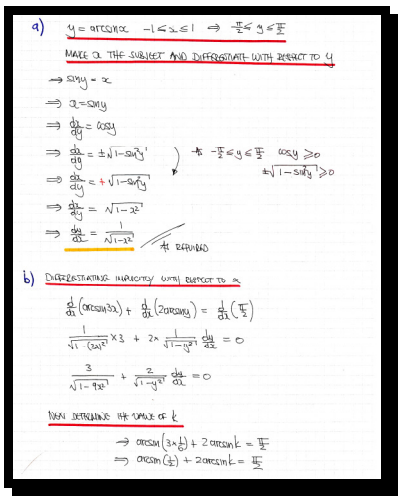
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

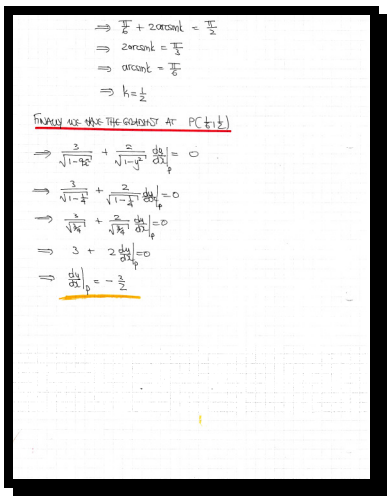
The point $P\left(\frac{1}{6}, k\right)$, where k is a constant lies on the curve with equation

$$\arcsin 3x + 2\arcsin y = \frac{\pi}{2}, \ |x| \le \frac{1}{3}, \ |y| \le 1.$$

b) Find the value of the gradient at P.

$$\boxed{}, \quad \boxed{-\frac{3}{2}}$$





Question 3 (****)

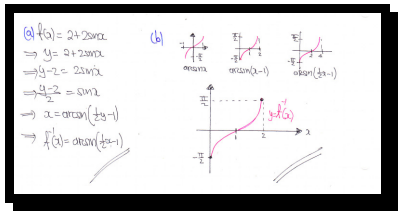
$$f(x)=2+2\sin x\,,\ -\pi\le x\le\pi\,.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include the coordinates of any points where the graph of $f^{-1}(x)$ meet the coordinate axes as well as the coordinates of its endpoints.

$$\boxed{f^{-1}(x)=\arcsin\left(\frac{1}{2}x-1\right)}$$

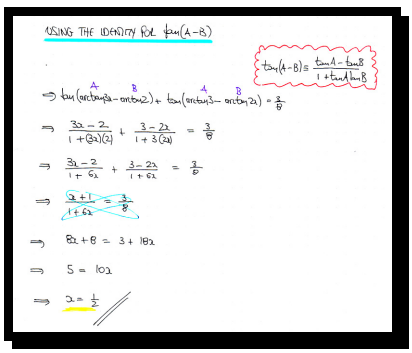


Question 4 (****)

Solve the following trigonometric equation

$$\tan\left(\arctan 3x-\arctan 2\right)+\tan\left(\arctan 3-\arctan 2x\right)=\frac{3}{8}\,.$$

$$\boxed{x=\frac{1}{2}}$$



Question 5 (****)

A curve has equation

$$y=\pi-\arccos\left(x+1\right),\ -2\le x\le 0\,.$$

a) Describe geometrically the 3 transformations that map the graph of

$$y=\arccos x\,,\ -1\le x\le 1\,,$$

onto the graph of

$$y=\pi-\arccos\left(x+1\right),\ -2\le x\le 0\,.$$

b) Sketch the graph of

$$y=\pi-\arccos\left(x+1\right),\ -2\le x\le 0\,.$$

The sketch must include the coordinates of any points where the graph meets the coordinate axes.

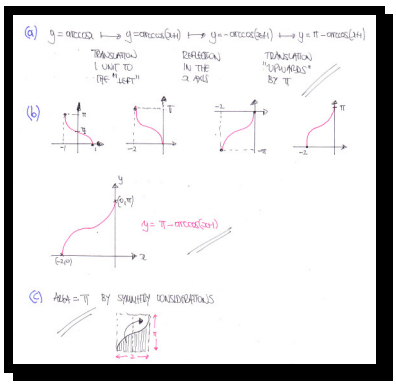
c) Use symmetry arguments to find the area of the finite region bounded by

$$y=\pi-\arccos\left(x+1\right),\ -2\le x\le 0\,,$$

and the coordinate axes.

$\boxed{}$, $\boxed{\text{translation by 1 unit to the right, followed by reflection in the } x \text{ axis}}$,

$\boxed{\text{area}=\pi}$

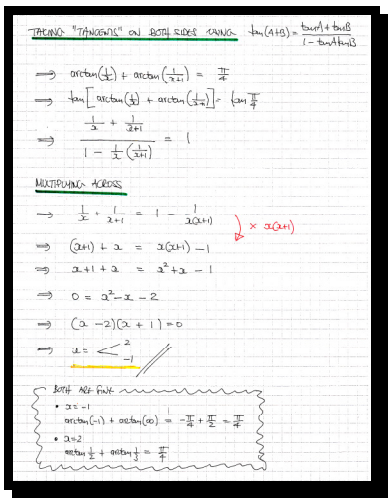


Question 6 (****)

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{4}.$$

$$\boxed{x=-1,\ 2}$$



Question 7 (****)

$$f(x)=-2+2\tan\left(\frac{1}{2}x\right),\ -\pi\le x\le\pi\,.$$

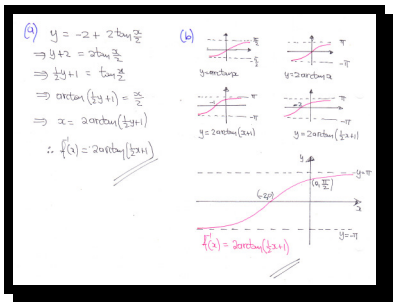
a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include …

- …the equations of the asymptotes of $f^{-1}(x)$
- …the coordinates of any points where the graph of $f^{-1}(x)$ meets the coordinate axes.

$$\boxed{f^{-1}(x)=2\arctan\left(\frac{1}{2}x+1\right)}$$

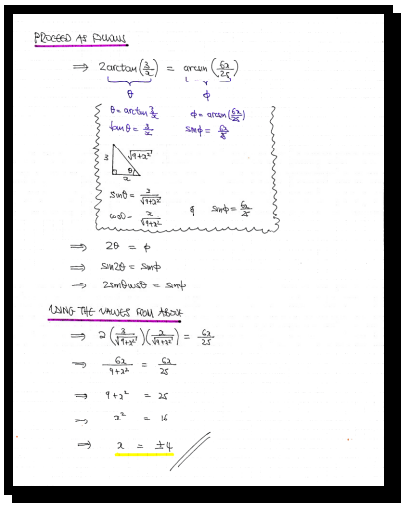


Question 8 (****)

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right)=\arcsin\left(\frac{6x}{25}\right).$$

$\boxed{}$, $\boxed{x=\pm 4}$



Question 9 **(****)**

The curves C_1 and C_2 have respective equations

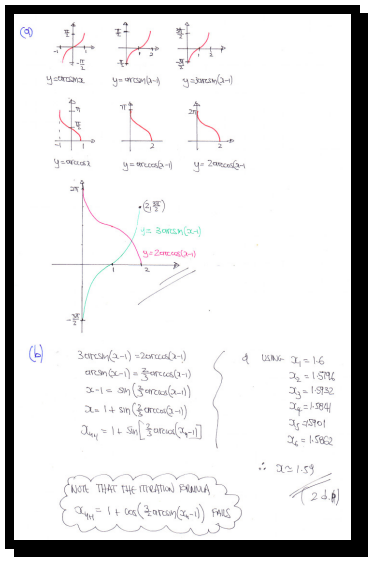
$$y_1 = 3\arcsin(x-1) \text{ and } y_1 = 2\arccos(x-1).$$

a) Sketch in the same diagram the graphs of C_1 and C_2.

The sketch must include the coordinates of any points where the graphs of C_1 and C_2 meet the coordinate axes as well as the coordinates of the endpoints of the curves.

b) Use a suitable iteration formula of the form $x_{n+1} = f(x_n)$ with $x_1 = 1.6$ to find the x coordinate of the point of intersection between the graphs of C_1 and C_2.

$$\boxed{x \approx 1.59}$$

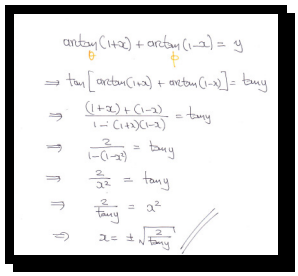


Question 10 (****)

Make x the subject of the equation

$$\arctan\left(1+x\right)+\arctan\left(1-x\right)=y\,.$$

$$\boxed{x=\pm\sqrt{\frac{2}{\tan y}}}$$



Question 11 (****)

It is given that

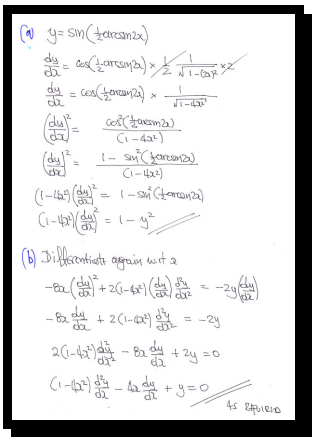
$$\frac{d}{du}\left(\arcsin u\right)=\frac{1}{\sqrt{1-u^2}}, \quad |u|\le 1.$$

Hence show that if $y=\sin\left(\frac{1}{2}\arcsin 2x\right)$, then …

a) … $\left(1-4x^2\right)\left(\frac{dy}{dx}\right)^2=1-y^2$.

b) … $\left(1-4x^2\right)\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+y=0$.

, proof



Question 12 **(****)**

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

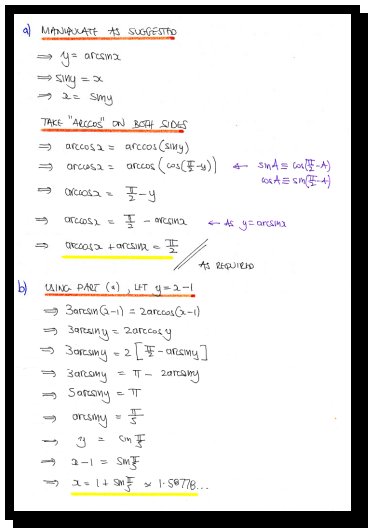
a) By expressing $\arccos x$ in terms of y, show that

$$\arcsin x + \arccos x = \frac{\pi}{2}\,.$$

b) Hence, or otherwise, solve the equation

$$3\arcsin(x-1) = 2\arccos(x-1)\,.$$

$$\boxed{x = 1 + \sin\left(\frac{\pi}{5}\right) \approx 1.5878}$$



Question 13 **(****)**

A curve has equation

$$y=\arcsin 2x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

a) By finding $\dfrac{dx}{dy}$ and using an appropriate trigonometric identity show that

$$\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}.$$

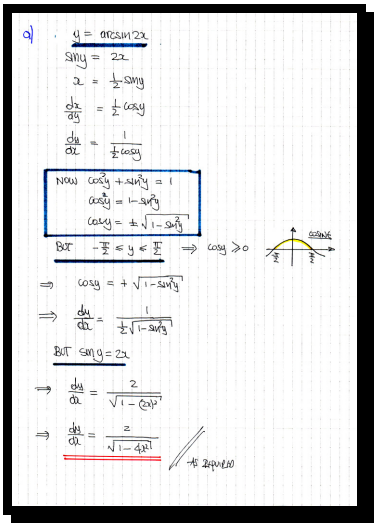
b) Show further that …

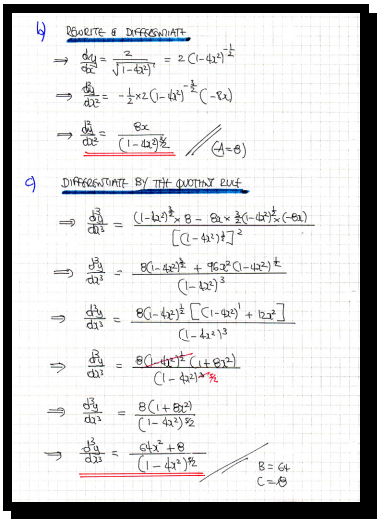
i. … $\dfrac{d^2y}{dx^2}=\dfrac{Ax}{\left(1-4x^2\right)^{\frac{3}{2}}}$,

ii. … $\dfrac{d^3y}{dx^3}=\dfrac{Bx^2+C}{\left(1-4x^2\right)^{\frac{5}{2}}}$,

where A, B and C are constants to be found.

$A=$ [ ], $B=$ [ ], $C=$ [ ], proof





Question 14 **(****)**

$$y=\arcsin x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

a) By finding $\dfrac{dx}{dy}$ and using an appropriate trigonometric identity show that

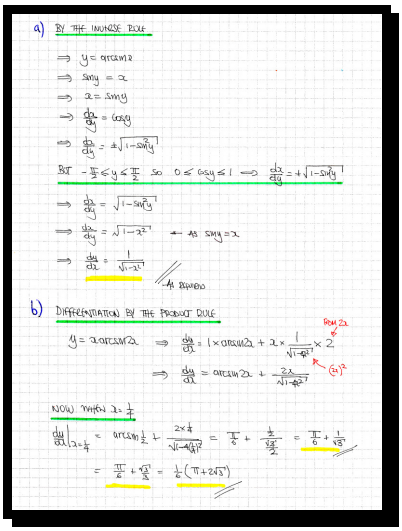
$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}.$$

A curve C has equation

$$y=x\arcsin 2x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

b) Find the exact value of $\dfrac{dy}{dx}$ at the point on C where $x=\dfrac{1}{4}$.

$\boxed{}$, $\boxed{\frac{1}{6}\left(\pi+2\sqrt{3}\right)}$



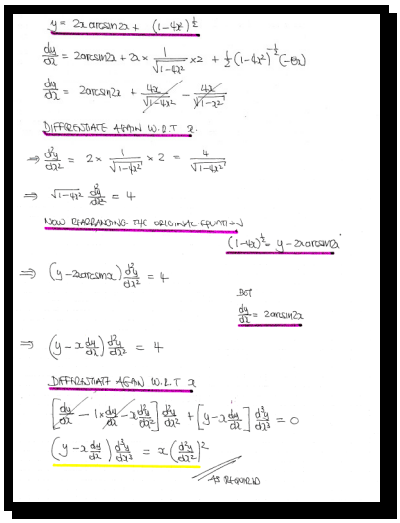
Question 15 (****)

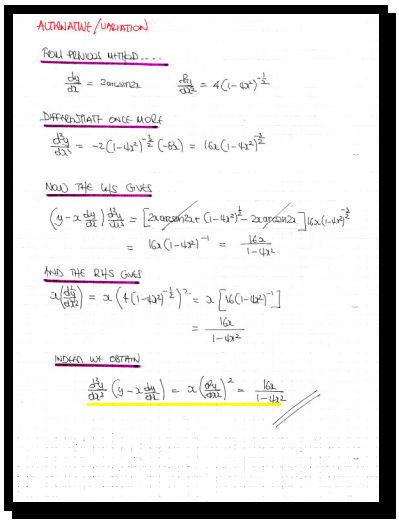
$$y=2x\arcsin 2x+\sqrt{1-4x^2}\ ,\ -\frac{1}{2}\le x\le\frac{1}{2}\ .$$

Show clearly that

$$\frac{d^3y}{dx^3}\left(y-x\frac{dy}{dx}\right)=x\left(\frac{d^2y}{dx^2}\right)^2\ .$$

$\boxed{}$, proof



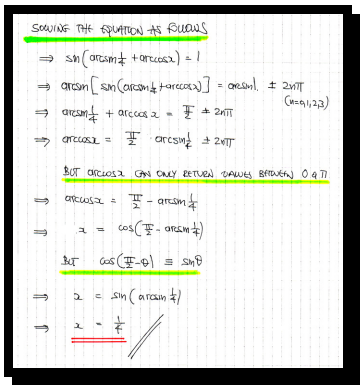


Question 16 **(****)**

Use trigonometric algebra to solve the equation

$$\sin\left[\arcsin\frac{1}{4}+\arccos x\right]=1.$$

$$\boxed{x=\frac{1}{4}}$$



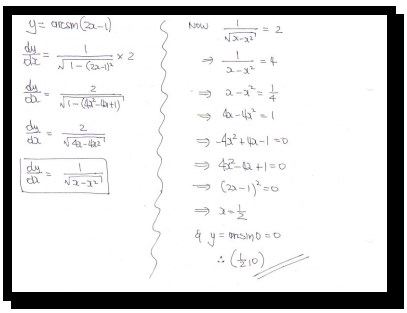
Question 17 **(****)**

The curve C has equation

$$y=\arcsin\left(2x-1\right),\ -0\le x\le 1.$$

Find the coordinates of the point on C, whose gradient is 2.

$$\boxed{\left(\frac{1}{2},0\right)}$$

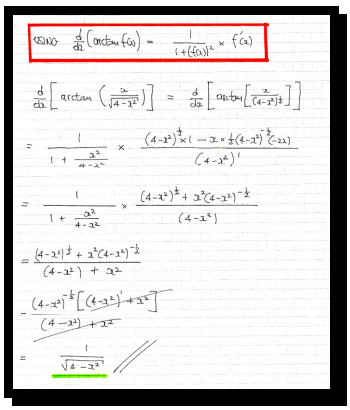


Question 18 **(****)**

Find a simplified expression for

$$\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right]$$

$\square$,
$$\boxed{\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right]=\frac{1}{\sqrt{4-x^2}}}$$

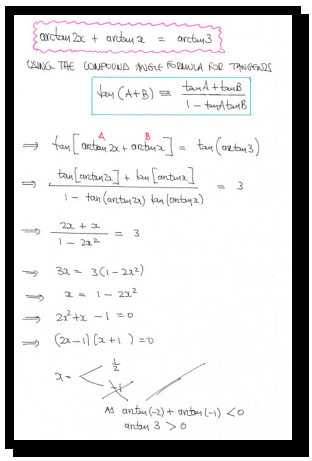


Question 19 **(****)**

Solve the following trigonometric equation.

$$\arctan 2x+\arctan x=\arctan 3\,,\qquad x\in\mathbb{R}\,.$$

$\square$, $\boxed{x=\frac{1}{2}}$



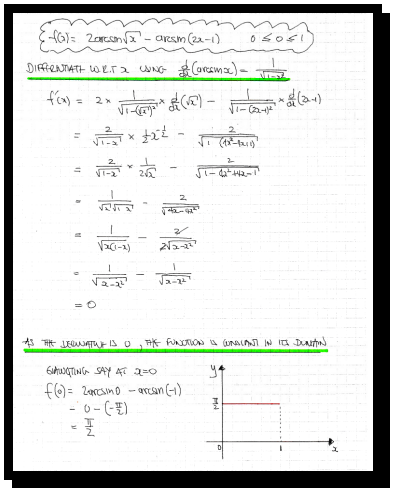
16
HARD QUESTIONS

Question 1 (**+)**

$$f(x)=2\arcsin\sqrt{x}-\arcsin(2x-1),\quad 0\le x\le 1.$$

By considering $f'(x)$ sketch the graph of $f(x)$.

[ ], [graph]



Question 2 (**+)**

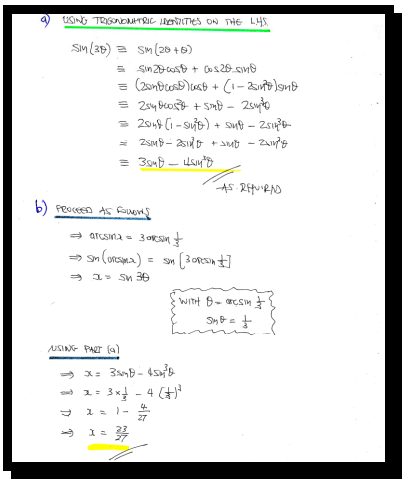
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

a) Prove the validity of the above trigonometric identity by considering the expansion of $\sin(2\theta+\theta)$.

b) Hence or otherwise solve the equation

$$\arcsin x = 3\arcsin\left(\frac{1}{3}\right).$$

$$\boxed{x = \frac{23}{27}}$$



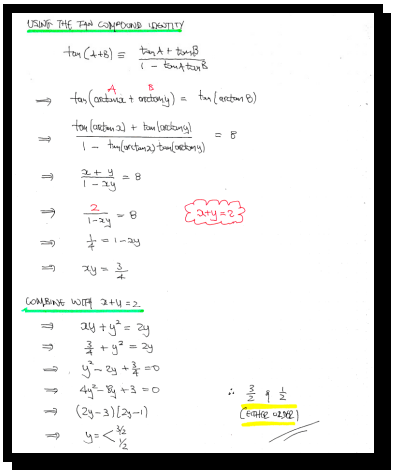
Question 3 (**+)**

Solve the following simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2 \, .$$

$$\boxed{x = \frac{1}{2}, y = \frac{3}{2}, \text{ in either order}}$$



Question 4 **(****+)**

$$f(x)=\sec x, \qquad 0\le x<\frac{\pi}{2}\cup\frac{\pi}{2}<x\le\pi .$$

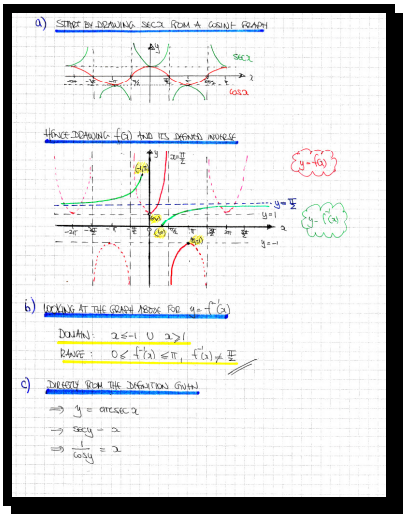
a) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)=\text{arcsec}\,x$.

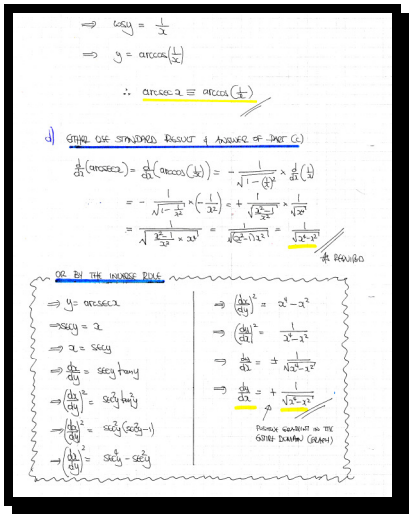
b) State the domain and range of $f^{-1}(x)=\text{arcsec}\,x$.

c) Show clearly that $\text{arcsec}\,x=\arccos\left(\frac{1}{x}\right)$.

d) Show further that $\frac{d}{dx}(\text{arcsec}\,x)=\frac{1}{\sqrt{x^4-x^2}}$.

$\boxed{}$, $\boxed{\text{domain: } x\le -1\cup x\ge 1}$, $\boxed{\text{range: } 0\le f^{-1}(x)\le\pi,\ f^{-1}(x)\ne\frac{\pi}{2}}$



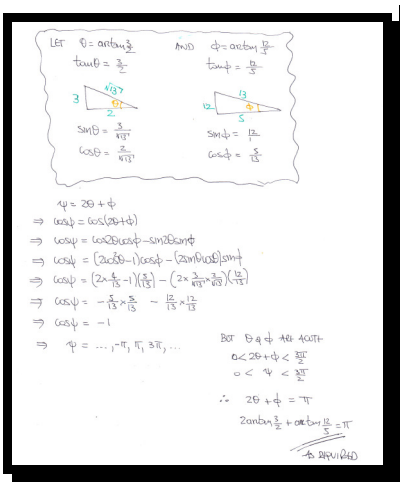


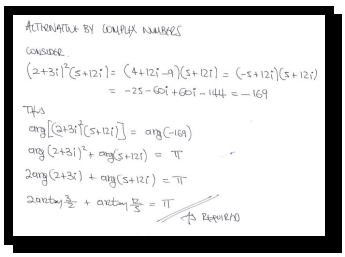
Question 5 (**+)**

Show clearly that

$$2\arctan\left(\frac{3}{2}\right)+\arctan\left(\frac{12}{5}\right)=\pi\,.$$

V, proof



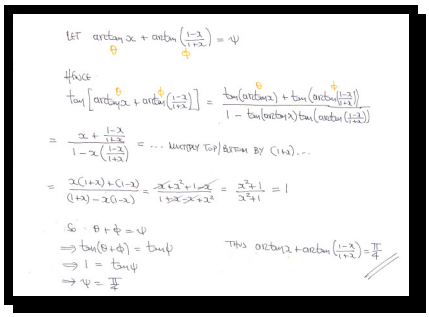


Question 6 (**+)**

Show clearly that

$$\arctan x+\arctan\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}\,.$$

proof

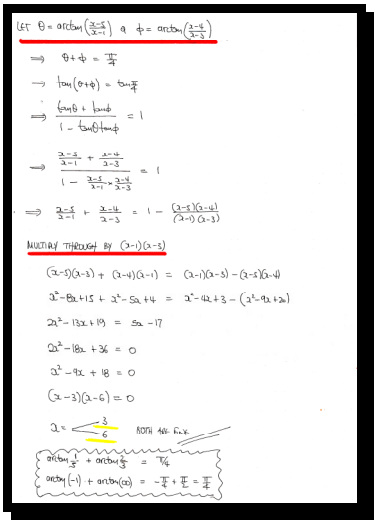


Question 7 (**+)**

Solve the following trigonometric equation.

$$\arctan\left(\frac{x-5}{x-1}\right)+\arctan\left(\frac{x-4}{x-3}\right)=\frac{\pi}{4}\,,\ x\in\mathbb{R}\,.$$

$$\boxed{x=3\ \cup\ x=6}$$



Question 8 **(****+)**

$$y=\arccos x\,,\ -1\le x\le 1\,,\ 0\le y\le \pi\,.$$

a) By writing $y=\arccos x$ as $x=\cos y$, show that

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}\,.$$

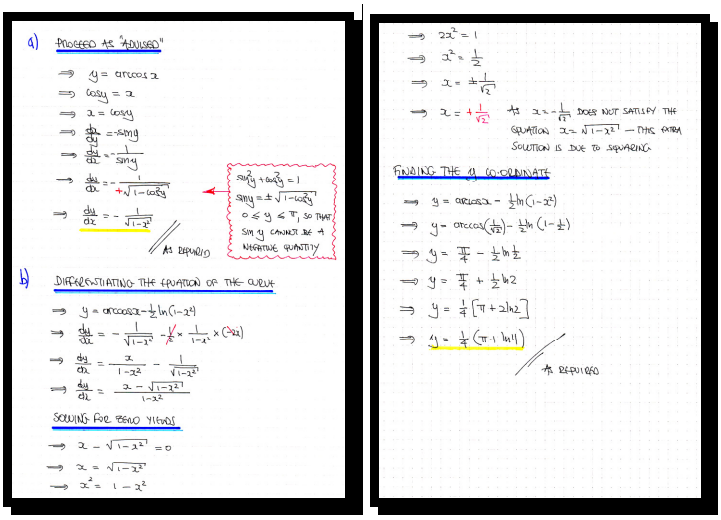
The curve C has equation

$$y=\arccos x-\frac{1}{2}\ln\left(1-x^2\right),\ x>0\,.$$

b) Show that the y coordinate of the stationary point of C is

$$\frac{1}{4}\left(\pi+\ln 4\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

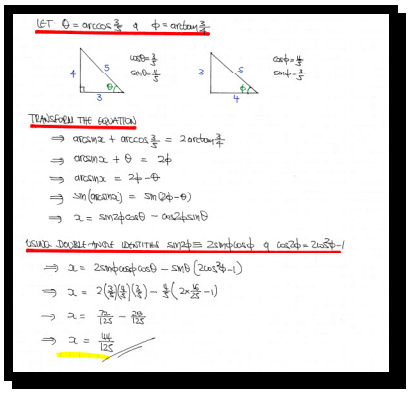


Question 9 (**+)**

Solve the following trigonometric equation

$$\arcsin x+\arccos\frac{3}{5}=2\arctan\frac{3}{4}.$$

$\boxed{\phantom{x=\frac{44}{125}}}$, $\boxed{x=\frac{44}{125}}$

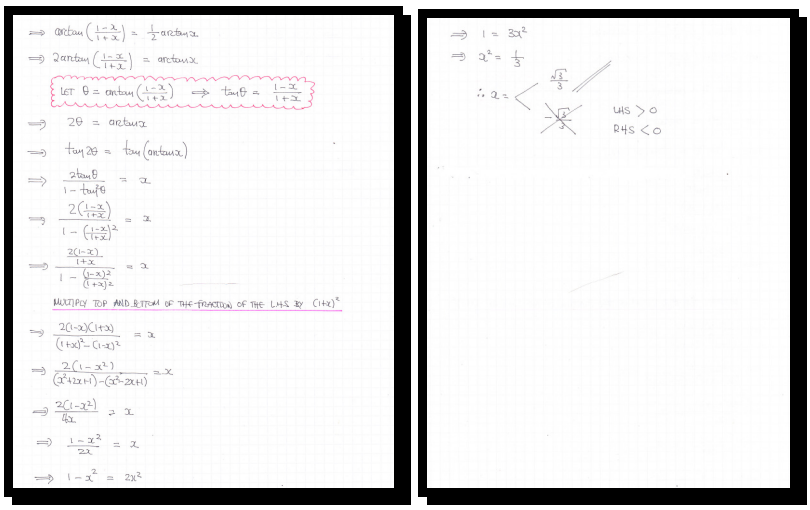


Question 10 (**+)**

Find the solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\arctan x.$$

$\boxed{\phantom{x=\frac{\sqrt{3}}{3}}}$, $\boxed{x=\frac{\sqrt{3}}{3}}$



Question 11 **(****+)**

The functions f and g are defined by

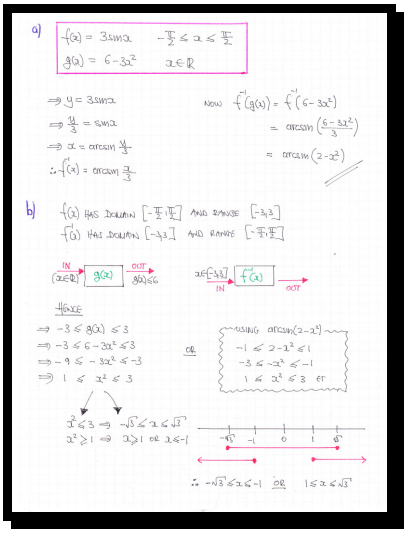
$$f(x) \equiv 3\sin x,\ x \in \mathbb{R},\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2,\ x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}g(x)$.

b) Determine the domain of $f^{-1}g(x)$.

$\square$, $\boxed{f^{-1}g(x) = \arcsin\left(2 - x^2\right)}$, $\boxed{-\sqrt{3} \le x \le -1 \text{ or } 1 \le x \le \sqrt{3}}$



Question 12 **(****+)**

$$y = \arctan x\,,\ \ x \in \mathbb{R}\,.$$

a) By writing the above equation in the form $x = g(y)$, show that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}\,.$$

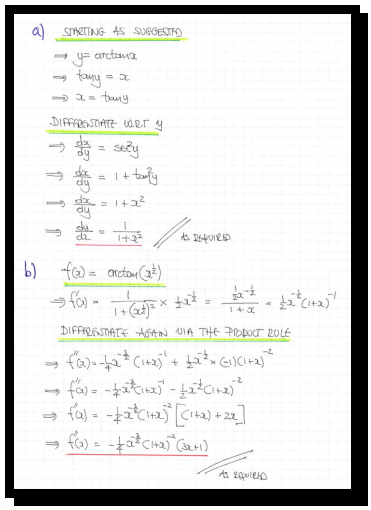
The function f is defined as

$$f(x) = \arctan\sqrt{x}\,,\ \ x \in \mathbb{R}\,,\ \ x \geq 0\,.$$

b) Show further that

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(3x+1)(x+1)^{-2}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

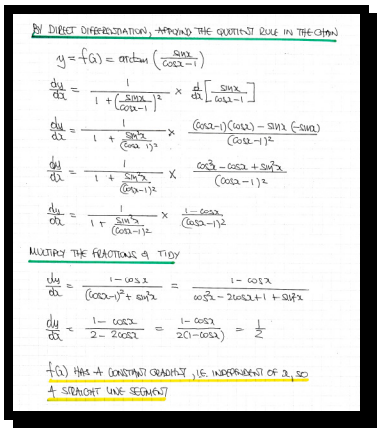


Question 13 (**+)**

$$f(x) \equiv \arctan\left(\frac{\sin x}{\cos x - 1}\right), \ 0 < x < 2\pi .$$

Show that $f(x)$ represents a straight line segment.

V, [ ], proof

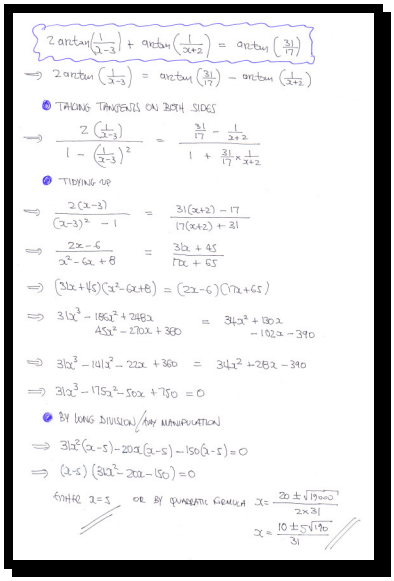


Question 14 **(****+)**

$$2\arctan\left[\frac{1}{x-3}\right]+\arctan\left[\frac{1}{x+2}\right]=\arctan\left[\frac{31}{17}\right].$$

Show that $x=5$ is one of the solutions of the above trigonometric equation, and find in exact surd form the other two solutions.

$$\boxed{x=\frac{10\pm5\sqrt{190}}{31}}$$



Question 15 **(****+)**

$$y=\arccos x\,,\ x\in\mathbb{R}\,,\ -1\le x\le 1\,.$$

a) By writing the above equation in the from $x=f\left(y\right)$, show that

$$\frac{d}{dx}\left(\arccos x\right)=-\frac{1}{\sqrt{1-x^2}}\,.$$

A curve has equation

$$y=\arccos\left(1-x^2\right),\ \ x\in\mathbb{R}\,,\ 0<x\le\sqrt{2}\,.$$

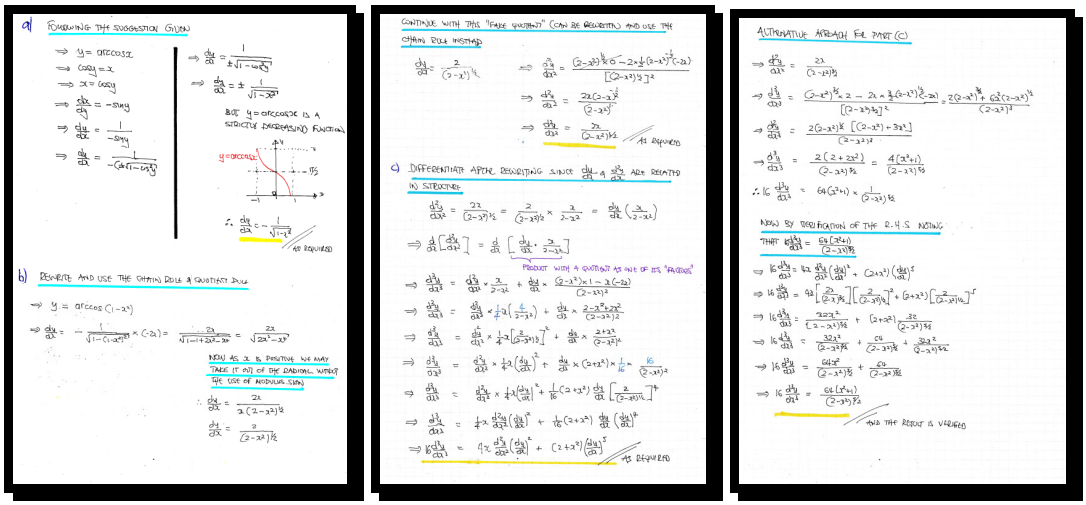
b) Show further that

$$\frac{d^2y}{dx^2}=\frac{2x}{\left(2-x^2\right)^{\frac{3}{2}}}\,.$$

c) Show clearly that

$$16\frac{d^3y}{dx^3}=4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2+\left(2+x^2\right)\left(\frac{dy}{dx}\right)^5\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



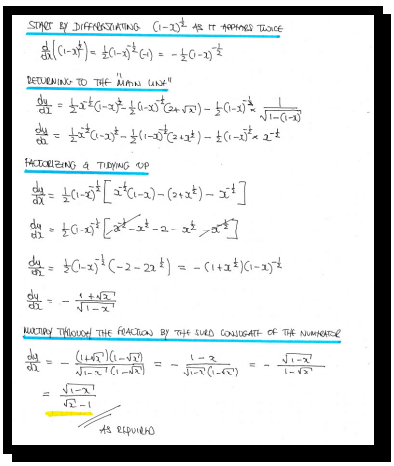
Question 16 **(****+)**

$$y = \left(2+\sqrt{x}\right)\sqrt{1-x}+\arcsin\sqrt{1-x}\ ,\ 0\le x\le 1\,.$$

Show with detailed workings that

$$\frac{dy}{dx} = \frac{\sqrt{1-x}}{\sqrt{x}-1}\,.$$

V , [ ] , [proof]



36 ENRICHMENT QUESTIONS

Question 1 (*****)

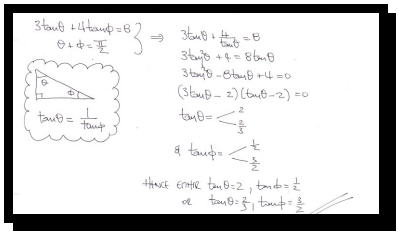
Given the simultaneous equations

$$3\tan\theta + 4\tan\varphi = 8$$

$$\theta + \varphi = \frac{\pi}{2},$$

find the possible value of $\tan\theta$ and the possible value of $\tan\varphi$.

$$\boxed{[\tan\theta, \tan\varphi] = \left[2, \frac{1}{2}\right] = \left[\frac{2}{3}, \frac{3}{2}\right]}$$



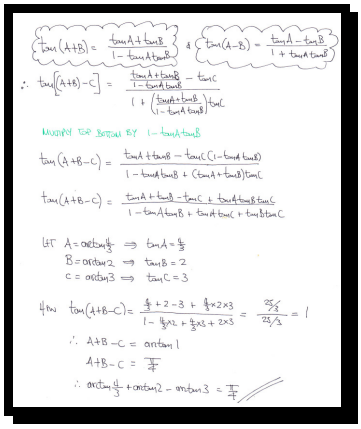
Question 2 (*****)

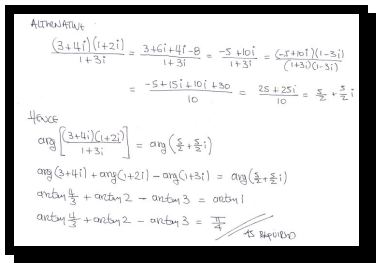
Simplify, showing all steps in the calculation, the expression

$$\arctan\frac{4}{3}+\arctan 2-\arctan 3,$$

giving the answer in terms of π.

$$\boxed{\frac{\pi}{4}}$$





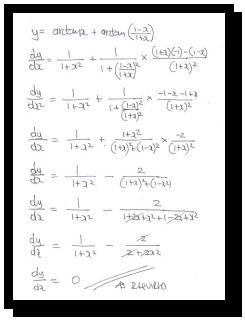
Question 3 (*****)

$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right), \ x \in \mathbb{R}.$$

Without simplifying the above expression, use differentiation to show that for all values of x

$$\frac{dy}{dx} = 0.$$

proof



Question 4 (*****)

A curve C has equation

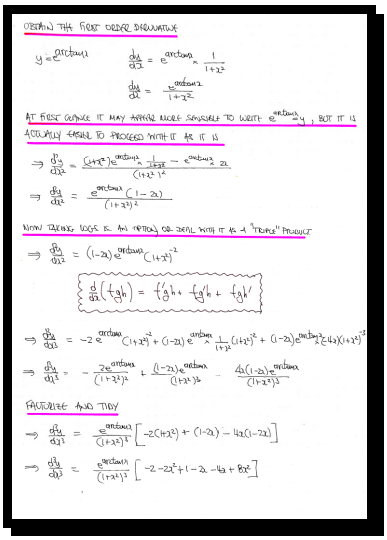
$$y = \mathrm{e}^{\arctan x}, \quad x \in \mathbb{R}.$$

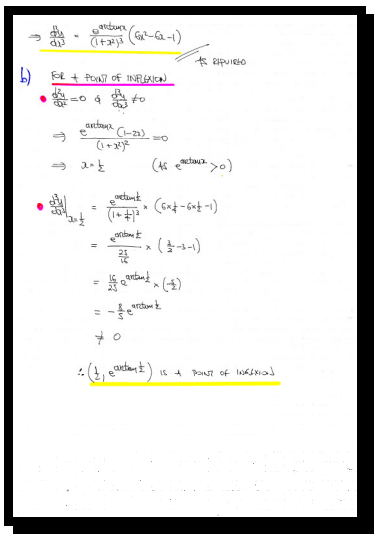
a) Show, with detailed workings, that

$$\frac{d^3y}{dx^3} = \frac{\left(6x^2 - 6x - 1\right)\mathrm{e}^{\arctan x}}{\left(1+x^2\right)^3}.$$

b) Deduce that C has a point of inflection, stating its coordinates.

$\square$, $\boxed{\left(\frac{1}{2}, \mathrm{e}^{\arctan\frac{1}{2}}\right)}$



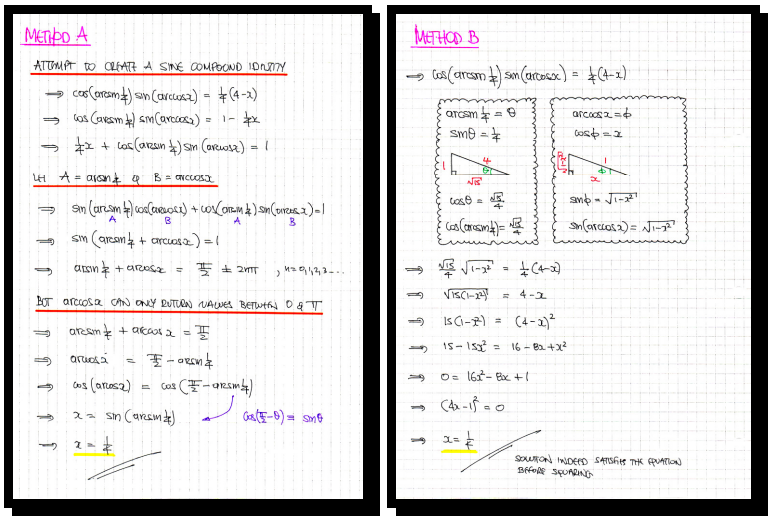


Question 5 (*****)

Solve the following trigonometric equation

$$\cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=\frac{1}{4}\left(4-x\right)\ ,\quad x\in\mathbb{R}\ .$$

$\square$, $\boxed{x=\frac{1}{4}}$



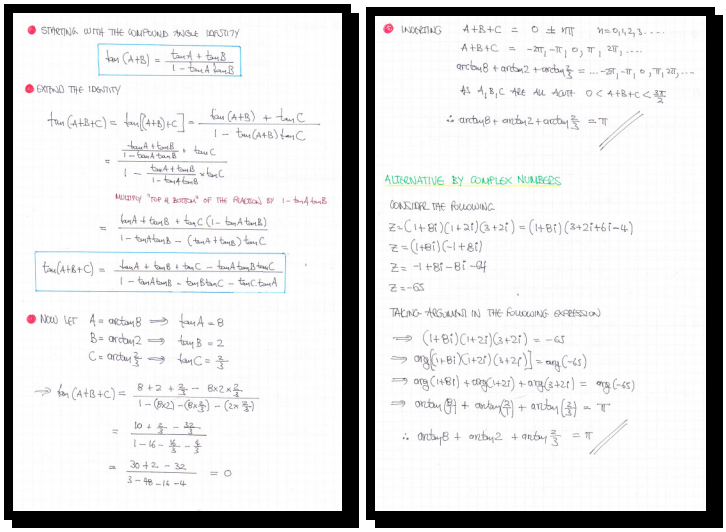
Question 6 **(*****)**

Simplify, showing all steps in the calculation, the expression

$$\arctan 8+\arctan 2+\arctan\frac{2}{3},$$

giving the answer in terms of π.

$\square$, $\boxed{\pi}$



Question 7 **(*****)**

It is given that

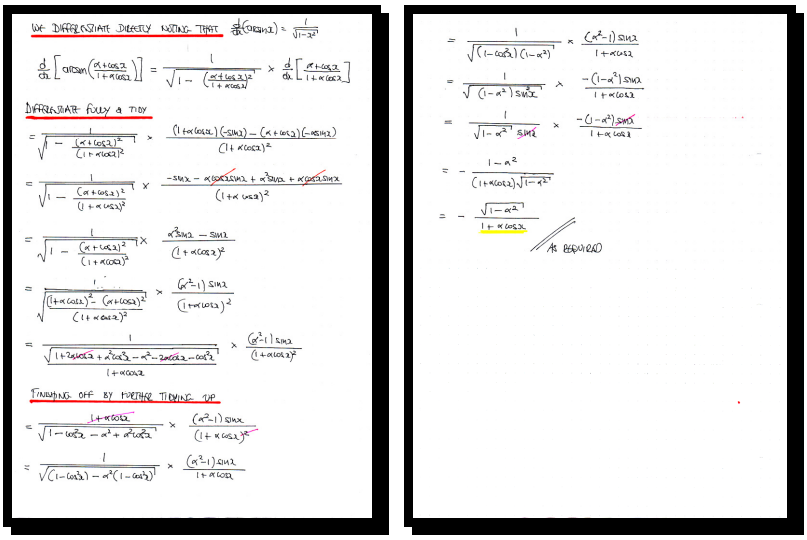
$$y=\arcsin\left[\frac{\alpha+\cos x}{1+\alpha\cos x}\right],$$

where α is a constant.

Show that

$$\frac{dy}{dx}=-\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}.$$

[ ], [proof]



Question 8 (*****)

The functions f and g are defined by

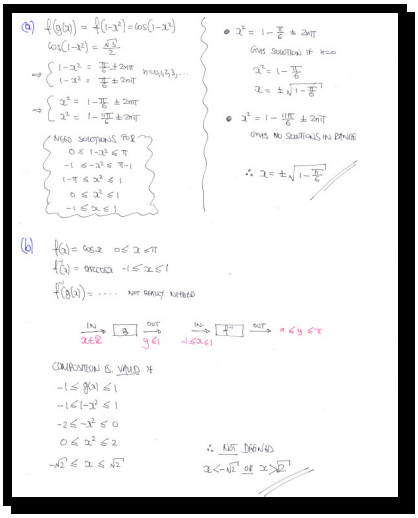
$$f(x) \equiv \cos x,\ x \in \mathbb{R},\ 0 \le x \le \pi$$

$$g(x) \equiv 1 - x^2,\ x \in \mathbb{R}.$$

a) Solve the equation $fg(x) = \frac{1}{2}$.

b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

$$\boxed{x = \pm\sqrt{1 - \frac{\pi}{6}}}, \quad \boxed{x < -\sqrt{2} \text{ or } x > \sqrt{2}}$$



Question 9 **(*****)**

The acute angles θ and φ satisfy the following equations

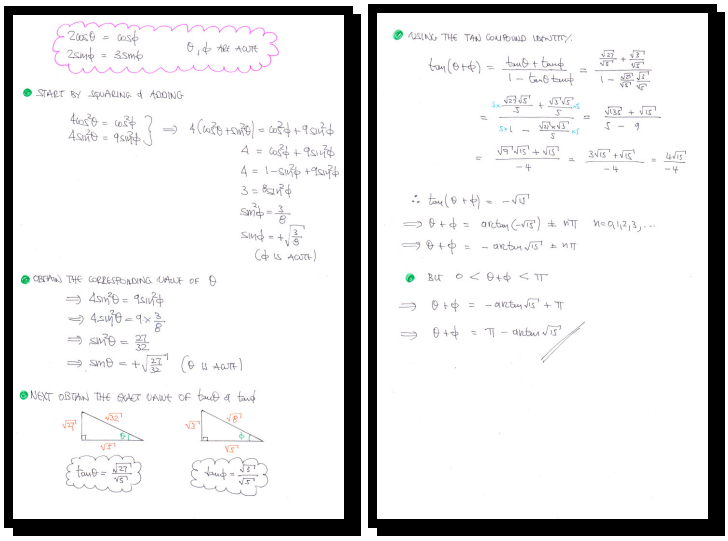
$$2\cos\theta = \cos\varphi$$

$$2\sin\theta = 3\sin\varphi .$$

Show clearly that

$$\theta + \varphi = \pi - \arctan\sqrt{15}$$

, proof

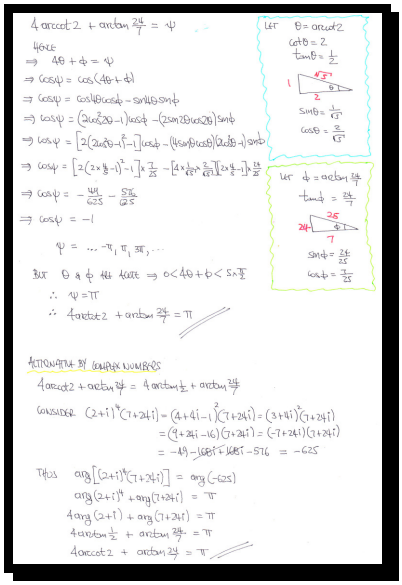


Question 10 (*****)

Show clearly that

$$4\operatorname{arccot}2+\arctan\left(\frac{24}{7}\right)=\pi\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

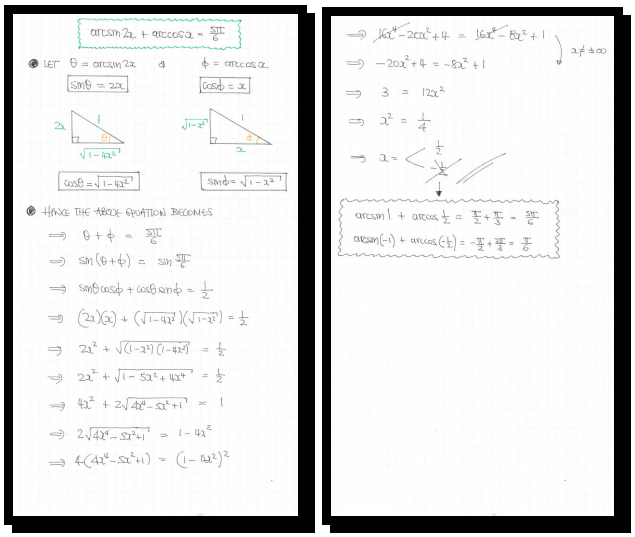


Question 11 (*****)

Solve the following trigonometric equation

$$\arcsin 2x+\arccos x=\frac{5\pi}{6}.$$

$\boxed{}$, $\boxed{x=\frac{1}{2}}$

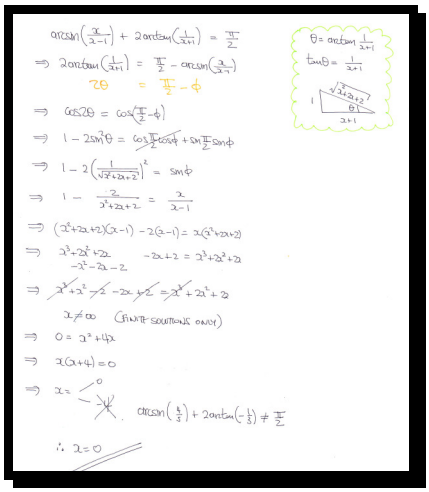


Question 12 (*****)

Find the only finite solution of the equation

$$\arcsin\left(\frac{x}{x-1}\right)+2\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{2}.$$

$\boxed{x=0}$

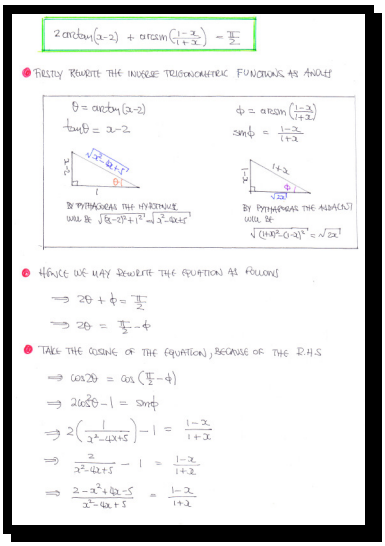


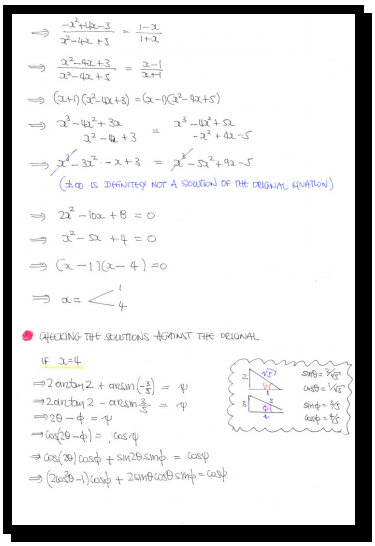
Question 13 (*****)

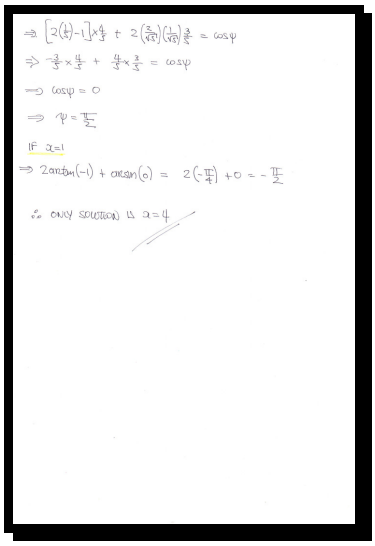
Solve the trigonometric equation

$$2\arctan(x-2)+\arcsin\left(\frac{1-x}{1+x}\right)=\frac{\pi}{2},\ x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=4}$







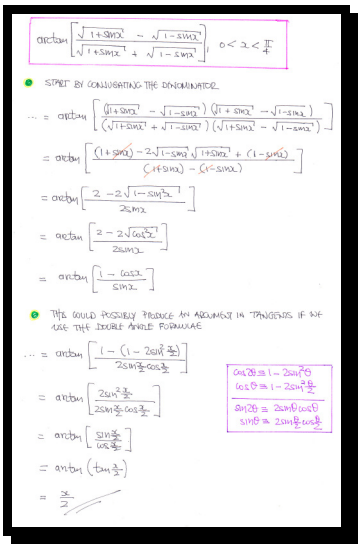
Question 14 **(*****)**

Use trigonometric algebra to fully simplify

$$\arctan\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{4},$$

giving the final answer in terms of x.

$\square$, $\boxed{\frac{1}{2}x}$



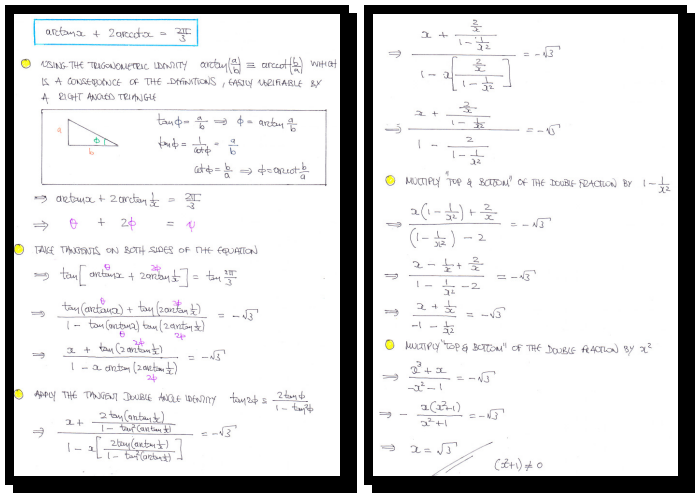
Question 15 **(*****)**

Use trigonometric algebra to solve the equation

$$\arctan x+2\operatorname{arccot} x=\frac{2\pi}{3}.$$

You may assume that $\operatorname{arccot} x$ is the inverse function for the part of $\cot x$ for which $0\le x\le\pi$.

$\square$, $\boxed{x=\sqrt{3}}$



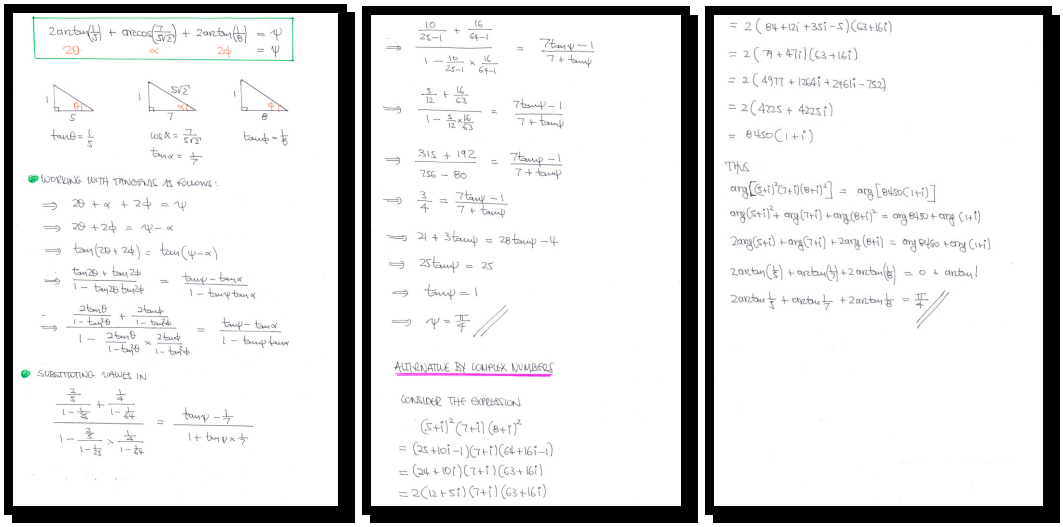
Question 16 **(*****)**

Use trigonometric algebra to fully simplify

$$2\arctan\left(\frac{1}{5}\right)+\arccos\left(\frac{7}{5\sqrt{2}}\right)+\arctan\left(\frac{1}{8}\right),$$

giving the final answer in terms of π.

$\square$, $\boxed{\frac{\pi}{4}}$



Question 17 **(*****)**

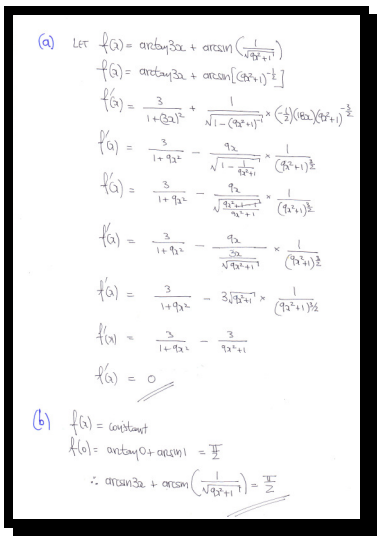
$$f(x)=\arctan(3x)+\arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right),\ x\in\mathbb{R}.$$

Show, by a detailed method, that …

a) … $f'(x)=0$.

b) … $\arctan(3x)+\arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right)\equiv k\pi$, stating the value of the constant k.

$\square$, $\boxed{k=\frac{1}{2}}$



Question 18 **(*****)**

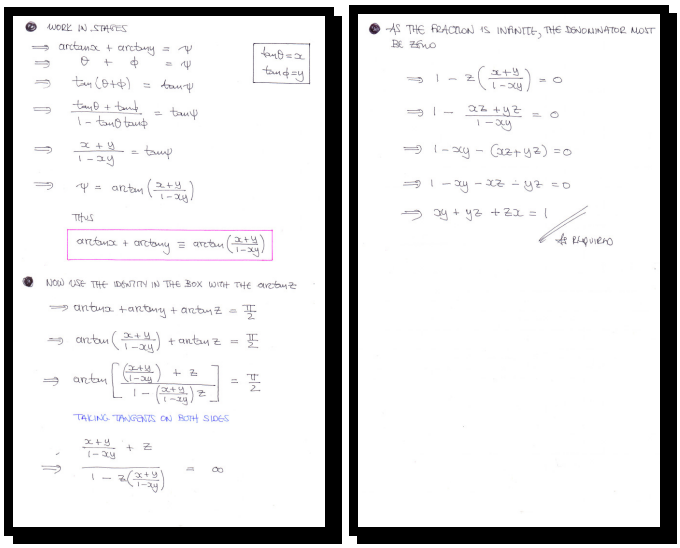
It given that

$$\arctan x+\arctan y+\arctan z=\frac{\pi}{2}.$$

Show that x, y and z satisfy the relationship

$$xy+yz+zx=1.$$

, proof



Question 19 **(*****)**

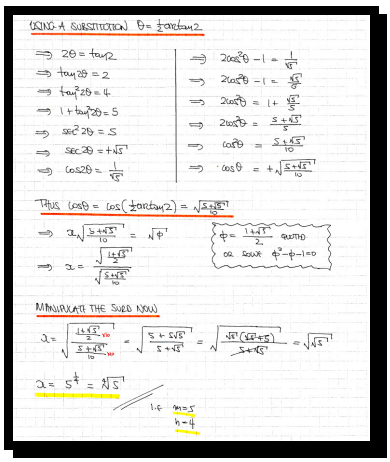
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

Solve the equation

$$x\cos\left(\frac{1}{2}\arctan 2\right) = \sqrt{\phi}\,, \quad x \in \mathbb{R}\,.$$

Give the answer in the form $\sqrt[n]{m}$, where m and n are positive integers.

V, [ ], $\boxed{x = \sqrt[4]{5}}$

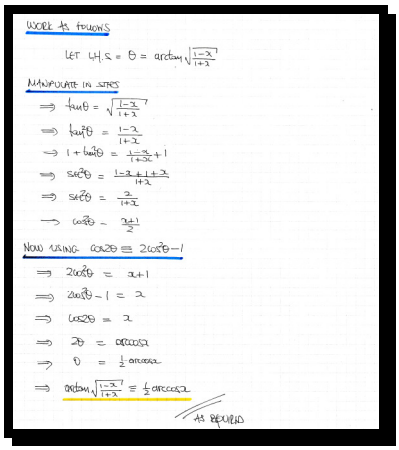


Question 20 (*****)

Prove that if $|x| \leq 1$

$$\arctan\left[\sqrt{\frac{1-x}{1+x}}\right] \equiv \frac{1}{2}\arccos x \,.$$

V, [ ], proof



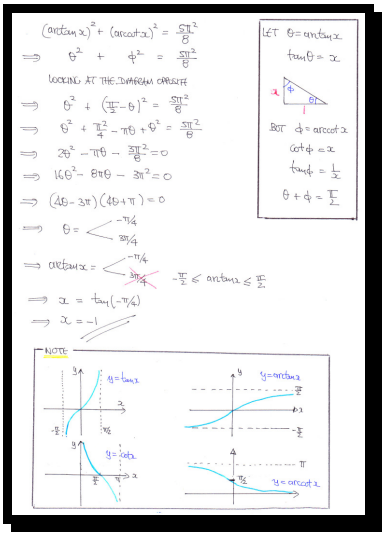
Question 21 (*****)

Use a trigonometric algebra to solve the following equation

$$(\arctan x)^2+(\operatorname{arccot} x)^2=\frac{5\pi^2}{8}.$$

You may assume that $y=\operatorname{arccot} x$ is the inverse function of $y=\cot x$, $0\le x\le\pi$

$\boxed{}$, $\boxed{x=-1}$

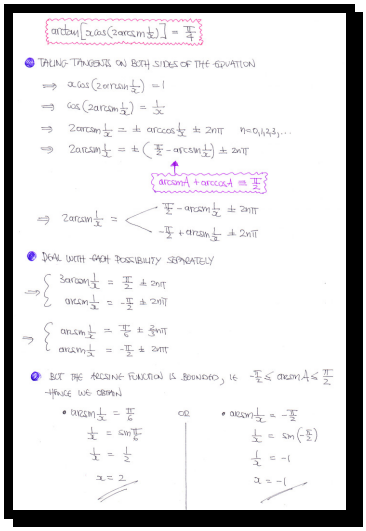


Question 22 (*****)

Solve the following trigonometric equation

$$\arctan\left[x\cos\left(2\arcsin\frac{1}{x}\right)\right]=\frac{1}{4}\pi\,.$$

$\boxed{}$, $\boxed{x=-1,\quad x=2}$

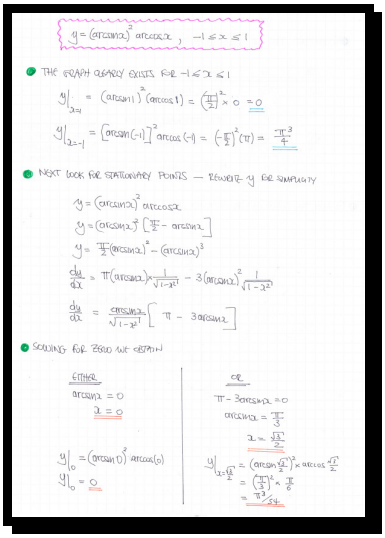


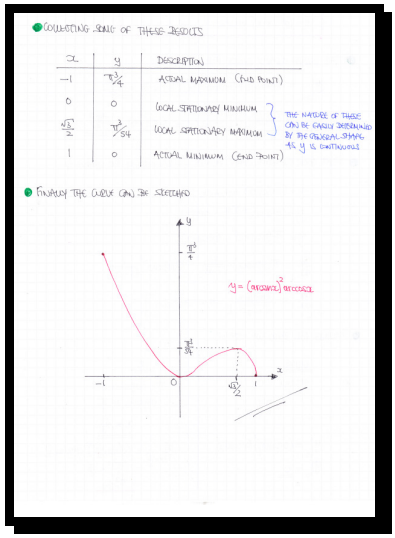
Question 23 (*****)

On a clearly labelled set of axes, draw a detailed sketch of the graph of

$$y=\left(\arcsin x\right)^2\arccos x\,,\ -1\leq x\leq 1\,.$$

[ ] , [graph]





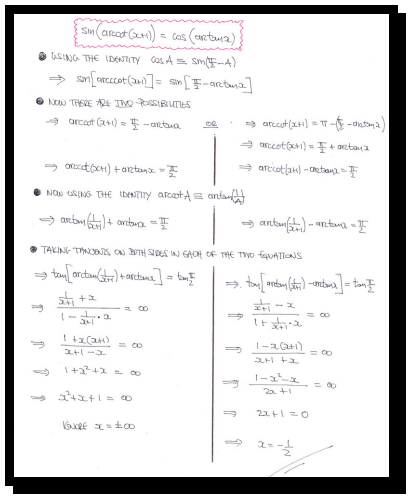
Question 24 (*****)

Solve the following trigonometric equation

$$\sin\left[\operatorname{arccot}(x+1)\right]=\cos\left(\arctan x\right).$$

You may assume that $y=\operatorname{arccot} x$ is the inverse function for $y=\cot x$, $0\le x\le\pi$.

$\square$, $\boxed{x=-\frac{1}{2}}$

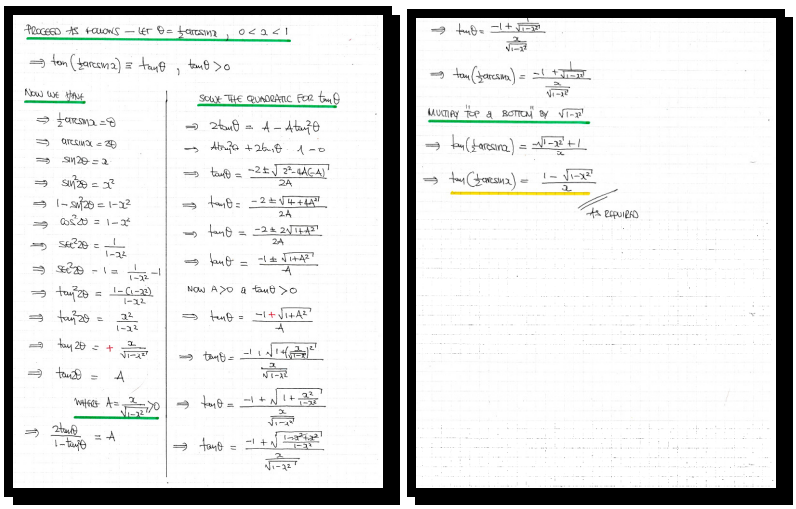


Question 25 **(*****)**

Prove that if $|x| \leq 1$

$$\tan\left[\frac{1}{2}\arcsin x\right] \equiv \frac{1-\sqrt{1-x^2}}{x}.$$

V, ☐, proof



Question 26 **(*****)**

It is given that

$$(\arcsin x)^3+(\arccos x)^3=k\pi^3,\ |x|\le 1,$$

for some constant k.

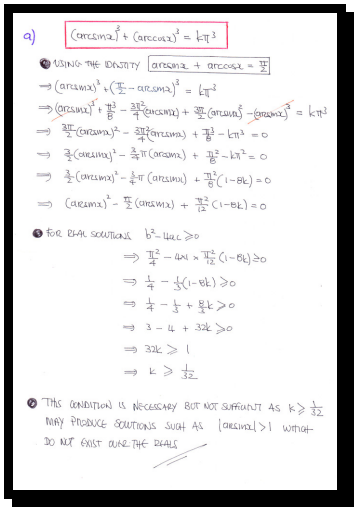
a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

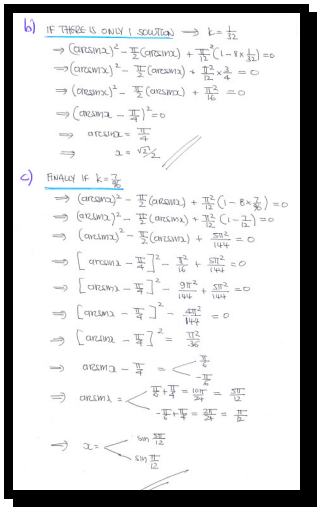
$$k\ge\frac{1}{32}.$$

b) Solve the equation given that it only has one solution.

c) Given instead that that $k=\frac{7}{96}$, find the two solutions of the equation, giving the answers in the form $x=\sin(a\pi)$, where $a\in\mathbb{Q}$.

$$\boxed{},\quad \boxed{x=\frac{\sqrt{2}}{2}},\quad \boxed{x=\sin\left(\frac{\pi}{12}\right),\quad x=\sin\left(\frac{5\pi}{12}\right)}$$





Question 27 (*****)

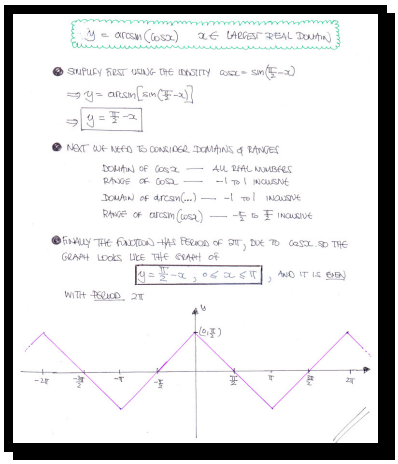
Sketch the graph of

$$f(x)=\arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

$\boxed{}$, $\boxed{\text{graph}}$

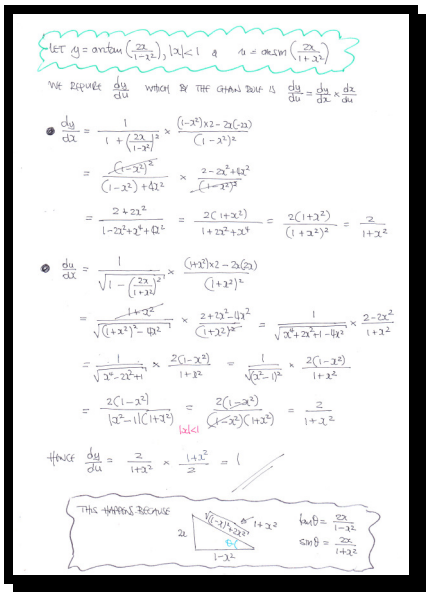


Question 28 **(*****)**

$$y=\arctan\left(\frac{2x}{1-x^2}\right),\ x\in\mathbb{R}.$$

Differentiate y with respect to $\arcsin\left(\frac{2x}{1+x^2}\right)$, fully simplifying the answer.

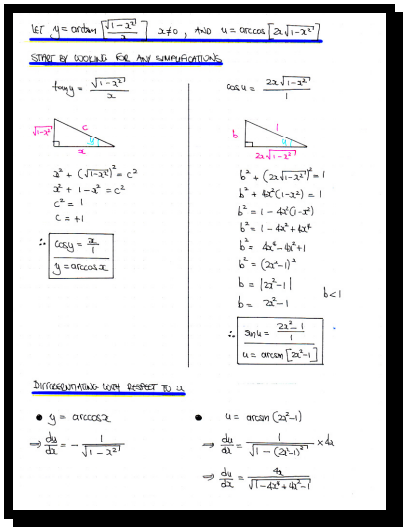
$\square$, $\boxed{1}$

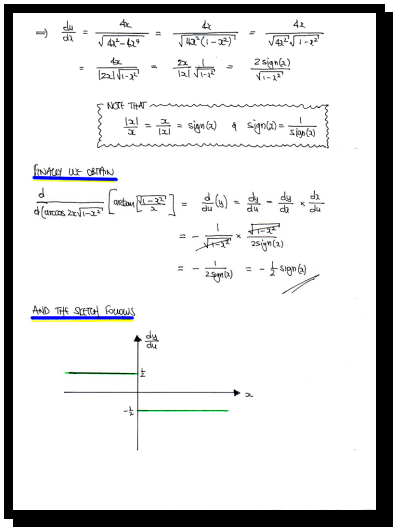


Question 29 (*****)

Differentiate $\arctan\left[\dfrac{\sqrt{1-x^2}}{x}\right]$ with respect to $\arccos\left[2x\sqrt{1-x^2}\right]$, and hence sketch the graph of the resulting gradient function.

$\boxed{}$, $\boxed{-\frac{1}{2}\text{sign}(x)}$





Question 30 (*****)

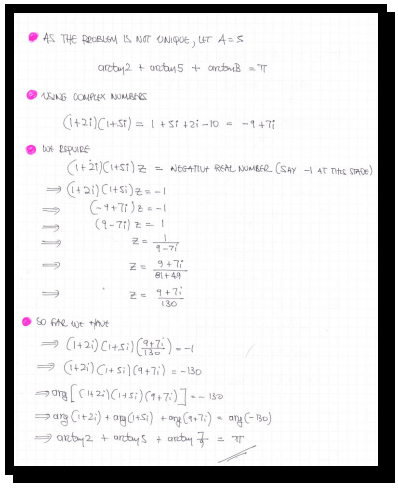
It is given that

$$\arctan 2 + \arctan A + \arctan B = \pi .$$

It is further given that A and B are distinct positive real numbers other than unity.

Determine a pair of possible values for A and B.

$\square$, $\boxed{5 \ \& \ \frac{7}{9}}$

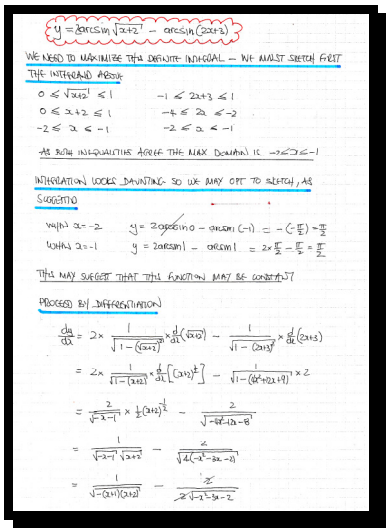


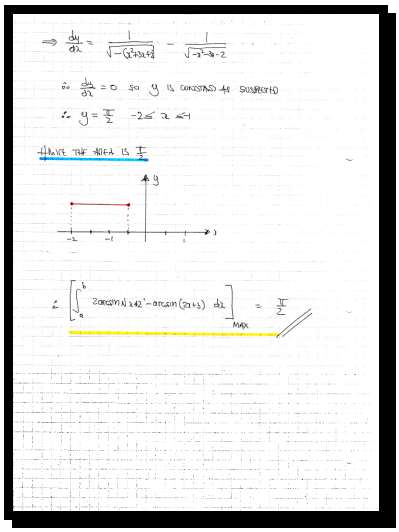
Question 31 **(*****)**

By sketching the graph of the integrand, or otherwise, determine the maximum value of the following function

$$F(a,b) \equiv \int_a^b 2\arcsin\sqrt{x+2} \; - \; \arcsin(2x+3) \; dx \, .$$

$\boxed{}$, $\boxed{\text{proof}}$



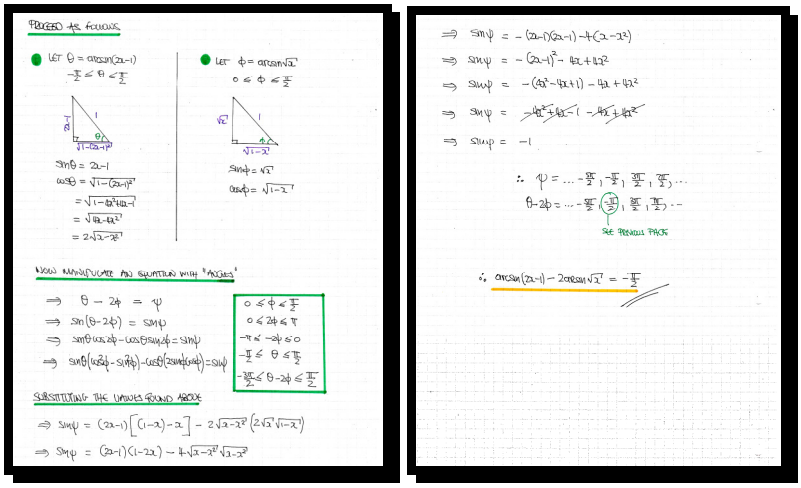


Question 32 (*****)

If $0 \leq x \leq 1$, simplify fully

$$\arcsin\left(2x-1\right)-2\arcsin\sqrt{x}\,.$$

V, $\square$, $\boxed{-\frac{1}{2}\pi}$

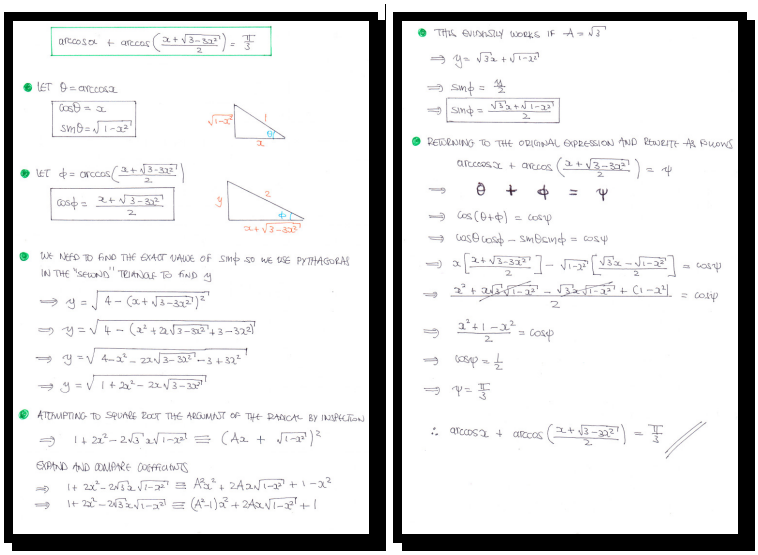


Question 33 (*****)

Prove that for all x such that $-1 \le x \le 1$

$$\arccos x + \arccos\left[\frac{1}{2}\left(x+\sqrt{3-3x^2}\right)\right] = \frac{\pi}{3}.$$

, proof



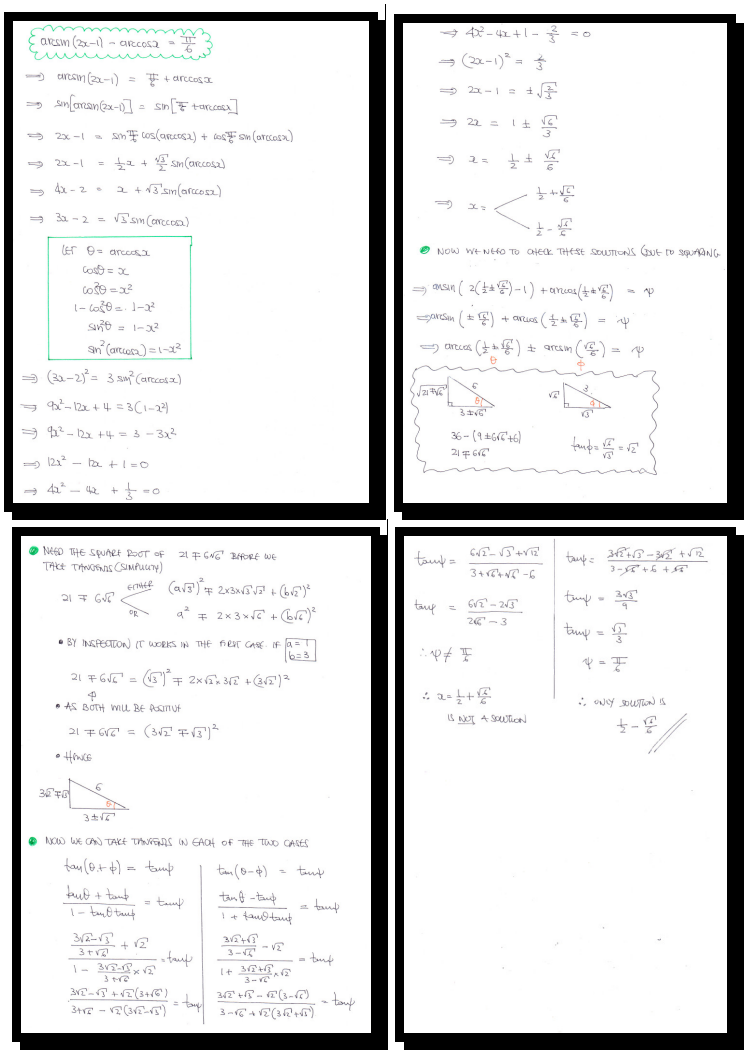
Question 34 (*****)

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1)-\arccos x=\frac{\pi}{6}.$$

The rejection of any additional solutions must be fully justified.

$\square$, $\boxed{x=\frac{1}{2}-\frac{1}{6}\sqrt{6}}$



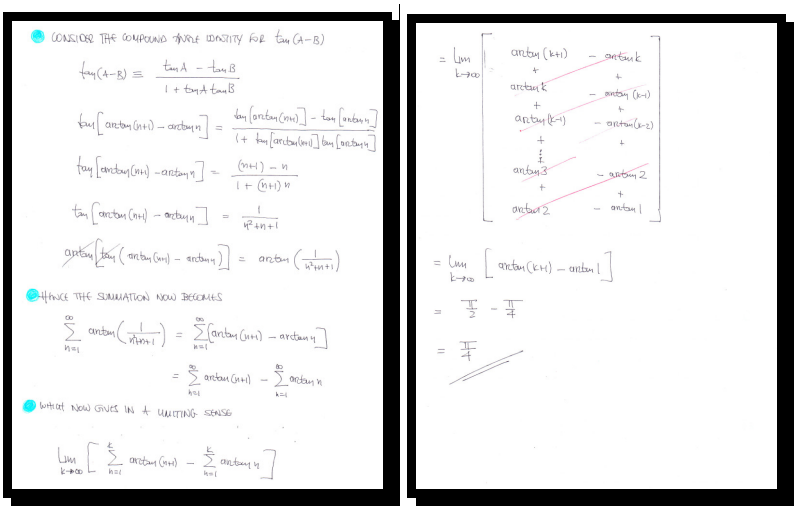
Question 35 **(*****)**

By considering the trigonometric identity for $\tan(A-B)$, with $A=\arctan(n+1)$ and $B=\arctan(n)$, sum the following series

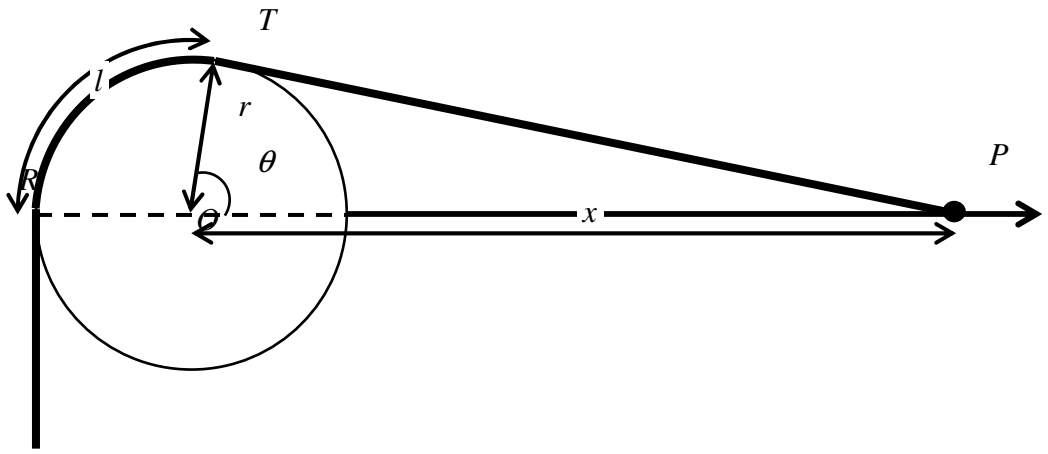
$$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\square$, $\boxed{\frac{\pi}{4}}$



Question 36 (*****)



A circular wheel of radius r and centre at the origin O of a positive x axis. A particle P is constrained to move on the positive x axis, so that the distance OP is x. The particle is connected to a taut cable which runs over the wheel and hangs vertically down on the other side of the wheel as shown in the figure above. The section of the cable RT, which is in contact with the wheel has length l. The section of the cable TP is a straight line.

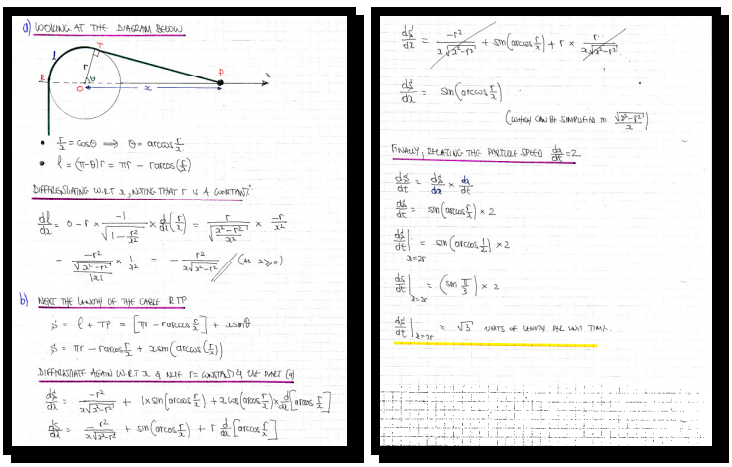
a) Given that the angle $TOP = \theta$ show that

$$\frac{dl}{dx} = -\frac{r^2}{x\sqrt{x^2 - r^2}}.$$

Let $s = l + |TP|$ and suppose that P is moving in the positive x direction with constant speed 2 units per unit time.

b) Find the rate at which s is increasing when P is at a distance of $2r$ from O.

$\square$, $\boxed{\sqrt{3}}$

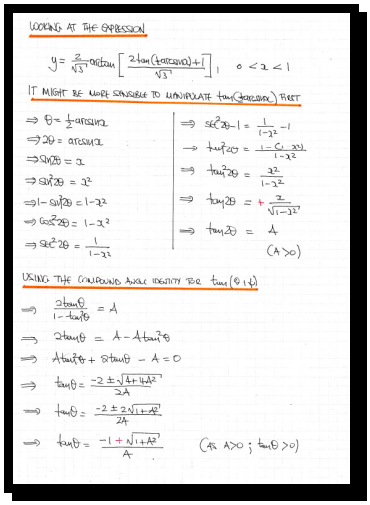


Question 37 **(*****)**

Prove that if $0 < x < 1$

$$\frac{d}{dx}\left[\frac{2}{\sqrt{3}}\arctan\left[\frac{1+2\tan\left(\frac{1}{2}\arcsin x\right)}{\sqrt{3}}\right]\right] \equiv \frac{1}{(2+x)\sqrt{1-x^2}}.$$

V, [ ], proof


LOOKING AT THE EXPRESSION

$y = \frac{2}{\sqrt{3}}\arctan\left[\frac{2\tan(\frac{1}{2}\arcsin x)+1}{\sqrt{3}}\right]$, $0 < x < 1$

IT MIGHT BE MORE SENSIBLE TO MANIPULATE $\tan(\frac{1}{2}\arcsin x)$ FIRST

$\Rightarrow \theta = \frac{1}{2}\arcsin x$

$\Rightarrow 2\theta = \arcsin x$

$\Rightarrow \sin 2\theta = x$

$\Rightarrow \sin^2 2\theta = x^2$

$\Rightarrow 1-\sin^2 2\theta = 1-x^2$

$\Rightarrow \cos^2 2\theta = 1-x^2$

$\Rightarrow \sec^2 2\theta = \frac{1}{1-x^2}$

$\Rightarrow \sec^2 2\theta - 1 = \frac{1}{1-x^2} - 1$

$\Rightarrow \tan^2 2\theta = \frac{1-(1-x^2)}{1-x^2}$

$\Rightarrow \tan^2 2\theta = \frac{x^2}{1-x^2}$

$\Rightarrow \tan 2\theta = + \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \tan 2\theta = A$ $(A > 0)$

USING THE COMPOUND ANGLE IDENTITY FOR $\tan(\theta+\theta)$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = A$

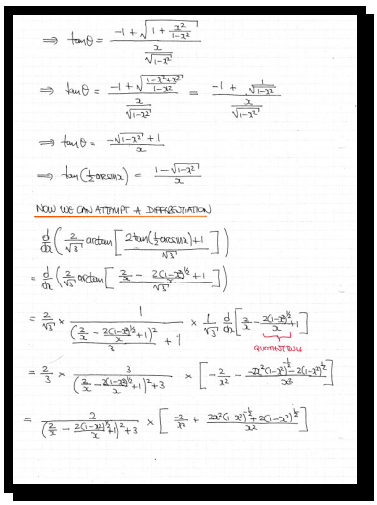
$\Rightarrow 2\tan\theta = A - A\tan^2\theta$

$\Rightarrow A\tan^2\theta + 2\tan\theta - A = 0$

$\Rightarrow \tan\theta = \frac{-2 \pm \sqrt{4+4A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-2 \pm 2\sqrt{1+A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+A^2}}{A}$ $(\text{as } A>0 \text{ , } \tan\theta > 0)$


$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+\frac{x^2}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{\frac{1-x^2+x^2}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}} = \frac{-1+\frac{1}{\sqrt{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-\sqrt{1-x^2}+1}{x}$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) = \frac{1-\sqrt{1-x^2}}{x}$

NOW WE CAN ATTEMPT A DIFFERENTIATION

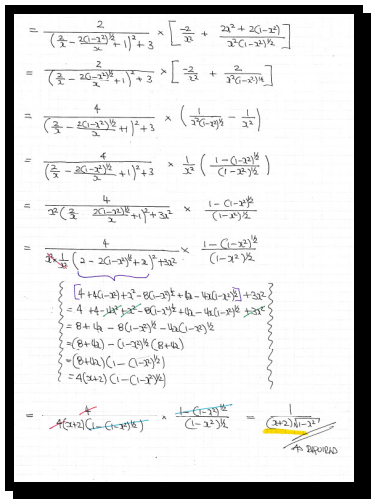
$\frac{d}{dx}\left(\frac{2}{\sqrt{3}}\arctan\left[\frac{2\tan(\frac{1}{2}\arcsin x)+1}{\sqrt{3}}\right]\right)$

$= \frac{d}{dx}\left(\frac{2}{\sqrt{3}}\arctan\left[\frac{\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1}{\sqrt{3}}\right]\right)$

$= \frac{2}{\sqrt{3}} \times \frac{1}{\left(\frac{\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1}{\sqrt{3}}\right)^2+1} \times \frac{1}{\sqrt{3}}\frac{d}{dx}\left[\frac{\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1}{\sqrt{3}}\right]$ (SIMPLIFY THIS)

$= \frac{2}{3} \times \frac{3}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[-\frac{2}{x^2} - \frac{-2x^2(1-x^2)^{-\frac{1}{2}}-2(1-x^2)^{\frac{1}{2}}}{x^2}\right]$

$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2} + \frac{2x^2(1-x^2)^{-\frac{1}{2}}+2(1-x^2)^{\frac{1}{2}}}{x^2}\right]$


$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2} + \frac{2x^2+2(1-x^2)}{x^2(1-x^2)^{\frac{1}{2}}}\right]$

$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2} + \frac{2}{x^2(1-x^2)^{\frac{1}{2}}}\right]$

$= \frac{4}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left(\frac{1}{x^2(1-x^2)^{\frac{1}{2}}} - \frac{1}{x^2}\right)$

$= \frac{4}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \frac{1}{x^2}\left(\frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\right)$

$= \frac{4}{x^2\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3x^2} \times \frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}$

$= \frac{4}{3x^2\left(\frac{2}{3x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3x^2} \times \frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}$

$= 4+4(1-x^2)+x^2-8(1-x^2)^{\frac{1}{2}}+4-4(1-x^2)^{\frac{1}{2}}+3x^2$

$= 4+4-4x^2+x^2-8(1-x^2)^{\frac{1}{2}}+4-4(1-x^2)^{\frac{1}{2}}+3x^2$

$= 8+4-8(1-x^2)^{\frac{1}{2}}-4(1-x^2)^{\frac{1}{2}}$

$= (8+4)-(1-x^2)^{\frac{1}{2}}(8+4)$

$= (8+4)(1-(1-x^2)^{\frac{1}{2}})$

$= 4(2+2)(1-(1-x^2)^{\frac{1}{2}})$

$= \frac{4}{4(2+2)(1-(1-x^2)^{\frac{1}{2}})} \times \frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} = \frac{1}{(2+2)(1-x^2)^{\frac{1}{2}}}$

AS REQUIRED