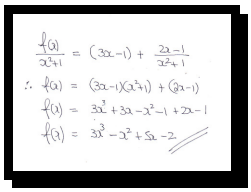
POLYNOMIALS

Question 1 (**)

When $f(x)$ is divided by $\left(x^2+1\right)$ the quotient is $(3x-1)$ and the remainder is $(2x-1)$.

Determine an expression for $f(x)$.

$$\boxed{f(x)=3x^3-x^2+5x-2}$$

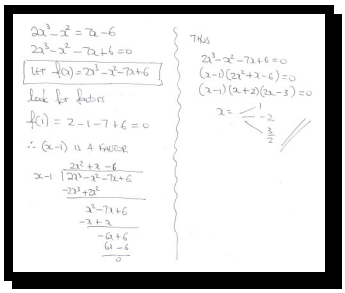


Question 2 (***)

Find the three solutions of the cubic equation

$$2x^3-x^2=7x-6\,.$$

$$\boxed{x=-2,1,\frac{3}{2}}$$

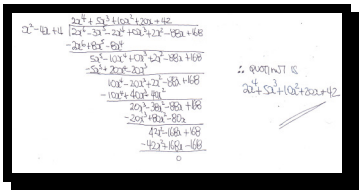


Question 3 (***)

Find the quotient of the division of

$$2x^6-3x^5-2x^4+2x^2-88x+168 \qquad \text{by} \qquad x^2-4x+4\,.$$

$$\boxed{2x^4+5x^3+10x^2+20x+42}$$

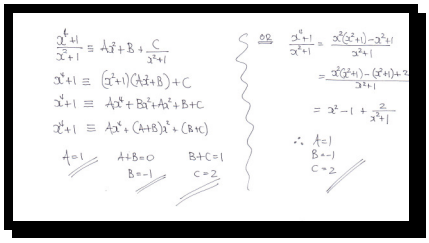


Question 4 (***)

$$\frac{x^4+1}{x^2+1}\equiv Ax^2+B+\frac{C}{x^2+1}\,.$$

Find the value of each of the constants A, B and C.

$$\boxed{A=1}\,,\ \boxed{B=-1}\,,\ \boxed{C=2}$$



Question 5 (*+)**

A quintic polynomial is defined, in terms of the constants a and b, by

$$f(x)=x^5+ax^4+bx^3-x^2+4x-3.$$

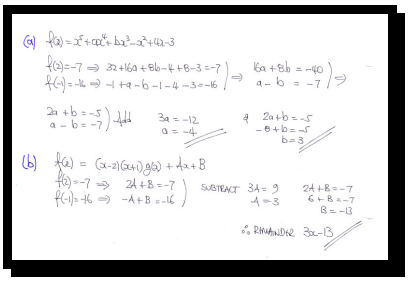
When $f(x)$ is divided by $(x-2)$ the remainder is -7.

When $f(x)$ is divided by $(x+1)$ the remainder is -16.

a) Determine in any order the value of a and the value of b.

b) Find the remainder when $f(x)$ is divided by $(x-2)(x+1)$.

$\boxed{}$, $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{3x-13}$



Question 6 (*+)**

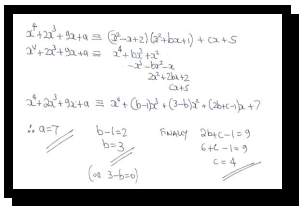
A polynomial $p(x)$ is defined, in terms of a constant a, by

$$p(x)=x^4+2x^3+9x+a\,.$$

When $p(x)$ is divided by x^2-x+2 the quotient is x^2+bx+1 and the remainder is $cx+5$, where b and c are constants.

Find the value of a, b and c.

$$\boxed{a=7,\ b=3,\ c=4}$$



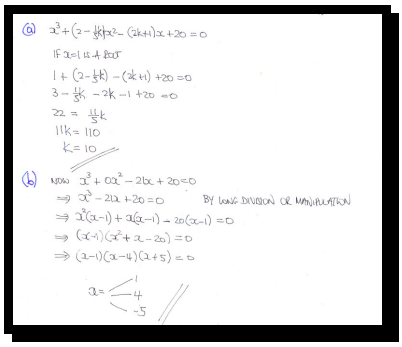
Question 7 (*+)**

$$x^3+\left(2-\frac{1}{5}k\right)x^2+(2k+1)x+20=0\,.$$

a) Determine the value of the real constant k, if the above equation is to have $x=1$ as one of its roots.

b) Solve the equation for the value of k, found in part **(a)**.

$\boxed{k=10}$, $\boxed{x=-5,4,1}$



Question 5 (*+)**

The following information is given for a polynomial $f(x)$.

- When $f(x)$ is divided by $(x-2)$ the remainder is 5.
- When $f(x)$ is divided by $(x+2)$ the remainder is -11.
- When $f(x)$ is divided by $(x+2)(x-2)$ the remainder is $ax+b$, and the quotient is $g(x)$, where a and b are constants.

a) Determine the value of a and the value of b.

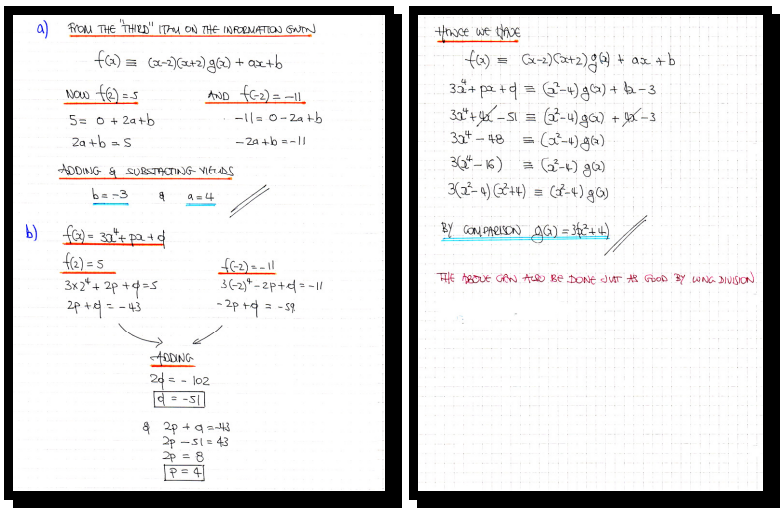
It is further given that

$$f(x)=3x^4+px+q,$$

where p and q are constants.

b) Find a simplified expression for $g(x)$.

$\boxed{}$, $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{g(x)=3\left(x^2+4\right)}$



Question 9 (****)

$$f(x)=x^3+(a+2)x^2-2x+b,$$

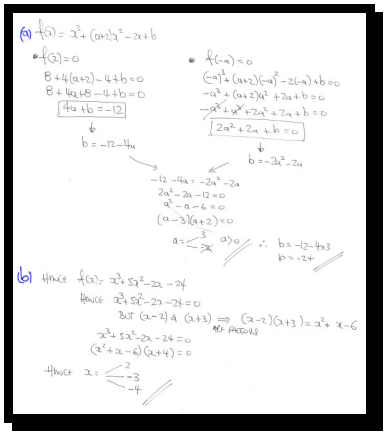
where a and b are non zero constants.

It is given that $(x-2)$ and $(x+a)$ are factors of $f(x)$, $a>0$.

a) By forming two equations show that $a=3$ and find the value of b.

b) Solve the equation $f(x)=0$.

$\boxed{b=-24}$, $\boxed{x=-4,-3,2}$



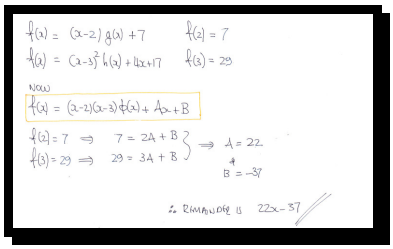
Question 10 (****)

When the polynomial $f(x)$ is divided by $(x-2)$ the remainder is 7.

When $f(x)$ is divided by $(x-3)^2$ the remainder is $(4x+17)$.

Find the remainder when $f(x)$ is divided by $(x-2)(x-3)$.

$\boxed{22x-37}$



Question 11 (****)

A polynomial $p(x)$ is given by

$$p(x)=4x^3-2x^2+x+5\,.$$

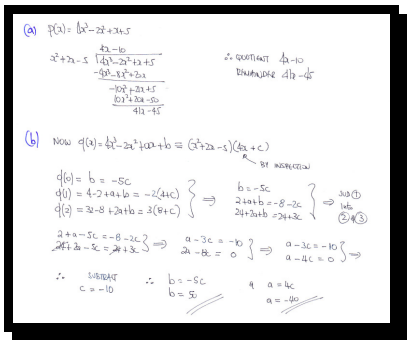
a) Find the remainder and the quotient when $p(x)$ is divided by x^2+2x-5.

A different polynomial $q(x)$ is defined as

$$q(x)=4x^3-2x^2+ax+b\,.$$

b) Find the value of each of the constants a and b so that when $q(x)$ is divided by x^2+2x-5 there is no remainder.

$\boxed{R=41x-45}$, $\boxed{Q=4x-10}$, $\boxed{a=-40,\ b=50}$



Question 12 (**+)**

$$f(x) \equiv x^4 - 9x^3 + 30x^2 - 44x .$$

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (x-3)(x-A)^3 + B ,$$

where A and B are constants.

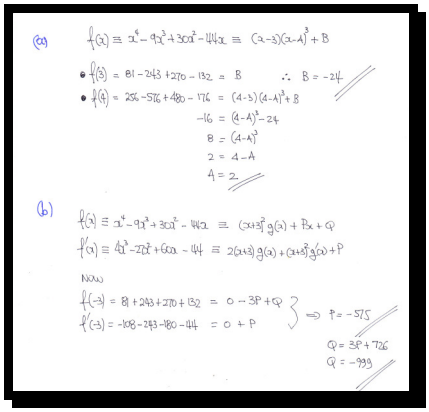
a) Find the value of A and the value of B.

The polynomial $f(x)$ also satisfies the relationship

$$f(x) \equiv (x+3)^2 g(x) + Px + Q , \text{ where } P \text{ and } Q \text{ are constants.}$$

b) Find the value of each of the constants P and Q.

$\boxed{A=2}$, $\boxed{B=-24}$, $\boxed{P=-575}$, $\boxed{Q=-999}$



Question 13 (**+)**

$$f(x)\equiv 9x^4+24x^3-32x-16\,.$$

The polynomial $f(x)$ has linear factors

$$(x+2) \quad \text{and} \quad (3x+2)$$

a) Show that the roots of the equation $f(x)=0$ are

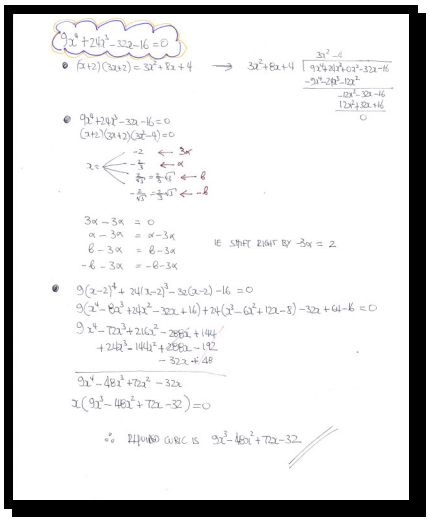
$$\alpha,\ 3\alpha,\ \beta \text{ and } -\beta,$$

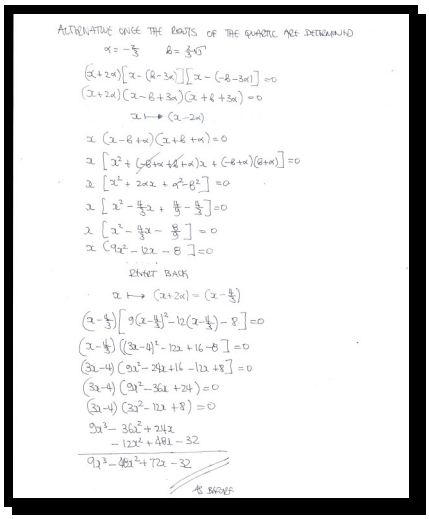
where α and β must be stated in exact form if appropriate.

b) Hence determine a cubic equation with integer coefficients with roots

$$-2\alpha,\ \beta-3\alpha \text{ and } -\beta-3a\,.$$

$$\boxed{\alpha=-\frac{2}{3}},\ \boxed{\beta=-\frac{2}{3}\sqrt{3}},\ \boxed{9x^3-48x^2+72x-32=0}$$





Question 14 (**+)**

A polynomial $f(x)$ is defined by

$$f(x) \equiv 2x^6 + ax^5 + bx^4 + 2x^2,$$

where a and b are constants.

When $f(x)$ is divided by $(x-2)(2x+1)$ the remainder is $(3x+2)$.

a) Determine the value of a and the value of b.

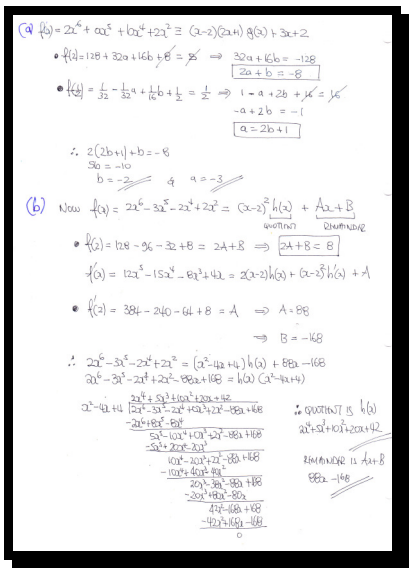
When $f(x)$ is divided by $(x-2)^2$ the quotient is $h(x)$ and the remainder is $(Ax+B)$, where A and B are constants..

b) Find …

i. … the value of A and the value of B.

ii. … an expression for $h(x)$.

$\boxed{a=-3}$, $\boxed{b=-2}$, $\boxed{A=88}$, $\boxed{B=-168}$, $\boxed{h(x)=2x^4+5x^3+10x^2+20x+42}$



Question 15 (*****)

A polynomial in x satisfies the relationship

$$f(x)=\left(x^2-4\right)g(x)+Ax+B,$$

where A and B are constants.

a) Find the value of A and the value of B, given that $f(2)=5$ and $f(-2)=-7$.

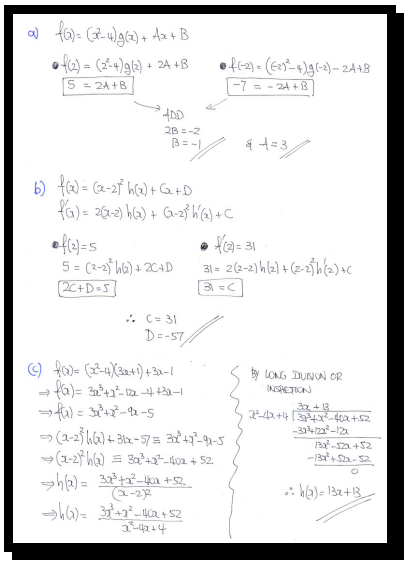
It is now given that the polynomial in x also satisfies the relationship

$$f(x)=(x-2)^2 h(x)+Cx+D.$$

b) Find the value of each of the constants C and D, given that $f'(2)=31$.

c) Given further that $g(x)=3x+1$, find $h(x)$.

$$\boxed{A=3},\ \boxed{B=-1},\ \boxed{C=31},\ \boxed{D=-57},\ \boxed{h(x)=3x+13}$$



Question 16 (*****)

A polynomial in x is given by

$$f(x)=x^8+kx^5-27x^2-13\text{, where } k \text{ is a constant.}$$

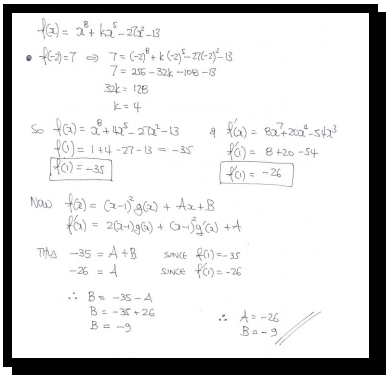
The polynomial also satisfies the relationship

$$f(x)=(x-1)^2 g(x)+Ax+B,$$

where A and B are constants.

Find the value of A and the value of B, given that $f(2)=7$

$$\boxed{A=-26}\text{, }\boxed{B=-9}$$



Question 17 (*****)

$$f(x) \equiv Ax^5 + Bx^4 + 8x^2,$$

where A and B are non zero constants.

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (2x-1)(x-2)g(x) + 169x - 82.$$

a) Find the value of A and the value of B.

b) Determine the polynomial $g(x)$.

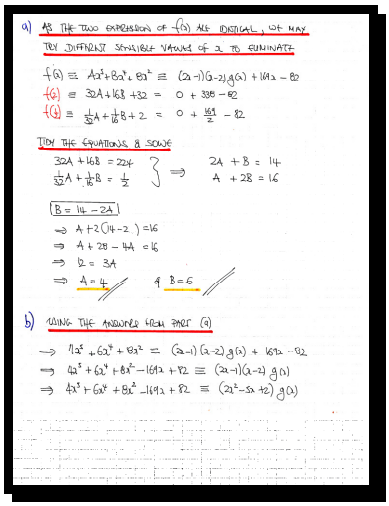
The polynomial $f(x)$ also satisfies the relationship

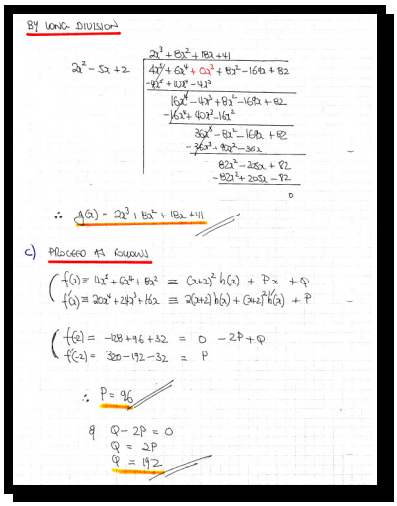
$$f(x) \equiv (x+2)^2 h(x) + Px + Q,$$

where P and Q are constants.

c) Find the value of each of the constants P and Q.

$\boxed{}$, $\boxed{A=4}$, $\boxed{B=6}$, $\boxed{g(x) \equiv 2x^3 + 8x^2 + 18x + 41}$, $\boxed{P=96}$, $\boxed{Q=192}$





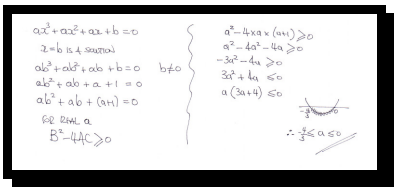
Question 18 (*****)

$$ax^3 + ax^2 + ax + b = 0,$$

where a and b are non zero real constants.

Given that $x = b$ is a root of the above cubic equation, determine the range of possible values of a.

$$\boxed{-\frac{4}{3} \leq a \leq 0}$$



Question 19 (*****)

$$f(x) \equiv x^5 + 3x^4 - 40x^3 - 47x^2 .$$

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (x-2)(x+A)(x+B)\left(x^2+3x-1\right) - 249x + 70 ,$$

where A and B are integer constants.

a) Find the value of A and the value of B.

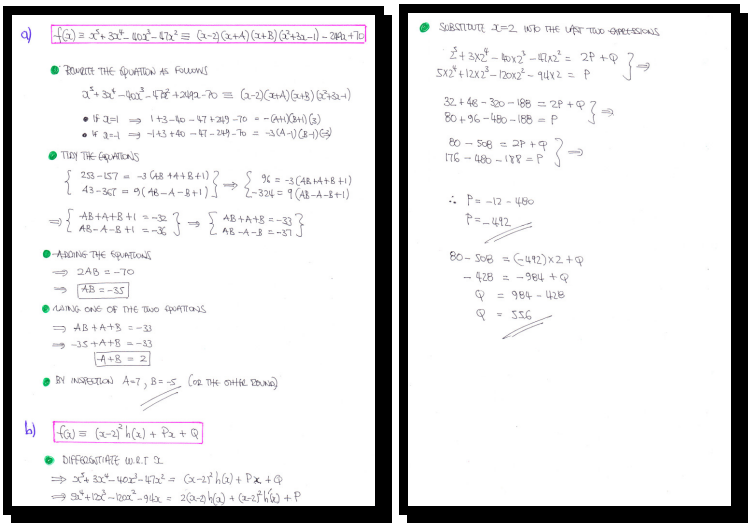
The polynomial $f(x)$ also satisfies the relationship

$$f(x) \equiv (x-2)^2 h(x) + Px + Q ,$$

where P and Q are constants.

b) Find the value of each of the constants P and Q.

$\boxed{}$, $\boxed{A=7}$, $\boxed{B=-5}$, $\boxed{P=-492}$, $\boxed{Q=556}$

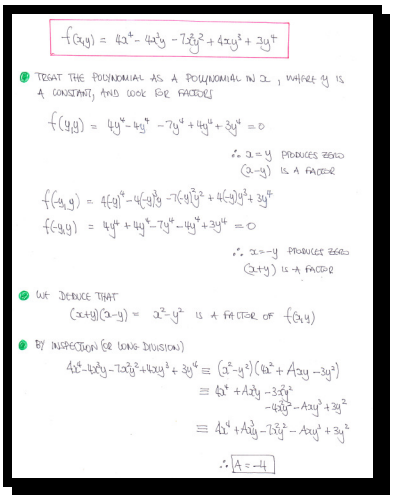


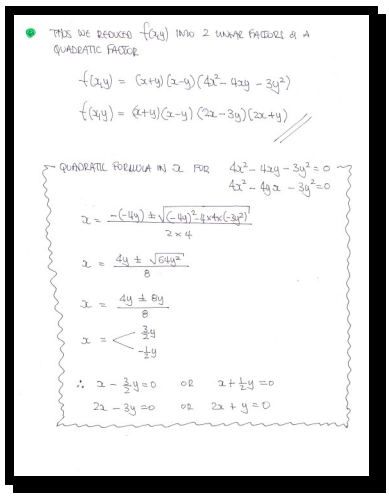
Question 20 (*****)

$$f(x,y)\equiv 4x^4-4x^3y-7x^2y^2+4xy^3+3y^4\,.$$

Express $f(x,y)$ as a product of 4 linear factors.

$\boxed{}$, $\boxed{f(x,y)\equiv(x+y)(x-y)(2x+y)(2x-3y)}$



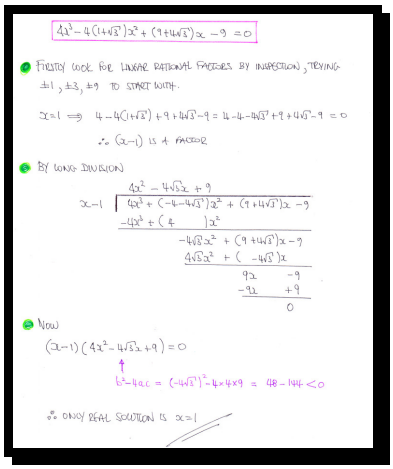


Question 21 (*****)

Solve the cubic equation

$$4x^3-4\left(1+\sqrt{3}\right)x^2+\left(9+4\sqrt{3}\right)x-9=0,\ x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=1}$

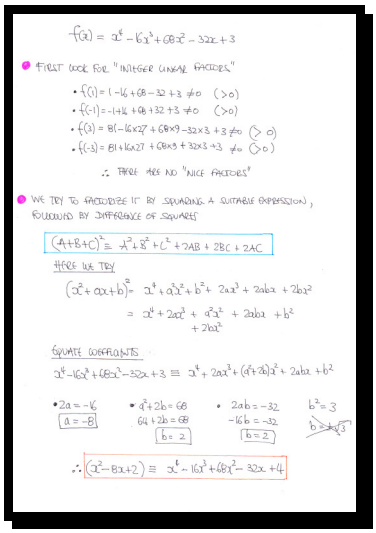


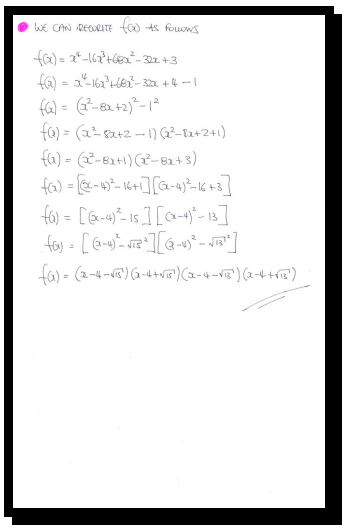
Question 22 (*****)

$$f(x) \equiv x^4 - 16x^3 + 68x^2 - 32x + 3, \quad x \in \mathbb{R}.$$

Factorize $f(x)$ into a product of 4 linear factors.

$\square$, $\boxed{f(x) \equiv \left(x-4-\sqrt{15}\right)\left(x-4+\sqrt{15}\right)\left(x-4-\sqrt{13}\right)\left(x-4+\sqrt{13}\right)}$



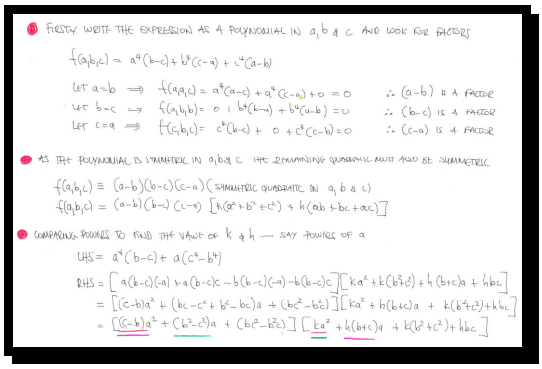


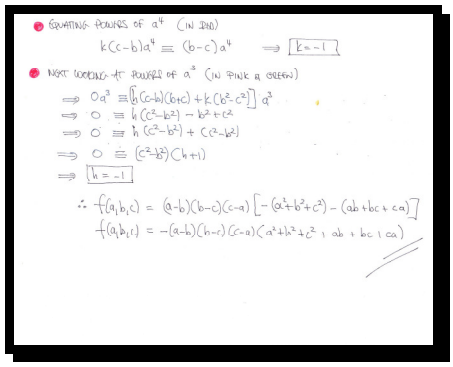
Question 23 **(*****)**

$$f(a,b,c) \equiv a^4(b-c)+b^4(c-a)+c^4(a-b).$$

Factorize $f(a,b,c)$ into a product of 3 linear factors and 1 quadratic factor.

$\square$, $\boxed{f(a,b,c) \equiv -(a-b)(b-c)(c-a)\left(a^2+b^2+c^2+ab+bc+ca\right)}$





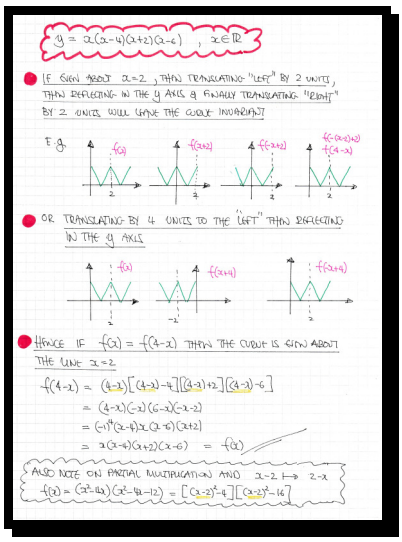
Question 24 (*****)

A quartic curve C has the following equation.

$$y=x(x-4)(x+2)(x-6), \quad x\in\mathbb{R}.$$

By considering suitable transformations, show that C is even about the straight line with equation $x=2$.

[ ], [proof]



Question 25 (*****)

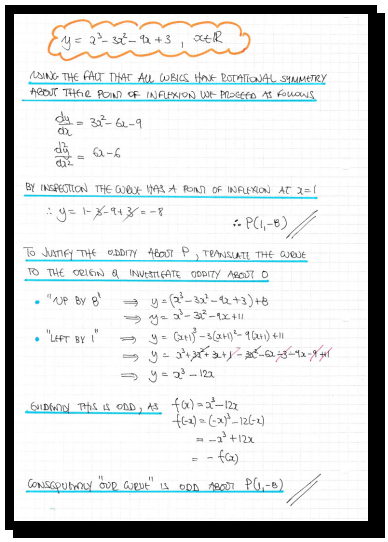
A cubic curve with equation

$$y = x^3 - 3x^2 - 9x + 3, \quad x \in \mathbb{R},$$

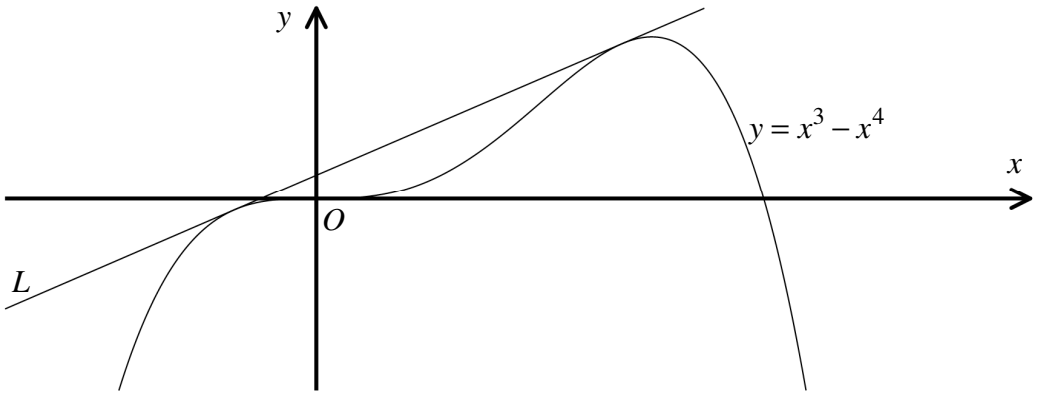
is odd about some point P.

Find the coordinates of P and use transformation arguments to justify the assertion that the curve is odd about P.

$\boxed{}$, $\boxed{P(1,-8)}$



Question 26 (*****)



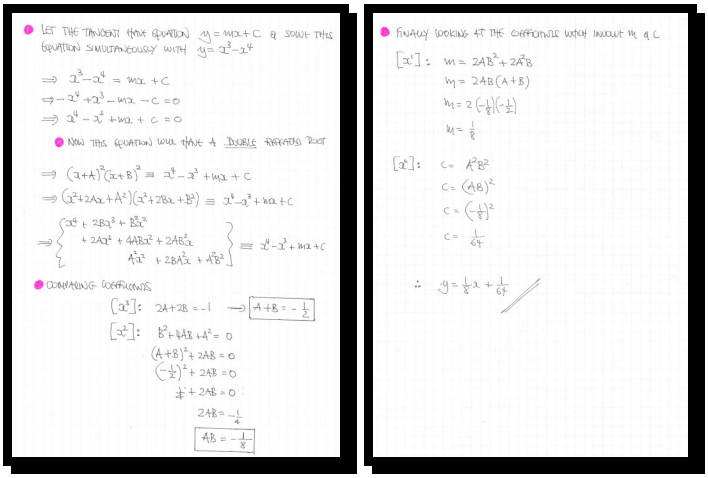
The figure above shows the curve C with equation

$$y = x^3 - x^4 .$$

The straight line L is a tangent to the C, at two distinct points.

Determine an equation of L.

$\square$, $\boxed{y = \frac{1}{8}x + \frac{1}{64}}$



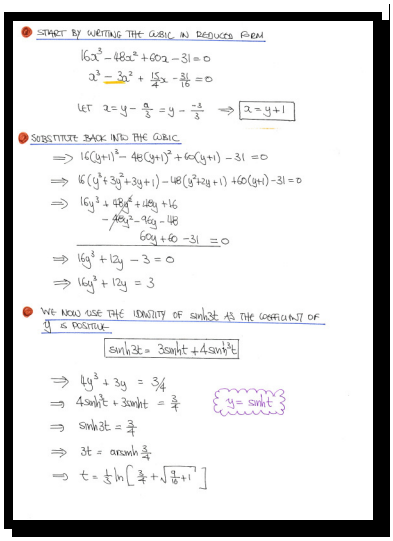
Question 27 **(*****)**

Solve the cubic equation

$$16x^3-48x^2+60x-31=0,\ x\in\mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\square$, $\boxed{x=1+2^{-\frac{2}{3}}-2^{-\frac{4}{3}}}$


START BY WRITING THE CUBIC IN REDUCED FORM

$16x^3-48x^2+60x-31=0$

$x^3-3x^2+\frac{15}{4}x-\frac{31}{16}=0$

LET $x=y-\frac{a}{3}=y-\frac{-3}{3}\Rightarrow \boxed{x=y+1}$

SUBSTITUTE BACK INTO THE CUBIC

$\Rightarrow 16(y+1)^3-48(y+1)^2+60(y+1)-31=0$

$\Rightarrow 16(y^3+3y^2+3y+1)-48(y^2+2y+1)+60(y+1)-31=0$

$\Rightarrow 16y^3+48y^2+48y+16$
$-48y^2-96y-48$
$60y+60-31=0$

$\Rightarrow 16y^3+12y-3=0$

$\Rightarrow 16y^3+12y=3$

WE NOW USE THE IDENTITY OF $\sinh 3t$ AS THE COEFFICIENT OF y IS POSITIVE

$\boxed{\sinh 3t=3\sinh t+4\sinh^3 t}$

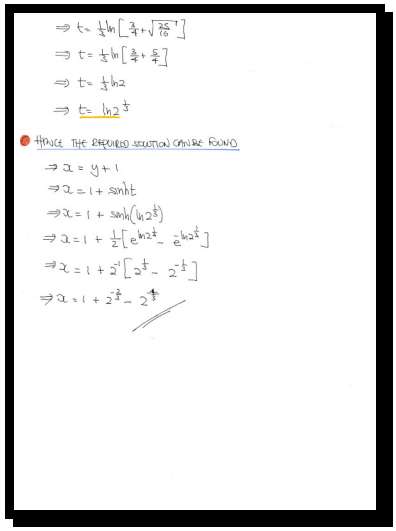
$\Rightarrow 4y^3+3y=3$

$\Rightarrow 4\sinh^3 t+3\sinh t=3$ $\quad y=\sinh t$

$\Rightarrow \sinh 3t=3$

$\Rightarrow 3t=\operatorname{arsinh}3$

$\Rightarrow t=\frac{1}{3}\ln\left[3+\sqrt{9+1}\right]$


$\Rightarrow t=\frac{1}{3}\ln\left[3+\sqrt{\frac{25}{16}}\right]$

$\Rightarrow t=\frac{1}{3}\ln\left[3+\frac{5}{4}\right]$

$\Rightarrow t=\frac{1}{3}\ln 2$

$\Rightarrow t=\ln 2^{\frac{1}{3}}$

HENCE THE REQUIRED SOLUTION CAN BE FOUND

$\Rightarrow x=y+1$

$\Rightarrow x=1+\sinh t$

$\Rightarrow x=1+\sinh\left(\ln 2^{\frac{1}{3}}\right)$

$\Rightarrow x=1+\frac{1}{2}\left[e^{\ln 2^{\frac{1}{3}}}-e^{-\ln 2^{\frac{1}{3}}}\right]$

$\Rightarrow x=1+2^{-1}\left[2^{\frac{1}{3}}-2^{-\frac{1}{3}}\right]$

$\Rightarrow x=1+2^{-\frac{2}{3}}-2^{-\frac{4}{3}}$

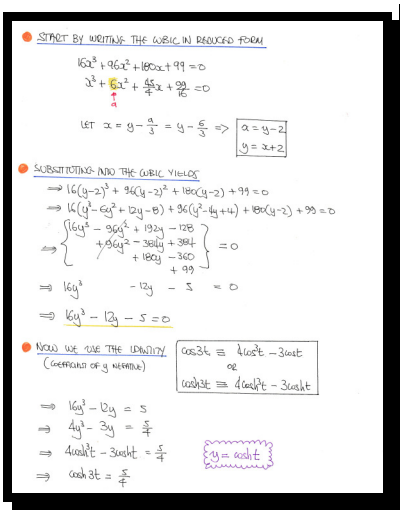
Question 28 **(*****)**

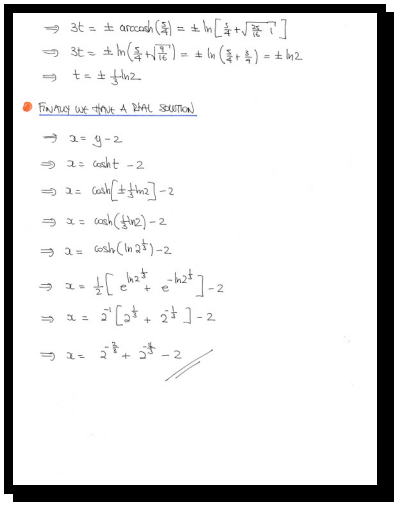
Solve the cubic equation

$$16x^3+96x^2+180x+99=0,\ x\in\mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\square$, $\boxed{x=-2+2^{-\frac{2}{3}}-2^{-\frac{4}{3}}}$





Question 29 **(*****)**

Solve the cubic equation

$$x^3-9x^2+3x-3=0,\ x\in\mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\square$, $\boxed{x=3+2^{\frac{5}{3}}+2^{\frac{4}{3}}}$

