DIFFERENTIAL EQUATIONS

(by separation of variables)

GENERAL SOLUTIONS

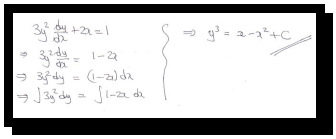
Question 1 (**)

Find a general solution of the differential equation

$$3y^2\frac{dy}{dx}+2x=1$$

giving the answer in the form $y^3=f(x)$.

$\boxed{}$, $\boxed{y^3=A-x^2+x}$



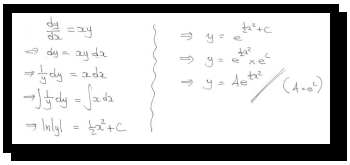
Question 2 (**)

Find a general solution of the differential equation

$$\frac{dy}{dx}=xy\,,\ x\neq 0\,,\ y\neq 0\,,$$

giving the answer in the form $y=f(x)$.

$$\boxed{y=A\mathrm{e}^{\frac{1}{2}x^2}}$$



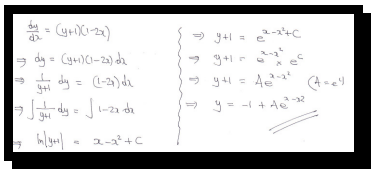
Question 3 (+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}=(y+1)(1-2x),\ y\neq -1.$$

giving the answer in the form $y=f(x)$.

$$\boxed{y=A\mathrm{e}^{x-x^2}-1}$$



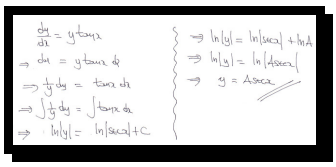
Question 4 (+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}=y\tan x\,,\ y>0$$

giving the answer in the form $y=f(x)$.

$$\boxed{y=A\sec x}$$



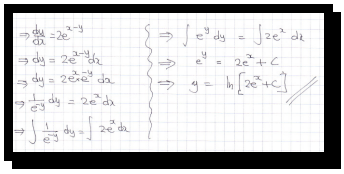
Question 5 (+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}=2\mathrm{e}^{x-y}$$

giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\ln\left(2\mathrm{e}^{x}+C\right)}$$



Question 6 (***)

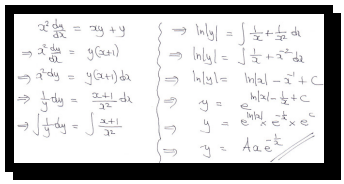
Find a general solution of the differential equation

$$x^2\frac{dy}{dx}=xy+y\,,\quad x\neq 0\,,\quad y\neq 0\,,$$

giving the answer in the form $y=f\left(x\right)$.

[*the final answer may not contain natural logarithms*]

$$\boxed{}\,,\quad \boxed{y=Ax\mathrm{e}^{-\frac{1}{x}}}$$



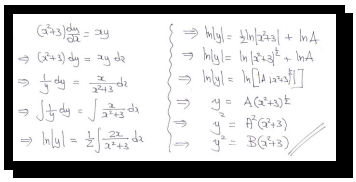
Question 7 (*+)**

Find a general solution of the differential equation

$$\left(x^2+3\right)\frac{dy}{dx}=xy\,,\ \ y>0\,,$$

giving the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=A\left(x^2+3\right)}$$



Question 8 (*+)**

Show that a general solution of the differential equation

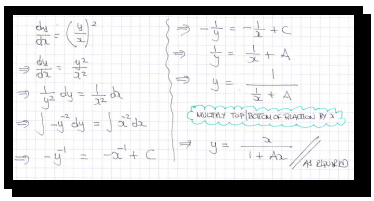
$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^2,$$

is given by

$$y=\frac{x}{1+Ax},$$

where A is an arbitrary constant.

proof



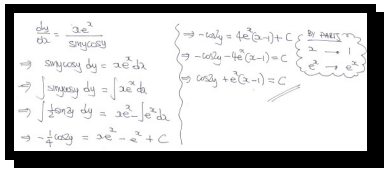
Question 9 (*+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}=\frac{x\mathrm{e}^{x}}{\sin y\cos y},$$

giving the answer in the form $f\left(x,y\right)=\text{constant}$.

$$\boxed{\cos 2y+4\mathrm{e}^{x}\left(x-1\right)=C\ \text{ or }\ \mathrm{e}^{x}\left(x-1\right)-\sin^{2}y=C\ \text{ or }\ \mathrm{e}^{x}\left(x-1\right)+\cos^{2}y=C}$$



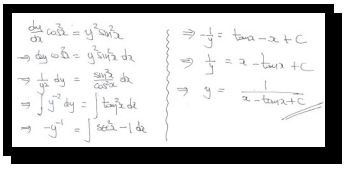
Question 10 (*+)**

Find a general solution of the differential equation

$$\frac{dy}{dx}\cos^{2}x=y^{2}\sin^{2}x$$

giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{C+x-\tan x}}$$



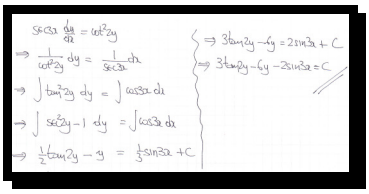
Question 11 (*+)**

Find a general solution of the differential equation

$$\sec 3x \frac{dy}{dx} = \cot^2 2y$$

giving the answer in the form $f(x, y) = c$.

$$\boxed{3\tan 2y - 6y - 2\sin 3x = C}$$



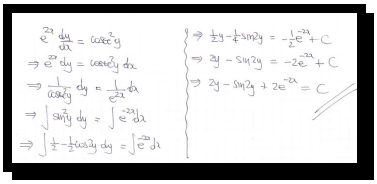
Question 12 (*+)**

Find a general solution of the differential equation

$$e^{2x} \frac{dy}{dx} = \operatorname{cosec}^2 y$$

giving the answer in the form $f(x, y) = c$.

$$\boxed{2x + 2e^{-2x} - \sin 2y = C}$$



Question 13 (****)

Show that a general solution of the differential equation

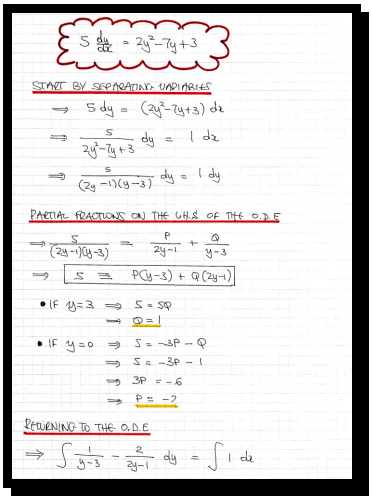
$$5\frac{dy}{dx}=2y^2-7y+3$$

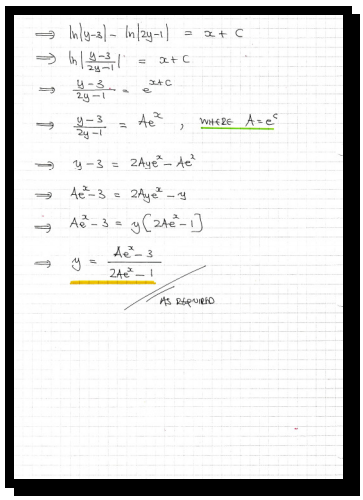
is given by

$$y=\frac{A\mathrm{e}^x-3}{2A\mathrm{e}^x-1},$$

where A is an arbitrary constant.

$\boxed{}$, proof


$5\frac{dy}{dx}=2y^2-7y+3$
START BY SEPARATING VARIABLES
$\Rightarrow 5\,dy=(2y^2-7y+3)\,dx$
$\Rightarrow \frac{5}{2y^2-7y+3}\,dy=1\,dx$
$\Rightarrow \frac{5}{(2y-1)(y-3)}\,dy=1\,dy$
PARTIAL FRACTIONS ON THE LHS OF THE O.D.E
$\Rightarrow \frac{5}{(2y-1)(y-3)}\equiv\frac{P}{2y-1}+\frac{Q}{y-3}$
$\Rightarrow 5\equiv P(y-3)+Q(2y-1)$
• IF $y=3$ $\Rightarrow 5=5P$ $\Rightarrow Q=1$
• IF $y=0$ $\Rightarrow 5=-3P-Q$ $\Rightarrow 5=-3P-1$ $\Rightarrow 3P=-6$ $\Rightarrow P=-2$
RETURNING TO THE O.D.E
$\Rightarrow \int\frac{1}{y-3}-\frac{2}{2y-1}\,dy=\int 1\,dx$



Question 14 (**+)**

Show that a general solution of the differential equation

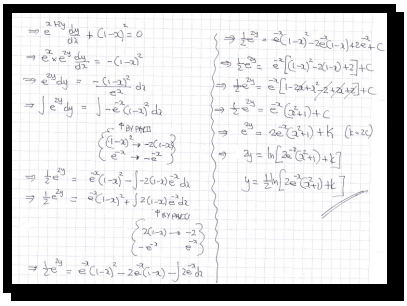
$$\mathrm{e}^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

is given by

$$y=\frac{1}{2}\ln\left[2\mathrm{e}^{-x}\left(x^2+1\right)+K\right],$$

where K is an arbitrary constant.

proof

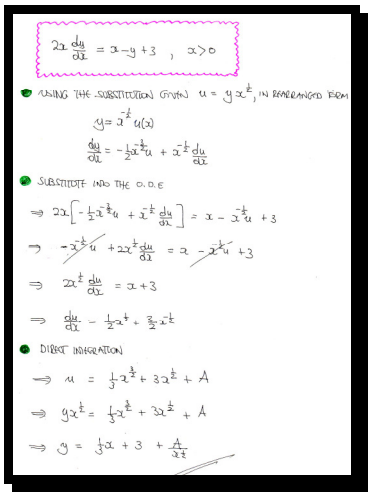


Question 15 **(*****)**

$$2x\frac{dy}{dx}=x-y+3\,,\ x>0\,.$$

Determine a general solution of the above differential equation, by using the substitution $u=y\sqrt{x}$.

$\boxed{}$, $\boxed{y=\frac{1}{3}x+3+Cx^{-\frac{1}{2}}}$

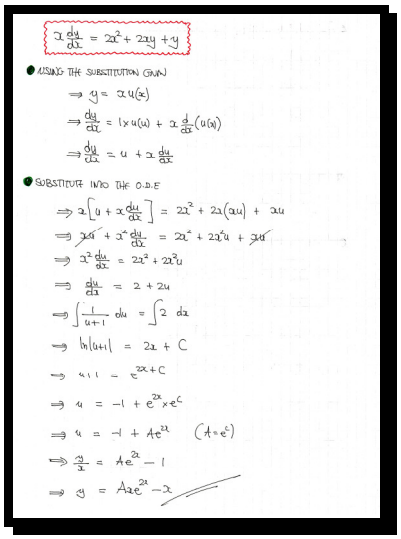


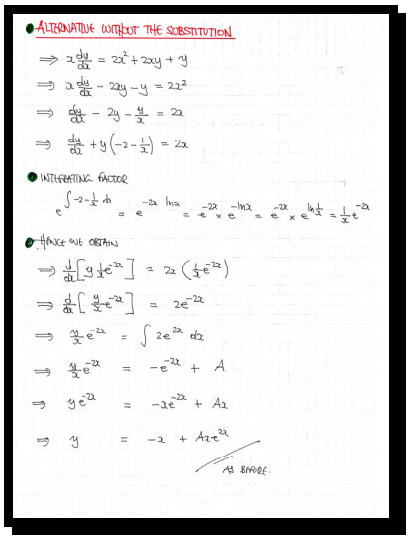
Question 16 (*****)

By using the substitution $y = xu$, where $u = f(x)$, or otherwise, find a simplified general solution for the following differential equation.

$$x\frac{dy}{dx} = 2x^2 + 2xy + y\,.$$

$\square$, $\boxed{y = Axe^{-x} - x}$



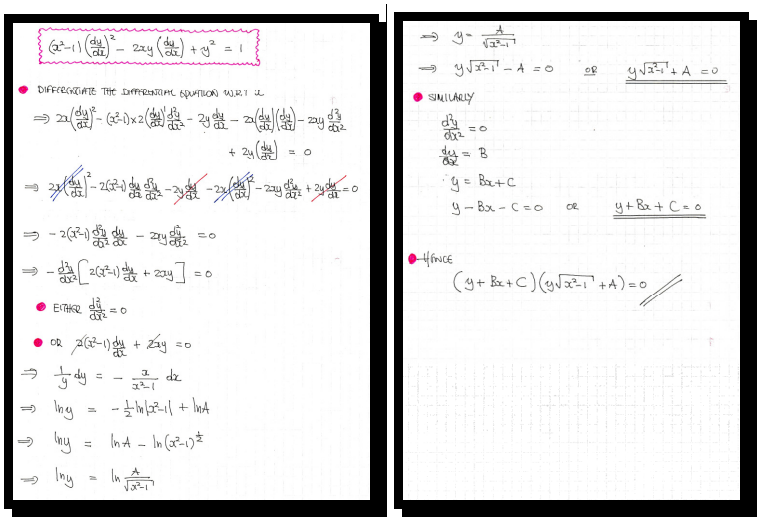


Question 17 (*****)

Use differentiation to find a simplified general solution for the following differential equation.

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1\,.$$

$\square$, $\boxed{\left(A+y\sqrt{x^2-1}\right)\left(y+Bx+C\right)=0}$



Question 18 (*****)

A circle touches the x axis at the origin O.

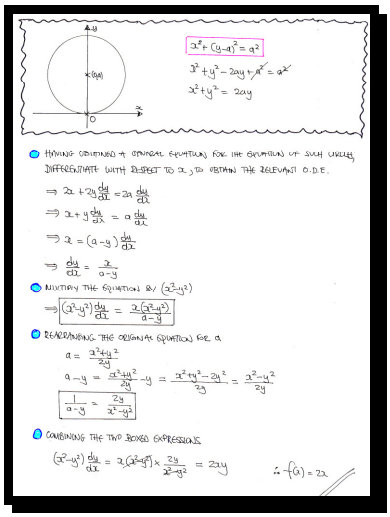
It is further given that the equation of such a circle satisfies the differential equation

$$\left(x^2-y^2\right)\frac{dy}{dx}=y\,f\left(x\right),$$

for some function f.

Use an algebraic method to find an expression for $f\left(x\right)$.

$\boxed{}$, $\boxed{f\left(x\right)=2x}$



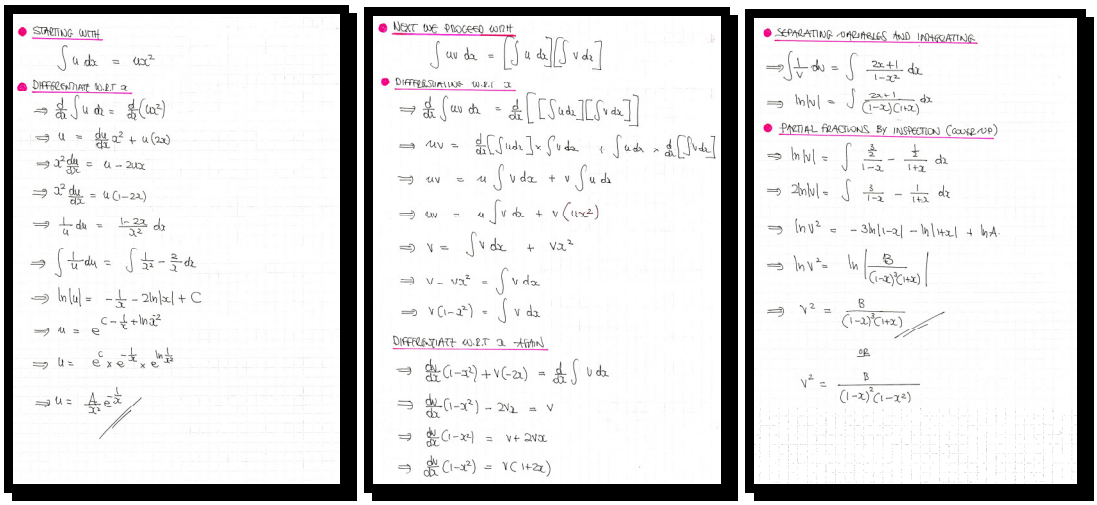
Question 19 (*****)

The non zero functions $u(x)$ and $v(x)$ satisfy the integral equations

$$\int u(x)\,dx = ux^2 \qquad \text{and} \qquad \int u(x)v(x)\,dx = \left[\int u(x)\,dx\right]\left[\int v(x)\,dx\right].$$

Determine, in terms of an arbitrary constant, a simplified expression for $u(x)$ and a similar expression for $\left[v(x)\right]^2$.

$$\square\,, \quad \boxed{u(x)=\frac{Ae^{-\frac{1}{x}}}{x^2}}\,, \quad \boxed{\left[v(x)\right]^2=\frac{B}{(1-x)^2\left(1-x^2\right)}}$$



Question 20 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi) x^{g(\phi)}$ is a solution of the differential equation,

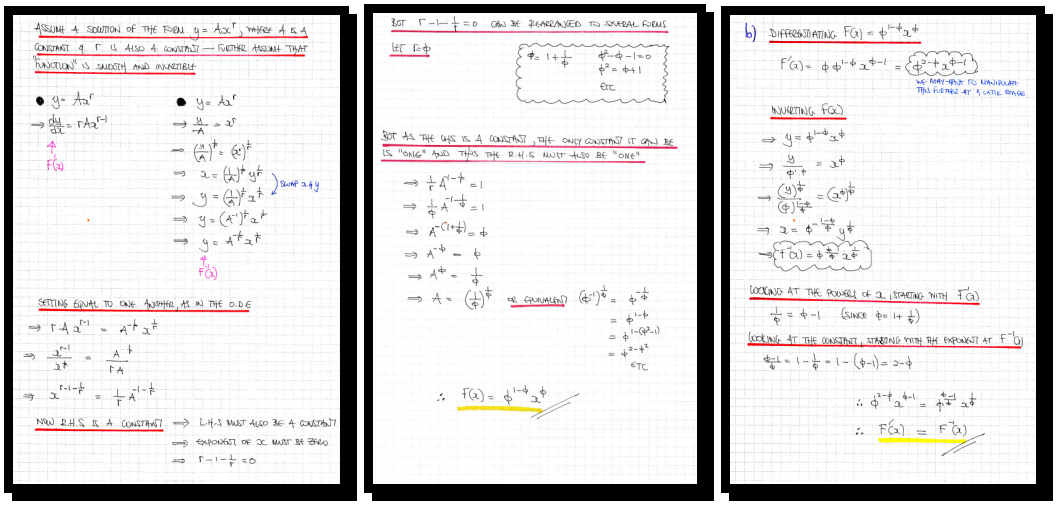
$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

$\square$, $\boxed{F(x) = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi} x^{\phi}}$



SPECIFIC SOLUTIONS

Question 1 (**)

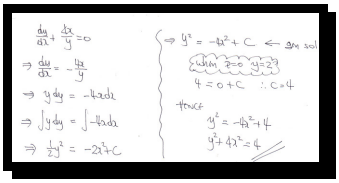
Solve the differential equation

$$\frac{dy}{dx}+\frac{4x}{y}=0,\ y\neq 0,$$

subject to the condition $y=2$ at $x=0$.

Give the answer in the form $f(x,y)=\text{constant}$.

$$\boxed{4x^2+y^2=4}$$



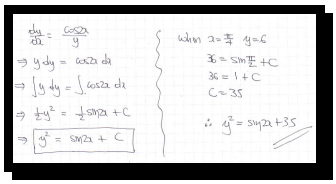
Question 2 (**)

Solve the differential equation

$$\frac{dy}{dx}=\frac{\cos 2x}{y},\quad x>0,\quad y>0,$$

subject to the condition $y=6$ at $x=\dfrac{\pi}{4}$, giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2=35+\sin 2x}$$



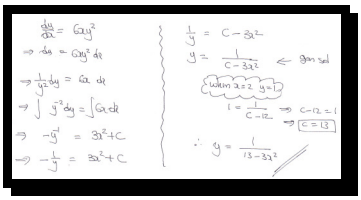
Question 3 (**)

Solve the differential equation

$$\frac{dy}{dx}=6xy^2,$$

with $y=1$ at $x=2$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{13-3x^2}}$$



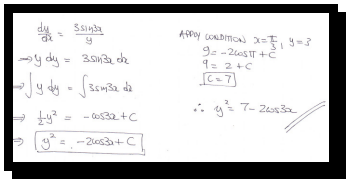
Question 4 (**)

Solve the differential equation

$$\frac{dy}{dx}=\frac{3\sin 3x}{y},$$

subject to the condition $y=3$ at $x=\frac{\pi}{3}$, giving the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=7-2\cos 3x}$$



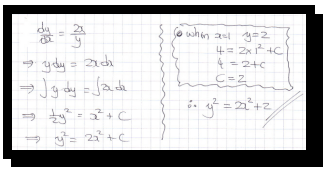
Question 5 (**)

Solve the differential equation

$$\frac{dy}{dx}=\frac{2x}{y},$$

with $y=2$ at $x=1$, giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2=2x^2+2}$$



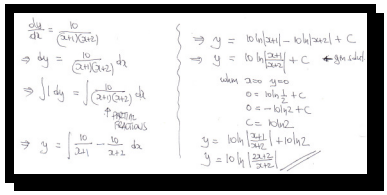
Question 6 (**)

Solve the differential equation

$$\frac{dy}{dx}=\frac{10}{(x+1)(x+2)},$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=10\ln\left|\frac{2x+2}{x+2}\right|}$$



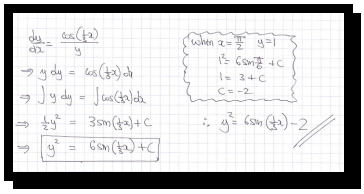
Question 7 (**)

Solve the differential equation

$$\frac{dy}{dx}=\frac{\cos\left(\frac{1}{3}x\right)}{y},$$

subject to the condition $y=1$ at $x=\frac{\pi}{2}$, giving the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=6\sin\left(\frac{1}{3}x\right)-2}$$



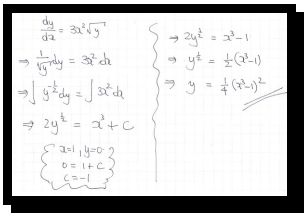
Question 8 (**)

Solve the differential equation

$$\frac{dy}{dx}=3x^2\sqrt{y}$$

subject to the condition $y=0$ at $x=1$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{4}\left(x^3-1\right)^2}$$



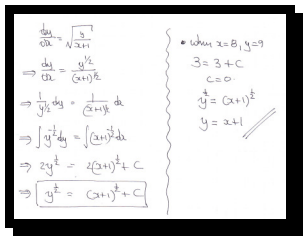
Question 9 (**)

Solve the differential equation

$$\frac{dy}{dx}=\sqrt{\frac{y}{x+1}}, \qquad y\neq 0,\ x\neq -1,$$

subject to the condition $y=9$ at $x=8$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=x+1}$$



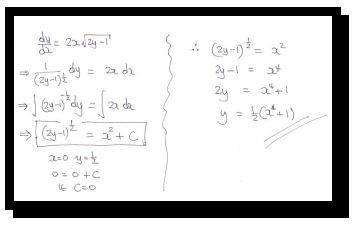
Question 10 (**)

Solve the differential equation

$$\frac{dy}{dx}=2x\sqrt{2y-1}, \quad y>\frac{1}{2}$$

subject to the condition $y=\frac{1}{2}$ at $x=0$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{2}\left(x^4+1\right)}$$



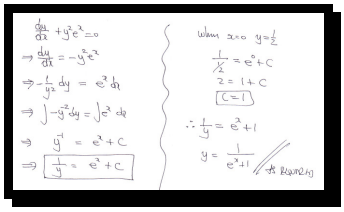
Question 11 (**)

Solve the differential equation

$$\frac{dy}{dx}+y^2\,\mathrm{e}^x=0,$$

subject to the condition $y=\frac{1}{2}$ at $x=0$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{\mathrm{e}^x+1}}$$



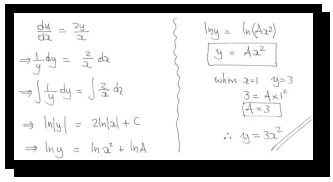
Question 12 (**)

$$\frac{dy}{dx}=\frac{2y}{x},\ x>0,\ y>0.$$

Show that the solution of the above differential equation subject to the boundary condition $y=3$ at $x=1$, is given by

$$y=3x^2.$$

$$\boxed{\text{proof}}$$



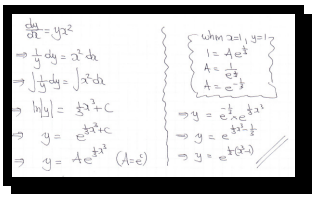
Question 13 (+)**

Solve the differential equation

$$\frac{dy}{dx} = yx^2, \ x \neq 0, \ y \neq 0,$$

subject to the condition $y = 1$ at $x = 1$, giving the answer in the form $y = f(x)$.

$$\boxed{y = \mathrm{e}^{\frac{1}{3}\left(x^3 - 1\right)}}$$



Question 14 (+)**

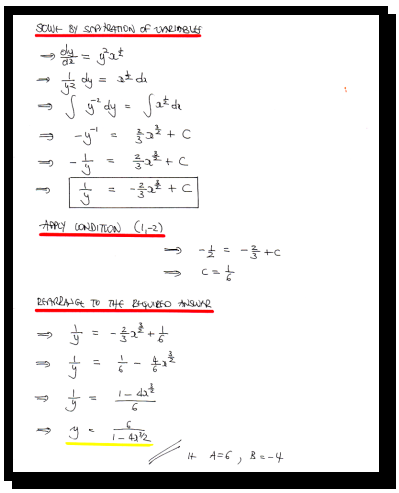
Solve the differential equation

$$\frac{dy}{dx}=y^2\sqrt{x}\,,\ x\neq 0\,,\ y\neq 0\,,$$

with $y=-2$ at $x=1$.

Give the answer in the form $y=\dfrac{A}{1+Bx^{\frac{3}{2}}}$, where A and B are integers.

$\square$, $\boxed{y=\dfrac{6}{1-4x^{\frac{3}{2}}}}$



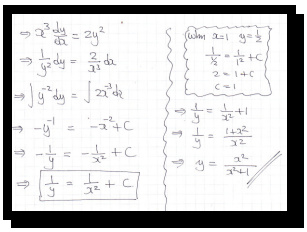
Question 15 (+)**

Solve the differential equation

$$x^3\frac{dy}{dx}=2y^2$$

subject to the condition $y=\frac{1}{2}$ at $x=1$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{x^2}{x^2+1}}$$



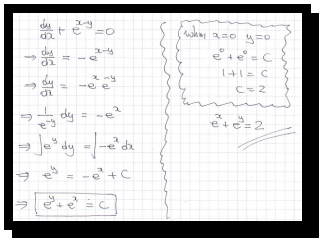
Question 16 (+)**

Solve the differential equation

$$\frac{dy}{dx}+\mathrm{e}^{x-y}=0,$$

subject to $y=0$ at $x=0$, giving the answer in the form $f\left(x,y\right)=\text{constant}$.

$$\boxed{\mathrm{e}^x+\mathrm{e}^y=2}$$



Question 17 (+)**

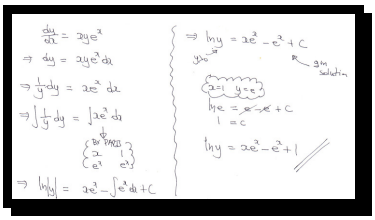
Solve the differential equation

$$\frac{dy}{dx}=xy\mathrm{e}^{x},\quad x>0,\quad y>0,$$

subject to boundary condition $y=\mathrm{e}$ at $x=1$.

Give the answer in the form $\ln y=f\left(x\right)$.

$$\boxed{\ln y=x\mathrm{e}^{x}-\mathrm{e}^{x}+1}$$



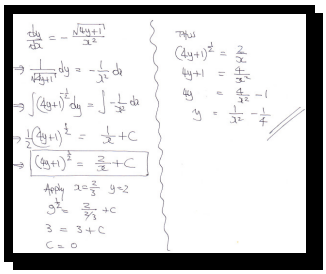
Question 18 (+)**

Solve the differential equation

$$\frac{dy}{dx}=-\frac{\sqrt{4y+1}}{x^{2}}$$

subject to the condition, $y=2$ at $x=\frac{2}{3}$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{x^{2}}-\frac{1}{4}}$$



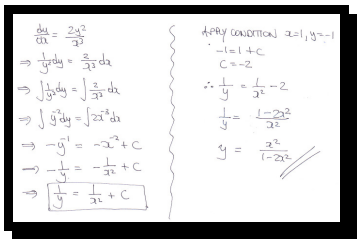
Question 19 (+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{2y^2}{x^3}$$

subject to the condition $y=-1$ at $x=1$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{x^2}{1-2x^2}}$$



Question 20 (+)**

Given that $y=2$ at $x=0$, solve the differential equation

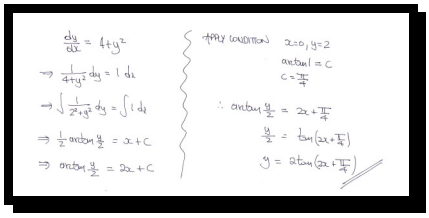
$$\frac{dy}{dx}=4+y^2,$$

giving the answer in the form $y=f(x)$.

You may assume that

$$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

$$\boxed{y=2\tan\left(2x+\frac{\pi}{4}\right)}$$



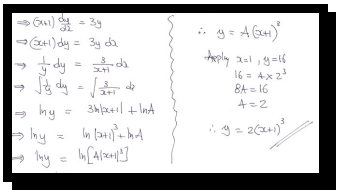
Question 21 (***)

$$(x+1)\frac{dy}{dx}=3y,\ \ y>0.$$

Solve the differential equation subject to the condition $y=16$ at $x=1$, to show that

$$y=2(x+1)^3.$$

proof



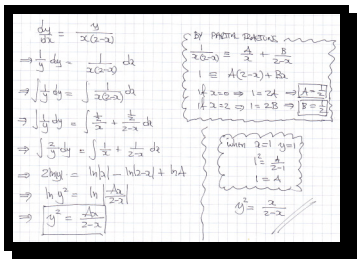
Question 22 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{x(2-x)},\ y>0$$

subject to the condition $y=1$ at $x=1$, giving the answer in the form $y^2=f(x)$.

$\square$, $\boxed{y^2=\dfrac{x}{2-x}}$



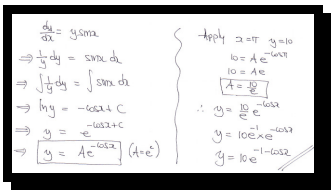
Question 23 (***)

Find the solution of the differential equation

$$\frac{dy}{dx}=y\sin x,\ y>0$$

subject to the condition $y=10$ at $x=\pi$, giving the answer in the form $y=f(x)$.

$$\boxed{y=10\mathrm{e}^{-1-\cos x}}$$



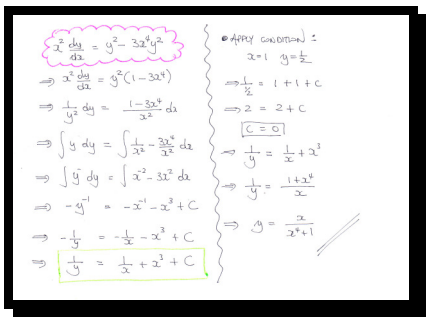
Question 24 (***)

Solve the differential equation

$$x^2\frac{dy}{dx}=y^2-3x^4y^2$$

subject to the condition $y=\frac{1}{2}$ at $x=1$, giving the answer in the form $y=f(x)$.

$\square$, $\boxed{y=\dfrac{x}{x^4+1}}$



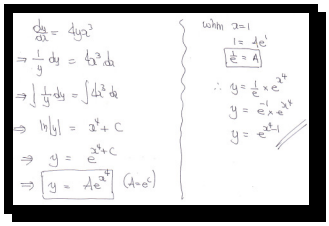
Question 25 (***)

Solve the differential equation

$$\frac{dy}{dx}=4yx^3,\quad y\neq 0$$

subject to the condition $y=1$ at $x=1$, giving the answer in the form $y=f(x)$.

$\boxed{y=\mathrm{e}^{x^4-1}}$



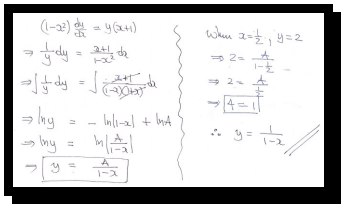
Question 26 (***)

Solve the differential equation

$$\left(1-x^2\right)\frac{dy}{dx}=y\left(x+1\right),\quad y\neq 0,\ x\neq\pm 1,$$

subject to the condition $y=2$ at $x=\frac{1}{2}$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\frac{1}{1-x}}$$



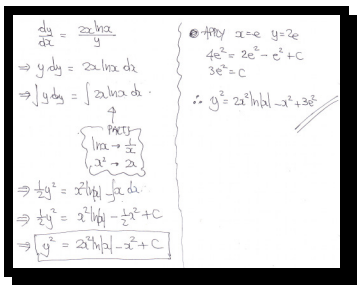
Question 27 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{2x\ln x}{y}\ ,\ x>0\,,\ y>0$$

subject to the condition $y=2\mathrm{e}$ at $x=\mathrm{e}$, giving the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=x^2\left(2\ln x-1\right)+3\mathrm{e}^2}$$



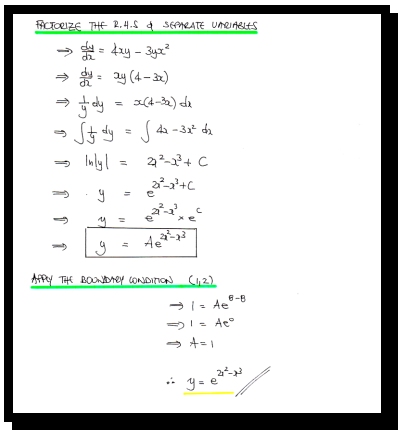
Question 28 (***)

Solve the differential equation

$$\frac{dy}{dx}=4xy-3yx^2$$

subject to the condition $y=1$ at $x=2$, giving the answer in the form $y=f\left(x\right)$.

$\boxed{}$, $\boxed{y=\mathrm{e}^{2x^2-x^3}}$



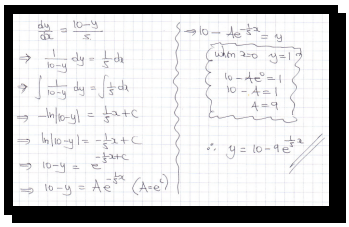
Question 29 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{10-y}{5}$$

subject to the condition $y=1$ at $x=0$, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=10-9\mathrm{e}^{-\frac{1}{5}x}}$$



Question 30 (***)

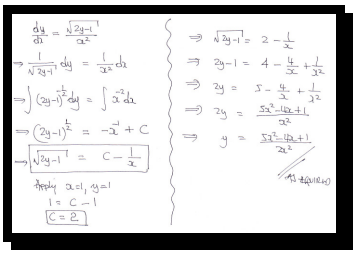
Show that the solution of the differential equation

$$\frac{dy}{dx}=\frac{\sqrt{2y-1}}{x^2},\quad x\neq 0,\, y>\frac{1}{2}$$

subject to the condition $y=1$ at $x=1$, is given by

$$y=\frac{5x^2-4x+1}{2x^2}.$$

proof



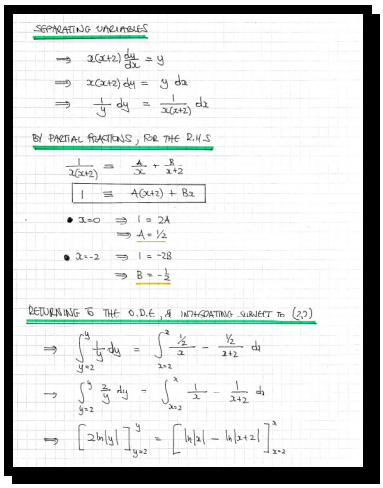
Question 31 (*+)**

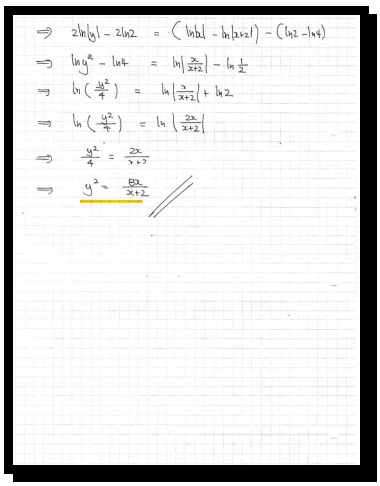
Solve the differential equation

$$x(x+2)\frac{dy}{dx}=y, \quad x>0, \quad y>0$$

subject to the condition $y=2$ at $x=2$, giving the answer in the form $y^2=f(x)$.

$\square$, $\boxed{y^2=\dfrac{8x}{x+2}}$





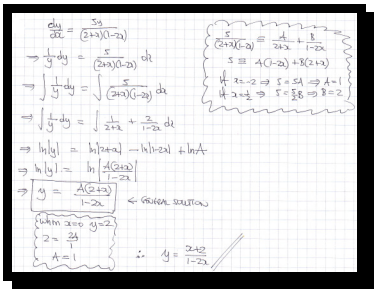
Question 32 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{5y}{(2+x)(1-2x)}$$

subject to the condition $y=2$ at $x=0$, giving the answer in the form $y=f(x)$.

$\square$, $\boxed{y=\dfrac{2+x}{1-2x}}$



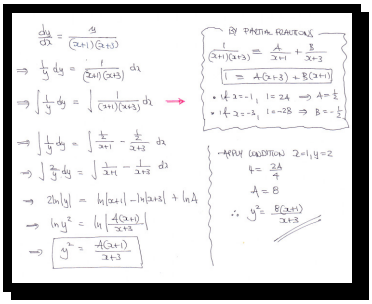
Question 33 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{(x+1)(x+3)},\ y>0,\ x>-1$$

subject to the condition $y=2$ at $x=1$, giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{8(x+1)}{x+3}}$$



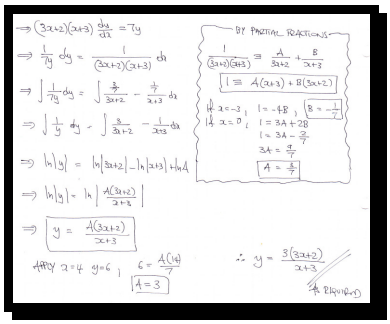
Question 34 (*+)**

$$(3x+2)(x+3)\frac{dy}{dx}=7y\,,\ y>0\,,\ x>-3\,.$$

Show that the solution of the above differential equation subject to the boundary condition, $y=6$ at $x=4$, is given by

$$y=\frac{3(3x+2)}{x+3}\,.$$

proof



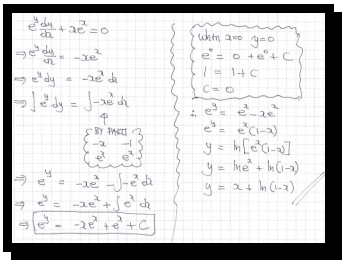
Question 35 (*+)**

Solve the differential equation

$$\mathrm{e}^{y}\frac{dy}{dx}+x\mathrm{e}^{x}=0\,,\quad x<1$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $y=f(x)$.

$$y=x+\ln(1-x)$$



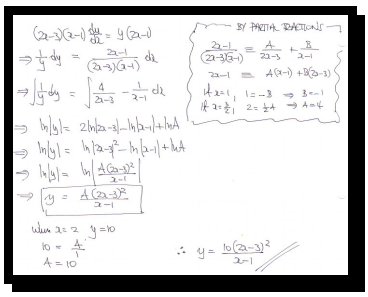
Question 36 (*+)**

Solve the differential equation

$$(2x-3)(x-1)\frac{dy}{dx}=y(2x-1),\ y\neq 0,$$

subject to the condition $y=10$ at $x=2$, giving the answer in the form $y=f(x)$.

$\square$, $\boxed{y=\dfrac{10(2x-3)^2}{x-1}}$



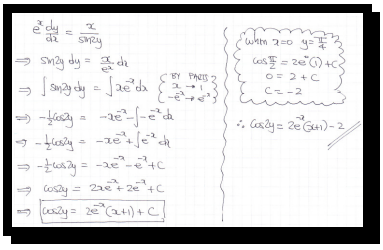
Question 37 (*+)**

Solve the differential equation

$$\mathrm{e}^x\frac{dy}{dx}=\frac{x}{\sin 2y},\ 0<y<\pi,$$

subject to $y=\dfrac{\pi}{4}$ at $x=0$, giving the answer in the form $\cos f(y)=g(x)$.

$\boxed{\cos 2y=2x\mathrm{e}^{-x}+2\mathrm{e}^{-x}-2}$



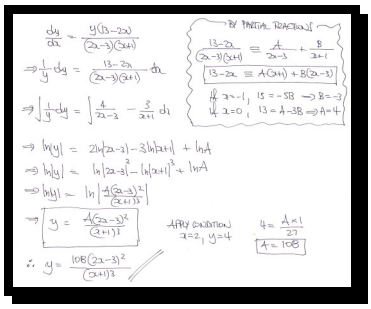
Question 38 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{y(13-2x)}{(2x-3)(x+1)},\ y\neq 0,$$

subject to the condition $y=4$ at $x=2$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{108(2x-3)^2}{(x+1)^3}}$$



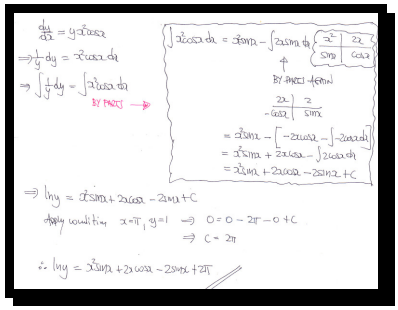
Question 39 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=yx^2\cos x,\quad x>0, y>0$$

subject to the condition $y=1$ at $x=\pi$, giving the answer in the form $\ln y=f(x)$.

$$\boxed{\ln y=x^2\sin x+2x\cos x-2\sin x+2\pi}$$



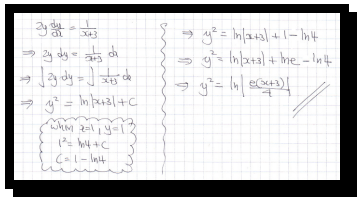
Question 40 (*+)**

$$2y\frac{dy}{dx}=\frac{1}{x+3},\ y\neq 0,$$

Show that the solution of the above differential equation, subject to the boundary condition $y=1$ at $x=1$, can be written as

$$y^2=\ln\left|\frac{\mathrm{e}(x+3)}{4}\right|.$$

proof



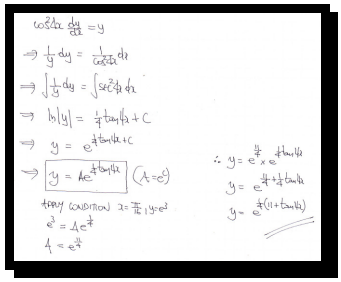
Question 41 (*+)**

$$\frac{dy}{dx}\cos^2 4x=y,\quad y>0.$$

Show that the solution of the above differential equation subject to the boundary condition $y=\mathrm{e}^3$ at $x=\frac{\pi}{16}$ is given by

$$y=\mathrm{e}^{\frac{1}{4}(11+\tan 4x)}.$$

, proof



Question 42 (*+)**

Show that if $y = a$ at $t = 0$, the solution of the differential equation

$$\frac{dy}{dt} = \omega\left(a^2 - y^2\right)^{\frac{1}{2}},$$

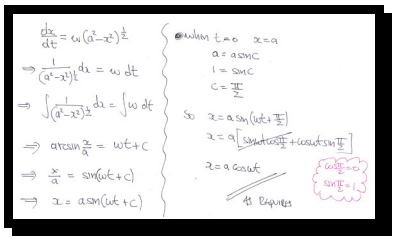
where a and ω are positive constants, can be written as

$$y = a\cos\omega t\,.$$

You may assume that

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C\,.$$

proof



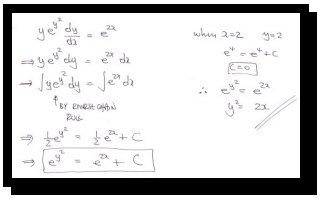
Question 43 (*+)**

Solve the differential equation

$$y\mathrm{e}^{y^2}\frac{dy}{dx} = \mathrm{e}^{2x}\,,\ x \neq 0\,,\ y \neq 0\,,$$

subject to the condition $y = 2$ at $x = 2$, giving the answer in the form $y^2 = f\left(x\right)$.

$y^2 = 2x$



Question 44 (*+)**

$$\frac{1}{(y-2)(y+1)} \equiv \frac{P}{(y-2)} + \frac{Q}{(y+1)}, \quad y \neq -1, \ 2$$

a) Find the value of each of the constants P and Q.

b) Hence, show that the solution of the differential equation

$$\frac{dy}{dx} = x^2(y-2)(y+1)$$

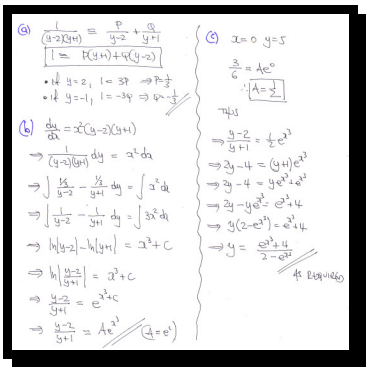
can be written as

$$\frac{y-2}{y+1} = Ae^{x^3}, \text{ where } A \text{ is a constant.}$$

c) Given further that $y = 5$ when $x = 0$, show clearly that

$$y = \frac{4+e^{x^3}}{2-e^{x^3}}.$$

$$\boxed{P = \frac{1}{3}, \ Q = -\frac{1}{3}}$$



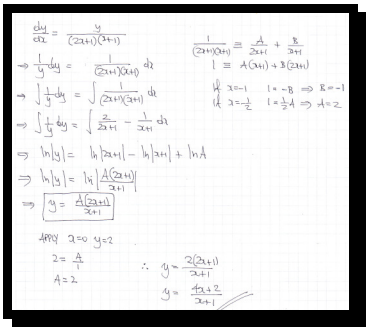
Question 45 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{(2x+1)(x+1)},\quad x>-\frac{1}{2},\quad y>0$$

subject to the condition $y=2$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{4x+2}{x+1}}$$



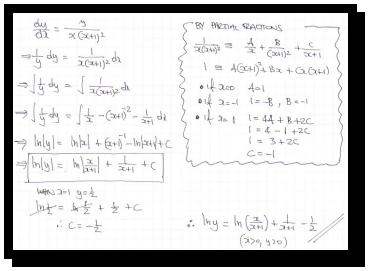
Question 46 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{x(x+1)^2},\quad x>0,\ y>0$$

subject to the condition $y=\frac{1}{2}$ at $x=1$, giving the answer in the form $\ln y=f(x)$.

$$\boxed{\ln y=\ln\left(\frac{x}{x+1}\right)+\frac{1}{x+1}-\frac{1}{2}}$$



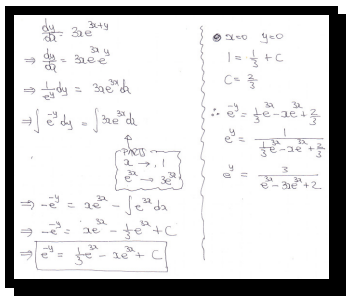
Question 47 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=3x\,\mathrm{e}^{3x+y}$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $\mathrm{e}^{y}=f\left(x\right)$.

$$\boxed{\mathrm{e}^{y}=\frac{3}{\mathrm{e}^{3x}-3x\,\mathrm{e}^{3x}+2}}$$



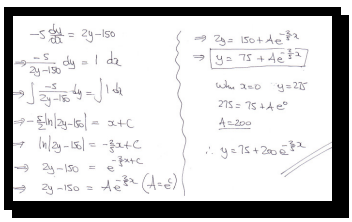
Question 48 (*+)**

$$-5\frac{dy}{dx}=2y-150\,,\quad y<75$$

Solve the above differential equation, given that when $x=0$, $y=275$.

Give the answer in the form $y=f\left(x\right)$.

$\boxed{}$, $\boxed{y=75+200\,\mathrm{e}^{-\frac{2}{5}x}}$



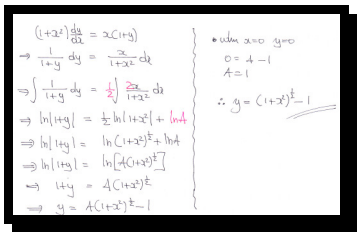
Question 49 (*+)**

$$\left(1+x^2\right)\frac{dy}{dx}=x\left(1+y\right),\text{ with }y=0\text{ at }x=0.$$

Show that the solution of the above differential equation is

$$y=\left(1+x^2\right)^{\frac{1}{2}}-1.$$

proof



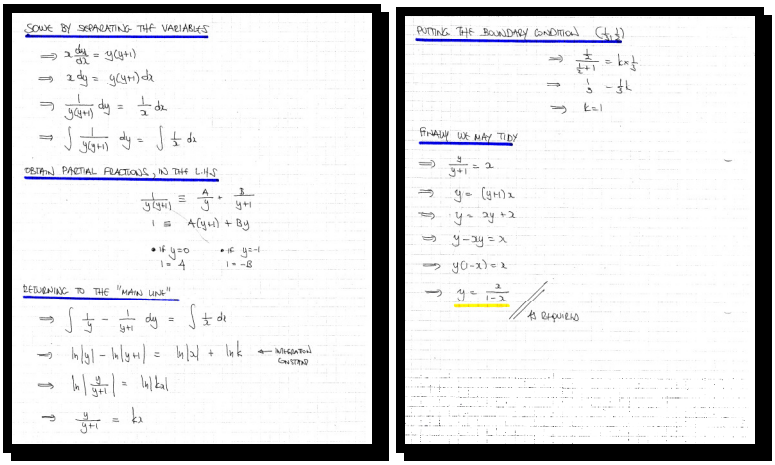
Question 50 **(***+)**

$$x\frac{dy}{dx}=y\left(y+1\right),\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to the boundary condition $y=\frac{1}{2}$ at $x=\frac{1}{3}$, is given by

$$y=\frac{x}{1-x}.$$

, proof



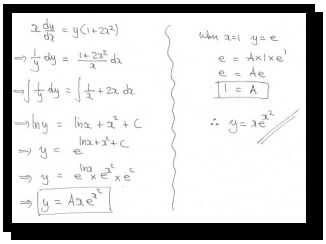
Question 51 (*+)**

$$x\frac{dy}{dx}=y\left(1+2x^2\right),\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to $y=\mathrm{e}$ at $x=1$, is

$$y=x\mathrm{e}^{x^2}.$$

proof



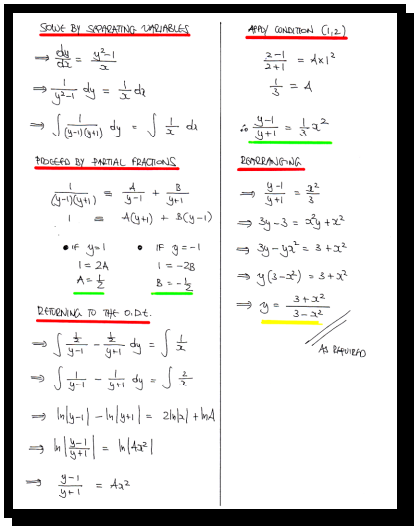
Question 52 (****)

$$\frac{dy}{dx}=\frac{y^2-1}{x},\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to $y=2$ at $x=1$, is

$$y=\frac{3+x^2}{3-x^2}.$$

[ ], [proof]



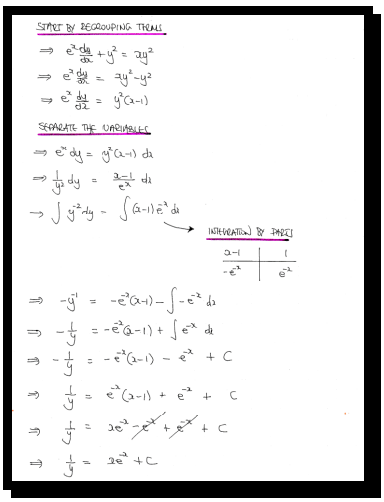
Question 53 (****)

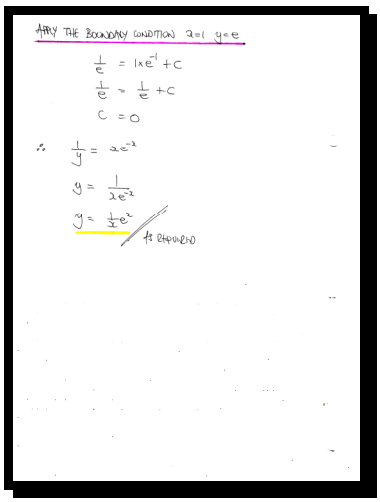
$$e^x \frac{dy}{dx} + y^2 = xy^2, \ x > 0, \ y > 0$$

Show that the solution of the above differential equation subject to $y = e$ at $x = 1$, is

$$y = \frac{1}{x} e^x .$$

[ ], proof





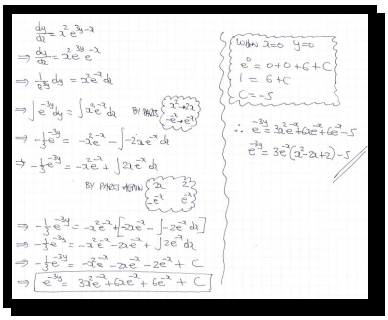
Question 54 (****)

Solve the differential equation

$$\frac{dy}{dx}=x^2\,\mathrm{e}^{3y-x}$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $\mathrm{e}^{f(y)}=g(x)$.

$$\boxed{\mathrm{e}^{-3y}=3\mathrm{e}^{-x}\left(x^2+2x+2\right)-5}$$



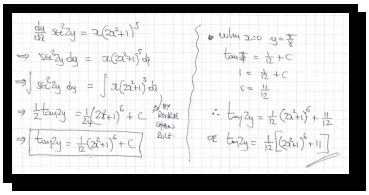
Question 55 (****)

Solve the differential equation

$$\frac{1}{x}\frac{dy}{dx}=\left(2x^2+1\right)^5\cos^2 2y$$

subject to $x=0$, $y=\dfrac{\pi}{8}$, giving the answer in the form $\tan f(y)=g(x)$

$$\boxed{\tan 2y=\frac{1}{12}\left[\left(2x^2+1\right)^6+11\right]}$$



Question 56 (****)

Solve the differential equation

$$(2x+1)-\frac{dy}{dx}(2x-1)^3=0$$

subject to the condition $y=0$ at $x=0$, giving the answer as $y=f(x)$.

$$\boxed{y=-\frac{x}{(2x-1)^2}}$$



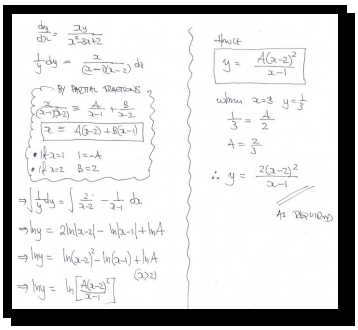
Question 57 (****)

$$\frac{dy}{dx}=\frac{xy}{x^2-3x+2},\ x,y>2$$

Solve the differential equation above, subject to the boundary condition $y=\frac{1}{3}$ at $x=3$, to show that

$$y=\frac{2(x-2)^2}{3(x-1)}.$$

proof



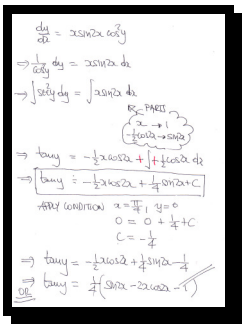
Question 58 (****)

Solve the differential equation

$$\frac{dy}{dx}=x\sin 2x\cos^2 y$$

subject to the condition $x=\frac{\pi}{4}$, $y=0$, giving the answer in the form $\tan y=f\left(x\right)$.

$$\boxed{\tan y=\frac{1}{4}\left(\sin 2x-2x\cos 2x-1\right)}$$



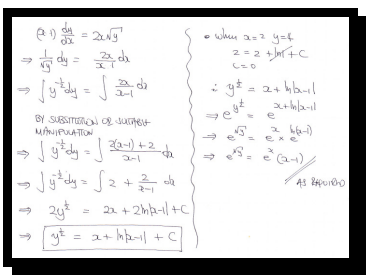
Question 59 (****)

$$\left(x-1\right)\frac{dy}{dx}=2x\sqrt{y}$$

Solve the differential equation above, subject to the boundary condition $y=4$ at $x=2$, to show that

$$\mathrm{e}^{\sqrt{y}}=\mathrm{e}^{x}\left(x-1\right).$$

$$\boxed{\text{proof}}$$



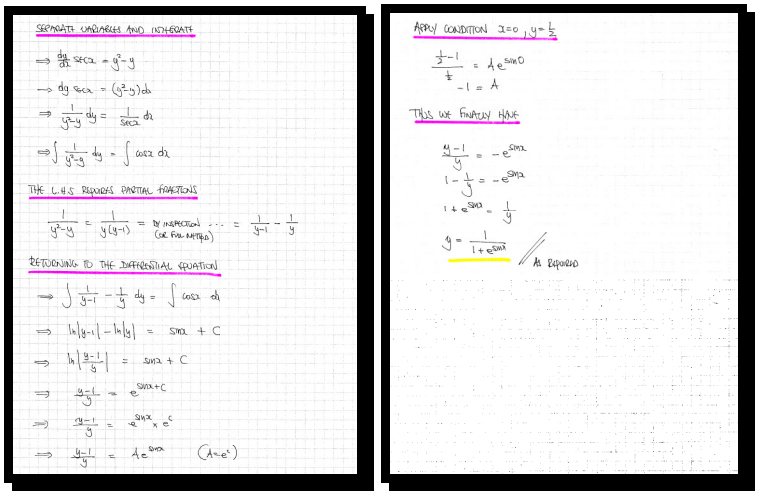
Question 60 (****)

$$\frac{dy}{dx}\sec x = y^2 - y$$

Solve the differential equation above, subject to the boundary condition, $y = \frac{1}{2}$ at $x = 0$, to show that

$$y = \frac{1}{1+\mathrm{e}^{\sin x}}.$$

[ ], [proof]



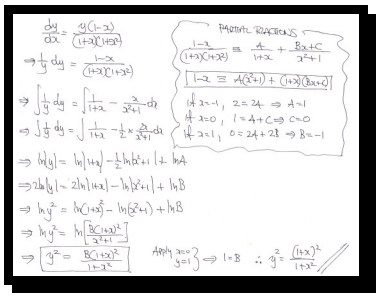
Question 61 (****)

Solve the differential equation

$$\frac{dy}{dx}=\frac{y(1-x)}{(1+x)(1+x^2)},$$

subject to the condition $x=0$, $y=1$, giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{(1+x)^2}{1+x^2}}$$



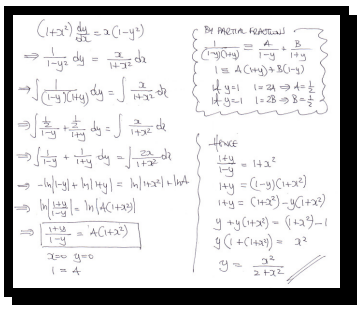
Question 62 (****)

Solve the differential equation

$$\left(1+x^2\right)\frac{dy}{dx}=x\left(1-y^2\right),\quad y\neq\pm1,$$

subject to the condition $y=0$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{x^2}{x^2+2}}$$



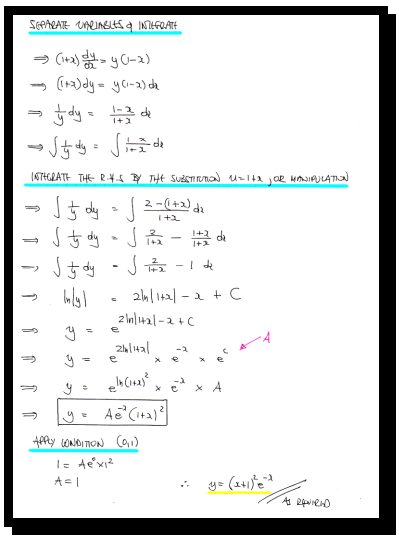
Question 63 (****)

$$(1+x)\frac{dy}{dx}=y(1-x),\ y>0,\ x>-1.$$

Solve the above given differential equation, subject to the boundary condition $y=1$ at $x=0$, to show that

$$y=(x+1)^2\mathrm{e}^{-x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



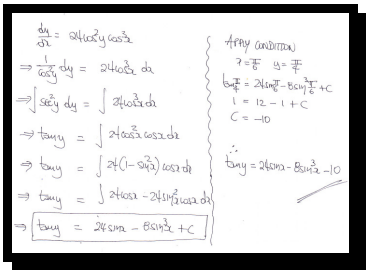
Question 64 (**+)**

Solve the differential equation

$$\frac{dy}{dx}=24\cos^2 y\cos^3 x$$

subject to the condition $y=\frac{\pi}{4}$ at $x=\frac{\pi}{6}$, giving the answer in the form $\tan y=f\left(x\right)$.

$$\boxed{\tan y=24\sin x-8\sin^3 x-10}$$



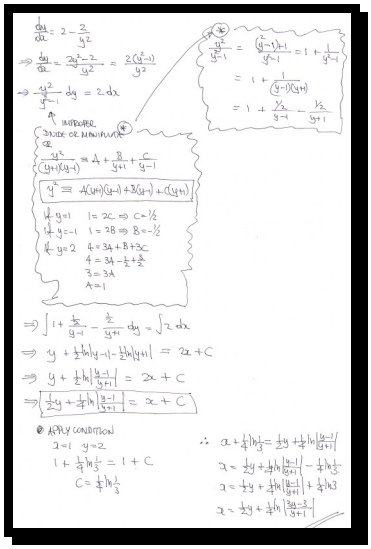
Question 65 (**+)**

Solve the differential equation

$$\frac{dy}{dx}=2-\frac{2}{y^2},$$

subject to the condition $y=2$ at $x=1$, giving the answer in the form $x=f\left(y\right)$.

$$\boxed{x=\frac{1}{2}y+\frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|}$$



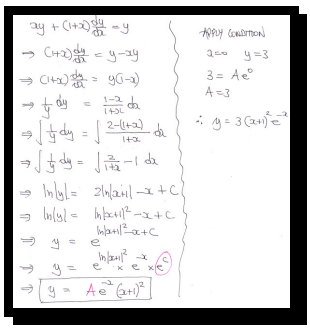
Question 66 (**+)**

$$xy+(1+x)\frac{dy}{dx}=y\,.$$

Solve the differential equation subject to the condition $y=3$ at $x=0$, to show that

$$y=3(1+x)^2\,\mathrm{e}^{-x}\,.$$

proof



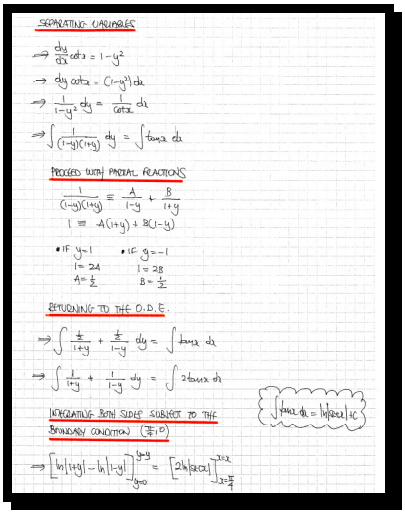
Question 67 (**+)**

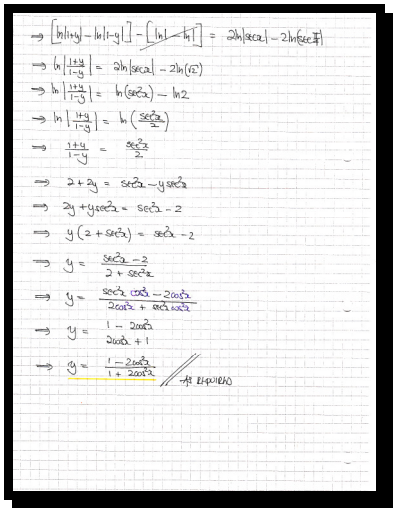
$$\frac{dy}{dx}\cot x = 1-y^2 .$$

Solve the differential equation above, subject to the boundary condition $y=0$ at $x=\frac{\pi}{4}$, to show that

$$y=\frac{1-2\cos^2 x}{1+2\cos^2 x} .$$

[ ], [proof]





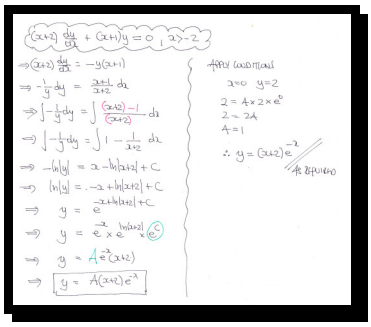
Question 68 (**+)**

$$(x+2)\frac{dy}{dx}+y(x+1)=0,\ x>-2.$$

Solve the differential equation above, subject to the boundary condition $y=2$ at $x=0$, to show that

$$y=(x+2)\mathrm{e}^{-x}.$$

proof



Question 69 (**+)**

A curve $y=f\left(x\right)$ satisfies the differential equation

$$y=1-\frac{dy}{dx}\frac{x+1}{\left(x-1\right)\left(x+2\right)},\ y>1, x>-1$$

a) Solve the differential equation to show that

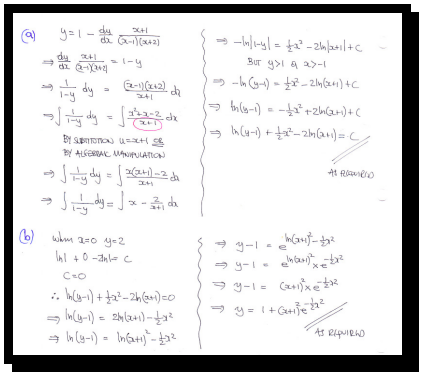
$$\ln\left(y-5\right)+\frac{1}{2}x^2+4x-2\ln\left(x+1\right)=C\,.$$

When $x=0$, $y=2$.

b) Show further that

$$y=1+\left(x+1\right)^2\mathrm{e}^{-\frac{1}{2}x^2}\,.$$

proof



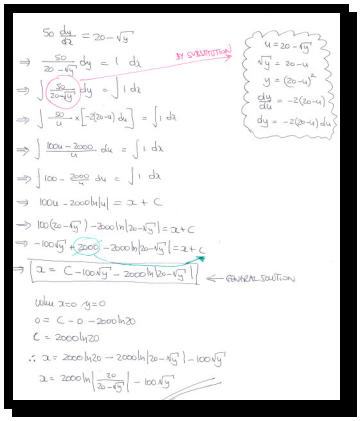
Question 70 (**+)**

Solve the differential equation

$$50\frac{dy}{dx}=20-\sqrt{y}\,,$$

given that when $x=0$, $y=0$, giving the answer in the form $x=f\left(y\right)$.

$$\boxed{x=2000\ln\left|\frac{20}{20-\sqrt{y}}\right|-100\sqrt{y}}$$



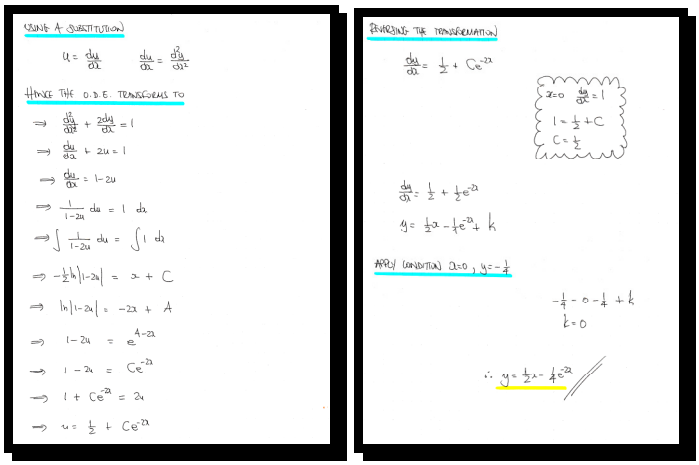
Question 71 (**+)**

Solve the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}=1,$$

given that $y=-\frac{1}{4}$ and $\frac{dy}{dx}=1$ at $x=0$, giving the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=\frac{1}{2}\left[2x-\mathrm{e}^{-2x}\right]}$



Question 72 (**+)**

A curve $y=f\left(x\right)$ satisfies the differential equation

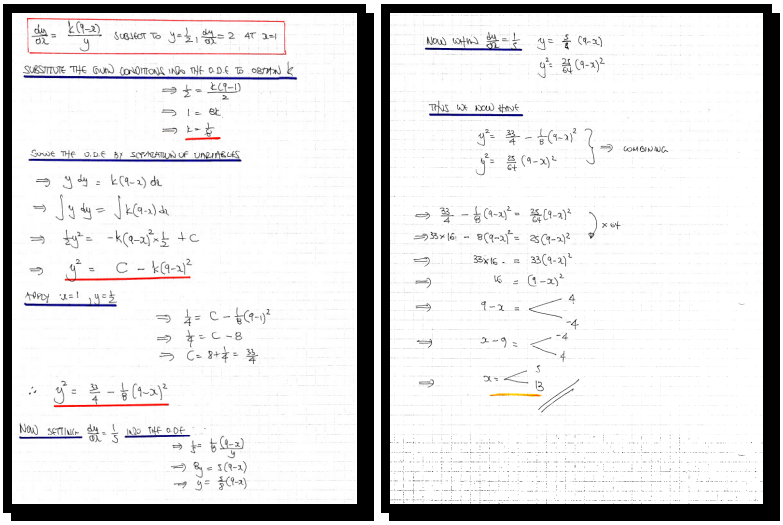
$$\frac{dy}{dx}=\frac{k\left(9-x\right)}{y},\quad y>0,\quad 0\leq x\leq 9,$$

where k is a positive constant.

It is further given that $y=\frac{1}{2}$, $\frac{dy}{dx}=2$ at $x=1$.

Find the possible values of x when $\frac{dy}{dx}=\frac{1}{5}$.

$\boxed{}$, $\boxed{x=5\ \cup\ x=13}$



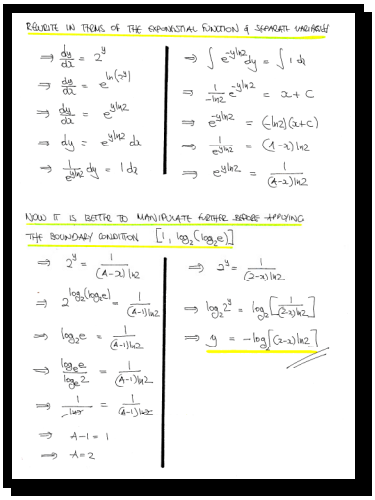
Question 73 (**+)**

A curve passes through the point with coordinates $\left[1\ ,\ \log_2\left(\log_2 \mathrm{e}\right)\right]$ and its gradient function satisfies

$$\frac{dy}{dx}=2^y\ ,\quad x\in\mathbb{R}\ ,\quad x<2\ .$$

Find the equation of the curve in the form $y=f\left(x\right)$

$\square$, $\boxed{y=-\log_2\left[\left(2-x\right)\ln 2\right]}$



Question 74 (*****)

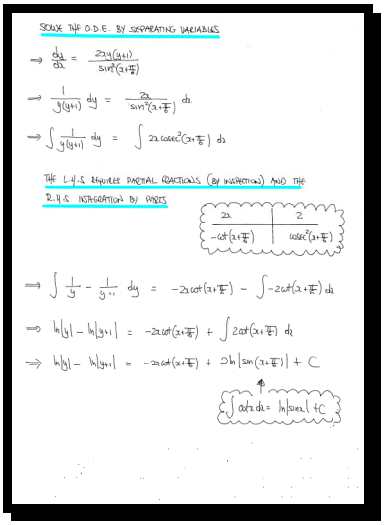
The function $y = f(x)$ satisfies the differential equation

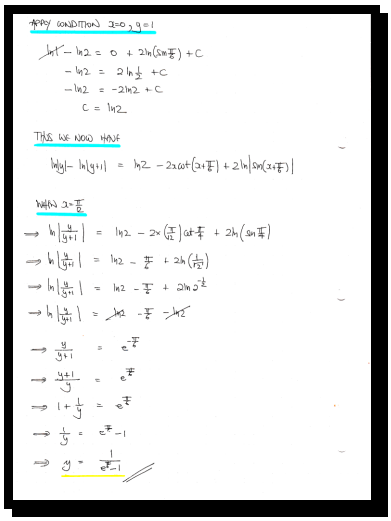
$$\frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2\left(x+\frac{1}{6}\pi\right)},$$

subject to the condition $y = 1$ at $x = 0$.

Find the exact value of y when $x = \frac{\pi}{12}$.

$\square$, $\boxed{y = \dfrac{1}{e^{\frac{1}{6}\pi} - 1}}$



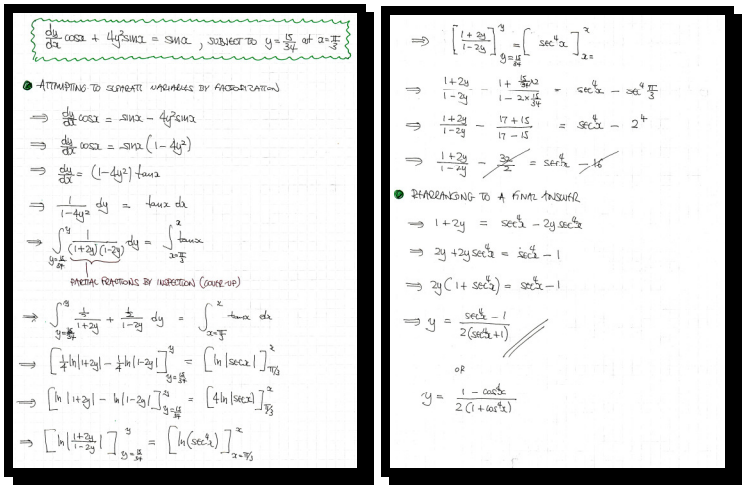


Question 75 (*****)

Determine, in the form $y = f(x)$, a simplified solution for the following differential equation.

$$\frac{dy}{dx}\cos x + 4y^2 \sin x = \sin x, \quad y = \frac{15}{34} \text{ at } x = \frac{1}{3}\pi .$$

$\square$, $\boxed{y = \dfrac{\sec^4 x - 1}{2\left(\sec^4 x + 1\right)}}$



Question 76 (*****)

The function $v = f(t)$ satisfies the differential equation

$$\frac{dv}{dt} = k\left[\frac{1}{v} - \frac{1}{h}\right],$$

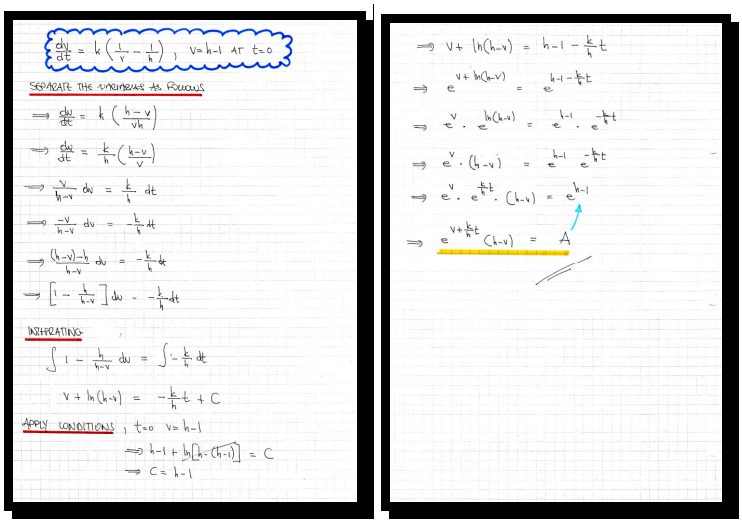
where k and h are non zero constants.

Given that $v = h - 1$ at $t = 0$, solve the differential equation to show that

$$(h - v)e^{\frac{k}{h}t + v} = A$$

where A is a non zero constant.

[ ], [proof]



Question 77 (*****)

The function $y=f\left(x\right)$ satisfies the differential equation

$$\frac{dy}{dx}=k\left[\frac{1}{h}-\frac{1}{x}\right],$$

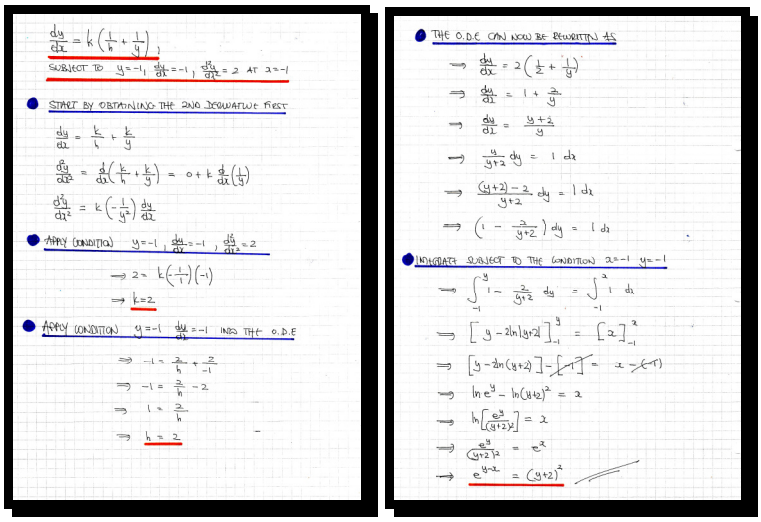
where k and h are non zero constants.

It is further given that $y=-1$, $\frac{dy}{dx}=-1$, $\frac{d^2y}{dx^2}=2$ at $x=-1$.

Solve the differential equation to show that

$$\mathrm{e}^{y-x}=\left(y+2\right)^2.$$

, proof



Question 78 (*****)

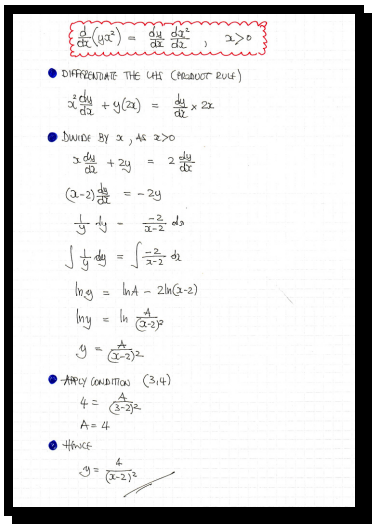
The function $y = f(x)$ satisfies the differential equation

$$\frac{d}{dx}\left(yx^2\right) = \frac{dy}{dx}\frac{d}{dx}\left(x^2\right), \quad x > 0$$

subject to the condition $y = 4$ at $x = 3$.

Find a simplified expression for $y = f(x)$.

$\square$, $\boxed{y = \dfrac{4}{(x-2)^2}}$

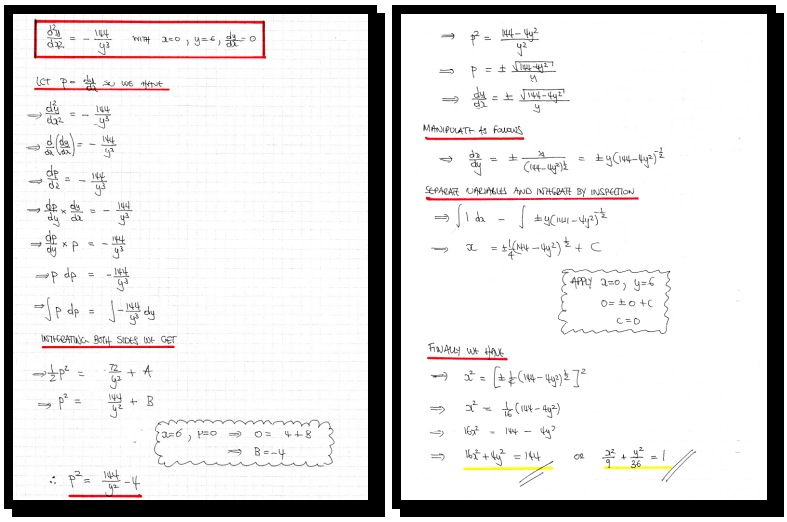


Question 79 (*****)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2}=-\frac{144}{y^3}, \qquad y(0)=6, \qquad \left.\frac{dy}{dx}\right|_{x=0}=0.$$

$$\boxed{}, \quad \boxed{\frac{x^2}{9}+\frac{y^2}{36}=1}$$



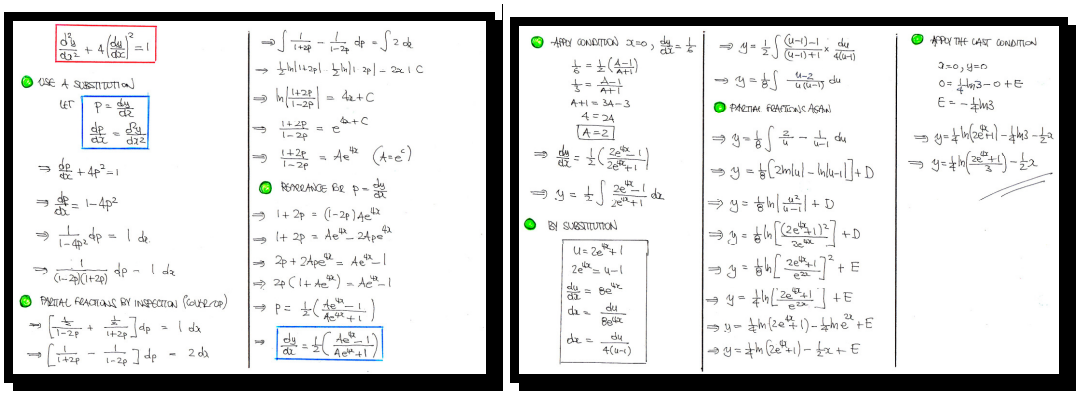
Question 80 (*****)

Solve the differential equation

$$\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1,$$

given that $y=0$ and $\dfrac{dy}{dx}=\dfrac{1}{6}$ at $x=0$, giving the answer in the form $y=f\left(x\right)$.

$\square$, $\boxed{y=\dfrac{1}{4}\ln\left[\dfrac{1+2\mathrm{e}^{4x}}{3}\right]-\dfrac{1}{2}x}$



Question 81 (*****)

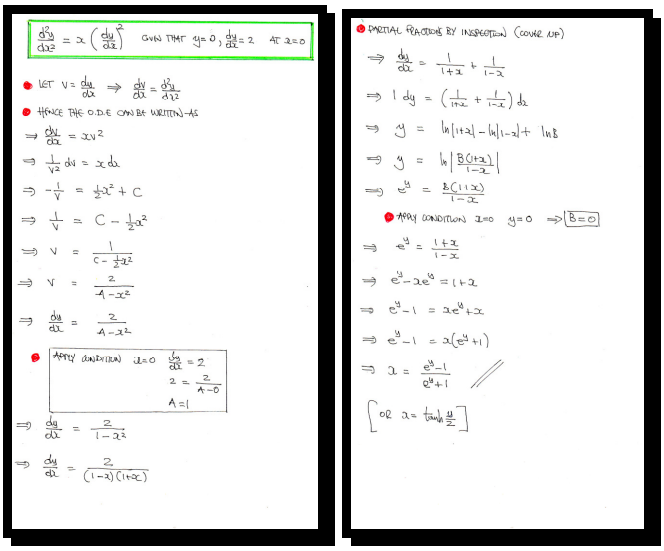
Solve the following differential equation

$$\frac{d^2y}{dx^2}=x\left(\frac{dy}{dx}\right)^2,$$

given further $y=0$, $\dfrac{dy}{dx}=2$ at $x=0$.

Give the answer in the form $x=f\left(y\right)$.

$\boxed{}$, $\boxed{x=\dfrac{\mathrm{e}^y-1}{\mathrm{e}^y+1}=\tanh\left(\dfrac{1}{2}x\right)}$



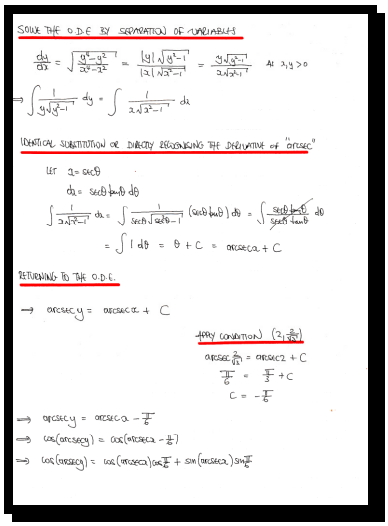
Question 82 (*****)

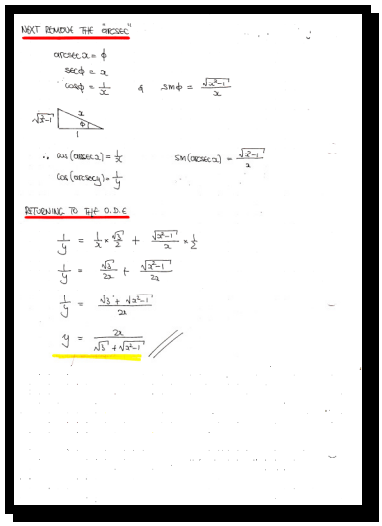
$$\frac{dy}{dx}=\sqrt{\frac{y^4-y^2}{x^4-x^2}}\,,\ x>0\,,\ y>0\,.$$

Find the solution of the above differential equation subject to the boundary condition $y=\dfrac{2}{\sqrt{3}}$ at $x=2$.

Give the answer in the form $y=\dfrac{2x}{f(x)}$, where $f(x)$ is a function to be found.

$\square$, $\boxed{f(x)=\sqrt{3}+\sqrt{x^2-1}}$





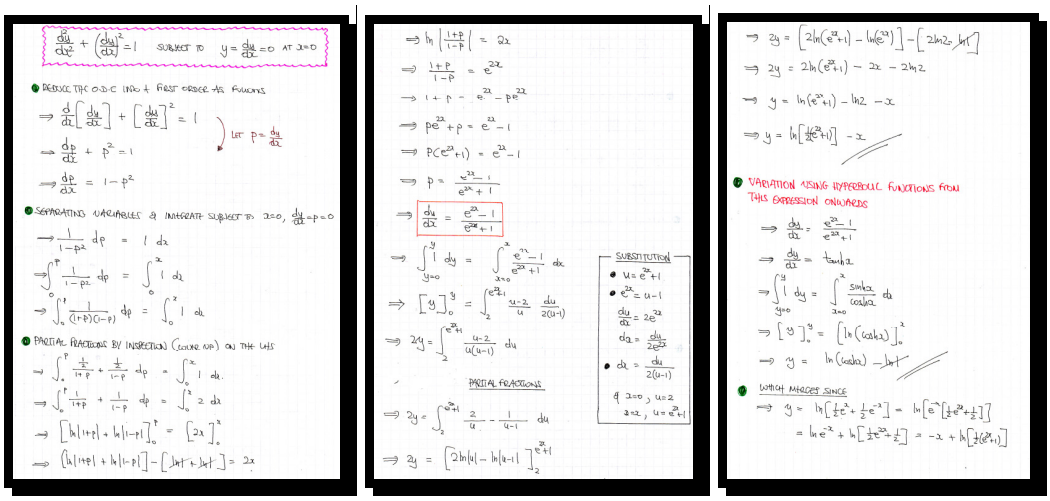
Question 83 **(*****)**

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1\,.$$

Given that $y=\dfrac{dy}{dx}=0$ at $x=0$, show that

$$y=-x+\ln\left[\frac{1}{2}\left(1+\mathrm{e}^{2x}\right)\right].$$

[ ], [proof]



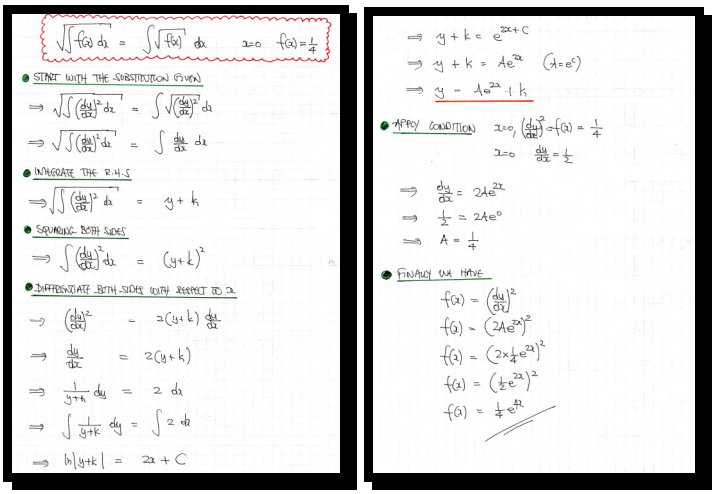
Question 84 (*****)

The non zero function $f(x)$ satisfies the integral equation

$$\sqrt{\int f(x)\,dx} = \int \sqrt{f(x)}\,dx\,, \qquad f(0)=\frac{1}{4}\,.$$

Use the substitution $f(x)=\left(\frac{dy}{dx}\right)^2$, to find a simplified expression for $f(x)$.

$\square$, $\boxed{f(x)=\frac{1}{4}e^{4x}}$



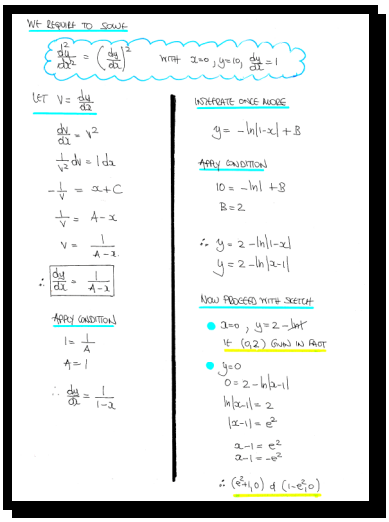
Question 85 **(*****)**

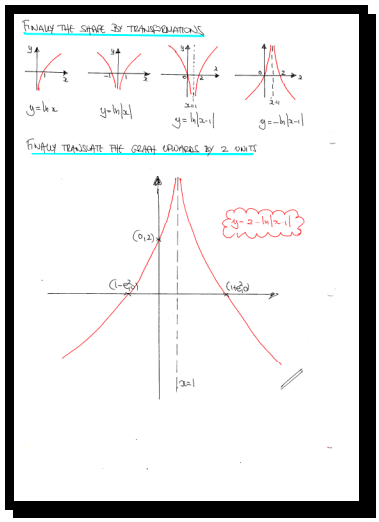
It is required to sketch the curve with equation $y=f(x)$, defined over the set of real numbers, in the greatest domain.

The curve has the property that at every point on the curve, the second derivative equals to the first derivative **squared**.

Showing all the relevant details, sketch the graph of $y=f(x)$, given further that the curve passes through the point $(0,2)$ and the gradient at that point is 1.

$\boxed{}$, $\boxed{\text{graph}}$





Question 86 (*****)

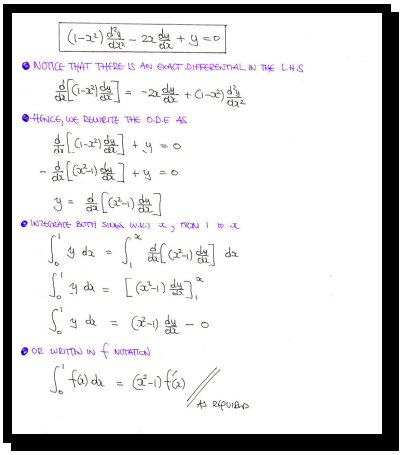
It is given that a function with equation $y=f(x)$ is a solution of the following differential equation.

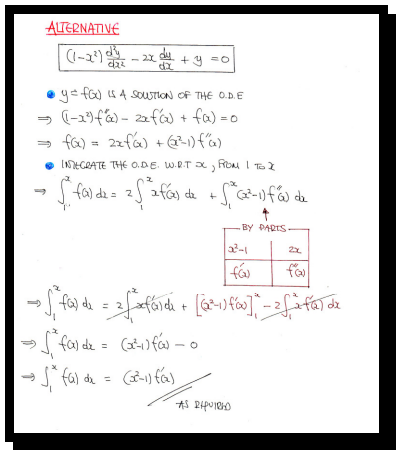
$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+y=0\,.$$

Show with a clear method that

$$\int_1^x f(x)\ dx=\left(x^2-1\right)f'(x)\,.$$

[ ], [proof]





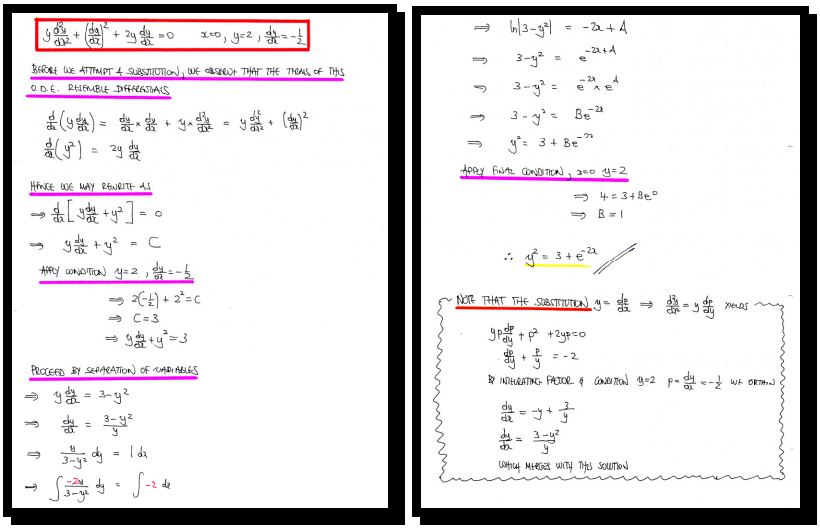
Question 87 **(*****)**

Solve the following differential equation

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}=0,\quad y(0)=2,\quad \frac{dy}{dx}(0)=-\frac{1}{2}.$$

Give the answer in the form $y^2=f(x)$.

$\boxed{}$, $\boxed{y^2=3+\mathrm{e}^{-2x}}$



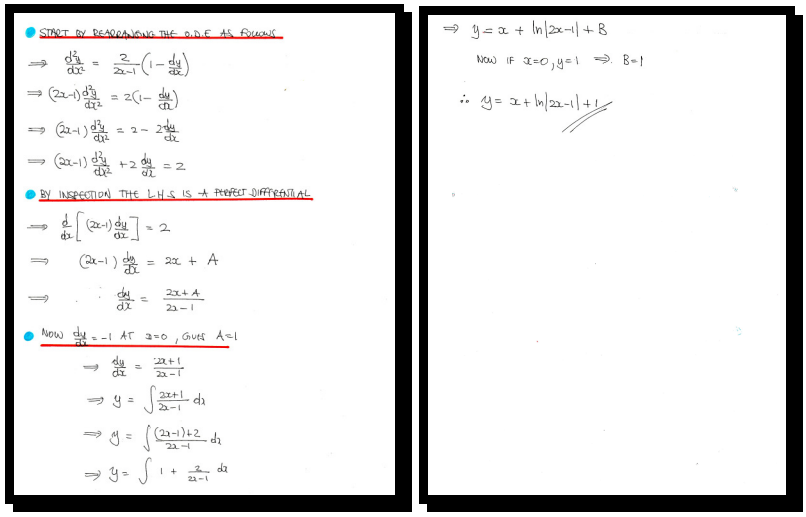
Question 88 (*****)

The function with equation $y=f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}=\frac{2}{2x-1}\left(1-\frac{dy}{dx}\right), \quad y(0)=1, \quad \frac{dy}{dx}(0)=-1.$$

Solve the above differential equation giving the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=x+1+\ln|2x-1|}$



Question 89 **(*****)**

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi) x^{g(\phi)}$ is a solution of the differential equation,

$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

V, , $\boxed{F(x) = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi} x^{\phi}}$

