# NUMERICAL SOLUTIONS OF EQUATIONS

# ITERATIVE METHODS

**Question 1 (**)**

$$x^3+10x-4=0\,.$$

**a)** Show that the above equation has a root $\alpha$, which lies between $0$ and $1$.

The recurrence relation

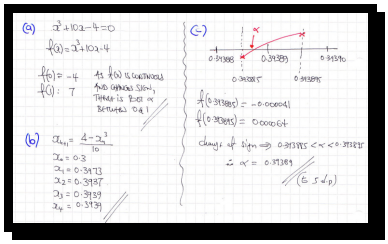
$$x_{n+1}=\frac{4-x_n^3}{10}$$

starting with $x_0=0.3$ is to be used to find $\alpha$.

**b)** Find, to $4$ decimal places, the value of $x_1$, $x_2$, $x_3$ and $x_4$.

**c)** By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha=0.39389$, correct to $5$ decimal places.

$\square$, $\boxed{x_1=0.3973,\quad x_2=0.3937,\quad x_3=0.3939,\quad x_4=0.3939}$



**Question 2 (**)**

$$\mathrm{e}^{-x}+\sqrt{x}=2$$

**a)** Show that the above equation has a root $\alpha$, which lies between $3$ and $4$.

The recurrence relation

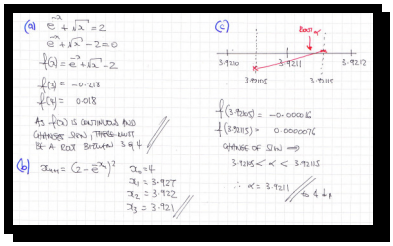
$$x_{n+1}=\left(2-\mathrm{e}^{-x_n}\right)^2$$

starting with $x_0=4$ is to be used to find $\alpha$.

**b)** Find, to $3$ decimal places, the value of $x_1$, $x_2$ and $x_3$.

**c)** By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha=3.9211$, correct to $4$ decimal places.

$\square$, $\boxed{x_1=3.927,\ x_2=3.922,\ x_3=3.921}$



**Question 3 (**)**

$$e^{3x} = x + 20$$

**a)** Show that the above equation has a root $\alpha$ between $1$ and $2$.

The recurrence relation

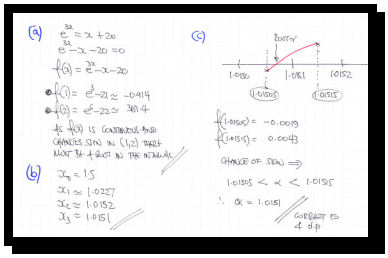
$$x_{n+1} = \frac{1}{3}\ln\left(x_n + 20\right)$$

starting with $x_0 = 1.5$ is to be used to find $\alpha$.

**b)** Find to $4$ decimal places, the value of $x_1$, $x_2$ and $x_3$.

**c)** By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha = 1.0151$, correct to $4$ decimal places.

$\square$, $\boxed{x_1 = 1.0227, \quad x_2 = 1.0152, \quad x_3 = 1.0151}$



**Question 4** **(**+)**

$$f(x)=4x-3\sin x-1,\quad 0\le x\le 2\pi .$$

**a)** Show that the equation $f(x)=0$ has a solution $\alpha$ in the interval $(0.7,0.8)$.

An iterative formula, of the form given below, is used to find $\alpha$.

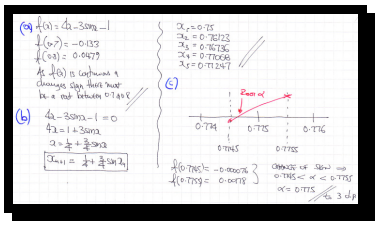
$$x_{n+1}=A+B\sin x_n,\quad x_1=0.75,$$

where $A$ and $B$ are constants.

**b)** Find, to 5 decimal places, the value of $x_2$, $x_3$, $x_4$ and $x_5$.

**c)** By considering the sign of $f(x)$ in a suitable interval show clearly that $\alpha=0.775$, correct to 3 decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_2=0.76123,\ x_3=0.76736,\ x_4=0.77068,\ x_5=0.77247}$



**Question 5** **(**+)**

$$x^3 - x^2 = 6x + 6, \quad x \in \mathbb{R}.$$

**a)** Show that the above equation has a root $\alpha$ in the interval $(3,4)$.

**b)** Show that the above equation can be written as

$$x = \sqrt{\frac{6x+6}{x-1}}.$$

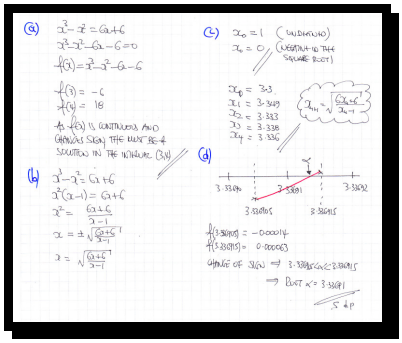
An iterative formula of the form given in part **(b)**, starting with $x_0$ is used to find $\alpha$.

**c)** Give two different values for $x_0$ that would not produce an answer for $x_1$.

**d)** Starting with $x_0 = 3.3$ find the value of $x_1$, $x_2$, $x_3$ and $x_4$, giving each of the answers correct to $3$ decimal places.

**e)** By considering the sign of an appropriate function in a suitable interval, show clearly that $\alpha = 3.33691$, correct to $5$ decimal places.

$\boxed{\quad}$, $\boxed{x_0 \neq 1,\ 0,\ 0.5 \text{ etc}}$, $\boxed{x_1 = 3.349,\ x_2 = 3.333,\ x_3 = 3.338,\ x_4 = 3.336}$



**Question 6** **(***)**

The curves $C_1$ and $C_2$ have respective equations

$$y=9-x^2,\ x\in\mathbb{R} \quad \text{and} \quad y=\mathrm{e}^x,\ x\in\mathbb{R}.$$

**a)** Sketch in the same diagram the graph of $C_1$ and the graph of $C_2$.

The sketch must include the coordinates of the points where each of the curves meet the coordinate axes.

**b)** By considering the graphs sketched in part (a), show clearly that the equation

$$\left(9-x^2\right)\mathrm{e}^{-x}=1$$

has exactly one positive root and one negative root.

To find the negative root of the equation the following iterative formula is used

$$x_{n+1}=-\sqrt{9-\mathrm{e}^{x_n}},\ x_1=-3.$$

**c)** Find, to $5$ decimal places, the value of $x_2$, $x_3$ and $x_4$.

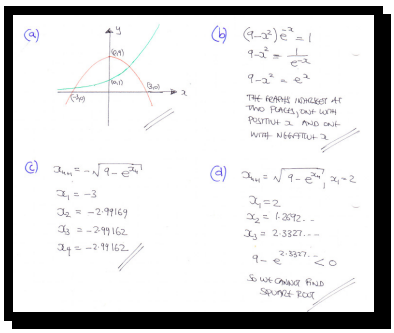
To find the positive root of the equation the following iterative formula is used

$$x_{n+1}=\sqrt{9-\mathrm{e}^{x_n}},\ x_1=2.$$

**d)** Explain, clearly but briefly, why do these iterations fail.

$\boxed{\phantom{x}}$, $\boxed{C_1:(-3,0),(3,0),(0,9)}$, $\boxed{C_2:(0,1)}$,

$\boxed{x_2=-2.99169,\quad x_3=-2.99162,\quad x_4=-2.99162}$



**Question 7 (***)**

$$x^3+3x=5,\ x\in\mathbb{R}.$$

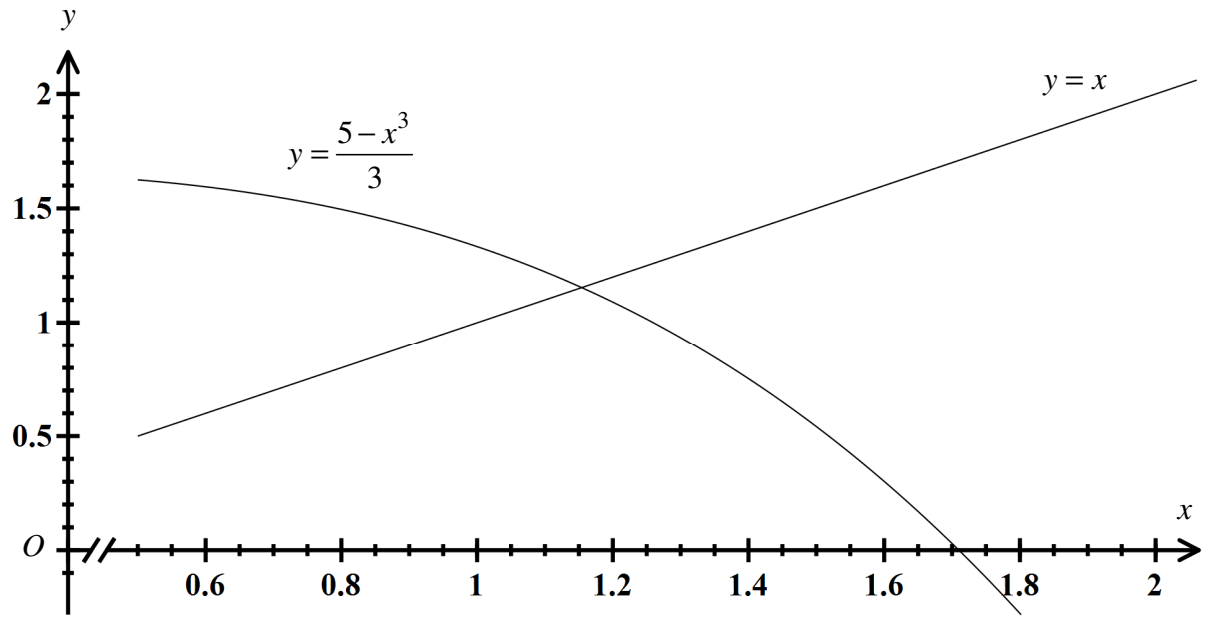
**a)** Show that the above equation has a root $\alpha$, between 1 and 2.

An attempt is made to find $\alpha$ using the iterative formula

$$x_{n+1}=\frac{5-x_n^{\,3}}{3},\ x_1=1.$$

**b)** Find, to 2 decimal places, the value of $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$.

The diagram below is used to investigate the results of these iterations.




**[continues overleaf]**

**[continued from overleaf]**

**c)** On a copy of this diagram draw a "staircase" or "cobweb" pattern marking the position of $x_1$, $x_2$, $x_3$ and $x_4$, further stating the results of these iterations.

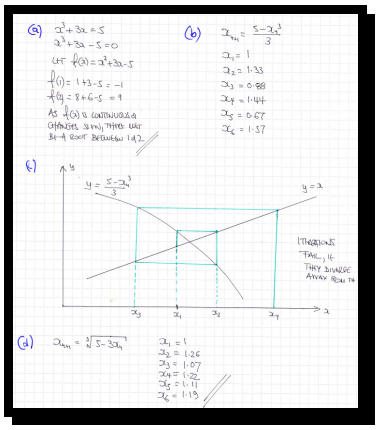
**d)** Use the iterative formula

$$x_{n+1} = \sqrt[3]{5-3x_n}\ ,\ x_1 = 1,$$

to find, to $2$ decimal places, the value of $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$.

$\boxed{\phantom{x_2 = 1.33,\quad x_3 = 0.88,\quad x_4 = 1.44,\quad x_5 = 0.67,\quad x_6 = 1.57}}$, $\boxed{x_2 = 1.33,\quad x_3 = 0.88,\quad x_4 = 1.44,\quad x_5 = 0.67,\quad x_6 = 1.57}$,

$\boxed{x_2 = 1.26,\quad x_3 = 1.07,\quad x_4 = 1.22,\quad x_5 = 1.11,\quad x_6 = 1.19}$



**Question 8 (***)**

$$x^3 = 5x + 1, \ x \in \mathbb{R}.$$

**a)** Show that the above equation has a root $\alpha$ between $2$ and $3$.

The iterative formula

$$x_{n+1} = \sqrt[3]{5x_n + 1}, \ x_1 = 2,$$

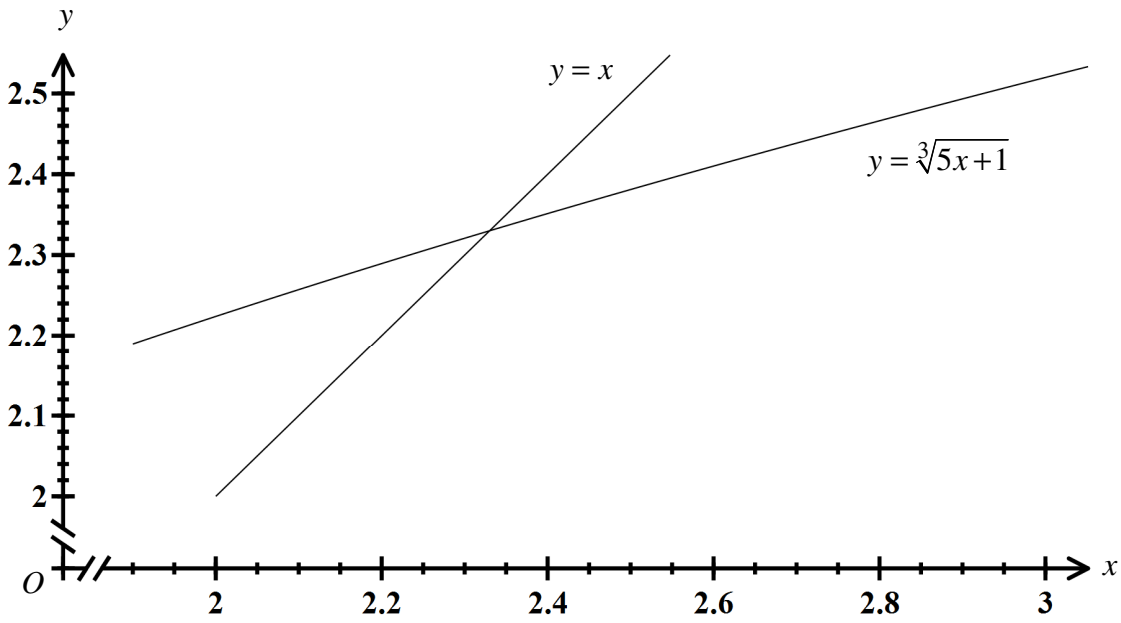
is to be used to find $\alpha$

**b)** Find, to $2$ decimal places, the value of $x_2$, $x_3$ and $x_4$.

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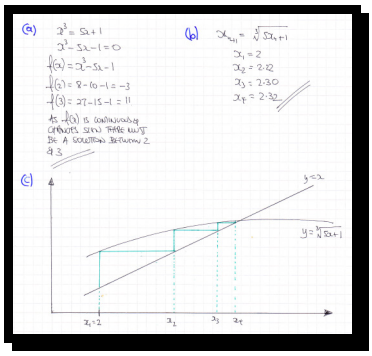
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The diagram below is used to investigate the results of these iterations.




**c)** On a copy of this diagram draw a "staircase" or "cobweb" pattern showing how these iterations converge to $\alpha$, marking the position of $x_1$, $x_2$, $x_3$ and $x_4$.

$x_2 = 2.22, \quad x_3 = 2.30, \quad x_4 = 2.32$



**Question 9 (***)**

A cubic equation has the following equation.

$$x^3+1=4x,\ x\in\mathbb{R}.$$

**a)** Show that the above equation has a root $\alpha$, which lies between $0$ and $1$.

**b)** Show further that the above equation can be written as

$$x=\frac{1}{4-x^2}.$$

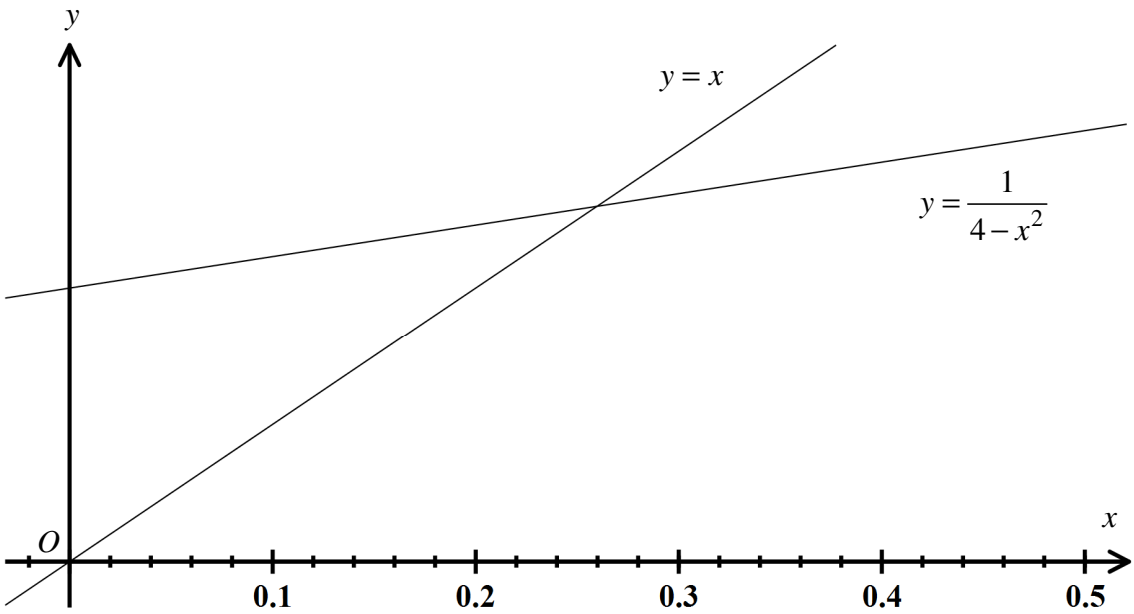
An iterative formula, based on the rearrangement of part **(b)**, is to be used to find $\alpha$.

**c)** Starting with $x_1=0.1$, find to $4$ decimal places, the value of $x_2$, $x_3$ and $x_4$.

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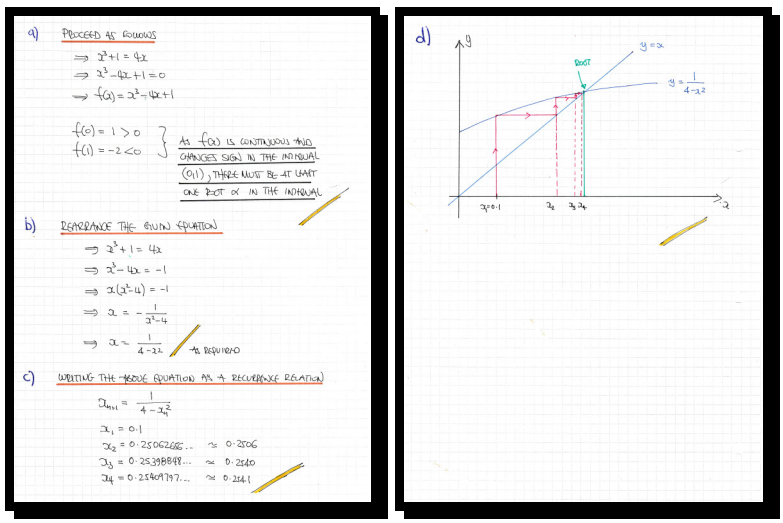
**[continued from overleaf]**

The diagram below is used to show the convergence of these iterations.




**d)** Draw on a copy of this diagram a "staircase" or "cobweb" pattern showing how these iterations converge to $\alpha$, marking the position of $x_1$, $x_2$, $x_3$ and $x_4$.

$\boxed{\phantom{x_1}}$, $\boxed{\phantom{x_1}}$, $\boxed{x_2 = 0.2506, \quad x_3 = 0.2540, \quad x_4 = 0.2541}$



**Question 10 (***)**

$$f(x)=x^3-6x^2+12x-11,\ x\in\mathbb{R}.$$

**a)** Show that the equation $f(x)=0$ has a root $\alpha$ between $3$ and $4$.

An approximation to the value of $\alpha$ is to be found by using the iterative formula

$$x_{n+1}=\sqrt[3]{6x_n{}^2-12x_n+11},\ x_1=3.$$

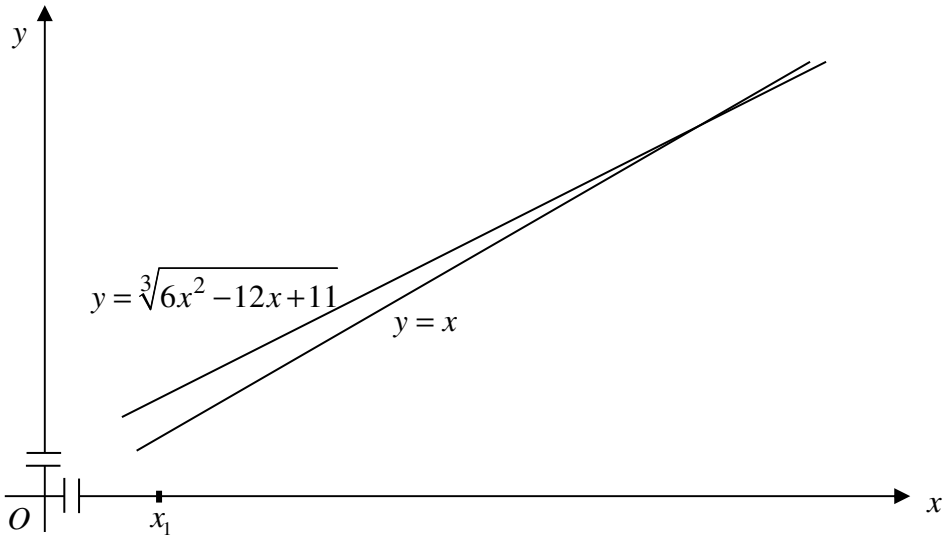
**b)** Find, to $3$ decimal places, the value of $x_2$, $x_3$ and $x_4$.

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**[continued from overleaf]**

The diagram below shows the graphs of

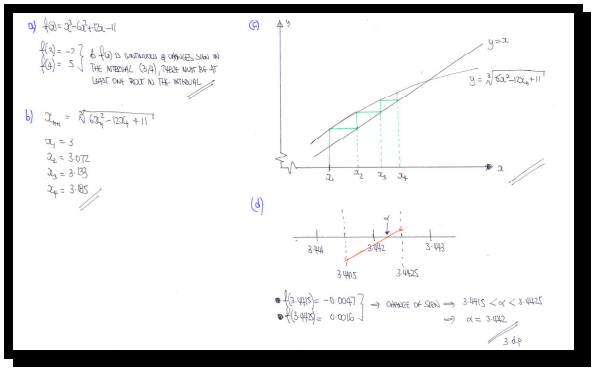
$$y = x \qquad \text{and} \qquad y = \sqrt[3]{6x^2 - 12x + 11}\,.$$




**c)** Draw on a copy of the diagram a "staircase" or "cobweb" pattern showing how these iterations converge to $\alpha$, marking the position of $x_2$, $x_3$ and $x_4$.

**d)** By considering the sign of $f(x)$ in a suitable interval show that $\alpha = 3.442$, correct to $3$ decimal places.

$\boxed{\phantom{x_2 = 3.072,\ x_3 = 3.133,\ x_4 = 3.185}}$, $\boxed{x_2 = 3.072,\ x_3 = 3.133,\ x_4 = 3.185}$



**Question 11 (***)**

$$4\arccos x = x + 1,\ x \in \mathbb{R},\ -1 \le x \le 1.$$

The above equation has a single root $\alpha$.

**a)** Show that $0.5 < \alpha < 1$.

An iterative formula of the form $x_{n+1} = \cos\left(f\left(x_n\right)\right)$ is used to find $\alpha$.

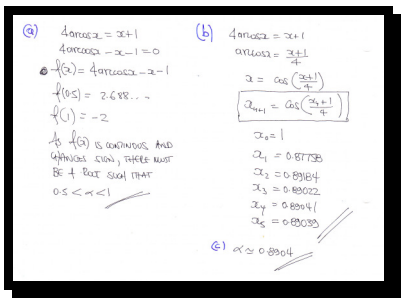
**b)** Use this iterative formula, starting with $x_0 = 1$, to find the value of $x_1$, $x_2$, $x_3$, $x_4$ and $x_5$.

Give the answers correct to $5$ decimal places.

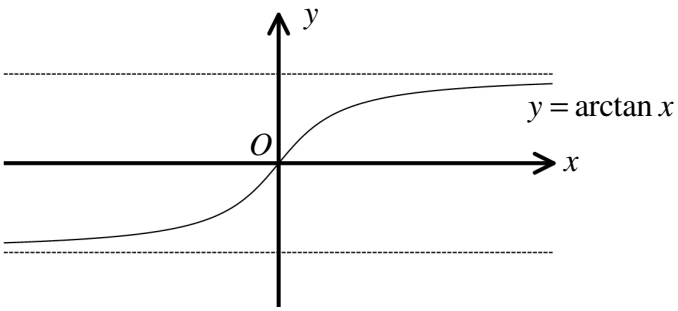
**c)** Write down the value of $\alpha$ to an appropriate accuracy.

$\square$, $\boxed{x_1 = 0.87758,\ \ x_2 = 0.89184,\ \ x_3 = 0.89022,\ \ x_4 = 0.89041,\ \ x_5 = 0.89039}$,

$\boxed{\alpha = 0.8904}$



**Question 12 (***)**




The diagram above shows the graph of

$$y = \arctan x \, .$$

**a)** Write down the equations of the two horizontal asymptotes.

**b)** Copy the diagram above and use it to show that the equation

$$3x - \arctan x = 1$$

has only one positive real root.

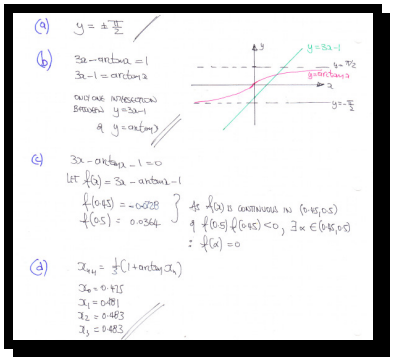
**c)** Show that this root lies between $0.45$ and $0.5$.

**d)** Use the iterative formula

$$x_{n+1} = \frac{1}{3}\left(1 + \arctan x_n\right), \quad x_0 = 0.475$$

to find, to $3$ decimal places, the value of $x_1$, $x_2$ and $x_3$.

$$\boxed{\phantom{x}}, \quad \boxed{y = \pm\frac{\pi}{2}}, \quad \boxed{x_1 = 0.481, \quad x_2 = 0.483, \quad x_3 = 0.483}$$



**Question 13 (***+)**

The curve $C$ has equation

$$y = x^3 - 3x^2 - 3,$$

and crosses the $x$ axis at the point $A(\alpha, 0)$.

**a)** Show that $\alpha$ lies between $3$ and $4$.

**b)** Show further that the equation $x^3 - 3x^2 - 3 = 0$ can be rearranged to

$$x = 3 + \frac{3}{x^2}, \ x \neq 0.$$

The equation rearrangement of part **(b)** is written as the following recurrence relation

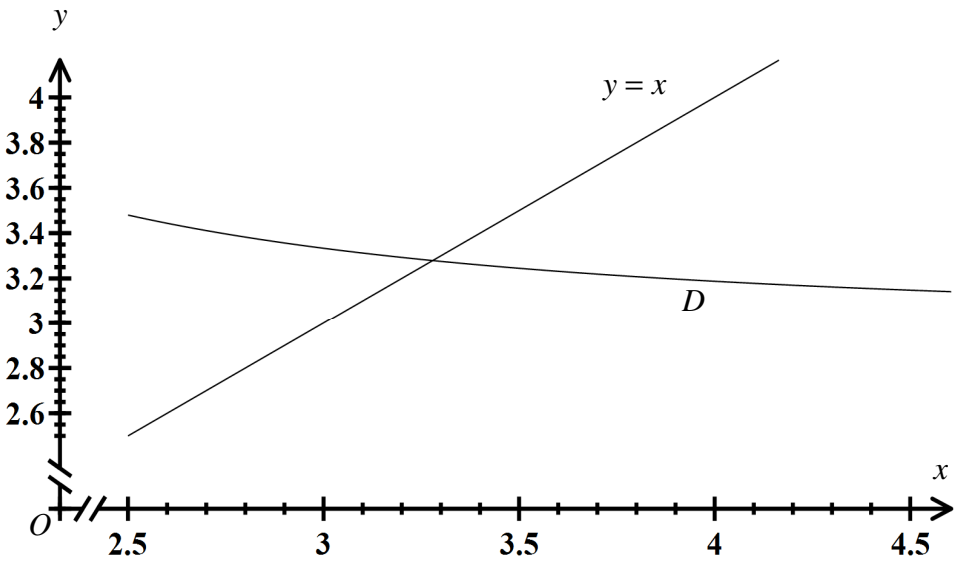
$$x_{n+1} = 3 + \frac{3}{x_n^2}, \ x_1 = 4.$$

**c)** Use the above iterative formula to find, to $4$ decimal places, the value of $x_2$, $x_3$, $x_4$ and $x_5$.

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**[continued from overleaf]**

The diagram below is used to describe how the iteration formula converges to $\alpha$, and shows the graph of $y = x$ and another curve $D$.

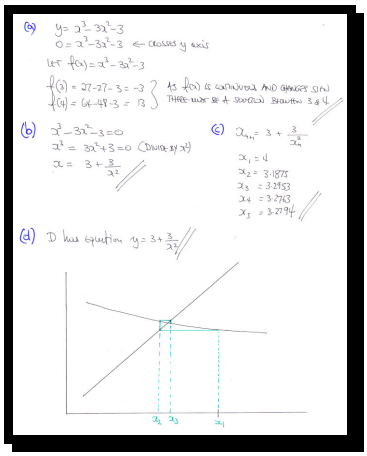



**d)** Write down the equation of $D$.

**e)** On a copy of the diagram draw a "staircase" or a "cob-web" pattern to show how the convergence to the root $\alpha$ is taking place, marking clearly the position of $x_1$, $x_2$ and $x_3$.

$\boxed{\phantom{x}}$, $\boxed{x_1 = 3.1875, \quad x_2 = 3.2953, \quad x_3 = 3.2763, \quad x_4 = 3.2794}$,

$\boxed{D : y = 3 + \dfrac{3}{x^2}}$



**Question 14** **(***+)**

$$x^3-1-\frac{1}{x}=0,\ x\neq 0.$$

**a)** By sketching two suitable graphs in the same diagram, show that the above equation has one positive root $\alpha$ and one negative root $\beta$.

The sketch must include the coordinates of the points where the curves meet the coordinate axes.

**b)** Explain why $\alpha>1$ .

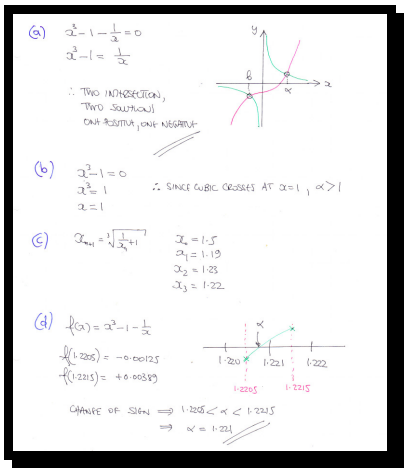
To find $\alpha$ the following iterative formula is used

$$x_{n+1}=\sqrt[3]{\frac{1}{x_n}+1},\ x_0=1.5.$$

**c)** Find, to $2$ decimal places, the value of $x_1$, $x_2$ and $x_3$.

**d)** By considering the sign of an appropriate function $f(x)$ in a suitable interval, show clearly that $\alpha=1.221$, correct to $3$ decimal places.

$\square$, $\boxed{x_1=1.19,\ \ x_2=1.23,\ \ x_3=1.22}$



**Question 15 (***+)**

$$f(x)=x^4+3x-1,\ x\in\mathbb{R}.$$

**a)** By sketching two suitable graphs in the same set of axes, determine the number of real roots of the equation $f(x)=0$.

**b)** Show that the equation $f(x)=0$ has a root $\alpha$ between $0$ and $1$.

The recurrence relation

$$x_{n+1}=\frac{1-x_n^4}{3}$$

starting with $x_0=0.3$ is to be used to find $\alpha$.

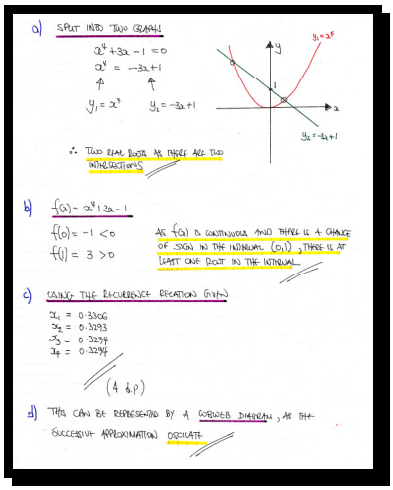
**c)** Find to $4$ decimal places, the value of $x_1$, $x_2$, $x_3$ and $x_4$.

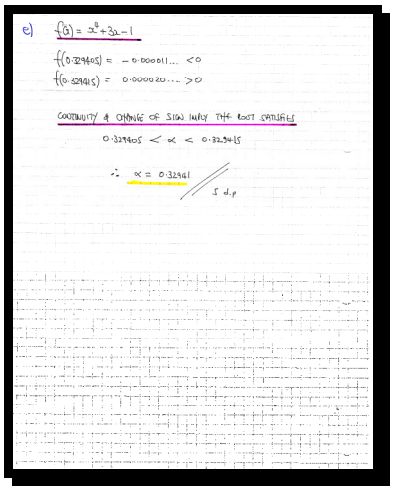
**d)** Explain whether the convergence to the root $\alpha$, can be represented by a cobweb or a staircase diagram.

**e)** By considering the sign of $f(x)$ in a suitable interval, show that $\alpha=0.32941$, correct to $5$ decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_1=0.3306,\ x_2=0.3293,\ x_3=0.3294,\ x_4=0.3294}$,

$\boxed{\text{cobweb, because of oscillation}}$



**Question 16 (***+)**

The curve with equation $y = 2^x$ intersects the straight line with equation $y = 3 - 2x$ at the point $P$, whose $x$ coordinate is $\alpha$.

**a)** Show clearly that …

**i.** … $0.5 < \alpha < 1$.

**ii.** … $\alpha$ is the solution of the equation

$$x = \frac{\ln(3-2x)}{\ln 2}.$$

An iterative formula based on the equation of part **(aii)** is used to find $\alpha$.

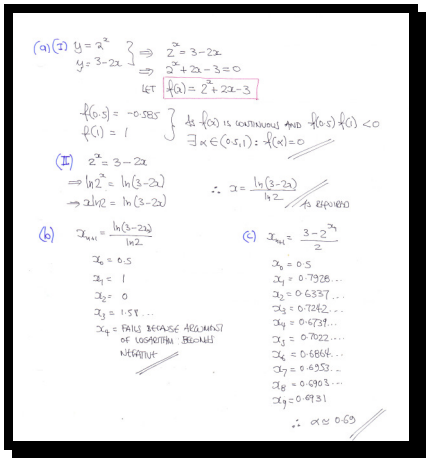
**b)** Starting with $x_0 = 0.5$, find the value of $x_1$, $x_2$ and $x_3$, explaining why a valid value of $x_4$ cannot be produced.

**c)** Use the iterative formula

$$x_{n+1} = \frac{3-2^{x_n}}{2}, \ x_0 = 0.5,$$

with as many iterations as necessary, to determine the value of $\alpha$ correct to 2 decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_1 = 1, \ x_2 = 0, \ x_3 = 1.584}$, $\boxed{\alpha = 0.69}$



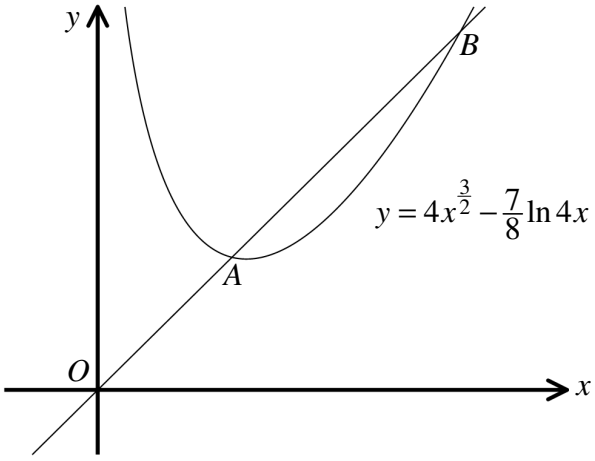
**Question 17 (***+)**

A curve $C$ has equation

$$y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x\,, \quad x \in \mathbb{R}\,, \quad x > 0\,.$$

The point $A$ is on $C$, where $x = \frac{1}{4}$.

**a)** Find an equation of the normal to the curve at $A$.




This normal meets the curve again at the point $B$, as shown in the figure above.

**b)** Show that the $x$ coordinate of $B$ satisfies the equation

$$x = \left(\frac{16x + 7\ln 4x}{32}\right)^{\frac{2}{3}}.$$

**[continues overleaf]**

**[continued from overleaf]**

The recurrence relation

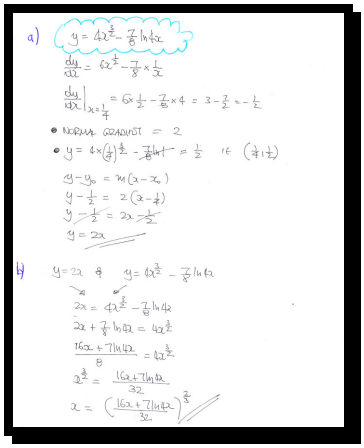
$$x_{n+1} = \left( \frac{16x_n + 7\ln 4x_n}{32} \right)^{\frac{2}{3}}, \quad x_0 = 0.7$$

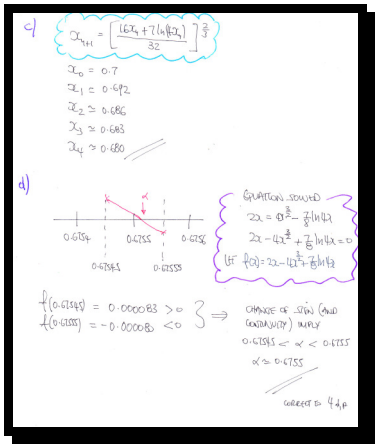
is to be used to find the $x$ coordinate of $B$.

**c)** Find, to 3 decimal places, the value of $x_1$, $x_2$, $x_3$ and $x_4$.

**d)** Show that the $x$ coordinate of $B$ is 0.6755, correct to 4 decimal places.

$\boxed{\phantom{x}}$, $\boxed{y = 2x}$, $\boxed{x_1 = 0.692, \quad x_2 = 0.686, \quad x_3 = 0.683, \quad x_4 = 0.680}$





**Question 18 (****)**

The curve $C$ has equation

$$y = x\cos x\,,\ 0 \le x \le \frac{\pi}{2}\,.$$

The curve has a single turning point at $M$.

**a)** Show that $x$ coordinate of $M$ is a solution of the equation

$$x = \arctan\left(\frac{1}{x}\right).$$

**b)** Show further that the equation

$$x = \arctan\left(\frac{1}{x}\right)$$

has root $\alpha$ between $0.8$ and $1$.

The iterative formula

$$x_{n+1} = \arctan\left(\frac{1}{x_n}\right) \text{ with } x_1 = 0.9$$

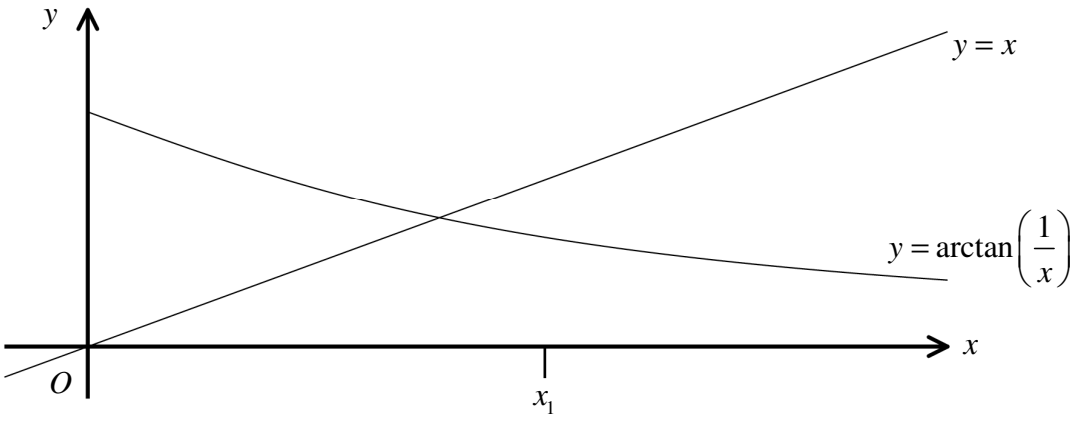
is to be used to find $\alpha$.

**c)** Find, to $3$ decimal places, the value of $x_2$, $x_3$ and $x_4$.

**[continues overleaf]**

**[continued from overleaf]**

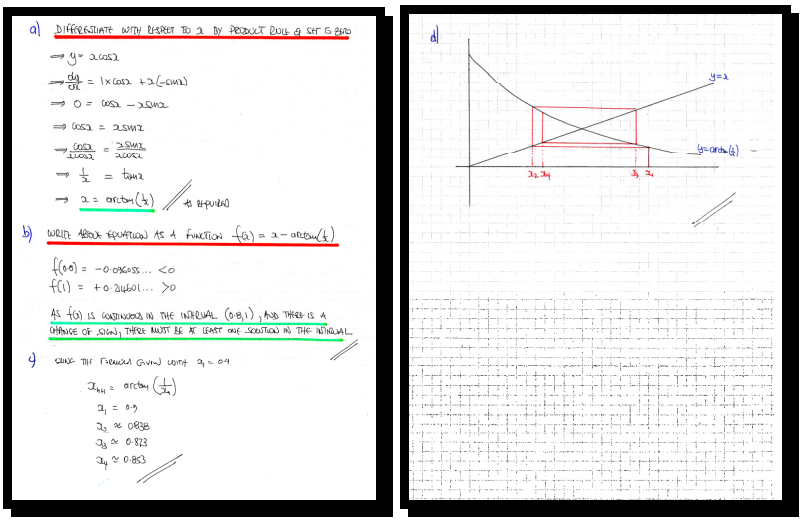
The diagram below shows the graphs of $y = x$ and $y = \arctan\left(\frac{1}{x}\right)$.




**d)** Use a copy of the above diagram to show how the convergence to the root $\alpha$ takes place, by constructing a staircase or cobweb pattern.

Indicate clearly the positions of $x_2$, $x_3$ and $x_4$.

$\boxed{\phantom{x}}$, $\boxed{x_2 = 0.838, \quad x_3 = 0.873, \quad x_4 = 0.853}$



**Question 19** **(****)**

$$f(x)\equiv\left|4e^{2x}-28\right|,\ x\in\mathbb{R}.$$

**a)** Sketch the graph of $f(x)$.

The sketch must include the coordinates of any intersections with the coordinate axes and the equations of any asymptotes.

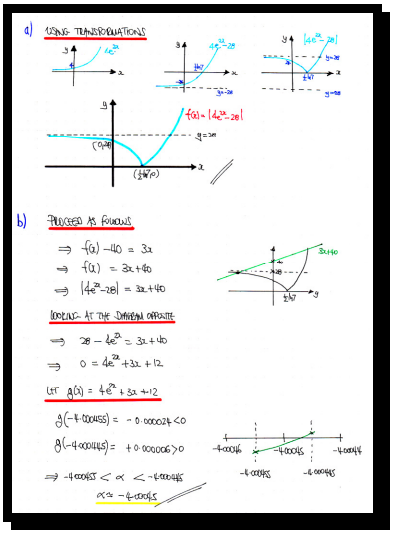
The equation $f(x)-40=3x$ has a negative root $\alpha$.

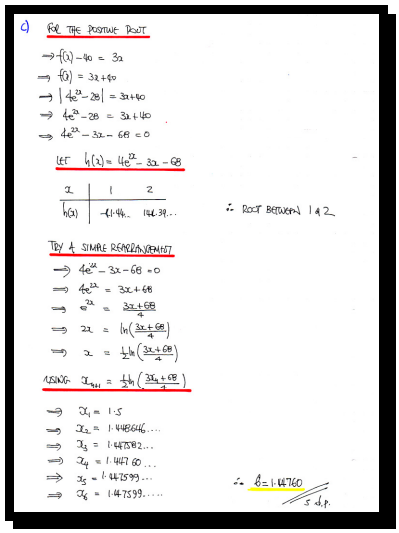
**b)** Show that $\alpha=-4.00045$, correct to $5$ decimal places.

The equation $f(x)-40=3x$ also has a positive root $\beta$.

**c)** Use a numerical method, based on an iterative method, to determine the value of $\beta$ correct to $5$ decimal places.

$\boxed{\phantom{00}}$, $\boxed{\beta\approx 1.44760}$



**Question 20 (****)**

The curve $C$ has equation

$$y=\frac{3x+1}{x^3-x^2+5}.$$

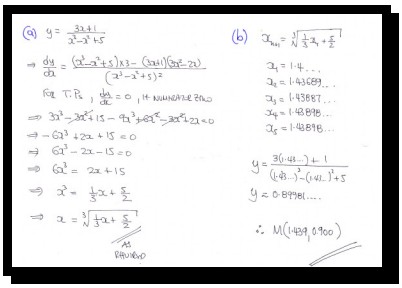
The curve has a single stationary point at $M$, with approximate coordinates $(1.4,0.9)$.

**a)** Show that the $x$ coordinate of $M$ is a solution of the equation

$$x=\sqrt[3]{\frac{1}{3}x+\frac{5}{2}}.$$

**b)** By using an iterative formula based on the equation of part **(a)**, determine the coordinates of $M$ correct to three decimal places.

$\boxed{\phantom{00}}$, $\boxed{M(1.439,0.900)}$



**Question 21** **(****)**

The curves $C_1$ and $C_2$ have respective equations

$$y_1 = 3\arcsin(x-1) \quad \text{and} \quad y_1 = 2\arccos(x-1).$$

**a)** Sketch in the same set of axes the graph of $C_1$ and the graph of $C_2$.

The sketch must include the coordinates.

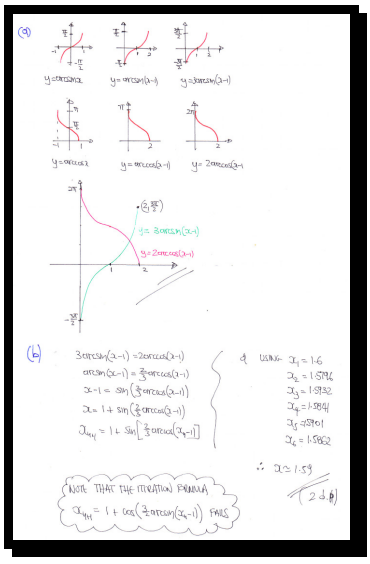
- … of any points where each of the graphs meet the coordinate axes.
- … of the endpoints of each of the graphs.

**b)** Use a suitable iteration formula of the form

$$x_{n+1} = f(x_n) \text{ with } x_1 = 1.6,$$

to find an approximate value for the $x$ coordinate of the point of intersection between the graph of $C_1$ and the graph of $C_2$.

$\boxed{\phantom{x}}$, $\boxed{x \approx 1.59}$



**Question 22 (****)**

At the point $P$ which lies on the curve with equation

$$x=\ln\left(y^3-4y\right),$$

the gradient is $2$.

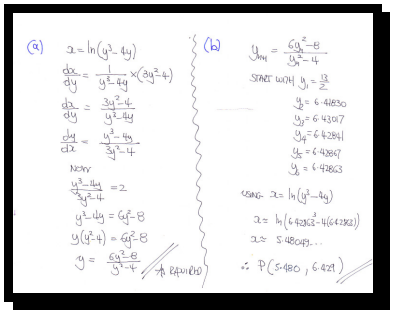
The point $P$ is close to the point with coordinates $\left(\frac{11}{2},\frac{13}{2}\right)$.

**a)** Show that the $y$ coordinate of $P$ is a solution of the equation

$$y=\frac{6y^2-8}{y^2-4}.$$

**b)** By using an iterative formula based on the equation of part **(a)**, determine the coordinates of $P$ correct to three decimal places.

$\boxed{\phantom{00}}$, $\boxed{P\left(5.480,6.429\right)}$



**Question 23 (****)**

At the point $P$ which lies on the curve with equation

$$y=\frac{x}{y+\ln y},$$

the gradient is $2$.

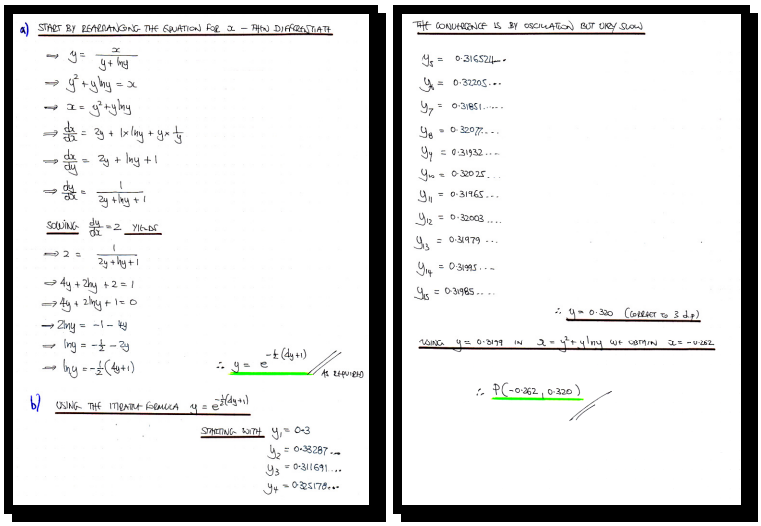
The point $P$ is close to the point with coordinates $(-0.3,0.3)$.

**a)** Show that the $y$ coordinate of $P$ is a solution of the equation

$$y=\mathrm{e}^{-\frac{1}{2}(4y+1)}.$$

**b)** By using an iterative formula based on the equation of part **(a)**, determine the coordinates of $P$ correct to three decimal places.

$\boxed{\phantom{xxxxxx}}$, $\boxed{P(-0.262,0.320)}$



**Question 24 (****)**

A curve has equation

$$y=\frac{x+1}{x^3+2x+1}.$$

It is given that the curve has a local maximum at the point $M$, whose approximate coordinates are $(-1.7,0.1)$.

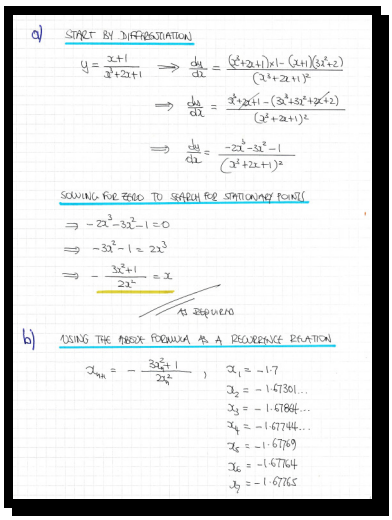
**a)** Show that the $x$ coordinate of $M$ is a solution of the equation

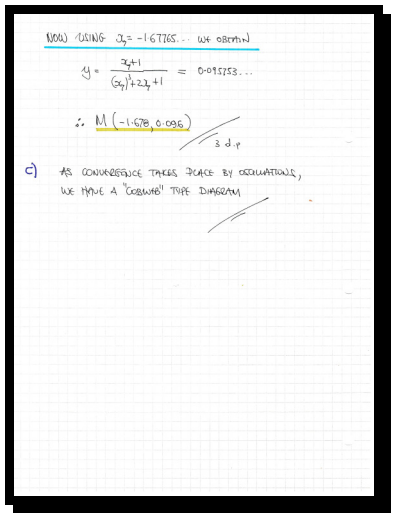
$$x=-\frac{3x^2+1}{2x^2}.$$

**b)** By using an iterative formula based on the equation of part **(a)**, determine the coordinates of $M$ correct to three decimal places.

**c)** State, with a reason, whether the convergence taking place using the formula of part **(b)** is of the "cobweb" type or the "staircase" type.

$\boxed{\phantom{00}}$, $\boxed{M(-1.678,0.096)}$



**Question 25 (****)**

It is required to find the approximate coordinates of the points of intersection between the graphs of

$$y_1 = 9 - x^2,\ x \in \mathbb{R} \qquad \text{and} \qquad y_2 = \ln(x-1),\ x \in \mathbb{R},\ x > 1.$$

**a)** Show that the two graphs intersect at a single point $P$.

**b)** Explain why the $x$ coordinate of $P$ lies between $2$ and $3$.

The recurrence formula

$$x_{n+1} = \sqrt{9 - \ln(x_n - 1)},$$

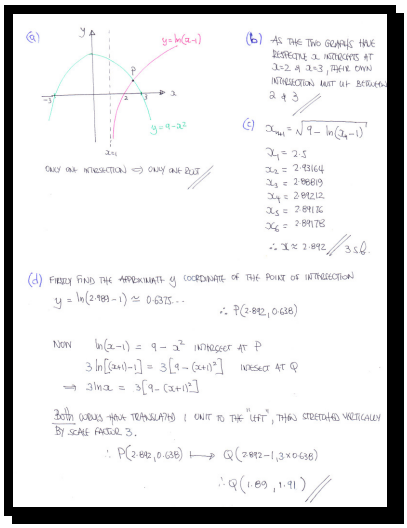
starting with a suitable value for $x_1$, is to be used to find the $x$ coordinate of $P$.

**c)** Calculate the $x$ coordinate of $P$, correct to three decimal places.

**d)** By considering two suitable transformations, determine correct to two decimal places the coordinates of the points of intersection between the graph of

$$y_3 = 3\left[9 - (x+1)^2\right],\ x \in \mathbb{R} \qquad \text{and} \qquad y_4 = 3\ln x,\ x \in \mathbb{R},\ x > 0.$$

$\square$, $\boxed{x \approx 2.892}$, $\boxed{(1.89, 1.91)}$



**Question 26 (****)**

$$y = \arctan x\,,\ x \in \mathbb{R}\,.$$

**a)** By writing $y = \arctan x$ as $x = \tan y$ show that

$$\frac{dy}{dx} = \frac{1}{1+x^2}\,.$$

The curve $C$ has equation

$$y = \arctan x - 4\ln\left(1+x^2\right) - 3x^2\,,\ x \in \mathbb{R}\,.$$

**b)** Show that the $x$ coordinate of the stationary point of $C$ is a root of the equation

$$6x^3 + 14x - 1 = 0\,.$$

**c)** Show that the above equation has a root $\alpha$ in the interval $(0,1)$.

The iterative formula

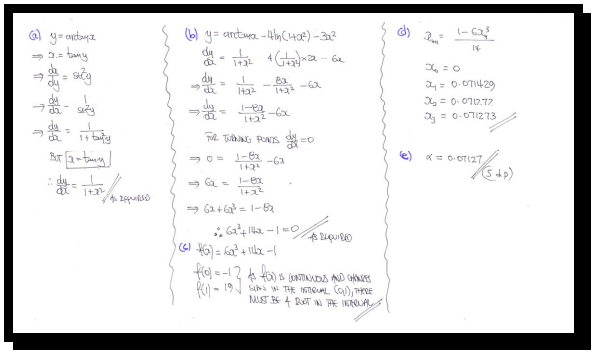
$$x_{n+1} = \frac{1-6x_n^3}{14} \quad \text{with } x_0 = 0\,,$$

is used to find this root.

**d)** Find, correct to $6$ decimal places, the value of $x_1$, $x_2$ and $x_3$.

**e)** Hence write down the value of $\alpha$, correct to $5$ decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_1 = 0.071429,\quad x_2 = 0.071272,\quad x_3 = 0.071273}$, $\boxed{\alpha = 0.07127}$



**Question 27 (****)**

$$y=\arctan x\,,\ x\in\mathbb{R}\,.$$

**a)** By writing $y=\arctan x$ as $x=\tan y$ show that

$$\frac{dy}{dx}=\frac{1}{1+x^2}\,.$$

The curve $C$ has equation

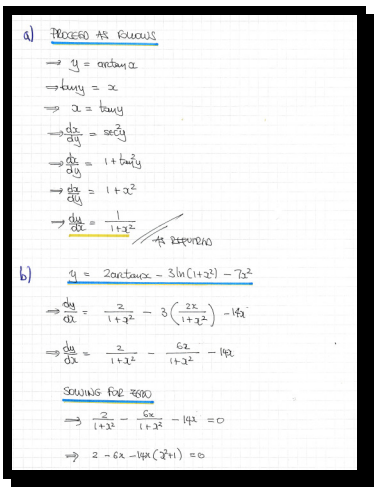
$$y=2\arctan x-3\ln\left(1+x^2\right)-7x^2\,,\ x\in\mathbb{R}\,.$$

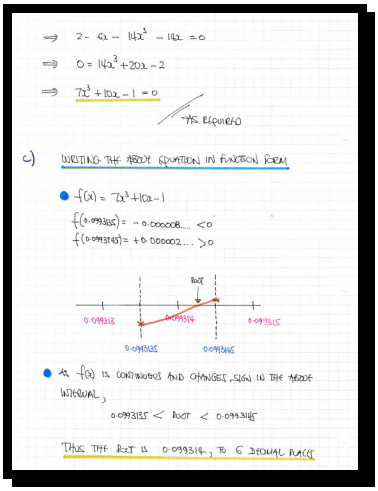
**b)** Show that the $x$ coordinate of the stationary point of $C$ is a solution of the cubic equation

$$7x^3+10x-1=0\,.$$

**c)** Hence show further that the $x$ coordinate of the stationary point of $C$ is $0.099314$, correct to $6$ decimal places.

[ ], [proof]



**Question 28** **(****)**

$$y=\arcsin x\,,\ -1\le x\le 1\,.$$

**a)** By writing $y=\arcsin x$ as $x=\sin y$ show that

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}\,.$$

The curve $C$ has equation

$$y=3\arcsin x-4x^{\frac{3}{2}}+5\,,\ 0\le x\le 1\,.$$

**b)** Show that the $x$ coordinates of the stationary points of $C$ are the solutions of the equation

$$4x^3-4x+1=0\,.$$

**c)** Show that the above equation has a root $\alpha$ in the interval $(0,0.5)$.

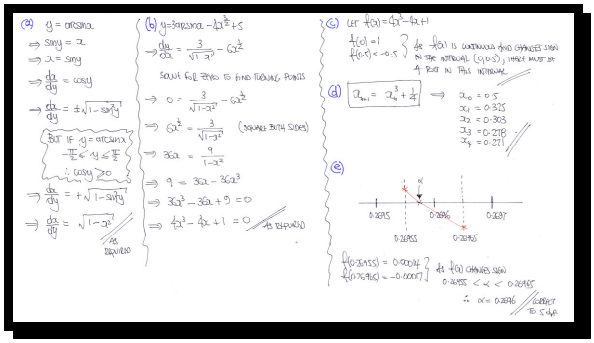
The root $\alpha$ can be found by using the iterative formula

$$x_{n+1}=x_n^3+\frac{1}{4}\,,\ \text{with}\ x_0=0.5\,.$$

**d)** Find, correct to $3$ decimal places, the value of $x_1$, $x_2$, $x_3$ and $x_4$.

**e)** By considering the sign of an appropriate function $f(x)$ in a suitable interval, show that $\alpha=0.2696$, correct to $4$ decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_1=0.375,\quad x_2=0.303,\quad x_3=0.278,\quad x_4=0.271}$, $\boxed{\alpha=0.07127}$



**Question 29 (****)**

It is known that the cubic equation

$$x^3 - 2x = 5, \quad x \in \mathbb{R},$$

has a single real solution $\alpha$, which is close to $2.1$.

Four iterative formulas based on rearrangements of this equation are being considered for their suitability to approximate the value of $\alpha$ to greater accuracy.

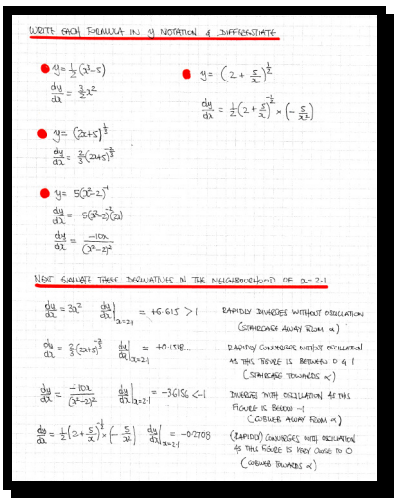
- $x_{n+1} = \frac{1}{2}\left(x_n^3 - 5\right)$
- $x_{n+1} = \sqrt[3]{2x_n + 5}$
- $x_{n+1} = \dfrac{5}{x_n^2 - 2}$
- $x_{n+1} = \sqrt{2 + \dfrac{5}{x_n}}$

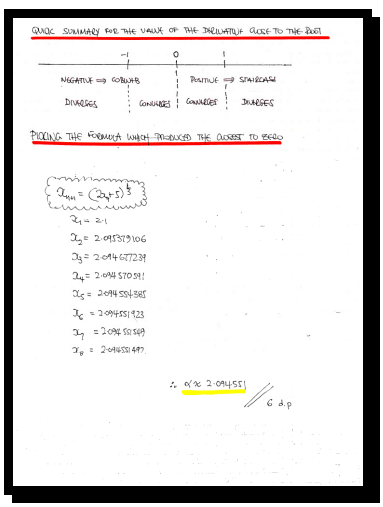
**a)** Use a differentiation method, and **without** carrying any direct iterations, briefly describe the suitability of these four formulas.

In these descriptions you must make a reference to rates of convergence or divergence, and cobweb or staircase diagrams.

**b)** Use one of these four formulas to approximate the value of $\alpha$, correct to 6 decimal places

$\boxed{\phantom{00}}$, $\boxed{\alpha \approx 2.094551}$



**Question 30** **(****+)**

A function $f$ is defined in the largest real domain by the equation

$$f(x)\equiv\frac{50x^2-142x+95}{2x-5}.$$

**a)** State the domain of $f$.

**b)** Evaluate $f(1)$, $f(2)$ and $f(3)$, and hence briefly discuss the number of possible intersections of $f$ with the coordinate axes, ...

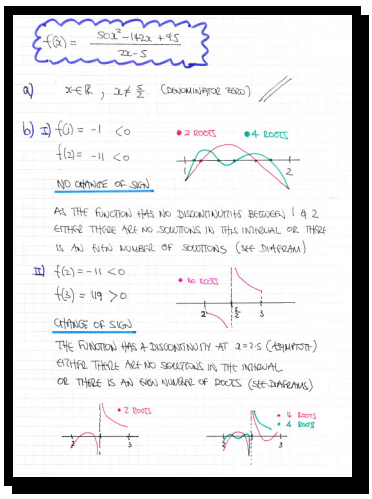
**i.** ... in the interval $[1,2]$.

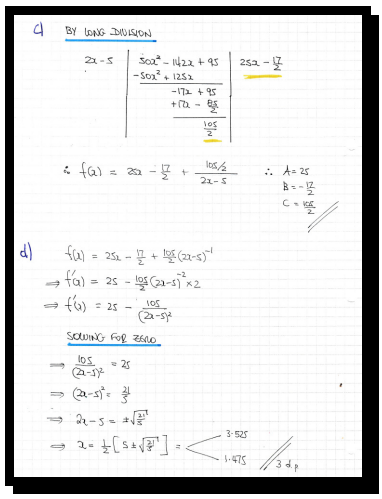
**ii.** ... in the interval $[2,3]$.

**c)** Express $f(x)$ in the form $Ax+B+\dfrac{C}{2x-5}$, where $A$, $B$ and $C$ are constants.

**d)** Calculate, correct to 3 decimal places, the $x$ coordinates of the stationary points of $f$.

$\boxed{\phantom{x}}$, $\boxed{x\in\mathbb{R},\ x\neq\frac{5}{2}}$, $\boxed{f(x)\equiv 25x-8.5+\frac{52.5}{2x-5}}$, $\boxed{x\approx 3.525,\quad x\approx 1.475}$





**Question 31 (****+)**

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

**a)** By writing $y = \arcsin x$ as $x = \sin y$ show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,.$$

The curve $C$ has equation

$$y = 2\arcsin x - 4x^{\frac{3}{2}}\,,\ 0 \le x \le 1\,.$$

**b)** Show that the $x$ coordinates of the stationary points of $C$ are the solutions of the equation

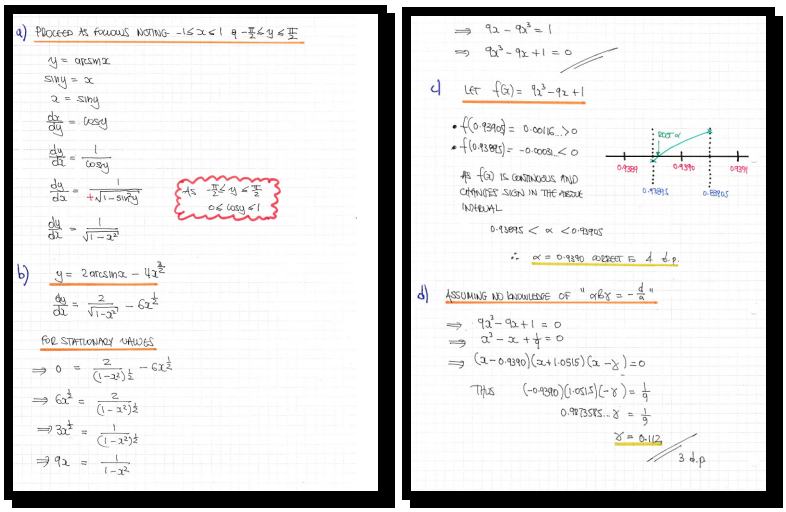
$$9x^3 - 9x + 1 = 0\,.$$

**c)** Show further that one of the roots, $\alpha$, of the equation of part **(b)** is $0.9390$, correct to $4$ decimal places.

It is further given that the equation of part **(b)** has $2$ more real roots, $\beta \approx -1.0515$, and $\gamma$.

**d)** Determine the value of $\gamma$, correct to $3$ places.

$\boxed{\phantom{00}}$, $\boxed{\gamma \approx 0.112}$



**Question 32 (****+)**

The curve $C$ has equation

$$f(x) \equiv 3x^4 + 8x^3 + 3x^2 - 12x - 6, \quad x \in \mathbb{R}.$$

The curve has a single stationary point whose $x$ coordinate lies in the interval $[n, n+1]$, where $n \in \mathbb{Z}$.

**a)** Determine with full justification the value of $n$.

A suitable equation is rearranged to produced three recurrence relations, each of which may be used to find the $x$ coordinate of the stationary point of $C$.

These recurrence relations, all starting with $x_0 = \frac{1}{2}n$ are shown below.

**(i)** $x_{n+1} = \dfrac{2}{2x_n^{\,2} + 4x_n + 1}$ **(ii)** $x_{n+1} = \dfrac{1}{x_n^{\,2}} - \dfrac{1}{2x_n} - 2$ **(iii)** $x_{n+1} = \sqrt{\dfrac{2 - x_n}{4 + 2x_n}}$

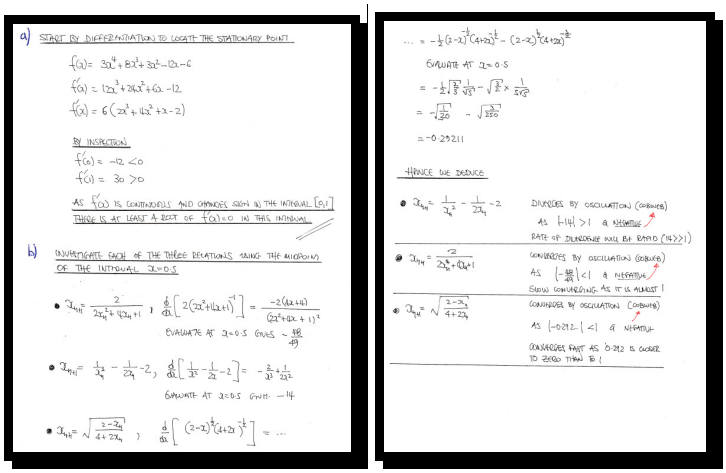
**b)** Use a differentiation method, to investigate the result in attempting to find an approximate value for the $x$ coordinate of the stationary point of $f(x)$, with each of these three recurrence relations.

The method must include ...

- ... whether the attempt is successful
- ... whether the convergence or divergence is a "cobweb" case or a "staircase" case.
- ... which recurrence relation converges at the fastest rate.

You may not answer part **(b)**, by simply generating sequences.

$\boxed{\phantom{00}}$, $\boxed{[0,1]}$



**Question 33 (****+)**

The curve $C$ has equation

$$y=\sqrt{\mathrm{e}^{2x}-2x}\,,\ \mathrm{e}^{2x}>2x\,.$$

The tangent to $C$ at the point $P$, where $x=p$, passes through the origin.

**a)** Show that $x=p$ is a solution of the equation

$$(1-x)\mathrm{e}^{2x}=x\,.$$

**b)** Show further that the equation of part **(a)** has a root between $0.8$ and $1$.

The iterative formula

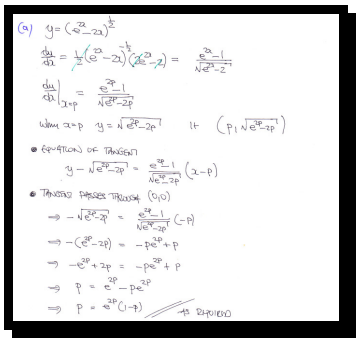
$$x_{n+1}=1-x_n\,\mathrm{e}^{-2x_n}$$

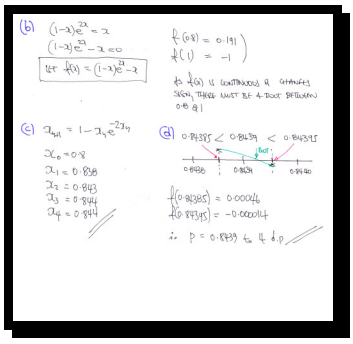
with $x_0=0.8$ is used to find this root.

**c)** Find, correct to $3$ decimal places, the value of $x_1$, $x_2$, $x_3$ and $x_4$.

**d)** Hence show that the value of $p$ is $0.8439$, correct to $4$ decimal places.

$\boxed{\phantom{x}}$, $\boxed{x_1=0.838,\quad x_2=0.843,\quad x_3=0.844,\quad x_4=0.844}$





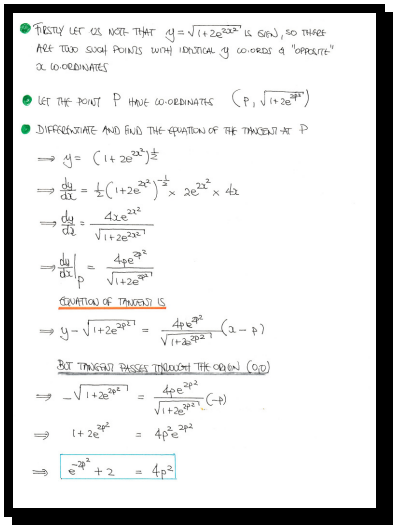
**Question 34 (*****)**

The point $P$ lies on the curve $C$ with equation

$$y=\sqrt{1+2\mathrm{e}^{2x^2}}\ ,\ x\in\mathbb{R}\ .$$

Given that the tangent to $C$ at $P$ passes through the origin, determine the coordinates of $P$, correct to $3$ significant figures.

$\boxed{\phantom{00}}$, $\boxed{P\left(\pm 0.761, 2.71\right)}$



FIRSTLY LET US NOTE THAT $y=\sqrt{1+2e^{2x^2}}$ IS EVEN, SO THERE ARE TWO SUCH POINTS WITH IDENTICAL $y$ CO-ORDS & "OPPOSITE" $x$ CO-ORDINATES

LET THE POINT $P$ HAVE CO-ORDINATES $(p, \sqrt{1+2e^{2p^2}})$

DIFFERENTIATE AND FIND THE EQUATION OF THE TANGENT AT $P$

$\Rightarrow y=(1+2e^{2x^2})^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}(1+2e^{2x^2})^{-\frac{1}{2}}\times 2e^{2x^2}\times 4x$

$\Rightarrow \frac{dy}{dx}=\frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}}$

$\Rightarrow \left.\frac{dy}{dx}\right|_P=\frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}$

EQUATION OF TANGENT IS

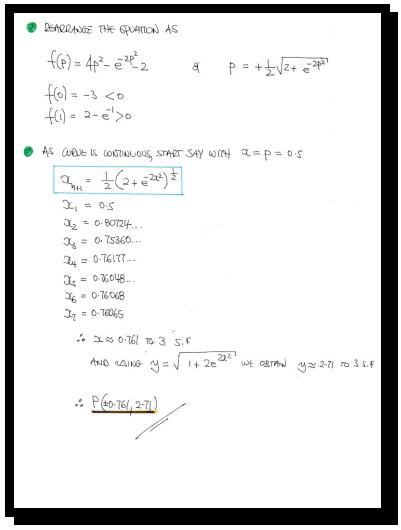
$\Rightarrow y-\sqrt{1+2e^{2p^2}}=\frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}(x-p)$

BUT TANGENT PASSES THROUGH THE ORIGIN $(0,0)$

$\Rightarrow -\sqrt{1+2e^{2p^2}}=\frac{4pe^{2p^2}}{\sqrt{1+2e^{2p^2}}}(-p)$

$\Rightarrow 1+2e^{2p^2}=4p^2e^{2p^2}$

$\Rightarrow \boxed{e^{2p^2}+2=4p^2}$



REARRANGE THE EQUATION AS

$f(p)=4p^2-e^{2p^2}-2$ & $p=+\frac{1}{2}\sqrt{2+e^{2p^2}}$

$f(0)=-3<0$

$f(1)=2-e^{-1}>0$

AS CURVE IS CONTINUOUS, START SAY WITH $x=p=0.5$

$x_{n+1}=\frac{1}{2}(2+e^{2x^2})^{\frac{1}{2}}$

$x_1=0.5$

$x_2=0.807214\ldots$

$x_3=0.75360\ldots$

$x_4=0.76177\ldots$

$x_5=0.76048\ldots$

$x_6=0.76068$

$x_7=0.76065$

$\therefore x\approx 0.761$ TO 3 S.F.

AND USING $y=\sqrt{1+2e^{2x^2}}$ WE OBTAIN $y\approx 2.71$ TO 3 S.F.

$\therefore P(\pm 0.761, 2.71)$

**Question 35 (*****)**

A curve $C$ is defined in the largest real domain by the equation

$$y = \log_x 2 .$$

**a)** Sketch a detailed graph of $C$.

The point $P$, where $x = 2$ lies on $C$.

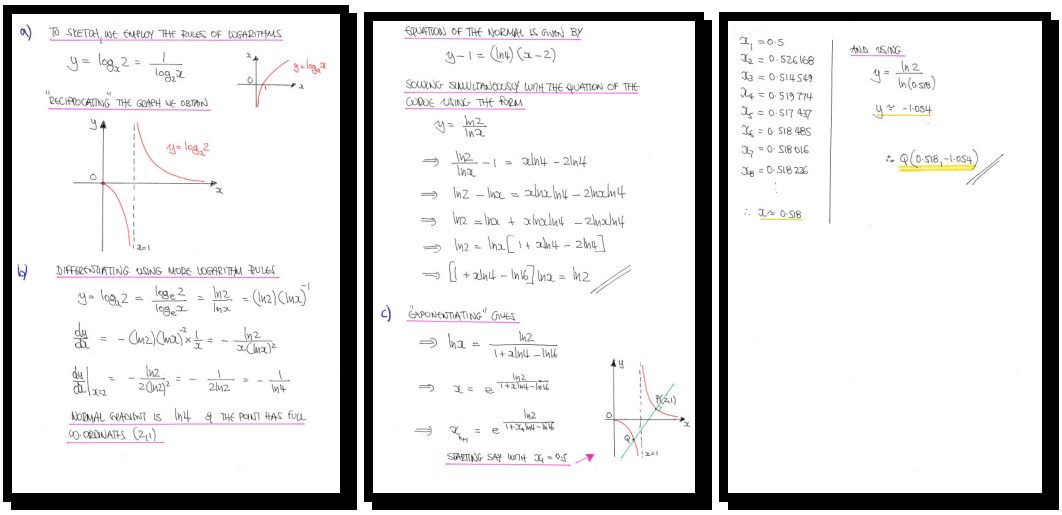
The normal to $C$ at $P$ meets $C$ again at the point $Q$.

**b)** Show that the $x$ coordinate of $Q$ is a solution of the equation

$$\left[1 + x\ln 4 - \ln 16\right]\ln x = \ln 2 .$$

**c)** Use an iterative formula of the form $x_{n+1} = \mathrm{e}^{f(x_n)}$, with a suitable starting value, to find the coordinates of $Q$, correct to $3$ decimal places.

$\boxed{\phantom{xxxxxx}}$, $\boxed{Q(0.518, -1.054)}$



# LINEAR INTERPOLATION

**Question 1 (**+)**

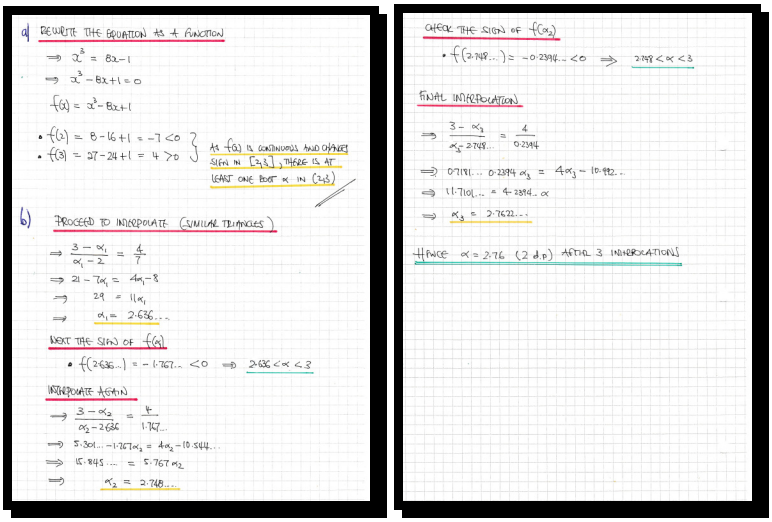
The following cubic equation is to be solved numerically.

$$x^3 = 8x - 1, \ x \in \mathbb{R}.$$

**a)** Show that the equation has a root $\alpha$, in the interval $[2,3]$.

**b)** Use linear interpolation three successive times, to find $\alpha$ correct to an appropriate degree of accuracy.

$\boxed{\phantom{00}}$, $\boxed{\alpha \approx 2.76}$



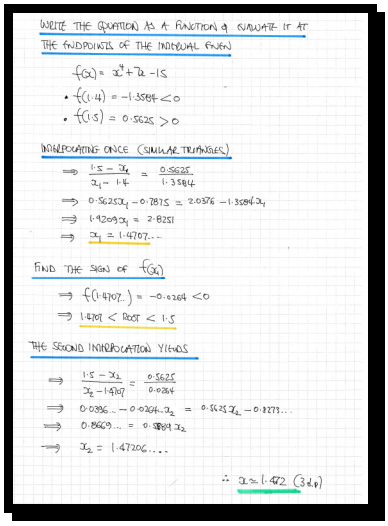
**Question 2 (**+)**

The following quartic equation is to be solved numerically.

$$x^4+7x-15=0,\ x\in\mathbb{R}.$$

Given that the above quartic has a real root $\alpha$ in the interval $[1.4,1.5]$, use linear interpolation twice, to find $\alpha$ correct to an appropriate degree of accuracy.

$\boxed{\phantom{xxxx}}$, $\boxed{\alpha\approx 1.472}$



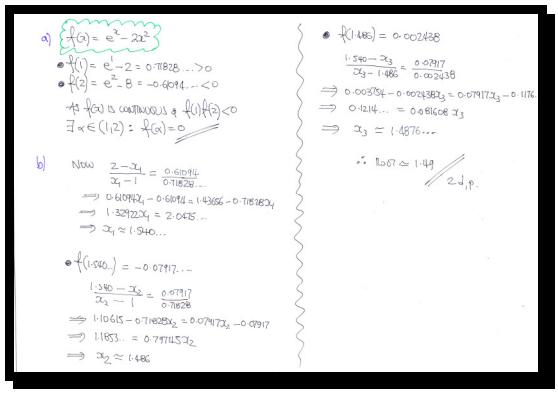
**Question 3 (***)**

$$e^x - 2x^2 = 0 .$$

**a)** Show that the above equation has a root $\alpha$, which lies between $1$ and $2$.

**b)** Use linear interpolation three times, starting in the interval $[1,2]$ to find, correct to $2$ decimal places the value of $\alpha$.

$$\boxed{x_1 \approx 1.540, \quad x_2 \approx 1.486, \quad x_3 \approx 1.488, \quad \alpha \approx 1.49}$$



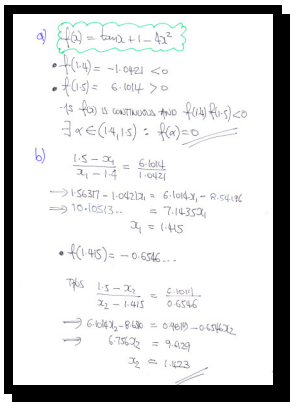
**Question 4 (***)**

$$\tan x = 4x^2 - 1 .$$

**a)** Show that the above equation has a root $\alpha$, which lies between $1.4$ and $1.5$.

**b)** Use linear interpolation twice, starting in the interval $[1.4,1.5]$ to find, correct to $3$ decimal places two approximations for $\alpha$.

$$\boxed{x_1 \approx 1.415, \quad x_2 \approx 1.423}$$



# NEWTON RAPHSON METHOD

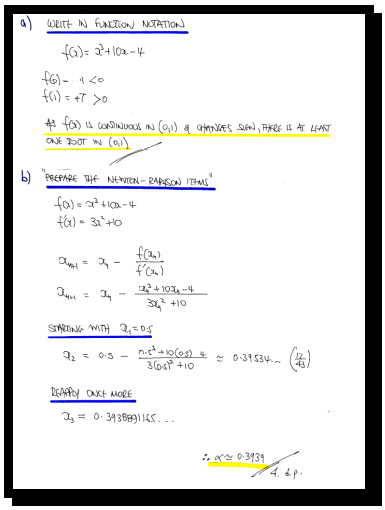
**Question 1 (**)**

$$x^3+10x-4=0\,.$$

**a)** Show that the above equation has a root $\alpha$, which lies between $0$ and $1$.

**b)** Use the Newton-Raphson method **twice**, starting with $x_1=0.5$ to find, correct to $4$ decimal places, an approximation for $\alpha$.

$\boxed{\phantom{x=0.3939}}$, $\boxed{x=0.3939}$



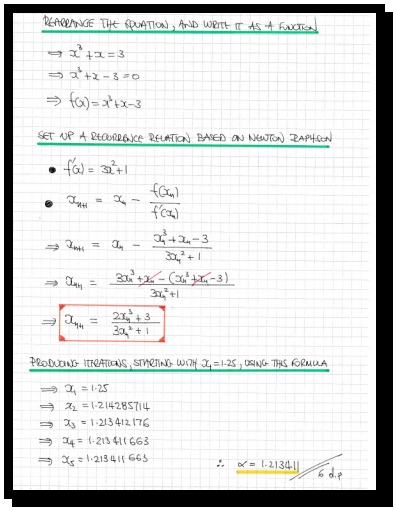
**Question 2** **(***+)**

It is known that the cubic equation below has a root $\alpha$, which is close to $1.25$.

$$x^3+x=3.$$

Use an iterative formula based on the Newton Raphson method to find the value of $\alpha$, correct to $6$ decimal places.

$\boxed{\phantom{x}}$, $\boxed{\alpha \approx 1.213411}$



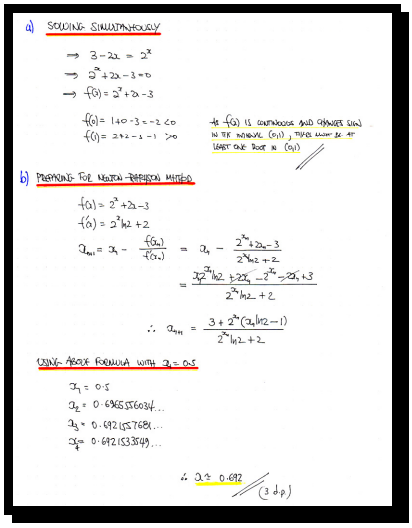
**Question 3** **(***+)**

The curve with equation $y=2^x$ intersects the straight line with equation $y=3-2x$ at the point $P$, whose $x$ coordinate is $\alpha$.

**a)** Show that $0<\alpha<1$.

**b)** Starting with $x=0.5$, use the Newton Raphson method to find the value of $\alpha$, correct to $3$ decimal places.

$\boxed{\phantom{0.692}}$, $\boxed{\alpha\approx 0.692}$



**Question 4 (***+)**

A curve has equation

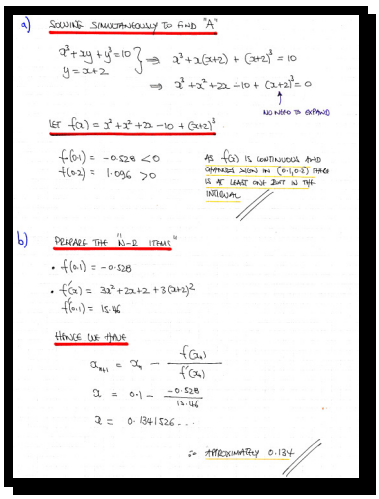
$$x^3+xy+y^3=10\,.$$

The straight line with equation $y=x+2$ meets this curve at the point $A$.

**a)** Show that the $x$ coordinate of $A$ lies in the interval $(0.1,0.2)$.

**b)** Use the Newton Raphson method once, starting with $x=0.1$, to find a better approximation for the $x$ coordinate of $A$.

$\boxed{\phantom{00}}$, $\boxed{x\approx 0.134}$



**Question 5** **(****)**

A curve has equation

$$x^3 + y = xy\,.$$

The straight line with equation $y+3x+1=0$ meets this curve at the point $A$.

**a)** Show that the $x$ coordinate of $A$ lies in the interval $(-0.4,-0.3)$.

**b)** If $P$ and $Q$ are integers, use an iterative procedure based on the formula

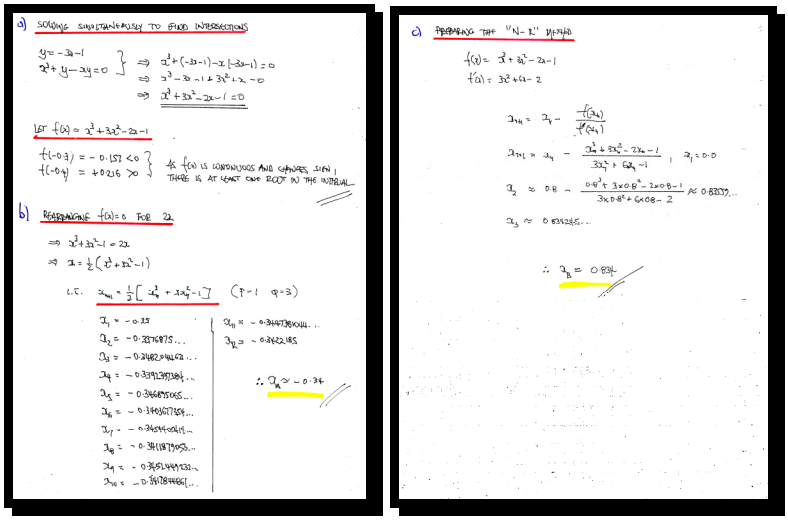
$$x_{n+1} = \frac{1}{2}\left[Px^3 + Qx^2 - 1\right],\ x_1 = -0.35\,,$$

to find the $x$ coordinate of $A$, correct to $2$ decimal places.

The straight line with equation $y+3x+1=0$ meets the above mentioned curve at another point $B$, whose the $x$ coordinate lies in the interval $(0.8,0.9)$.

**c)** Use the Newton Raphson method twice, starting with $x=0.8$, to find a better approximation for the $x$ coordinate of $B$.

$\boxed{\phantom{x}}$, $\boxed{x_A \approx 0.34}$, $\boxed{x_B \approx 0.834}$



**Question 6** **(****)**

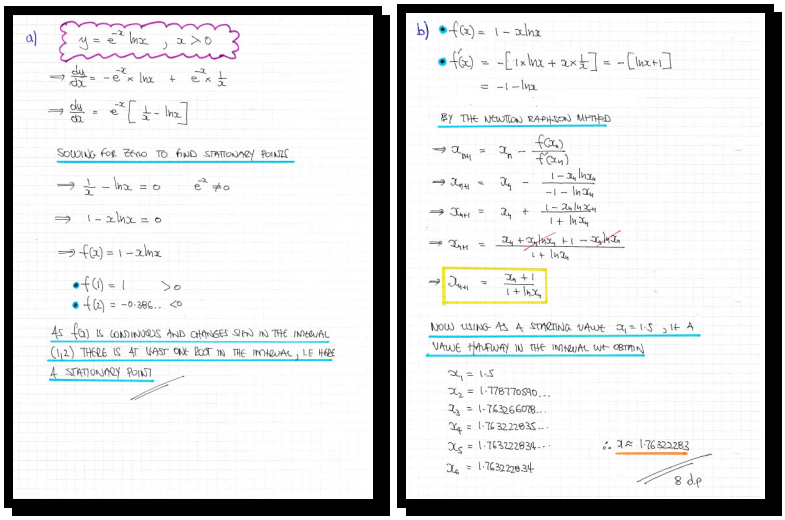
A curve $C$ has equation

$$y = \mathrm{e}^{-x} \ln x \, , \quad x > 0 \, .$$

**a)** Show that the $x$ coordinate of the stationary point of $C$ lies between 1 and 2.

**b)** Use an iterative formula based on the Newton Raphson method to find the $x$ coordinate of the stationary point of $C$, correct to 8 decimal places.

$\boxed{\phantom{x}}$, $\boxed{x \approx 1.76322283}$



**Question 7 (****)**

At the point $P$, which lies on the curve with equation

$$x = \ln\left(y^3 - y\right),$$

the gradient is $4$.

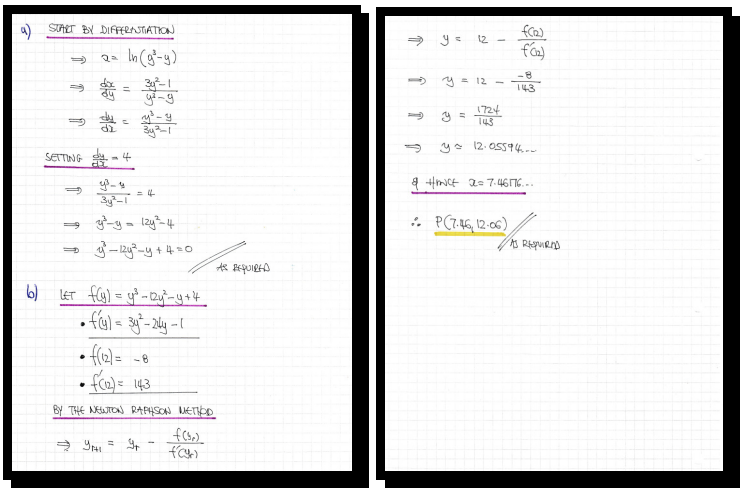
The point $P$ is close to the point with coordinates $(7.5\,,12)$.

**a)** Show that the $y$ coordinate of $P$ is a solution of the equation

$$y^3 - 12y^2 - y + 4 = 0\,.$$

**b)** Use the Newton Raphson method once on the equation of part **(a)**, in order to determine the coordinates of $P$, correct to two decimal places.

$\boxed{\phantom{00}}$, $\boxed{P(7.46, 12.06)}$



**Question 8** **(****)**

It is required to find the single real root $\alpha$ of the following equation

$$x^2 = \frac{2}{\sqrt{x}} + \frac{3}{x^2}, \ x > 0 .$$

**a)** Show that the $\alpha$ lies between $1$ and $2$.

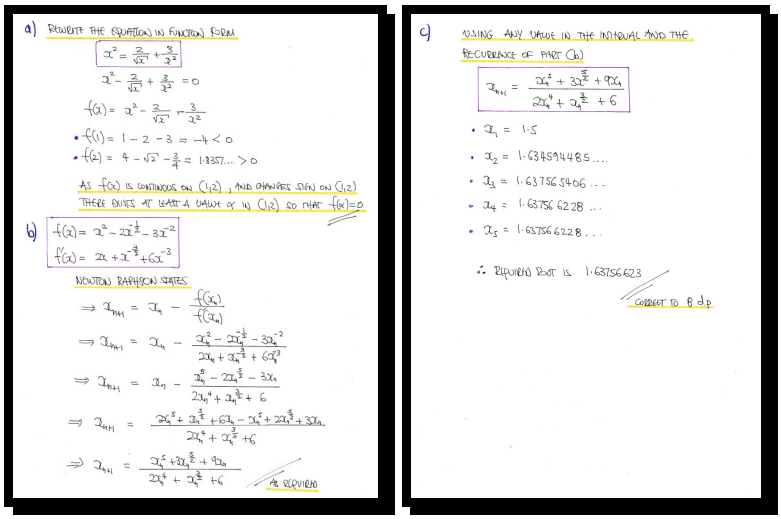
**b)** Use the Newton Raphson method to show that $\alpha$ can be found by the iterative formula

$$x_{n+1} = \frac{x_n^5 + 3x_n^{\frac{5}{2}} + 9x_n}{2x_n^4 + x_n^{\frac{3}{2}} + 6},$$

starting with a suitable value for $x_1$.

**c)** Hence find the value of $\alpha$, correct to $8$ decimal places.

$\boxed{\phantom{00}}$, $\boxed{\alpha \approx 1.63756623...}$



**Question 9** **(****)**

It is required to find the approximate coordinates of the points of intersection between the graphs of

$$y_1 = 1 - x^2,\ x \in \mathbb{R} \qquad \text{and} \qquad y_2 = \ln(x+1),\ x \in \mathbb{R},\ x > -1.$$

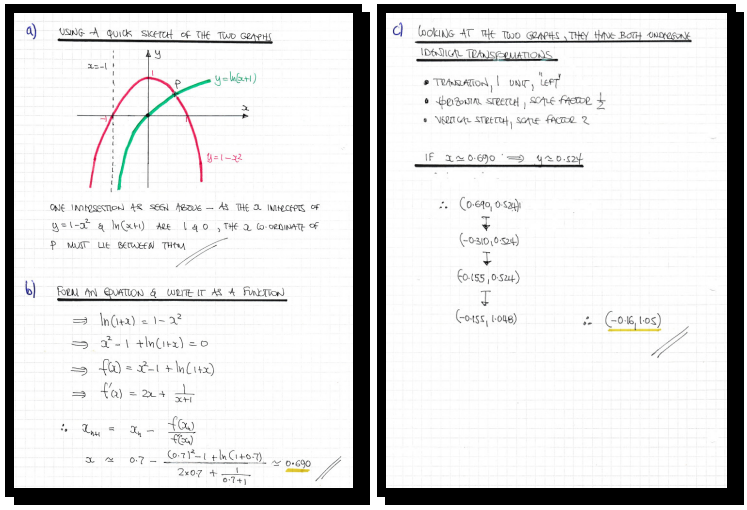
**a)** Show that the two graphs intersect at a single point $P$, explaining further why the $x$ coordinate of $P$ lies between $0$ and $1$.

**b)** Use the Newton Raphson method once, starting $x = 0.7$, to calculate the $x$ coordinate of $P$, giving the answer correct to $3$ decimal places.

**c)** By considering two suitable transformations, determine correct to $2$ decimal places the coordinates of the points of intersection between the graph of

$$y_3 = 2\left[1-(2x+1)^2\right],\ x \in \mathbb{R} \qquad \text{and} \qquad y_4 = 2\ln(2x+2),\ x \in \mathbb{R},\ x > -\frac{1}{2}.$$

$\boxed{\phantom{x}}$, $\boxed{x \approx 0.690}$, $\boxed{(-0.16, 1.05)}$



**Question 10 (****)**

An arithmetic series has first term $2$ and common difference $X$.

A geometric series has first term $2$ and common ratio $X$.

The sum of the $11^{\text{th}}$ term of the arithmetic series and the $11^{\text{th}}$ term of the geometric series is $900$.

**a)** Show that $X$ is a solution of the equation

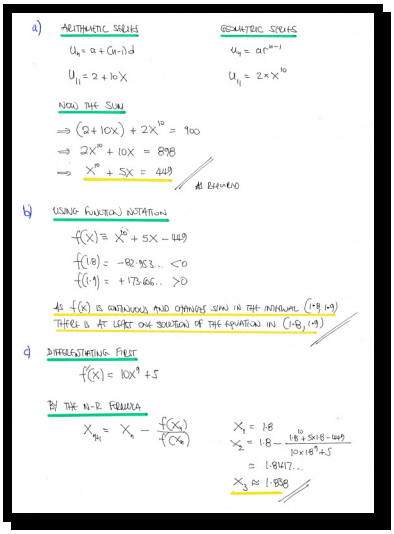
$$X^{10}+5X=449.$$

**b)** Show further that

$$1.8<X<1.9.$$

**c)** Use the Newton Raphson approximation method **twice**, with a starting value of $1.8$, to find an approximate value for $X$, giving the answer correct to 3 decimal places.

$\boxed{\phantom{00}}$, $\boxed{X\approx 1.838}$



**Question 11 (****+)**

The curve $C$ has equation

$$y=\sqrt{\mathrm{e}^{2x}+1}\,,\ x\in\mathbb{R}\,.$$

The tangent to the curve at the point $P$, where $x=p$, passes through the origin.

**a)** Show that $x=p$ is a solution of the equation

$$(x-1)\mathrm{e}^{2x}=1\,.$$

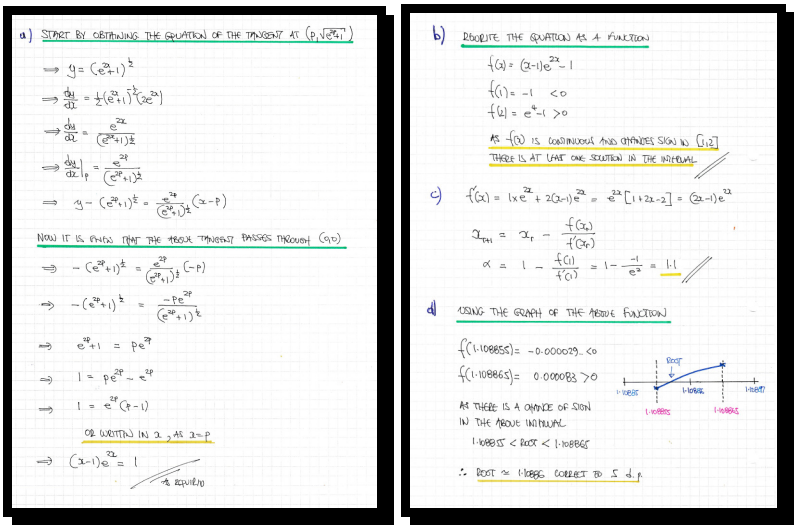
**b)** Show further that the equation of part **(a)** has a root between 1 and 2.

**c)** By using the Newton Raphson method once, starting with $x=1$, find an approximation for this root, correct to 1 decimal place.

It is further given that the Newton Raphson method fails on this occasion.

**d)** Use an appropriate method to verify that the root of the equation of part **(a)** is 1.10886 correct to 5 decimal places.

$\boxed{\phantom{x}}$, $\boxed{\alpha\approx 1.1}$



**Question 12** **(****+)**

The point $P$ has $x$ coordinate $2$ and lies on the curve with equation

$$xy = \mathrm{e}^x, \qquad xy > 0.$$

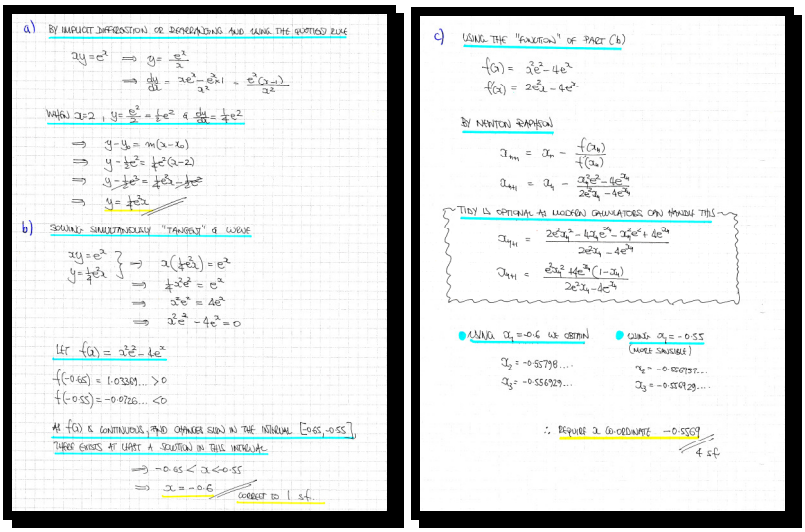
**a)** Determine an equation of the tangent to the curve at $P$.

The tangent to the curve found in part **(a)** meets the curve again at the point $Q$.

**b)** Show that the $x$ coordinate of $Q$ is $-0.6$, correct to one significant figure.

**c)** Use the Newton Raphson method **twice** to find a better approximation for the $x$ coordinate of $Q$, giving the answer correct to $4$ significant figures.

$\boxed{\phantom{x}}$, $\boxed{4y = \mathrm{e}^2 x}$, $\boxed{\approx -0.5569}$



**Question 13 (****+)**

It is required to find the real solutions of the equation

$$x^2 = 2^x .$$

**a)** State the $2$ integer solutions of the equation.

**b)** Sketch in the same set of axes the graph of $y = x^2$ and the graph of $y = 2^x$.

**c)** Use the Newton Raphson method, with a suitable function and an appropriate starting value, to find the third real root of this equation correct to $4$ decimal places.

You may use as many steps as necessary in part **(c)**, to obtain the required accuracy.

$\boxed{\phantom{x}}$, $\boxed{x_1 = 2 \ \cup \ x_2 = 4}$, $\boxed{x_3 \approx -0.7667}$

