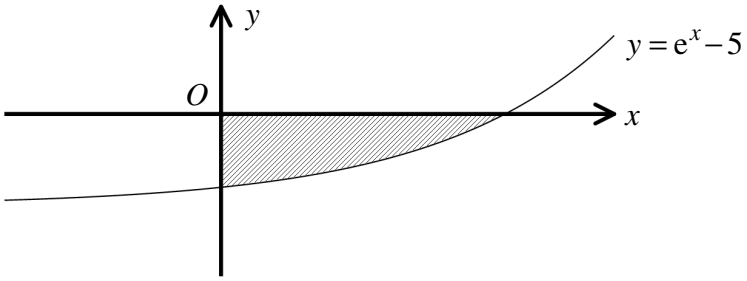
INTEGRATION
FINDING AREAS

Question 1 **(**)**

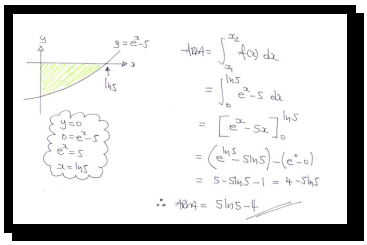


The diagram above shows the graph of the curve with equation

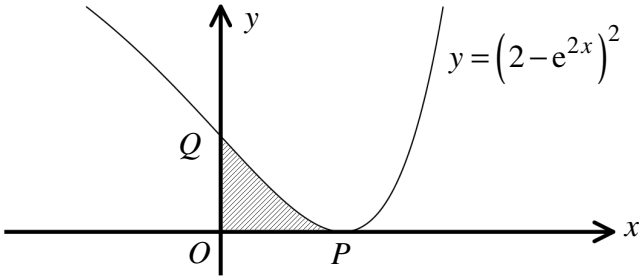
$$y = \mathrm{e}^x - 5, \ x \in \mathbb{R}.$$

Use integration to find the exact area of the finite region bounded by the curve and the coordinate axes.

$$\boxed{5\ln 5 - 4 \approx 4.05}$$



Question 2 **(**+)**



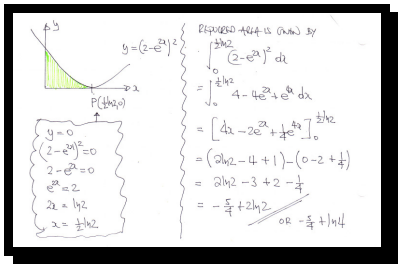
The figure above shows the graph of the curve with equation

$$y=\left(2-e^{2x}\right)^2,\ x\in\mathbb{R}.$$

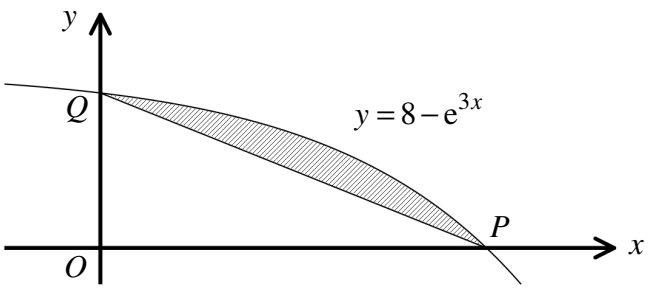
The curve touches the x axis at the point P and crosses the y axis at the point Q.

Find the exact area of the finite region bounded by the curve and the line segments OP and OQ.

$$\boxed{-\frac{5}{4}+\ln 4}$$



Question 3 (+)**



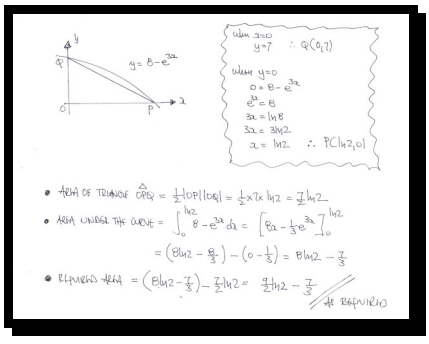
The diagram above shows the graph of the curve with equation

$$y = 8 - \mathrm{e}^{3x}, \ x \in \mathbb{R}.$$

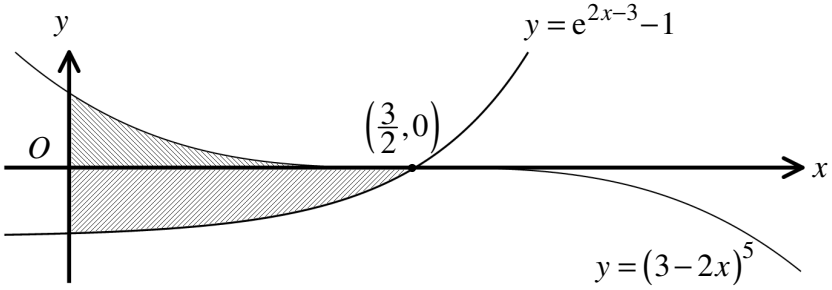
The curve meets the x axis at the point P and the y axis at the point Q.

Show that the area of the finite region bounded by the curve and the straight line segment PQ is exactly $\frac{9}{2}\ln 2 - \frac{7}{3}$.

proof



Question 4 (***)



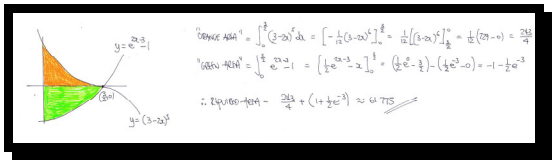
The figure above shows the graphs of the curves with equations

$$y=(3-2x)^5 \quad \text{and} \quad y=\mathrm{e}^{2x-3}-1\,.$$

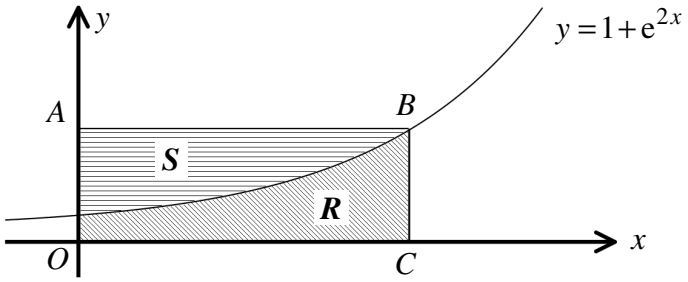
Both curves meet each other at the point with coordinates $\left(\frac{3}{2},0\right)$.

Calculate the area of the shaded region bounded by the two curves and the y axis giving the answer correct to three decimal places.

$$\approx 61.775$$



Question 5 (+)**



The figure above shows the graph of the curve with equation

$$y = 1 + \mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

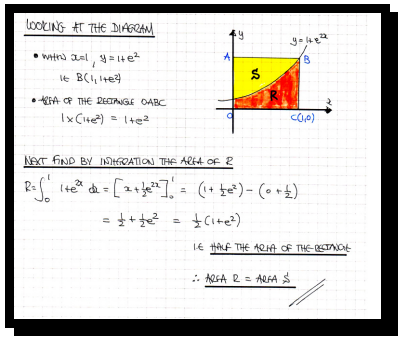
The point C has coordinates $(1,0)$. The point B lies on the curve so that BC is parallel to the y axis. The point A lies on the y axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the coordinate axes and the line BC.

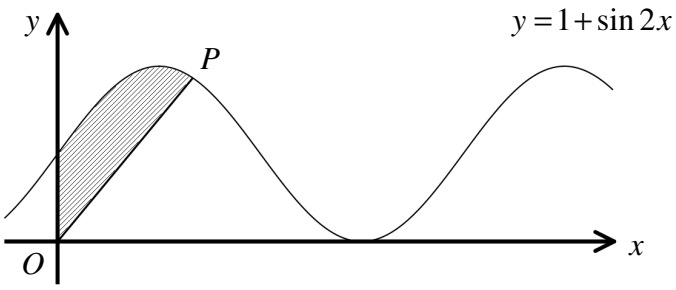
The region S is bounded by the curve, the y axis and the line AB.

Show that the area of R is equal to the area of S.

$\boxed{}$, proof



Question 6 **(**+)**



The figure above shows the graph of the curve with equation

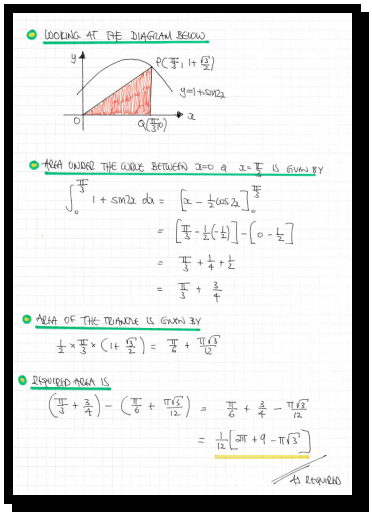
$$y = 1 + \sin 2x , \ x \in \mathbb{R} .$$

The point P lies on the curve where $x = \dfrac{\pi}{3}$.

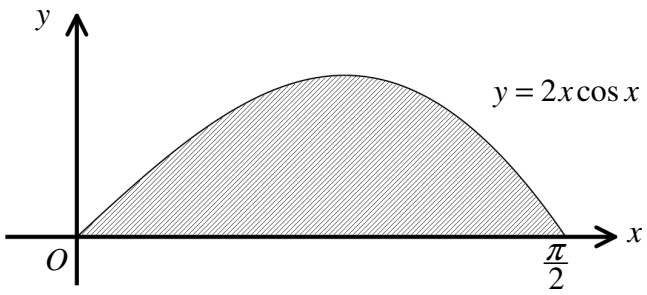
Show that the area of the finite region bounded by the curve, the y axis and the straight line segment OP is exactly

$$\frac{1}{12}\left(2\pi + 9 - \pi\sqrt{3}\right).$$

[ ], [proof]



Question 7 (***)

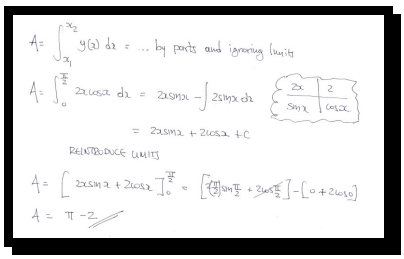


The figure above shows the graph of the curve with equation

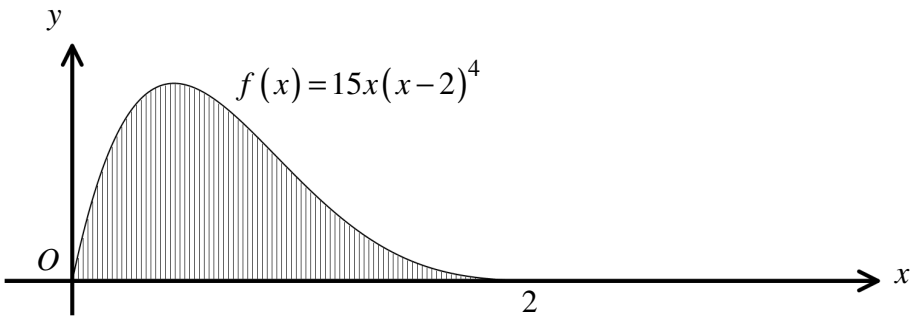
$$y = 2x\cos x\,,\ 0 \le x \le \frac{\pi}{2}\,.$$

Use integration by parts to find the exact area of the finite region bounded by the curve and the x axis.

$\boxed{\pi - 2}$



Question 8 **(***)**



The figure above shows the curve with equation

$$f(x)=15x(x-2)^4,\ 0\le x\le 2.$$

a) Express $f(x+2)$ in the form Ax^5+Bx^4, where A and B are constants.

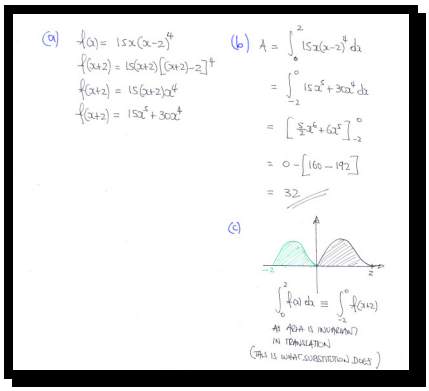
b) By using part (a), or otherwise find the value of

$$\int_0^2 f(x)\ dx.$$

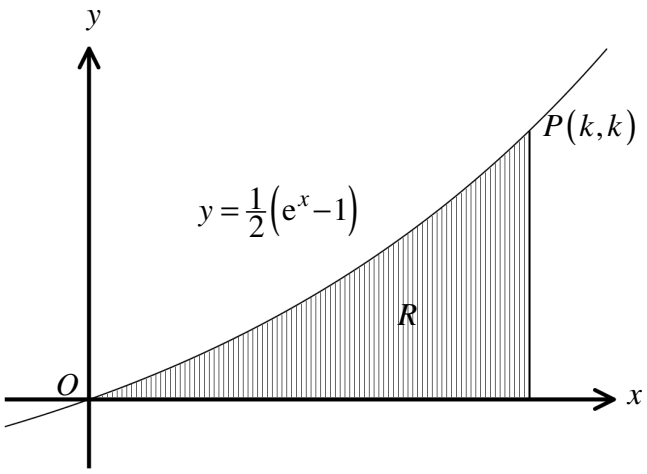
c) Explain geometrically the significance of

$$\int_0^2 f(x)\ dx \equiv \int_{-2}^0 f(x+2)\ dx.$$

$\boxed{f(x+2)=15x^5+30x^4}$, $\boxed{\int_0^2 f(x)\ dx=32}$



Question 9 **(***)**



The figure above shows the curve C with equation

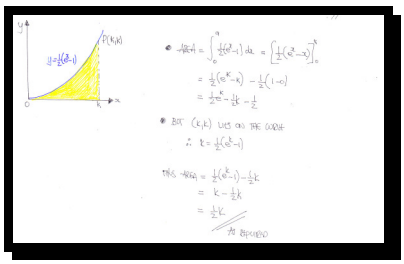
$$y=\tfrac{1}{2}\left(\mathrm{e}^{x}-1\right),\ x\in\mathbb{R}.$$

The point $P(k,k)$, where k is a positive constant, lies on C.

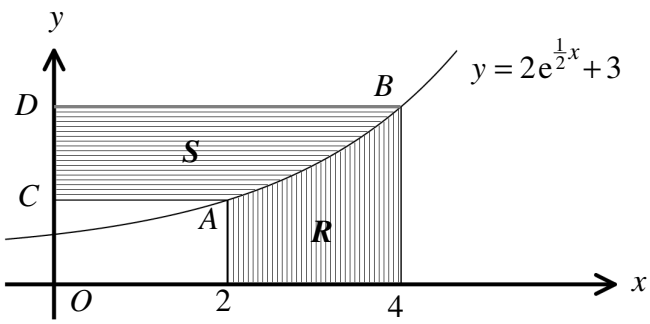
The finite region R, shown shaded in the figure, is bounded by C, the x axis and the line with equation $x=k$.

Show clearly that the area of R is $\tfrac{1}{2}k$.

proof



Question 10 **(***+)**



The figure above shows the graph of the curve with equation

$$y=2\mathrm{e}^{\frac{1}{2}x}+3,\ x\in\mathbb{R}.$$

The points A and B lie on the curve where $x=2$ and $x=4$, respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x=2$ and $x=4$.

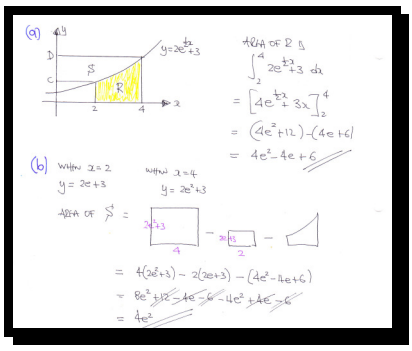
a) Determine the exact area of R.

The points C and D lie on the y axis so that the line segments CA and DB are parallel to the x axis.

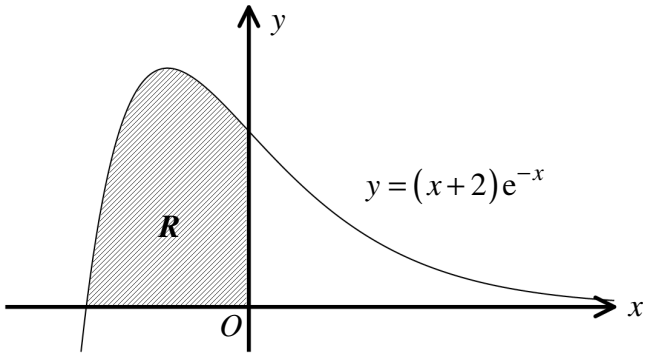
The region S is bounded by the curve, the y axis and the line segments CA and DB.

b) Determine the exact area of S.

$$\boxed{\text{area of } R=4\mathrm{e}^2-4\mathrm{e}+6},\ \boxed{\text{area of } S=4\mathrm{e}^2}$$



Question 11 (*+)**



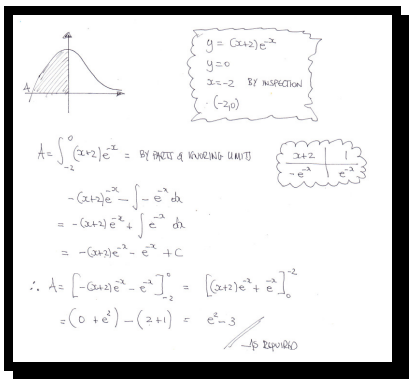
The figure above shows the graph of the curve with equation

$$y=(x+2)e^{-x},\ x\in\mathbb{R}.$$

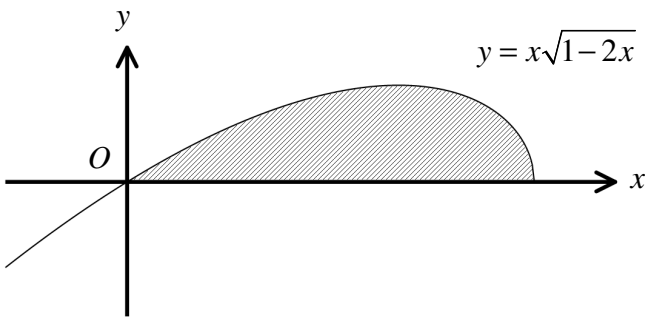
The finite region R, shown shaded in the figure, is bounded by the curve and the coordinate axes.

Use integration by parts to show that the area of R is e^2-3.

proof



Question 12 (*+)**

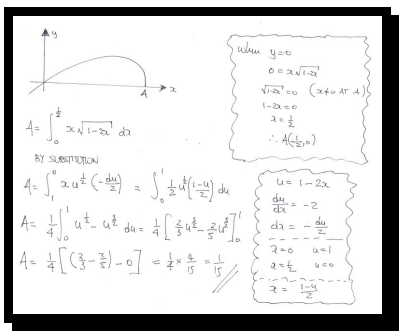


The figure above shows the graph of the curve with equation

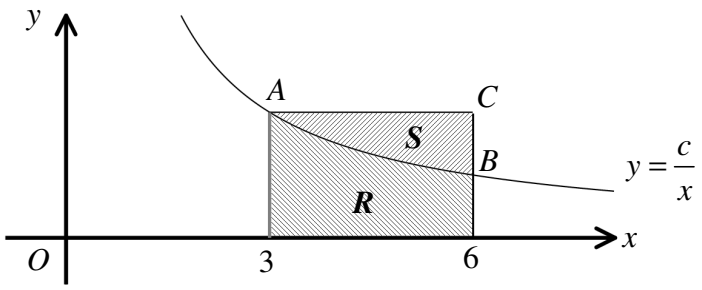
$$y = x\sqrt{1-2x}\ ,\ x \le \frac{1}{2}\ .$$

Use integration by substitution to find the area of the finite region bounded by the curve and the x axis.

$$\boxed{\frac{1}{15}}$$



Question 13 (*+)**



The figure above shows the graph of the curve with equation

$$y = \frac{c}{x},$$

where c is a positive constant.

The points A and B lie on the curve here $x = 3$ and $x = 6$ respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x = 3$ and $x = 6$.

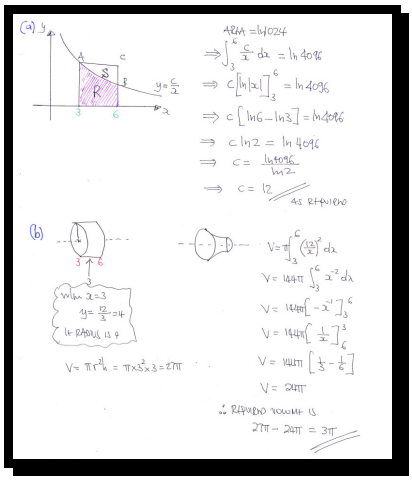
a) Given that the area of R is $\ln 4096$, show that $c = 12$.

The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis. The finite region S is bounded by the curve and the line segments AC and BC.

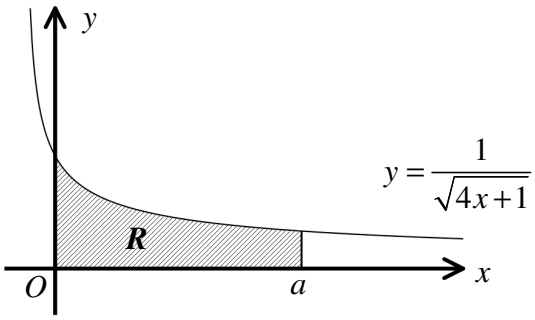
b) Find the volume of the solid produced when S is complete revolved about the x axis.

$\boxed{3\pi}$

[solution overleaf]


(a)
y
x
A
B
R
$y=\frac{c}{x}$
3
6
AREA = ln 2
$\Rightarrow \int_3^6 \frac{c}{x}\,dx = \ln 4096$
$\Rightarrow c\left[\ln|x|\right]_3^6 = \ln 4096$
$\Rightarrow c\left[\ln 6 - \ln 3\right] = \ln 4096$
$\Rightarrow c \ln 2 = \ln 4096$
$\Rightarrow c = \frac{\ln 4096}{\ln 2}$
$\Rightarrow c = 12$
AS required
(b)
3
6
WHEN $x=3$
$y = \frac{12}{3} = 4$
THE RADIUS IS 4
$V = \pi r^2 h = \pi \times 3^2 \times 3 = 27\pi$
$V = \pi\int_3^6 \left(\frac{12}{x}\right)^2 dx$
$V = 144\pi \int_3^6 x^{-2}\,dx$
$V = 144\pi\left[-x^{-1}\right]_3^6$
$V = 144\pi\left[\frac{1}{x}\right]_6^3$
$V = 144\pi\left[\frac{1}{3} - \frac{1}{6}\right]$
$V = 24\pi$
∴ required volume is
$27\pi - 24\pi = 3\pi$

Question 14 (*+)**



The figure above shows the graph of the curve with equation

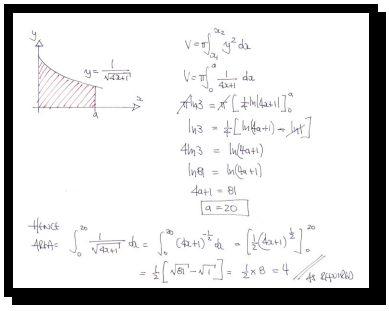
$$y=\frac{1}{\sqrt{4x+1}},\ x>-\frac{1}{4}.$$

The finite region R is bounded by the curve, the coordinate axes and the line with equation $x=a$, where a is a positive constant.

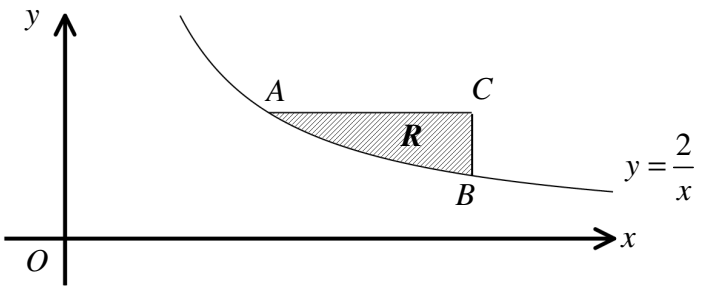
When R is revolved by 2π radians in the x axis the resulting solid of revolution has a volume of $\pi\ln 3$, in appropriate cubic units.

Determine the area of R.

$\boxed{\text{area}=4}$



Question 15 (*+)**



The figure above shows the graph of the curve with equation

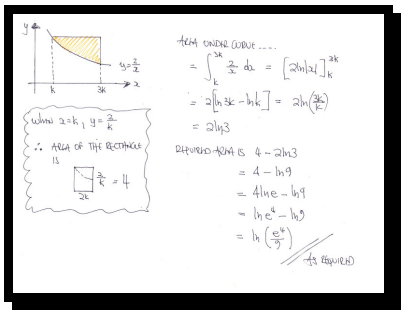
$$y=\frac{2}{x},\ x\in\mathbb{R},\ x\neq 0.$$

The points A and B lie on the curve. The respective x coordinates of these points are k and $3k$, where k is a positive constant. The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis.

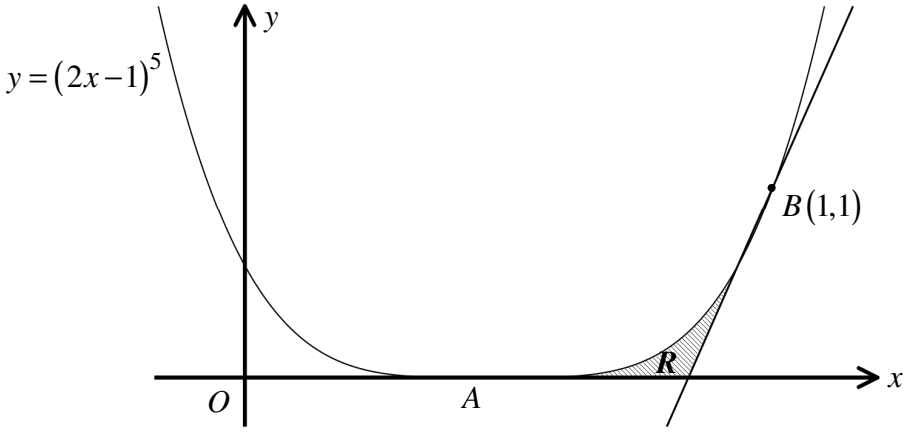
The finite region R is bounded by the curve and the line segments AC and BC.

Show that the area of R is $\ln\left(\frac{1}{9}e^4\right)$.

proof



Question 16 (*+)**



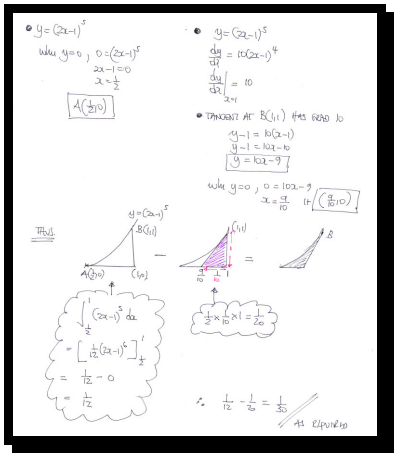
The figure above shows the graph of the curve with equation

$$y=(2x-1)^5,\ x\in\mathbb{R}.$$

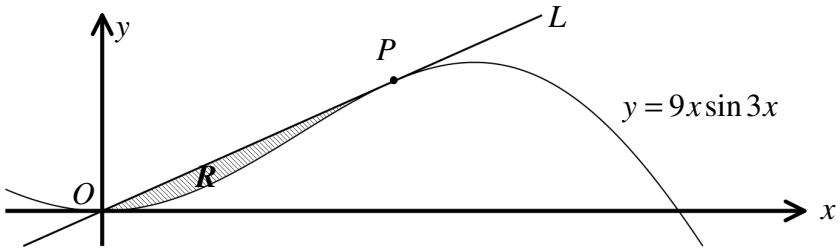
The curve meets the x axis at the point A and the point $B(1,1)$ lies on the curve.

Show that the area of the finite region R bounded by the curve, the x axis and the tangent to the curve at B is $\frac{1}{30}$.

proof



Question 17 **(****)**



The figure above shows the graph of the curve C with equation

$$y = 9x\sin 3x\,,\quad x\in\mathbb{R}\,.$$

The straight line L is the tangent to C at the point P, whose x coordinate is $\dfrac{\pi}{6}$.

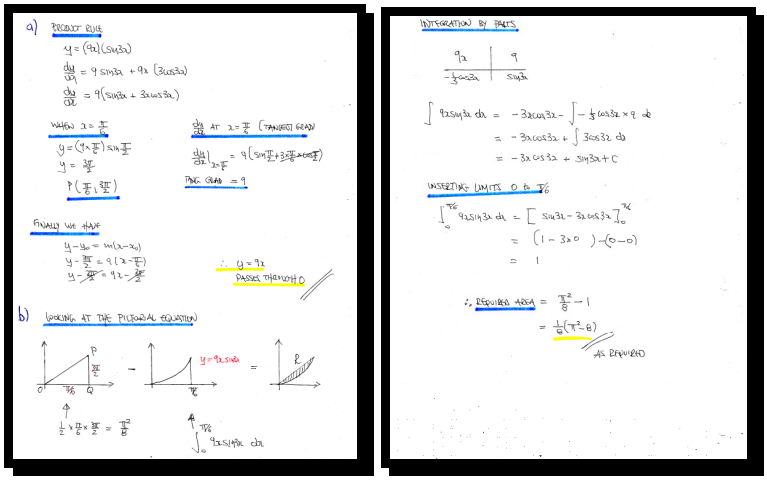
a) Show that L passes through the origin O.

The finite region R bounded by C and L.

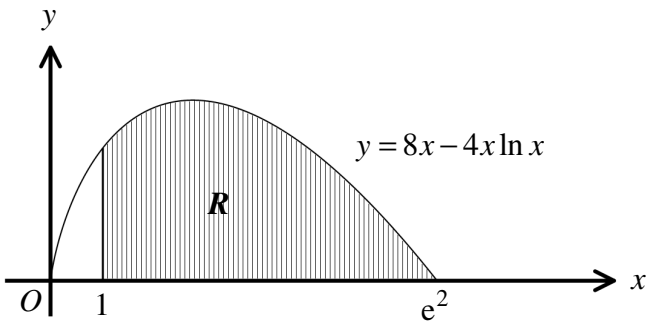
b) Show further that the area of R is

$$\frac{1}{8}\left(\pi^2-8\right).$$

, proof



Question 18 (****)



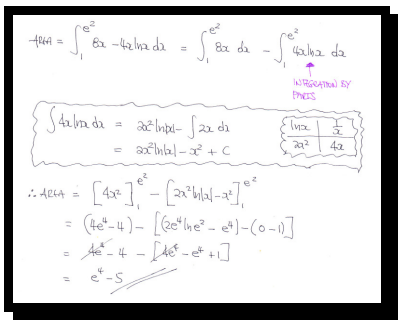
The figure above shows the graph of the curve with equation

$$y = 8x - 4x\ln x\,,\ 0 < x \le \mathrm{e}^2 .$$

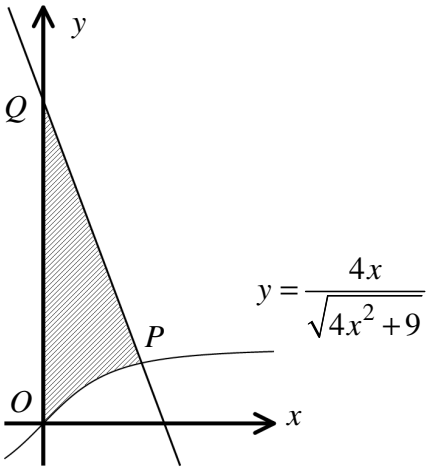
The region R is bounded by the curve, the x axis and the line with equation $x = 1$.

Determine the exact area of R.

$\boxed{\mathrm{e}^4 - 5}$



Question 19 (****)



The figure above shows the graph of the curve with equation

$$y=\frac{4x}{\sqrt{4x^2+9}}\,,\quad x\in\mathbb{R}\,.$$

The point P lies on the curve where $x=2$. The normal to the curve at P meets the y axis at Q.

a) Show that the y coordinate of Q is $\frac{769}{90}$.

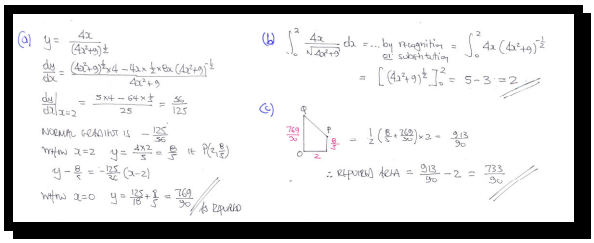
b) Find the value of

$$\int_0^2 \frac{4x}{\sqrt{4x^2+9}}\,dx\,.$$

c) Hence find the area of the finite region bounded by the curve, the y axis and the normal to the curve at P.

$\boxed{}$, $\boxed{2}$, $\boxed{\frac{733}{90}\approx 8.14}$

[solution overleaf]



(a) $y = \frac{4x}{(4x^2+9)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{(4x^2+9)^{\frac{1}{2}} \times 4 - 4x \times \frac{1}{2} \times 8x (4x^2+9)^{-\frac{1}{2}}}{4x^2+9}$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{5 \times 4 - 64 \times \frac{1}{5}}{25} = \frac{36}{125}$

NORMAL GRADIENT IS $-\frac{125}{36}$

WHEN $x=2 \quad y = \frac{4 \times 2}{5} = \frac{8}{5}$ IE $P\left(2, \frac{8}{5}\right)$

$y - \frac{8}{5} = -\frac{125}{36}(x-2)$

WHEN $x=0 \quad y = \frac{125}{18} + \frac{8}{5} = \frac{769}{90}$ AS REQUIRED

(b) $\int_0^2 \frac{4x}{\sqrt{4x^2+9}}\,dx = \ldots$ by recognition or substitution $= \int_0^2 4x\left(4x^2+9\right)^{-\frac{1}{2}}$

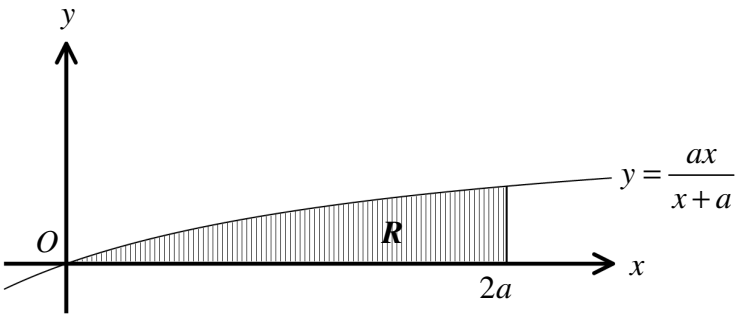
$= \left[\left(4x^2+9\right)^{\frac{1}{2}}\right]_0^2 = 5 - 3 = 2$

(c)

$= \frac{1}{2}\left(\frac{8}{5} + \frac{769}{90}\right) \times 2 = \frac{913}{90}$

$\therefore$ REQUIRED AREA $= \frac{913}{90} - 2 = \frac{733}{90}$

Question 20 **(****)**



The figure above shows the graph of the curve C with equation

$$y=\frac{ax}{x+a},\ x\neq -a,\text{ where } a \text{ is a positive constant.}$$

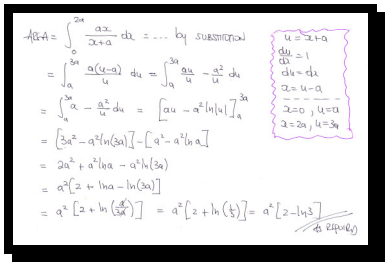
The curve passes through the origin O.

The finite region R is bounded by C, the x axis and the line with equation $x=2a$.

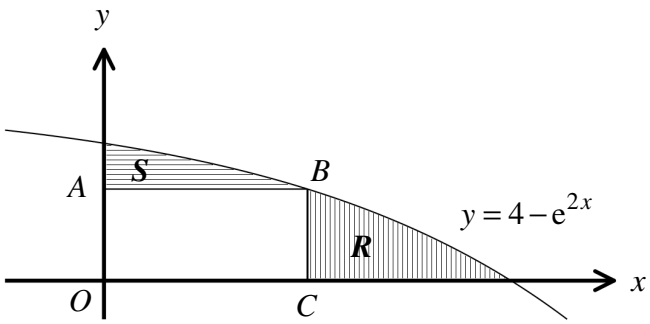
By using integration by substitution, or otherwise, show that the area of R is

$$a^2\left(2-\ln 3\right).$$

proof



Question 21 (****)



The figure above shows the graph of the curve with equation

$$y = 4 - \mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

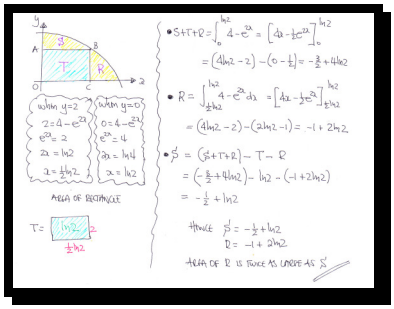
The point B lies on the curve where $y = 2$. The point A lies on the y axis and the point C lies on the x axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the x axis and the line BC.

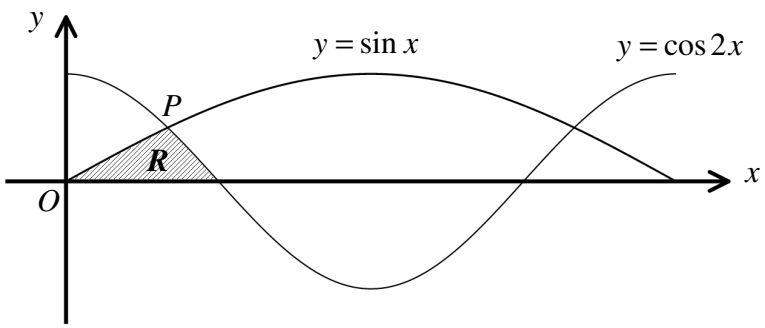
The region S is bounded by the curve, the y axis and the line AB.

Show, by exact calculations, that the area of R is twice as large as the area of S.

proof



Question 22 **(****)**



The figure above shows the graphs of the curves with equations

$$y=\cos 2x \text{ and } y=\sin x, \text{ for } 0\le x\le \pi .$$

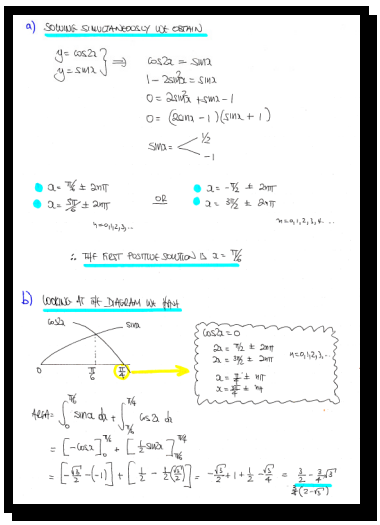
The point P is the first intersection between the graphs for which $x>0$.

a) Show clearly that the x coordinate of P is $\dfrac{\pi}{6}$.

The finite region R, shown shaded in the figure, is bounded by the two curves and the x axis, and includes the point P on its boundary.

b) Determine the exact area of R.

$\square$, $\boxed{\dfrac{3}{4}\left(2-\sqrt{3}\right)}$



Question 23 (****)

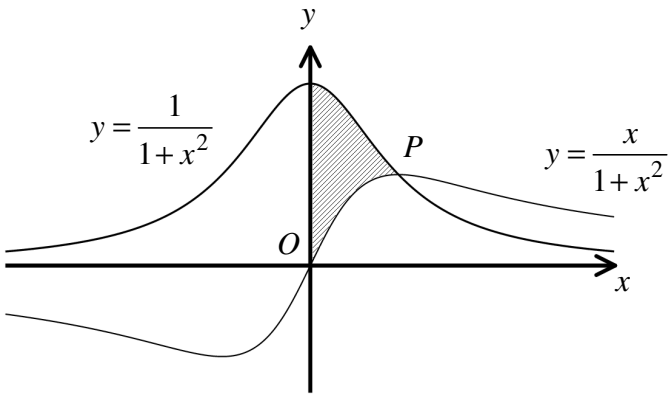
$$y = \arctan x\,,\ x \in \mathbb{R}\,.$$

a) By rewriting the above equation in the form $x = f(y)$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1+x^2}\,.$$

The figure below shows the graphs of the curves with equations

$$y = \frac{1}{1+x^2} \qquad \text{and} \qquad y = \frac{x}{1+x^2}\,.$$

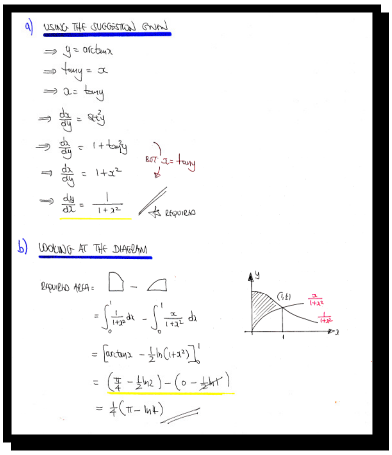


The two graphs intersect at the point $P\left(1, \frac{1}{2}\right)$.

b) Find the exact area of the finite region bounded by the two curves and the y axis, shown shaded in the figure.

$\boxed{\qquad}$, $\boxed{\frac{1}{4}(\pi - \ln 4)}$

[solution overleaf]



a) USING THE SUGGESTION GIVEN

$\Rightarrow y = \arctan x$

$\Rightarrow \tan y = x$

$\Rightarrow x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$ BUT $x = \tan y$

$\Rightarrow \frac{dx}{dy} = 1 + x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ AS REQUIRED

b) LOOKING AT THE DIAGRAM

REQUIRED AREA =

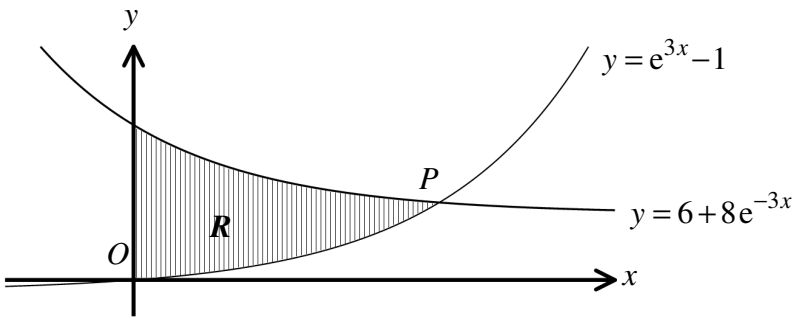
$= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$

$= \left[\arctan x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - \left(0 - \frac{1}{2}\ln 1\right)$

$= \frac{1}{4}(\pi - \ln 4)$

Question 24 **(****)**



The figure above shows the graphs of the curves with equations

$$y=e^{3x}-1 \quad \text{and} \quad y=6+8e^{-3x}.$$

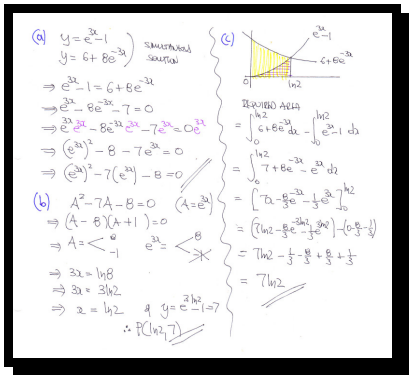
The curves intersect at the point P.

a) Find the exact coordinates of the point P.

The finite region R is bounded by the two curves and the y axis.

b) Determine the exact area of R.

$\boxed{(\ln 2, 7)}$, $\boxed{\text{area} = 7\ln 2}$



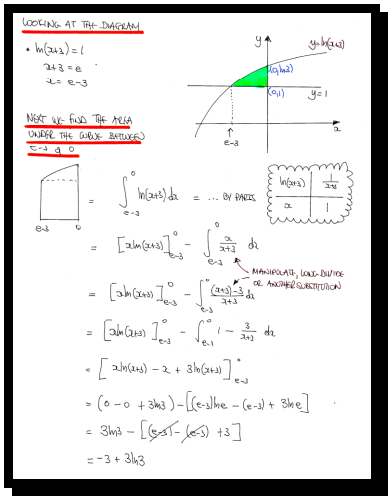
Question 25 (**+)**

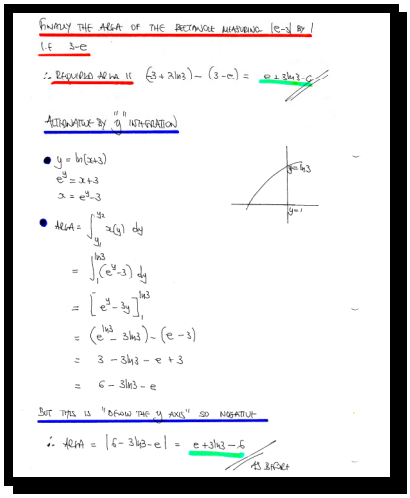
The finite region R is bounded by the curve with equation $y=\ln\left(x+3\right)$, the y axis and the straight line with equation $y=1$.

Show, with detailed workings, that the area of R is

$$\mathrm{e}-6+3\ln 3\,.$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 26 (**+)**

The function f is defined in the largest real domain by

$$f(x) \equiv (\ln x)^2, \quad x \in (0,\infty).$$

a) Sketch the graph of $f(x)$.

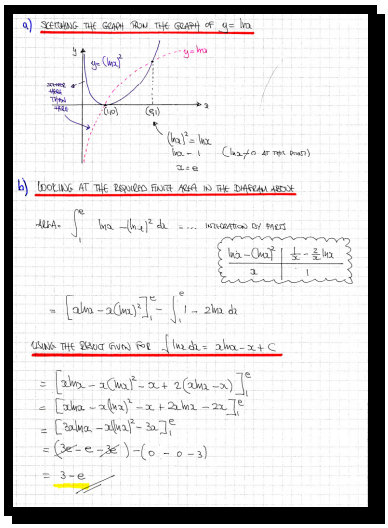
The function g is defined as

$$g(x) \equiv \ln x, \quad x \in (0,\infty).$$

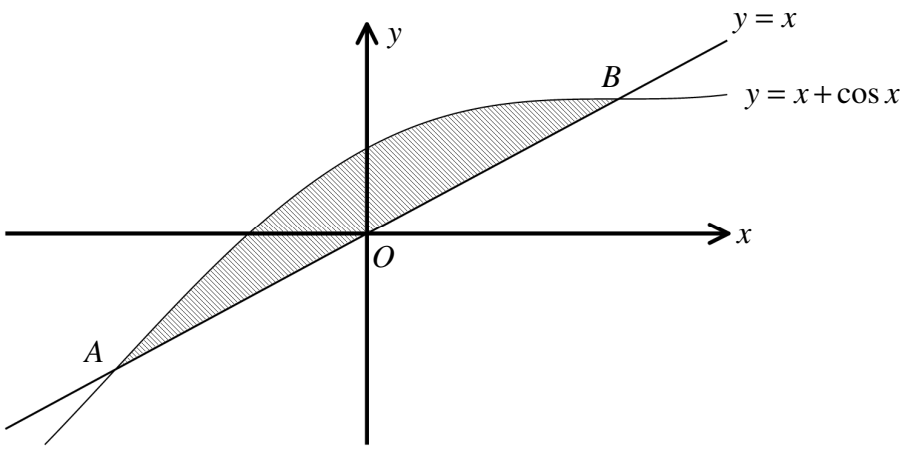
b) Determine in exact simplified form the area of the finite region bounded by the graph of f and the graph of g.

You may assume that $\int \ln x \, dx = x\ln x - x + \text{constant}$.

$\boxed{}$, $\boxed{3-\text{e}}$



Question 27 **(*****)**



The figure above shows the graph of the curve C with equation

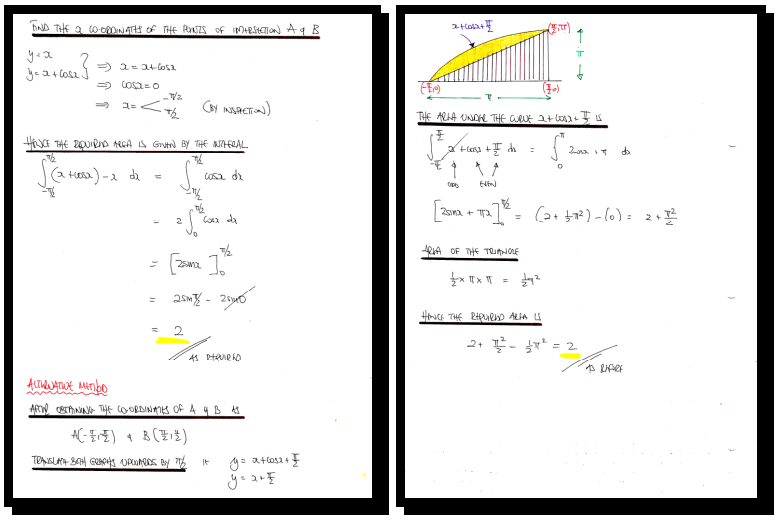
$$y = x + \cos x , \quad x \in \mathbb{R}$$

and the straight line L with equation

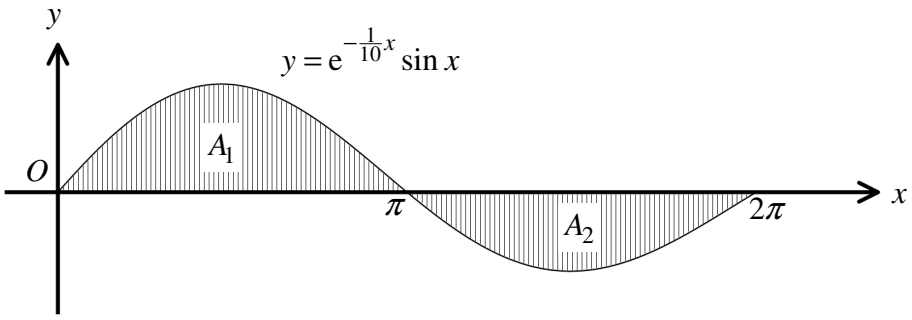
$$y = x , \quad x \in \mathbb{R} .$$

Show that the area of the finite region bounded by C and L is 2 square units.

[ ], proof



Question 28 (*****)



The figure above shows the curve with equation

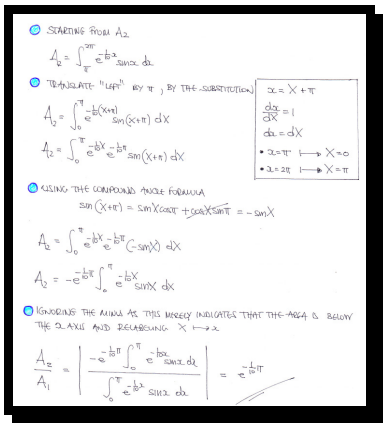
$$y = e^{-\frac{1}{10}x} \sin x , \ 0 \le x \le 2\pi .$$

The curve meets the x axis at the origin and at the points $(\pi, 0)$ and $(2\pi, 0)$.

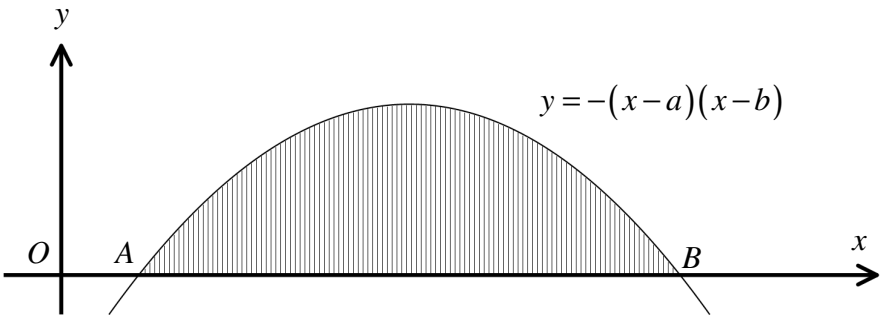
The finite region bounded by the curve for $0 \le x \le \pi$ and the x axis is denoted by A_1, and similarly A_2 denotes the finite region bounded by the curve for $\pi \le x \le 2\pi$ and the x axis.

By considering a suitable translation of the curve determine with justification the ratio of the areas of A_1 and A_2.

$\square$, $\boxed{\dfrac{A_1}{A_2} = e^{-\frac{\pi}{10}}}$



Question 29 **(*****)**



The figure above shows the parabola with equation

$$y=-(x-a)(x-b),\ b>a>0.$$

The curve meets the x axis at the points A and B.

a) Show that the area of the finite region R, bounded by the parabola and the x axis is

$$\frac{1}{6}(b-a)^3.$$

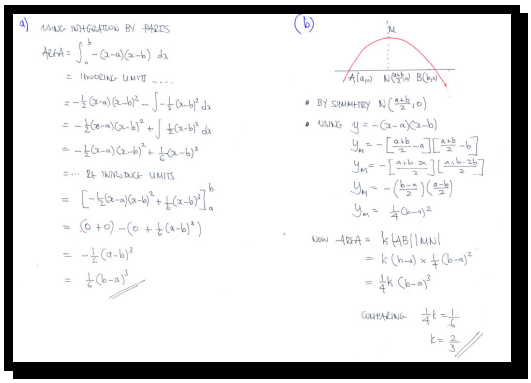
The midpoint of AB is N. The point M is the maximum point of the parabola.

b) Show clearly that the area of R is given by

$$k\left|AB\right|\left|MN\right|,$$

where k is a constant to be found.

$\square$, $\boxed{k=\frac{2}{3}}$



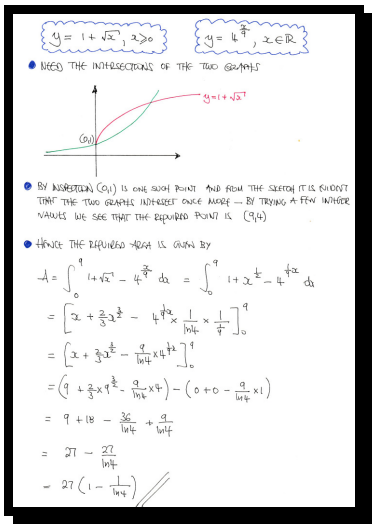
Question 30 (*****)

Determine, in exact simplified form, the area of the finite region bounded by the curves with equations

$$y=1+\sqrt{x}\,,\ x\in\mathbb{R}\,,\ x\geq 0\,.$$

$$y=4^{\frac{x}{9}}\,,\ x\in\mathbb{R}\,.$$

$$\square\,,\quad \boxed{27\left(1-\frac{1}{\ln 4}\right)}$$



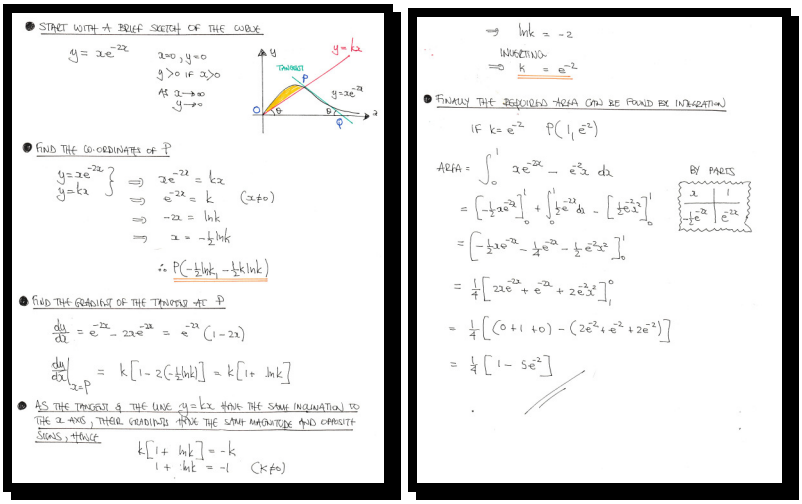
Question 31 (*****)

The straight line L with equation $y=kx$, where k is a positive constant, meets the curve C with equation $y=x\mathrm{e}^{-2x}$, at the point P.

The tangent to C at P meets the x axis at the point Q.

Given that $|OP|=|PQ|$, find in exact simplified form the area of the **finite** region bounded by C and L.

$\square$, $\boxed{\text{area}=\frac{1}{2}\left(1-5\mathrm{e}^{-2}\right)}$



Question 32 **(*****)**

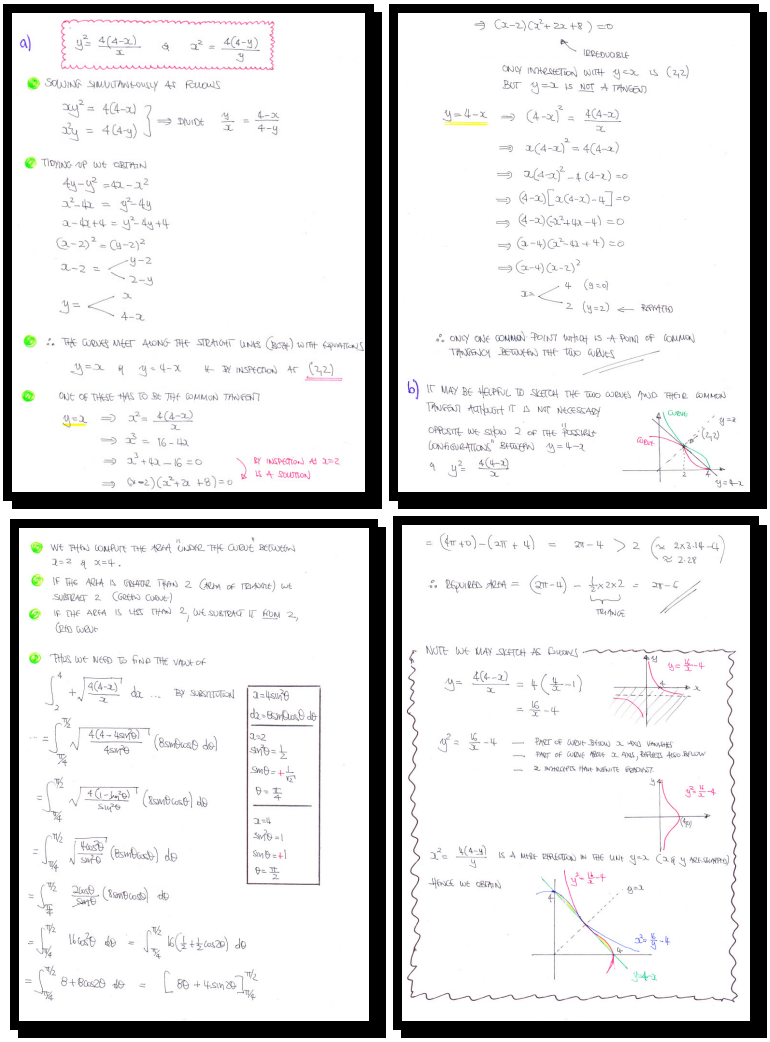
Two curves are defined in the largest possible real number domain and have equations

$$y^2=\frac{4(4-x)}{x} \qquad \text{and} \qquad x^2=\frac{4(4-y)}{y}.$$

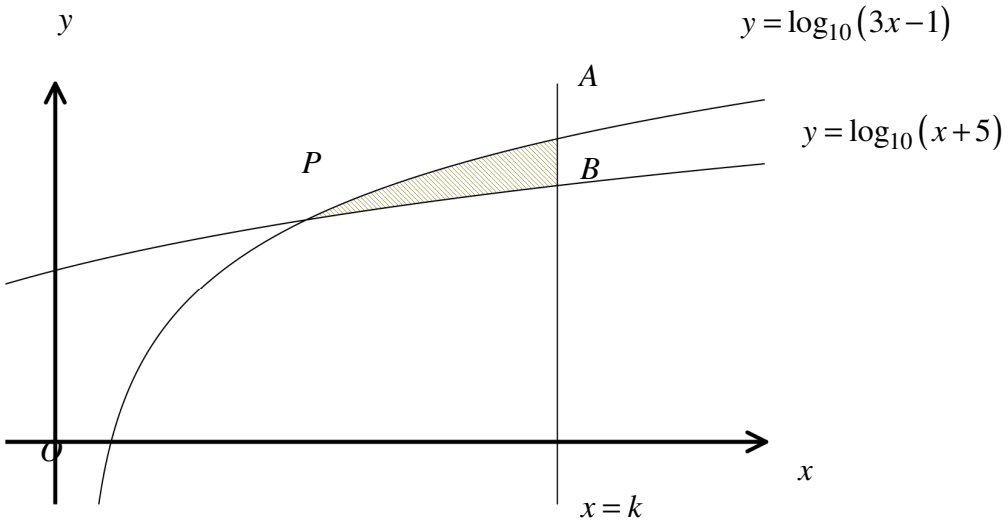
a) Show that the two curves have one, and only one, common point which is also a point of common tangency.

b) Find the exact value of the area enclosed by the common tangent to the curves, and either of the two curves.

$\square$, $\boxed{2(\pi-3)}$



Question 33 (*****)



The figure above shows the graphs of the curves with equations

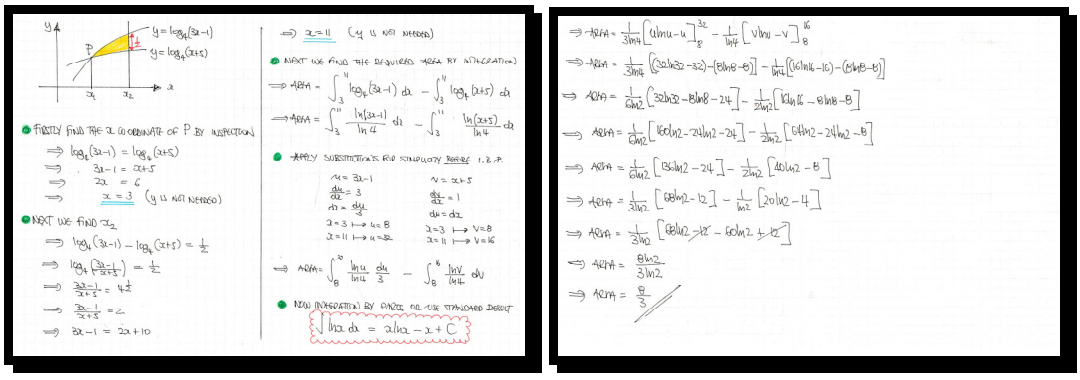
$$y=\log_{10}(3x-1), \quad \text{and} \quad y=\log_{10}(x+5).$$

The two curves intersect at the point P

The straight line with equation $x=k$, $k>3$, meets the graph of $y=\log_{10}(3x-1)$ at the point A and the graph of $y=\log_{10}(x+5)$ at the point B, so that $|AB|=\frac{1}{2}$.

Determine the value of the area of the finite region, bounded by the two curves and the straight line $x=k$, shown shaded in the above figure.

$\square$, $\boxed{\text{area}=\frac{8}{3}}$



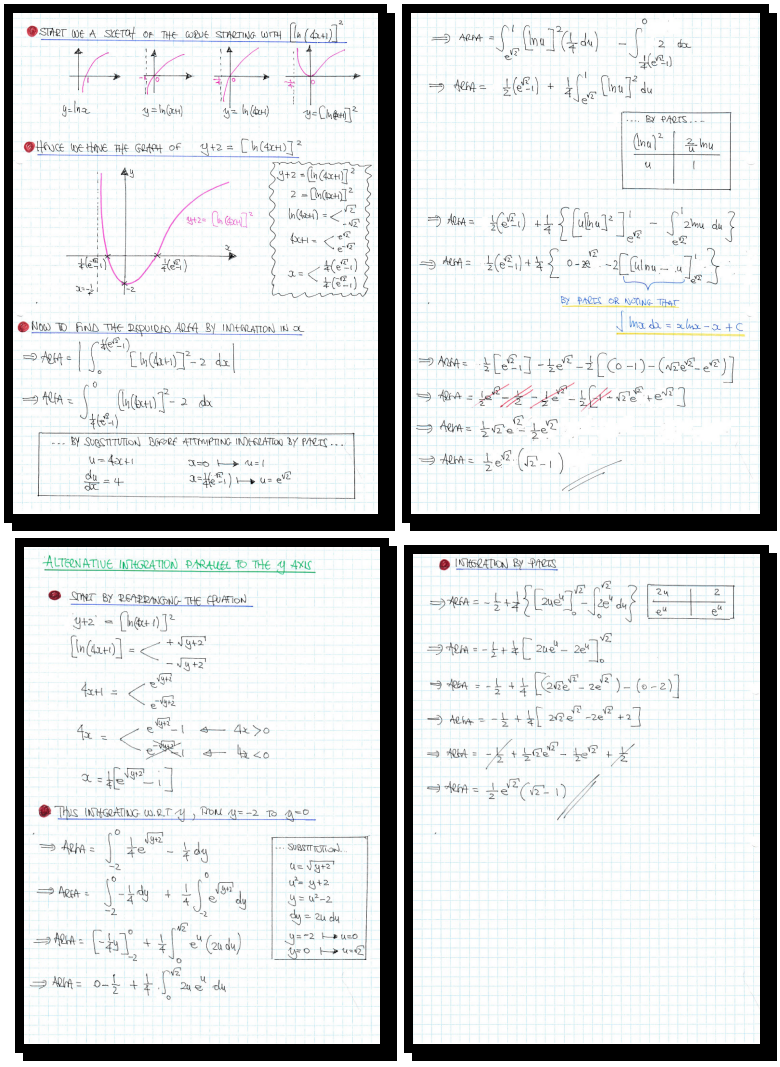
Question 34 **(*****)**

The curve C has equation

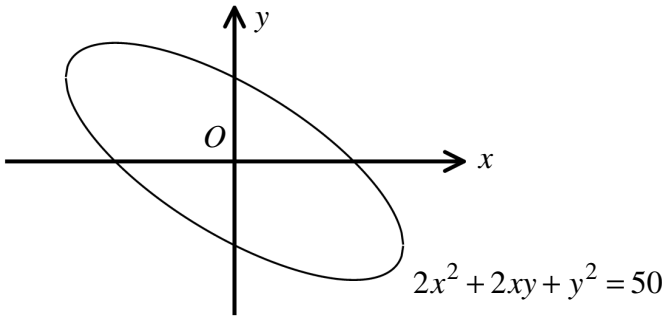
$$y+2=\left[\ln\left(4x+1\right)\right]^2, \quad x\in\mathbb{R}, \quad x\geq -\frac{1}{4}.$$

Sketch the graph of C and hence determine, in exact simplified form, the area of the finite region bounded by C, for which $x\geq 0$, and the coordinate axes.

$\boxed{}$, $\boxed{\text{area}=\frac{1}{2}\left(\sqrt{2}-1\right)\mathrm{e}^{\sqrt{2}}}$



Question 35 (*****)



The figure above shows the curve with equation

$$2x^2+2xy+y^2=50\,.$$

Determine the area of the finite region bounded by the x axis and the part of the curve for which $y\geq 0$.

$\boxed{}$, $\boxed{25\pi}$

