FUNCTIONS

EXAM QUESTIONS

Question 1 (**)

The function f is given by

$$f:x\mapsto\frac{x}{x+3},\ x\in\mathbb{R},\ x\neq-3.$$

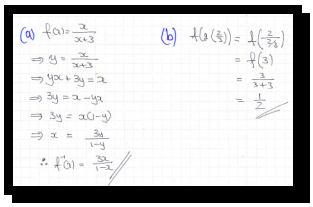
a) Find an expression for $f^{-1}(x)$.

The function g is defined as

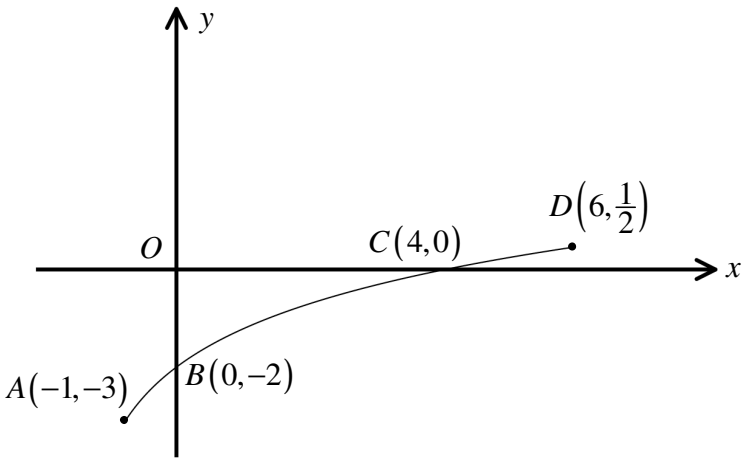
$$g:x\mapsto\frac{2}{x},\ x\in\mathbb{R},\ x\neq0.$$

b) Evaluate $fg\left(\frac{2}{3}\right)$.

$\square$, $\boxed{f^{-1}:x\mapsto\frac{3x}{1-x}}$, $\boxed{fg\left(\frac{2}{3}\right)=\frac{1}{2}}$



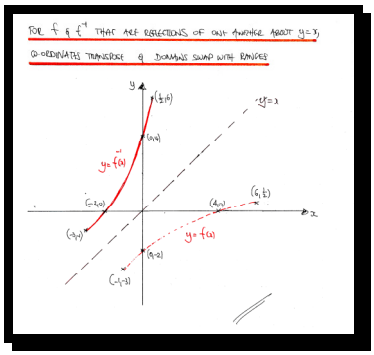
Question 2 **(**)**



The figure above shows the graph of the function $f(x)$, defined for $-1 \le x \le 6$.

Sketch the graph of $f^{-1}(x)$, marking clearly the end points of the graph and any points where it crosses the coordinate axes.

, graph



Question 3 (**)

The function f is given by

$$f(x)=\ln(4x-2),\ x\in\mathbb{R},\ x>\frac{1}{2}.$$

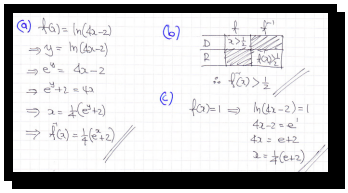
a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) State the range of $f^{-1}(x)$.

c) Solve the equation

$$f(x)=1.$$

$$\boxed{f^{-1}(x)=\frac{1}{4}\left(\mathrm{e}^{x}+2\right)},\ \boxed{f^{-1}(x)>\frac{1}{2}},\ \boxed{x=\frac{1}{4}(\mathrm{e}+2)}$$



Question 4 (**)

The function f is given by

$$f(x)=3-\ln x,\ x\in\mathbb{R},\ x>0.$$

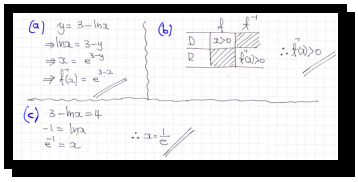
a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Solve the equation

$$f(x)=4.$$

$$\boxed{f^{-1}(x)=\mathrm{e}^{3-x}},\ \boxed{f^{-1}(x)>0},\ \boxed{x=\frac{1}{\mathrm{e}}}$$



Question 5 (+)**

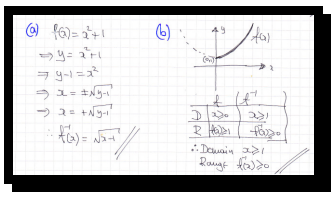
A function f is defined by

$$f(x)=x^2+1,\ x\in\mathbb{R},\ x\geq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=\sqrt{x-1}},\ \boxed{x\geq 1},\ \boxed{f^{-1}(x)\geq 0}$$



Question 6 (+)**

The functions f and g are given by

$$f(x)=x^2,\ x\in\mathbb{R}.$$

$$g(x)=\frac{1}{x+2},\ x\in\mathbb{R},\ x\neq -2.$$

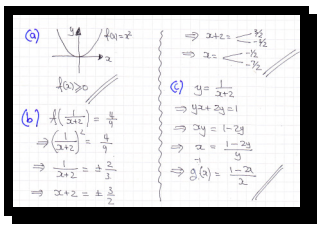
a) State the range of $f(x)$.

b) Solve the equation

$$fg(x)=\frac{4}{9}.$$

c) Find, in its simplest form, an expression for $g^{-1}(x)$.

$\square$, $\boxed{f(x)\geq 0}$, $\boxed{x=-\frac{1}{2},-\frac{7}{2}}$, $\boxed{g^{-1}(x)=\frac{1}{x}-2=\frac{1-2x}{x}}$



Question 7 (+)**

The functions f and g satisfy

$$f(x)=\ln(4-2x),\ x\in\mathbb{R},\ x<2.$$

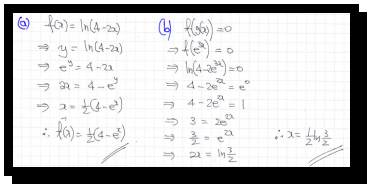
$$g(x)=\mathrm{e}^{3x},\ x\in\mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$.

b) Solve the equation

$$fg(x)=0.$$

$$\boxed{f^{-1}(x)=2-\frac{1}{2}\mathrm{e}^{x}},\ \boxed{x=\frac{1}{3}\ln\left(\frac{3}{2}\right)}$$



Question 8 (+)**

The functions f and g satisfy

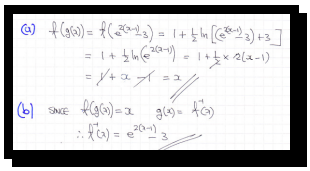
$$f(x)=1+\frac{1}{2}\ln(x+3),\ x\in\mathbb{R},\ x>3.$$

$$g(x)=\mathrm{e}^{2(x-1)}-3,\ x\in\mathbb{R}.$$

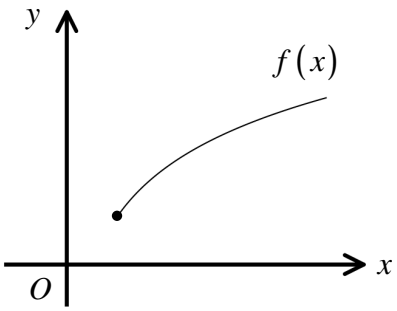
a) Find, in its **simplest** form, an expression for $fg(x)$.

b) Hence, or otherwise, write down an expression for $f^{-1}(x)$.

$$\boxed{fg(x)=x},\ \boxed{f^{-1}(x)=\mathrm{e}^{2(x-1)}-3}$$



Question 9 **(**+)**



The diagram above shows the graph of the function f, defined as

$$f(x) \equiv \frac{1}{1-x} + 4, \ x \in \mathbb{R}, \ x \geq 2 .$$

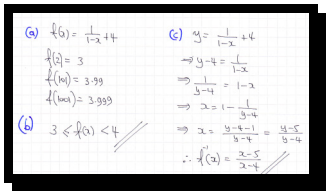
a) Evaluate $f(2)$, $f(101)$, $f(1001)$.

b) State the range of $f(x)$.

The inverse function is denoted by $f^{-1}(x)$.

c) Determine an expression for $f^{-1}(x)$, as a simplified fraction.

$\boxed{f(2)=3, \ f(101)=3.99, \ f(1001)=3.999}$, $\boxed{3 \leq f(x) < 4}$, $\boxed{f^{-1}(x) = \frac{x-5}{x-4}}$



Question 10 **(**+)**

The function f is given by

$$f(x)=4-\ln(2x-1),\ x\in\mathbb{R},\ x>\frac{1}{2}.$$

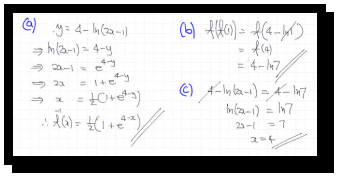
a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Determine the exact value of $ff(1)$.

c) Hence, or otherwise, solve the equation

$$f(x)=ff(1).$$

$\square$, $\boxed{f^{-1}(x)=\frac{1}{2}\left(1+\mathrm{e}^{4-x}\right)}$, $\boxed{ff(1)=4-\ln 7}$, $\boxed{x=4}$



Question 11 **(**+)**

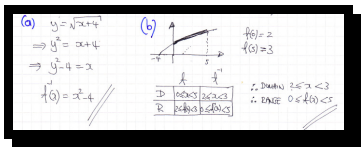
A function f is defined by

$$f(x)=\sqrt{x+4},\ x\in\mathbb{R},\ 0\le x<5.$$

a) Find an expression for $f^{-1}(x)$.

b) Determine the domain and the range of $f^{-1}(x)$.

$\boxed{f^{-1}(x)=x^2-4}$, $\boxed{2\le x<3}$, $\boxed{0\le f^{-1}(x)<5}$



Question 12 (+)**

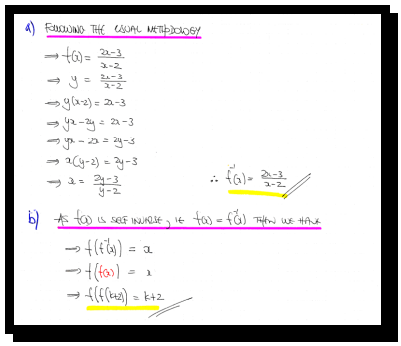
The function f is defined

$$f:x\mapsto\frac{2x-3}{x-2},\ x\in\mathbb{R},\ x\neq 2.$$

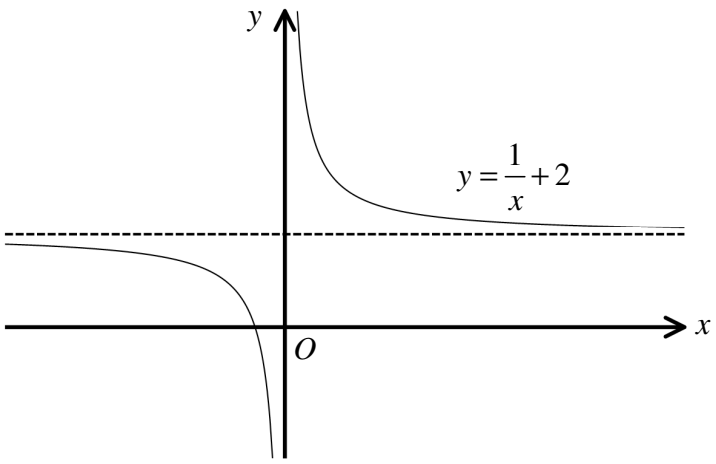
a) Find an expression for $f^{-1}(x)$ in its simplest form.

b) Hence, or otherwise, find in its simplest form $ff(k+2)$.

$\boxed{}$, $\boxed{f^{-1}:x\mapsto\frac{2x-3}{x-2}}$, $\boxed{k+2}$



Question 13 (***)



The figure above shows the graph of

$$y=\frac{1}{x}+2,\ x\neq 0.$$

a) State the equation of the horizontal asymptote to the curve, marked as a dotted line in the figure.

The function f is defined

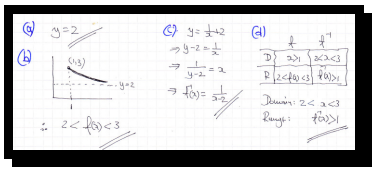
$$f(x)=\frac{1}{x}+2,\ x\in\mathbb{R},\ x>1.$$

b) State the range of $f(x)$.

c) Obtain an expression for $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{y=2}$, $\boxed{2<f(x)<3}$, $\boxed{f^{-1}(x)=\frac{1}{x-2}}$, $\boxed{2<x<3}$, $\boxed{f^{-1}(x)>1}$



Question 14 (***)

The functions f and g are given by

$$f(x)=\frac{2x+3}{2x-3},\ x\in\mathbb{R},\ x\neq\frac{3}{2}.$$

$$g(x)=x^2+2,\ x\in\mathbb{R}.$$

a) State the range of $g(x)$.

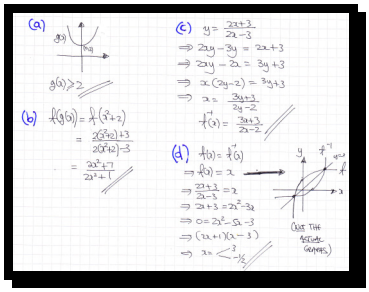
b) Find an expression, as a simplified algebraic fraction, for $fg(x)$.

c) Determine an expression, as a simplified algebraic fraction, for $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x)=f(x).$$

$\square$, $\boxed{g(x)\geq 2}$, $\boxed{fg(x)=\frac{2x^2+7}{2x^2+1}}$, $\boxed{f^{-1}(x)=\frac{3x+3}{2x-2}}$, $\boxed{x=-\frac{1}{2},3}$



Question 15 **(***)**

$$f(x)=\mathrm{e}^{2x}-4,\ x\in\mathbb{R}.$$

$$g(x)=\frac{1}{x-11},\ x\in\mathbb{R},\ x\neq 11.$$

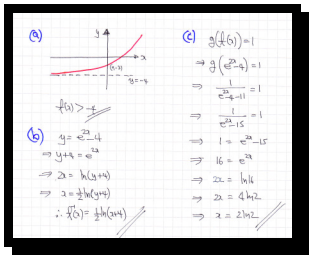
a) Determine the range of $f(x)$.

b) Find an expression for the inverse function $f^{-1}(x)$.

c) Solve the equation

$$gf(x)=1.$$

$$\boxed{f(x)>-4},\ \boxed{f^{-1}(x)=\tfrac{1}{2}\ln(x+4)},\ \boxed{x=2\ln 2}$$



Question 16 (***)

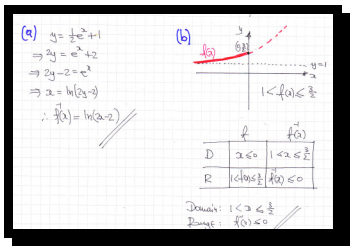
A function f is defined by

$$f(x)=\frac{1}{2}\mathrm{e}^{x}+1,\quad x\in\mathbb{R},\ x\leq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=\ln(2x-2)},\quad \boxed{1<x\leq\frac{3}{2}},\quad \boxed{f^{-1}(x)\leq 0}$$



Question 17 (***)

The function f is given by

$$f(x)=(x-3)^2+1,\ x\in\mathbb{R},\ x\geq 4.$$

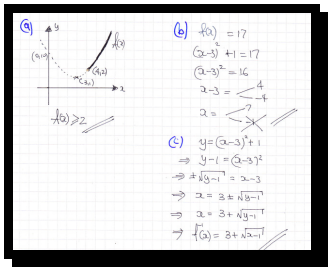
a) Sketch the graph of $f(x)$ and hence write down its range.

b) Solve the equation

$$f(x)=17.$$

c) Find an expression for $f^{-1}(x)$ in its simplest form.

$$\boxed{f(x)\geq 2},\quad \boxed{x=7,\ x\neq -1},\quad \boxed{f^{-1}(x)=3+\sqrt{x-1}},$$



Question 18 **(***)**

The functions f and g are given by

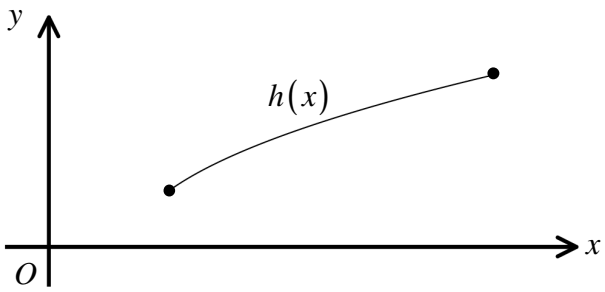
$$f(x)=\sqrt{x},\ x\in\mathbb{R},\ x\geq 0.$$

$$g(x)=x-2,\ x\in\mathbb{R}.$$

a) Find an expression for the function composition $fg(x)$.

The function h, whose graph is shown below, is defined by

$$h(x)=\sqrt{x-2},\ x\in\mathbb{R},\ 3\leq x\leq 11.$$



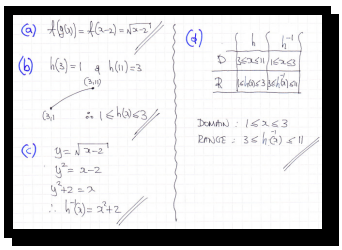
b) State the range of $h(x)$.

c) Determine an expression for the inverse function $h^{-1}(x)$.

d) State the domain and range of $h^{-1}(x)$.

$\square$, $\boxed{fg(x)=\sqrt{x-2}}$, $\boxed{1\leq h(x)\leq 3}$, $\boxed{h^{-1}(x)=x^2+2}$,

$\boxed{1\leq x\leq 3 \quad \& \quad 3\leq h^{-1}(x)\leq 11}$



Question 19 (***)

The functions f and g are defined by

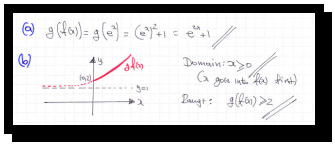
$$f(x)=\mathrm{e}^{x},\quad x\in\mathbb{R},\ x\geq 0.$$

$$g(x)=x^{2}+1,\ x\in\mathbb{R}.$$

a) Find an expression for $gf(x)$, in its simplest form.

b) Determine the domain and range of $gf(x)$.

$$\boxed{gf(x)=\mathrm{e}^{2x}+1},\quad \boxed{x\geq 0,\ \ gf(x)\geq 2}$$



Question 20 (***)

The functions f and g are defined by

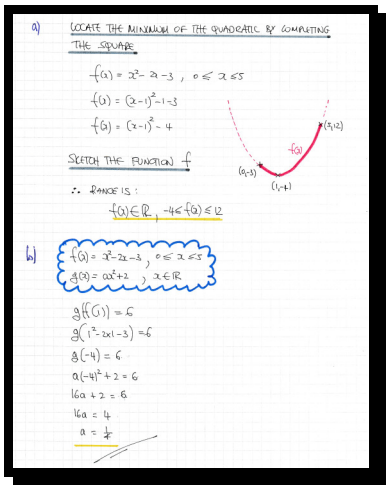
$$f:x\mapsto x^2-2x-3,\quad x\in\mathbb{R},\ 0\le x\le 5.$$

$$g:x\mapsto ax^2+2,\quad x\in\mathbb{R},\ a \text{ is a real constant.}$$

a) Find the range of f.

b) Determine the value of a, if $gf(1)=6$.

$\boxed{}$, $\boxed{-4\le f(x)\le 12}$, $\boxed{a=\frac{1}{4}}$



Question 21 (***)

The function f is defined as

$$f:x\mapsto \frac{2}{x-3}-\frac{4}{x^2-4x+3},\ x\in\mathbb{R},\ x>1.$$

a) Show clearly that

$$f:x\mapsto \frac{2}{x-1},\ x\in\mathbb{R},\ x>1.$$

b) Find an expression for f^{-1}, in its simplest form.

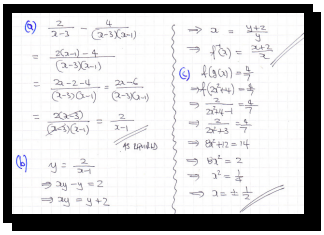
The function g is given by

$$g:x\mapsto 2x^2+4,\ x\in\mathbb{R}.$$

c) Solve the equation

$$fg(x)=\frac{4}{7}.$$

$\square$, $\boxed{f^{-1}(x)=\frac{x+2}{x}=1+\frac{2}{x}}$, $\boxed{x=\pm\frac{1}{2}}$



Question 22 (***)

The functions f and g are defined by

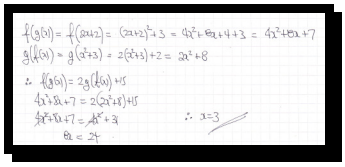
$$f:x\mapsto x^2+3,\quad x\in\mathbb{R}.$$

$$g:x\mapsto 2x+2,\quad x\in\mathbb{R}.$$

Solve the equation

$$fg(x)=2gf(x)+15.$$

$$\boxed{x=3}$$



Question 23 (***)

The functions f and g are defined as

$$f(x)=\frac{x+6}{x+2}, \quad x\in\mathbb{R},\ x\neq -2$$

$$g(x)=7-2x^2, \quad x\in\mathbb{R}.$$

a) State the range of $g(x)$.

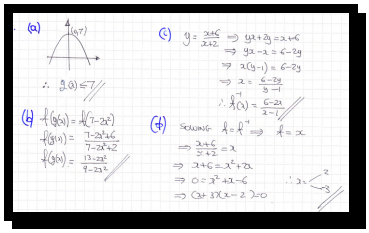
b) Find, as a simplified fraction, an expression for $fg(x)$.

c) Find, as a simplified fraction, an expression for $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x)=f(x).$$

$\square$, $\boxed{g(x)\leq 7}$, $\boxed{fg(x)=\frac{13-2x^2}{9-2x^2}}$, $\boxed{f^{-1}(x)=\frac{2x-6}{1-x}}$, $\boxed{x=-3,\ 2}$



Question 24 (***)

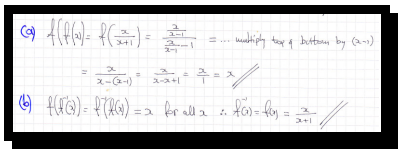
The function f is defined as

$$f:x\mapsto\frac{x}{x-1},\quad x\in\mathbb{R},\ x\neq 1$$

a) Find in its simplest form the composition $ff\left(x\right)$.

b) Find an expression for $f^{-1}\left(x\right)$ in its simplest form.

$\boxed{ff\left(x\right)=x}$, $\boxed{f:x\mapsto\frac{x}{x-1},\quad x\in\mathbb{R},\ x\neq 1}$



Question 25 (***)

The functions g and f are given by

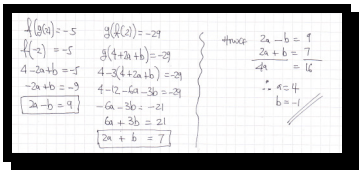
$$g:x\mapsto 4-3x,\quad x\in\mathbb{R}$$

$$f:x\mapsto x^2+ax+b,\quad x\in\mathbb{R},$$

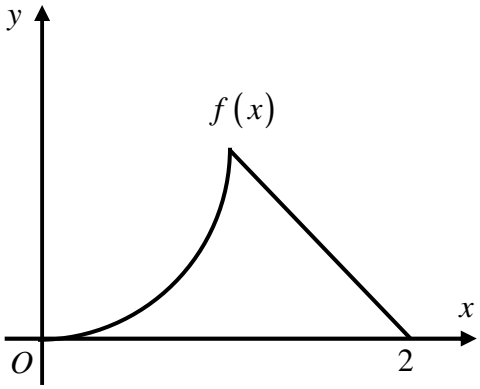
where a and b are non zero constants.

Given that $fg\left(2\right)=-5$ and $gf\left(2\right)=-29$, find the value a and the value of b.

$\boxed{a=4,b=-1}$



Question 26 (***)

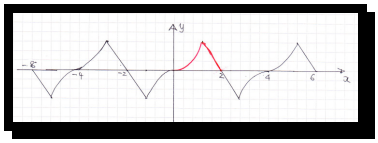


The figure above shows the **part** of the curve with equation

$$y = f(x), \text{ for } 0 \le x \le 2.$$

Given that the curve is odd and periodic with period 4, sketch the curve for $-6 \le x \le 6$.

graph



Question 27 (***)

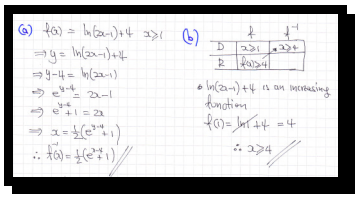
The function f is defined by

$$f(x)=\ln(2x-1)+4,\ x\in\mathbb{R},\ x\geq 1.$$

a) Find $f^{-1}(x)$ in its simplest form.

b) Determine the domain of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=\frac{1}{2}\left(1+\mathrm{e}^{x-4}\right)},\ \boxed{x\in\mathbb{R},\ x\geq 4}$$



Question 28 (***)

The function f is given by

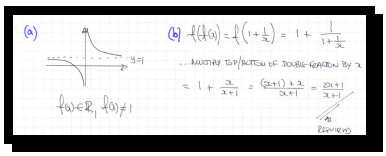
$$f:x\mapsto 1+\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

a) Find the range of f.

b) Show clearly that

$$ff:x\mapsto\frac{2x+1}{x+1}.$$

$$\boxed{f(x)\in\mathbb{R},\ f(x)\neq 1}$$



Question 29 (***)

The functions f and g are defined as

$$f(x)=2x-1, \quad x\in\mathbb{R}$$

$$g(x)=\mathrm{e}^{\frac{x}{2}}, \quad x\in\mathbb{R}.$$

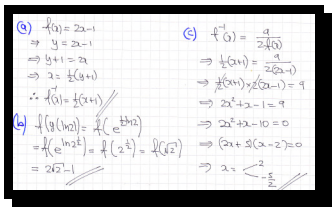
a) Find an expression for $f^{-1}(x)$.

b) Find, as an exact surd, the value of $fg(\ln 2)$.

c) Solve the equation

$$f^{-1}(x)=\frac{9}{2f(x)}.$$

$$\boxed{f^{-1}(x)=\frac{x+1}{2}}, \quad \boxed{fg(\ln 2)=2\sqrt{2}-1}, \quad \boxed{x=-\frac{5}{2},\ 2}$$



Question 30 (***)

The functions f, g and h are defined as

$$f(x)=x^2-1, \quad x\in\mathbb{R}$$

$$g(x)=\mathrm{e}^{\frac{3x}{2}}, \quad x\in\mathbb{R}$$

$$h(x)=fg(x), \quad x\in\mathbb{R}.$$

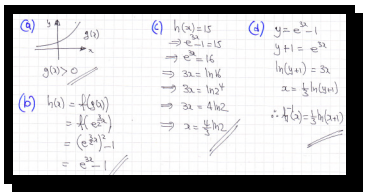
a) State the range of $g(x)$.

b) Find, in its simplest form, an expression for $h(x)$.

c) Solve the equation $h(x)=15$, giving the answer in terms of $\ln 2$.

d) Find an expression for $h^{-1}(x)$, the inverse of $h(x)$.

$\boxed{g(x)>0}$, $\boxed{fg(x)=\mathrm{e}^{3x}-1}$, $\boxed{x=\frac{4}{3}\ln 2}$, $\boxed{h^{-1}(x)=\frac{1}{3}\ln(x+1)}$



Question 31 (***)

The functions f and g are defined by

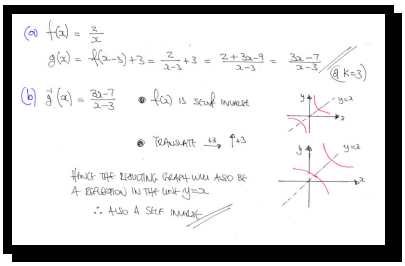
$$f(x)\equiv\frac{2}{x},\ x\in\mathbb{R},\ x\neq 0$$

$$g(x)\equiv f(x-3)+3,\ x\in\mathbb{R},\ x\neq k.$$

a) Find an expression for $g(x)$, as a simplified fraction stating the value of the constant k.

b) Find an expression for $g^{-1}(x)$.

$$\boxed{g(x)=\frac{3x-7}{x-3}},\ \boxed{k=3},\ \boxed{g^{-1}(x)=\frac{3x-7}{x-3},\ \text{self inverse}}$$



Question 32 (***)

The functions f and g are defined by

$$f:x\mapsto 4-x^2,\ x\in\mathbb{R}$$

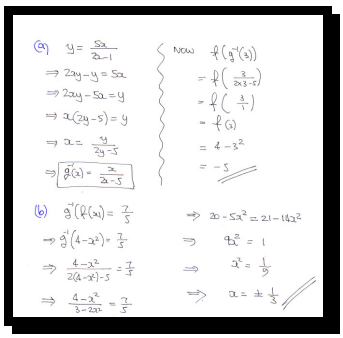
$$g:x\mapsto \frac{5x}{2x-1},\ x\in\mathbb{R},\ x\neq\frac{1}{2}.$$

a) Evaluate $fg^{-1}(3)$.

b) Solve the equation

$$g^{-1}f(x)=\frac{7}{5}.$$

$\square$, $\boxed{fg^{-1}(3)=-5}$, $\boxed{x=\pm\frac{1}{3}}$



Question 33 (***)

The function f is defined as

$$f:x\mapsto \frac{1}{x+2}+\frac{2x+11}{2x^2+x-6}\quad x\in\mathbb{R},\quad x>\frac{3}{2}.$$

a) Show clear that

$$f:x\mapsto \frac{4}{2x-3}\ ,\quad x\in\mathbb{R},\quad x>\frac{3}{2}.$$

b) Find an expression for f^{-1}, in its simplest form.

c) Find the domain of f^{-1}.

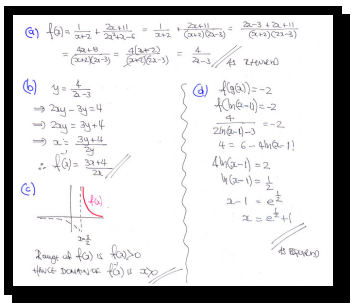
The function g is given by

$$g:x\mapsto \ln(x-1),\quad x\in\mathbb{R},\quad x>1.$$

d) Show that $x=1+\sqrt{\mathrm{e}}$ is the solution of the equation

$$fg(x)=-2.$$

$\square$, $\boxed{f^{-1}(x)=\dfrac{3x+4}{2x}}$, $\boxed{x>0}$



Question 34 **(***)**

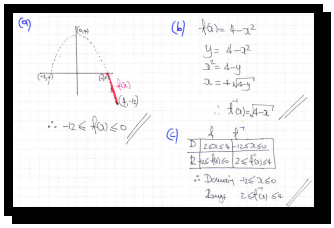
$$f(x)=4-x^2,\ x\in\mathbb{R},\ 2\le x\le 4.$$

a) Determine the range of $f(x)$.

b) Find an expression for the inverse function $f^{-1}(x)$.

c) State the domain and range of $f^{-1}(x)$.

$$\boxed{-12\le f(x)\le 0},\ \boxed{f^{-1}(x)=\sqrt{4-x}},\ \boxed{-12\le x\le 0},\ \boxed{2\le f^{-1}(x)\le 4}$$



Question 35 (*+)**

A function is defined by

$$f(x)=\sqrt{\mathrm{e}^{x}-1}\,,\ x\geq 0\,.$$

a) Find the values of …

i. … $f(\ln 5)$.

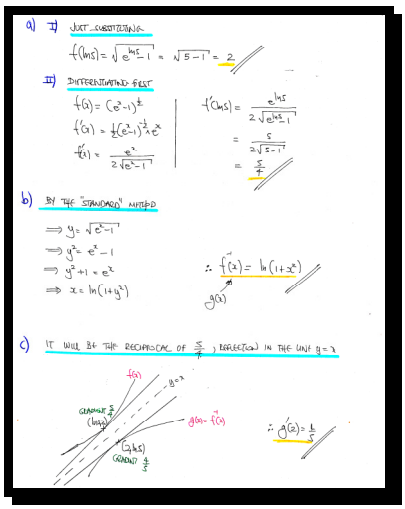
ii. … $f'(\ln 5)$.

The inverse function of $f(x)$ is $g(x)$.

b) Determine an expression for $g(x)$.

c) State the value of $g'(2)$.

$\square$, $\boxed{f(\ln 5)=2}$, $\boxed{f'(\ln 5)=\frac{5}{4}}$, $\boxed{g(x)=\ln\left(x^2+1\right)}$, $\boxed{g'(2)=\frac{4}{5}}$



Question 36 (*+)**

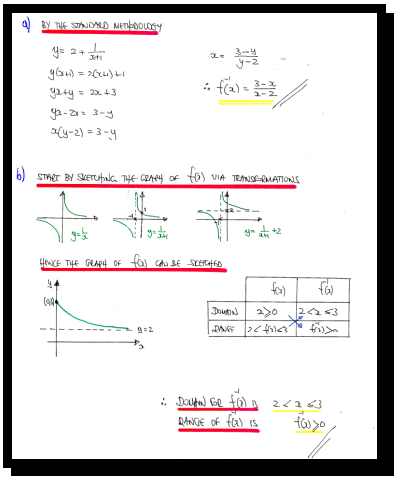
A function f is defined by

$$f(x)=2+\frac{1}{x+1},\quad x\in\mathbb{R},\ x\geq 0.$$

a) Find an expression for $f^{-1}(x)$, as a simplified fraction.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{f^{-1}(x)=\frac{3-x}{x-2}}$, $\boxed{2<x\leq 3}$, $\boxed{f^{-1}(x)\geq 0}$

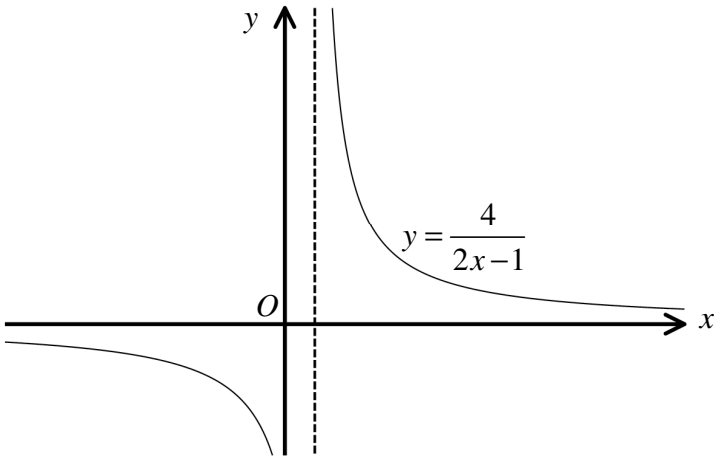


Question 37 (*+)**

$$y=\frac{2}{x-2}-\frac{6}{(x-2)(2x-1)}.$$

a) Show clearly that $y=\dfrac{4}{2x-1}$

The figure below shows the graph of $y=\dfrac{4}{2x-1}$, $x\neq a$.



b) State the equation of the vertical asymptote of the curve, shown dotted in the figure above.

The function f is defined

$$f(x)=\frac{4}{2x-1},\ x>1$$

c) State the range of $f(x)$.

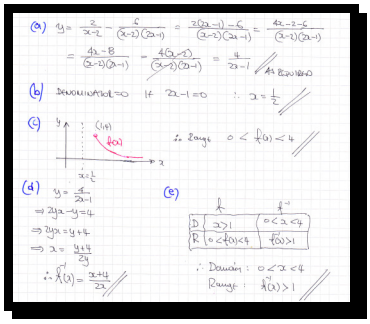
[continues overleaf]

[continued from overleaf]

d) Obtain an expression for the inverse of the function, $f^{-1}(x)$.

e) State the domain and range of $f^{-1}(x)$.

$\square$, $\boxed{x=\frac{1}{2}}$, $\boxed{0<f(x)<4}$, $\boxed{f^{-1}(x)=\frac{x+4}{2x}}$, $\boxed{0<x<4}$, $\boxed{f^{-1}(x)>1}$



Question 38 **(***+)**

The function f is defined by

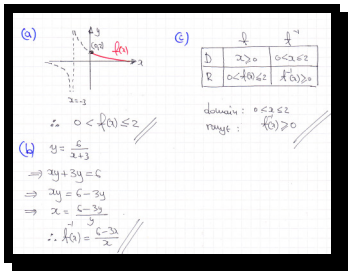
$$f(x)=\frac{6}{x+3},\ x\in\mathbb{R},\ x\geq 0.$$

a) Find the range of $f(x)$.

b) Determine an expression for $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

$$\boxed{0<f(x)\leq 2},\ \boxed{f^{-1}(x)=\frac{6-3x}{x}},\ \boxed{0<x\leq 2,\ f^{-1}(x)\geq 0}$$



Question 39 (*+)**

The function $f(x)$ is defined by

$$f(x)=\sqrt{x+1},\ x\in\mathbb{R},\ x\geq 0.$$

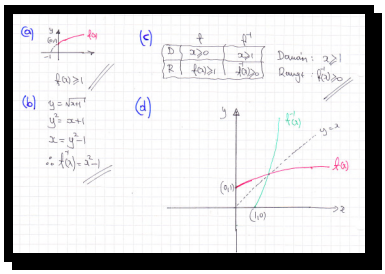
a) Find the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$ in its simplest form.

c) State the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$$\boxed{f(x)\geq 1},\quad \boxed{f^{-1}(x)=x^2-1},\quad \boxed{x\geq 1,\ \ f^{-1}(x)\geq 0}$$



Question 40 **(***+)**

The function f is given by

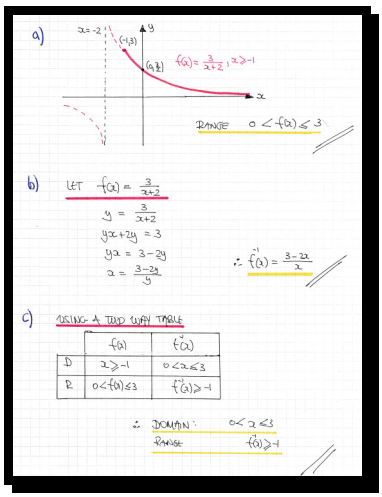
$$f:x\mapsto\frac{3}{x+2},\ x\in\mathbb{R},\ x\geq -1.$$

a) By sketching the graph of f, or otherwise, state its range.

b) Determine an expression for $f^{-1}(x)$, the inverse of f.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{f(x)\in\mathbb{R},\ 0<f(x)\leq 3}$, $\boxed{f^{-1}(x)=\frac{3}{x}-2=\frac{3-2x}{x}}$, $\boxed{0<x\leq 3,\ f^{-1}(x)\geq -1}$,



Question 41 **(***+)**

The function f is satisfies

$$f(x)=\sqrt{x}-3\,,\ x\in\mathbb{R}\,,\ 0\le x\le 9\,.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

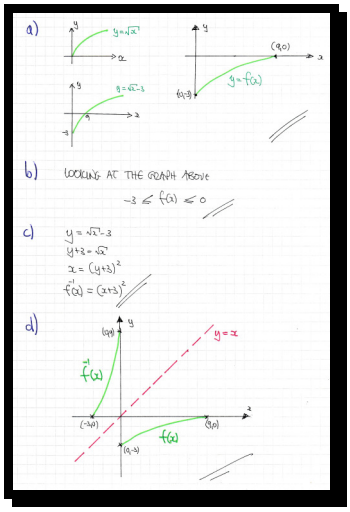
b) State the range of $f(x)$.

c) Find an expression for $f^{-1}(x)$.

d) Sketch in the same set of axes as that of part **(a)** the graph of $f^{-1}(x)$.

The sketch must include the coordinates of the points where the graph of $f^{-1}(x)$ meets the coordinate axes, and how $f^{-1}(x)$ is related graphically to $f(x)$.

$\boxed{}$, $\boxed{-3\le f(x)\le 0}$, $\boxed{f^{-1}(x)=(x+3)^2}$



Question 42 **(***+)**

The function f satisfies

$$f:x\mapsto\frac{3x+1}{x+4},\ x\in\mathbb{R},\ x>-4.$$

a) Find an expression for $f^{-1}(x)$ in its simplest form.

b) Determine the domain and the range of $f^{-1}(x)$.

The function g is given by

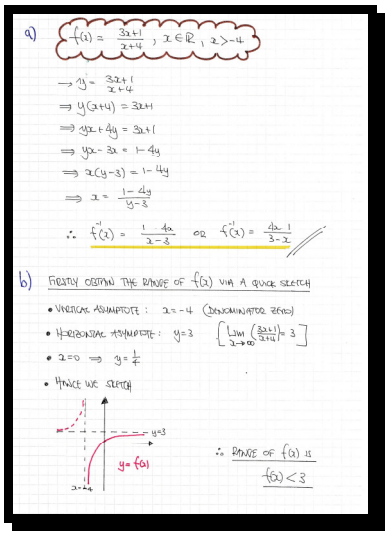
$$g:x\mapsto \mathrm{e}^{x}-3,\ \ x\in\mathbb{R}.$$

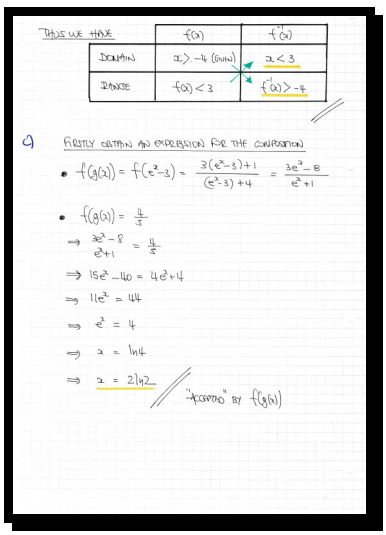
c) Solve the equation

$$fg(x)=\frac{4}{5}\ ,$$

giving exact answers in terms of $\ln 2$.

$\boxed{}$, $\boxed{f^{-1}:x\mapsto\dfrac{1-4x}{x-3}}$, $\boxed{x\in\mathbb{R},\ \ x<3}$, $\boxed{f^{-1}(x)\in\mathbb{R},\ \ f^{-1}(x)>-4}$, $\boxed{x=2\ln 2}$





Question 43 (*+)**

The function f is defined as

$$f:x\mapsto\frac{2x-1}{x^2-x-2}-\frac{1}{x-2},\ x\in\mathbb{R},\ x>4.$$

a) Show clearly that

$$f:x\mapsto\frac{1}{x+1},\ x\in\mathbb{R},\ x>4.$$

b) Find the range of f.

c) Determine an expression for the inverse function, $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

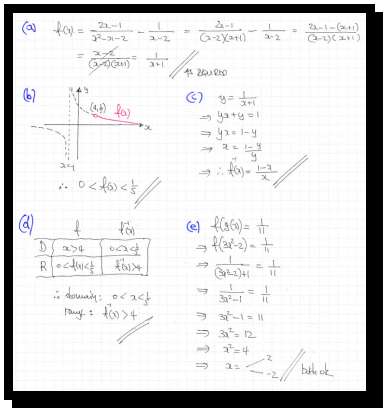
The function g is given by

$$g:x\mapsto 3x^2-2,\ \ x\in\mathbb{R}.$$

e) Solve the equation

$$fg(x)=\frac{1}{11}.$$

$\square$, $\boxed{0<f(x)<\frac{1}{5}}$, $\boxed{f^{-1}(x)=\frac{1-x}{x}}$, $\boxed{0<x<\frac{1}{5}}$, $\boxed{f^{-1}(x)>4}$, $\boxed{x=\pm 2}$



Question 44 **(***+)**

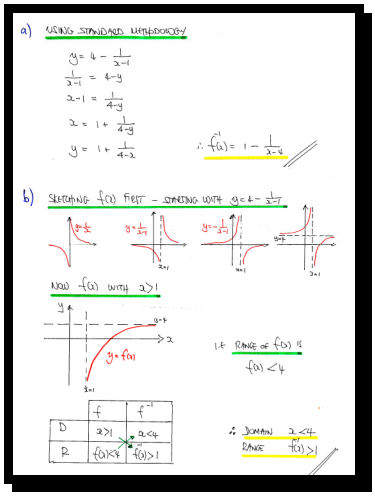
A function f is defined by

$$f(x)=4-\frac{1}{x-1},\quad x\in\mathbb{R},\ x>1.$$

a) Determine an expression for the inverse, $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\square$, $\boxed{f^{-1}(x)=1-\frac{1}{x-4}=\frac{x-5}{x-4}}$, $\boxed{x<4}$, $\boxed{f^{-1}(x)>1}$



Question 45 **(***+)**

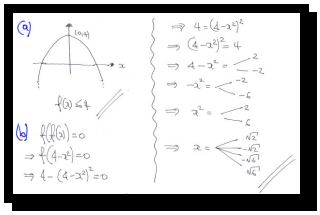
$$f(x)=4-x^2,\ x\in\mathbb{R}.$$

a) State the range of $f(x)$.

b) Solve the equation

$$ff(x)=0.$$

$\boxed{f(x)\le 4}$, $\boxed{x=\pm\sqrt{2},\pm\sqrt{6}}$



Question 46 (*+)**

$$f(x)=4(x+1)^2,\ x\in\mathbb{R},\ x\le -2.$$

a) State the range of $f(x)$.

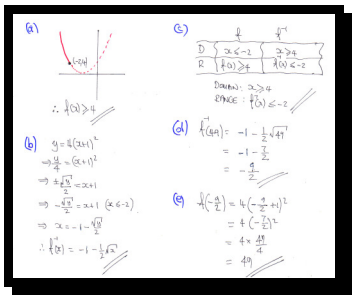
b) Find an expression for the inverse function $f^{-1}(x)$.

c) State the domain and range of $f^{-1}(x)$.

d) Evaluate $f^{-1}(49)$.

e) Verify that the answer to part (d) is correct by carrying an appropriate calculation involving $f(x)$.

$\boxed{f(x)\ge 4}$, $\boxed{f^{-1}(x)=-1-\frac{1}{2}\sqrt{x}}$, $\boxed{x\ge 4,\ f^{-1}(x)\le -2}$, $\boxed{f^{-1}(49)=-\frac{9}{2}}$, $\boxed{f\left(-\frac{9}{2}\right)=49}$



Question 47 (*+)**

The function f is given by

$$f:x\mapsto 3+\frac{2}{x-2},\ x\in\mathbb{R},\ x>2.$$

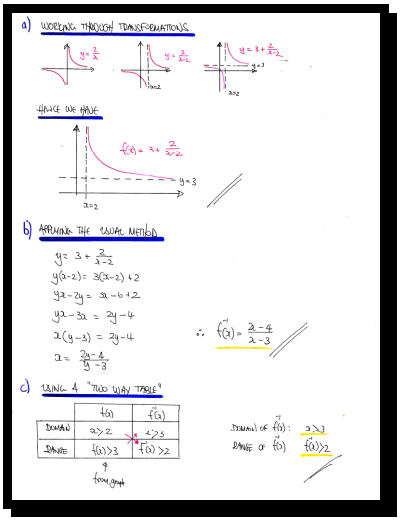
a) Sketch the graph of f.

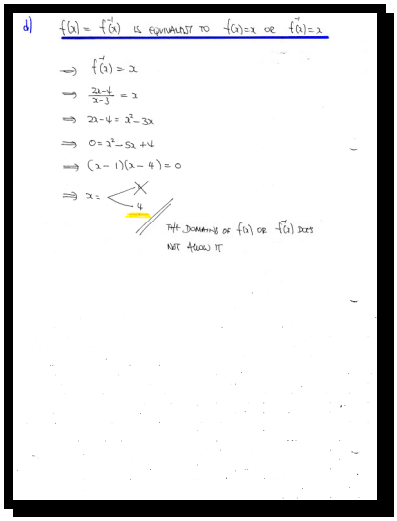
b) Find an expression for $f^{-1}(x)$ as a single fraction, in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Find the value of x that satisfy the equation $f(x)=f^{-1}(x)$

$$\boxed{},\ \boxed{f^{-1}(x)=\frac{2x-4}{x-3}},\ \boxed{x\in\mathbb{R},\ x>3},\ \boxed{f(x)\in\mathbb{R},\ f^{-1}(x)>2},\ \boxed{x=4, x\neq 1}$$





Question 48 (*+)**

The functions f and g are defined below

$$f(x)=x^2+2,\ \ x\in\mathbb{R},\ \ x>0$$

$$g(x)=3x-1,\ \ x\in\mathbb{R},\ \ x>4.$$

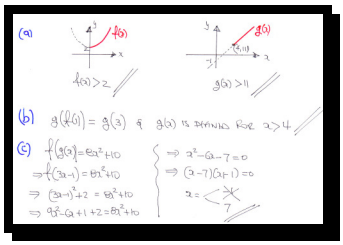
a) Write down the range of $f(x)$ and the range of $g(x)$.

b) Explain why $gf(1)$ cannot be evaluated.

c) Solve the equation

$$fg(x)=8x^2+10.$$

$\boxed{f(x)\in\mathbb{R},\ \ f(x)>2}$, $\boxed{g(x)\in\mathbb{R},\ \ g(x)>11}$ $\boxed{x=7,\ \ x\neq 1}$



Question 49 (*+)**

The functions f and g are defined below

$$f(x)=x^2-2,\ x\in\mathbb{R}$$

$$g(x)=2x+3,\ x\in\mathbb{R},\ x>0.$$

a) Write down the range of $f(x)$.

b) Find, in its simplest form, an expression for $fg(x)$.

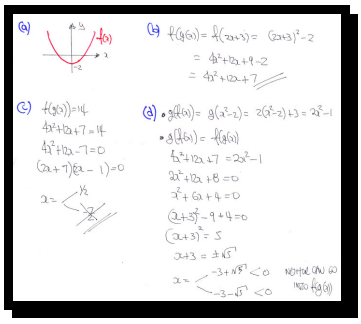
c) Solve the equation

$$fg(x)=14.$$

d) Show that there is no solution for the equation

$$fg(x)=gf(x).$$

$\boxed{f(x)\in\mathbb{R},\ f(x)\geq -2}$, $\boxed{fg(x)=4x^2+12x+7}$, $\boxed{x=\frac{1}{2},x\neq -\frac{7}{2}}$



Question 50 (*+)**

The functions f and g are defined as

$$f(x)=4+\ln x,\ x\in\mathbb{R},\ x>0.$$

$$g(x)=\mathrm{e}x^2,\ x\in\mathbb{R}.$$

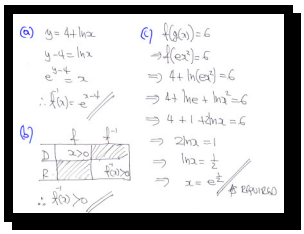
a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Show that $x=\sqrt{\mathrm{e}}$ is a solution of the equation

$$fg(x)=6.$$

$\boxed{f^{-1}(x)=\mathrm{e}^{x-4}}$, $\boxed{f^{-1}(x)>0}$



Question 51 **(***+)**

The function f is given by

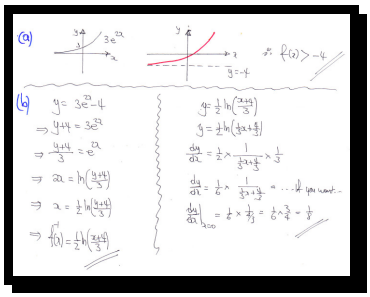
$$f(x)=3\mathrm{e}^{2x}-4,\ x\in\mathbb{R}.$$

a) State the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$.

c) Find the value of the gradient on $f^{-1}(x)$ at the point where $x=0$.

$$\boxed{f(x)>-4},\ \boxed{f^{-1}(x)=\frac{1}{2}\ln\left(\frac{x+4}{3}\right)},\ \boxed{\frac{1}{8}}$$



Question 52 (*+)**

The functions f and g are defined

$$f(x)=x^2-10x,\ x\in\mathbb{R}$$

$$g(x)=\mathrm{e}^x+5,\ x\in\mathbb{R}.$$

a) Find, showing all steps in the calculation, the value of $g(3\ln 2)$.

b) Find, in its simplest form, an expression for $fg(x)$.

c) Show clearly that

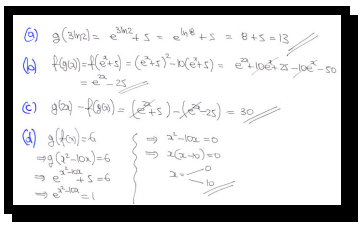
$$g(2x)-fg(x)=k,$$

stating the value of the constant k.

d) Solve the equation

$$gf(x)=6.$$

$\boxed{g(3\ln 2)=13}$, $\boxed{fg(x)=\mathrm{e}^{2x}-25}$, $\boxed{k=30}$, $\boxed{x=0,\ x=10}$



Question 53 **(***+)**

$$f(x)=\mathrm{e}^{x},\ x\in\mathbb{R},\ x>0.$$

$$g(x)=2x^{3}+11,\ x\in\mathbb{R}.$$

a) Find and simplify an expression for the composite function $gf(x)$.

b) State the domain and range of $gf(x)$.

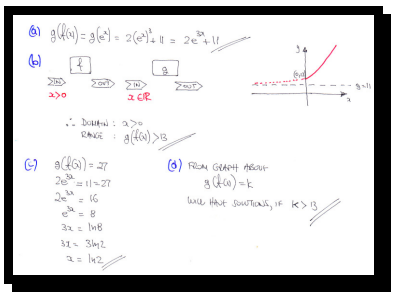
c) Solve the equation

$$gf(x)=27.$$

The equation $gf(x)=k$, where k is a constant, has solutions.

d) State the range of the possible values of k.

$\boxed{}$, $\boxed{gf(x)=2\mathrm{e}^{3x}+11}$, $\boxed{x>0,\ gf(x)>13}$, $\boxed{x=\ln 2}$, $\boxed{k>13}$



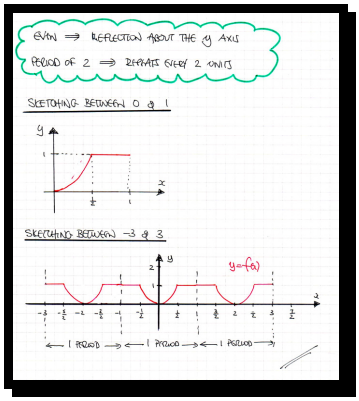
Question 54 **(***+)**

An **even** function f, of period 2 is defined by

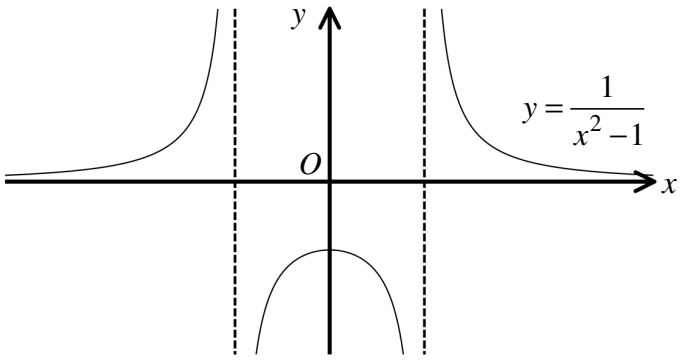
$$f(x)\equiv\begin{cases}4x^2 & 0\le x\le\frac{1}{2}\\ 1 & \frac{1}{2}\le x\le 1\end{cases}$$

Sketch the graph of $f(x)$ for $-3\le x\le 3$.

[ ], [graph]



Question 55 (*+)**



The figure above shows the graph of the curve C with equation

$$y=\frac{1}{x^2-1}\ ,\ x\neq\pm 1\,.$$

a) State the equations of the vertical asymptotes of the curve, marked with dotted lines in the diagram.

The function f is defined as

$$f\left(x\right)=\frac{1}{x^2-1},\ x\in\mathbb{R},\ x>1\,.$$

b) Write down the range of $f\left(x\right)$.

c) Find an expression for $f^{-1}\left(x\right)$.

[continues overleaf]

[continued from overleaf]

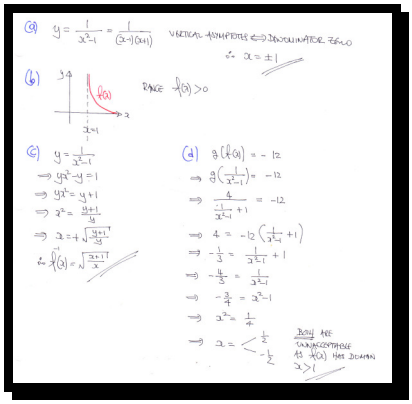
The function g is defined as

$$g(x)=\frac{4}{x+1},\ x\in\mathbb{R},\ x\neq -1.$$

d) Show no value of x satisfies the equation

$$gf(x)=-12.$$

$\boxed{x=\pm 1}$, $\boxed{f(x)\in\mathbb{R},\ f(x)>0}$, $\boxed{f^{-1}(x)=\sqrt{1+\frac{1}{x}}=\sqrt{\frac{x+1}{x}}}$, $\boxed{x=\pm\frac{1}{2}}$



Question 56 (*+)**

The functions f and g are defined by

$$f(x)=x-\frac{1}{x},\ x\in\mathbb{R},\ x\geq 1$$

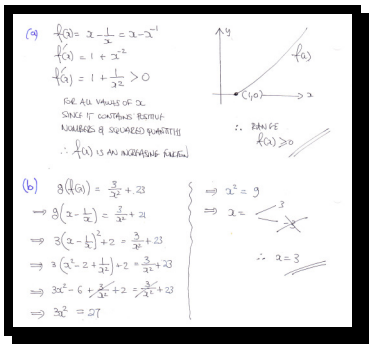
$$g(x)=3x^2+2,\ x\in\mathbb{R},\ x\geq 0.$$

a) By showing that $f(x)$ is an increasing function, find its range.

b) Solve the equation

$$gf(x)=\frac{3}{x^2}+23.$$

$\boxed{f(x)\geq 0}$, $\boxed{x=3}$



Question 57 (*+)**

The functions f and g are given by

$$f(x)=3x+\ln 2,\ x\in\mathbb{R}$$

$$g(x)=\mathrm{e}^{2x},\ x\in\mathbb{R}.$$

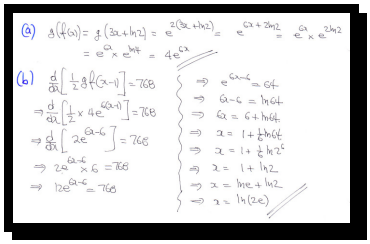
a) Show clearly that

$$gf(x)=4\mathrm{e}^{6x}.$$

b) Show further that $x=\ln(2\mathrm{e})$ is the solution of the equation

$$\frac{d}{dx}\left[\frac{1}{2}gf(x-1)\right]=768.$$

proof



Question 58 **(***+)**

The piecewise continuous function f is **odd** with domain all real numbers.

It is defined by

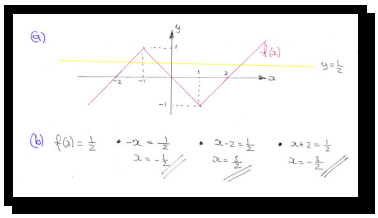
$$f(x)\equiv\begin{cases}-x & 0\le x\le 1\\ x-2 & x>1\end{cases}$$

a) Sketch the graph of f for all values of x.

b) Solve the equation

$$f(x)=\frac{1}{2}.$$

$\square$, $\boxed{x=-\frac{3}{2},-\frac{1}{2},\frac{5}{2}}$



Question 59 (*+)**

The functions f and g are given by

$$f(x)=x^2+2kx+4,\ x\in\mathbb{R}$$

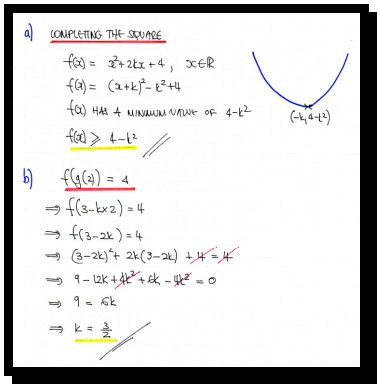
$$g(x)=3-kx,\ x\in\mathbb{R}.$$

where k is a non zero constant.

a) Find, in terms of k, the range of f.

b) Given further that $fg(2)=4$, determine the value of k.

$\square$, $\boxed{f(x)\geq 4-k^2}$, $\boxed{k=\frac{3}{2}}$



Question 60 (****)

The function f is defined by

$$f(x)=2+\sqrt{x}\,,\ x\in\mathbb{R},\ x\geq 0\,.$$

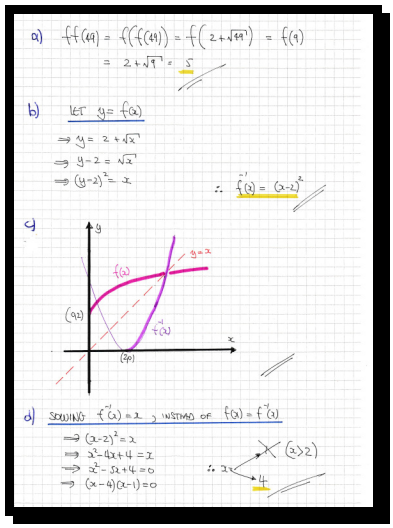
a) Evaluate $ff(49)$.

b) Find an expression for the inverse function, $f^{-1}(x)$.

c) Sketch in the same set of axes the graph of $f(x)$ and the graph of $f^{-1}(x)$, clearly marking the line of reflection between the two graphs.

d) Show that $x=4$ is the only solution of the equation $f(x)=f^{-1}(x)$.

$\boxed{}$, $\boxed{ff(49)=5}$, $\boxed{f^{-1}(x)=(x-2)^2}$



Question 61 (****)

The functions f and g are defined by

$$f(x)=x^2,\ x\in\mathbb{R},\ x\geq 1$$

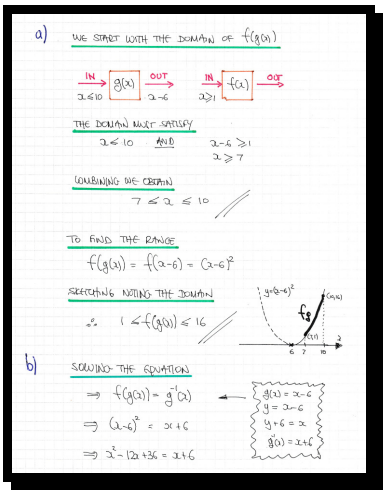
$$g(x)=x-6,\ x\in\mathbb{R},\ x\leq 10.$$

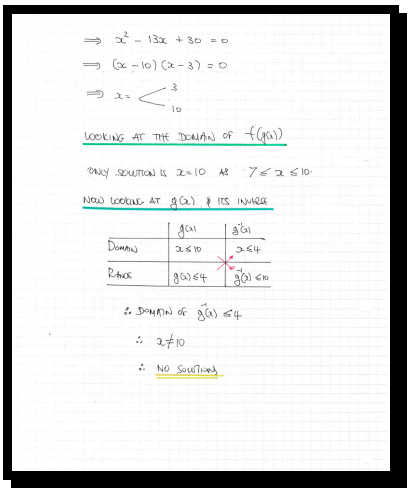
a) Find the domain and range of $fg(x)$.

b) Show the following equation has no solutions

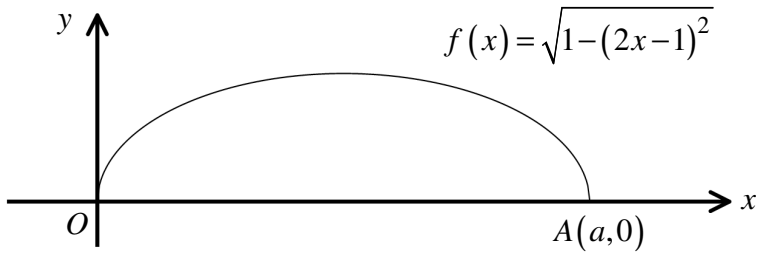
$$fg(x)=g^{-1}(x).$$

$\boxed{}$, $\boxed{7\leq x\leq 10}$, $\boxed{1\leq fg(x)\leq 16}$





Question 62 **(****)**



The figure above shows the graph of the function

$$f(x)\equiv\sqrt{1-(2x-1)^2}\ ,\ x\in\mathbb{R},\ 0\le x\le a\ .$$

a) Find the value of the constant a.

b) State the range of $f(x)$.

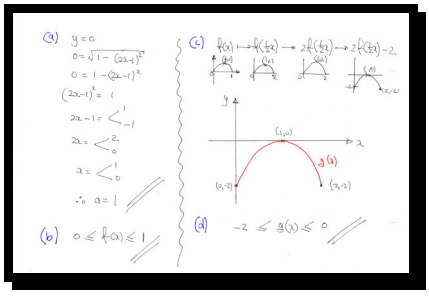
The function g is suitably defined by

$$g(x)=2f\left(\tfrac{1}{2}x\right)-2\ .$$

c) Sketch the graph of $g(x)$.

d) State the domain and range of $g(x)$.

$\boxed{}$, $\boxed{a=1}$, $\boxed{0\le f(x)\le 1}$, $\boxed{0\le x\le 2,\quad -2\le g(x)\le 0}$



Question 63 **(****)**

The function f is defined by

$$f(x)=1+\sqrt{x}\,,\ x\in\mathbb{R},\ x\geq 0\,.$$

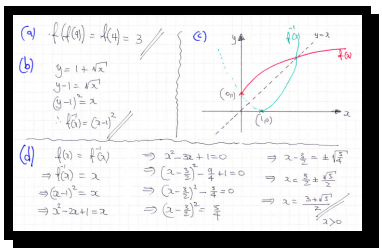
a) Evaluate $ff(9)$.

b) Find an expression for the inverse function, $f^{-1}(x)$.

c) Sketch in the same diagram the graph of $f(x)$ and the graph of $f^{-1}(x)$, clearly marking the line of reflection between the two graphs.

d) Show that $x=\dfrac{3+\sqrt{5}}{2}$ is the only solution of the equation $f(x)=f^{-1}(x)$.

$\square$, $\boxed{ff(9)=3}$, $\boxed{f^{-1}(x)=(x-1)^2}$



Question 64 **(****)**

The functions f and g are defined by

$$f(x)=x^2-4,\ x\in\mathbb{R},\ x>8$$

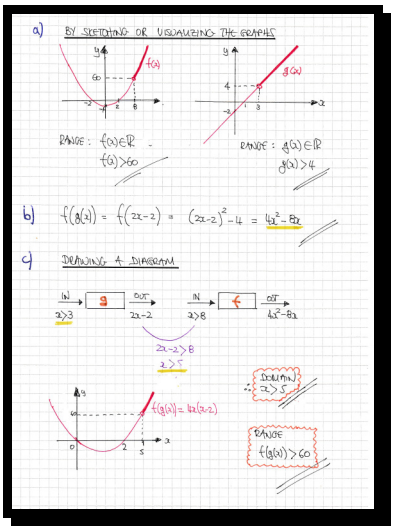
$$g(x)=2x-2,\ x\in\mathbb{R},\ x>3.$$

a) State the range of $f(x)$ and the range of $g(x)$.

b) Find a simplified expression for $fg(x)$.

c) Determine the domain and range of $fg(x)$.

$\boxed{}$, $\boxed{f(x)>60}$, $\boxed{g(x)>4}$, $\boxed{fg(x)=4x^2-8x}$, $\boxed{x>5}$, $\boxed{fg(x)>60}$



Question 65 (****)

The functions f and g are defined by

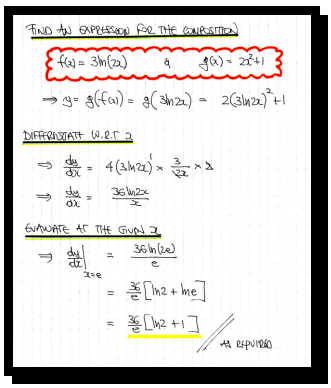
$$f(x)=3\ln 2x,\ x\in\mathbb{R},\ x>0$$

$$g(x)=2x^2+1,\ x\in\mathbb{R}.$$

Show that the value of the gradient on the curve $y=gf(x)$ at the point where $x=\mathrm{e}$ is

$$\frac{36}{\mathrm{e}}(1+\ln 2).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 66 **(****)**

$$f(x)=3x^2-18x+21,\ x\in\mathbb{R},\ x>4.$$

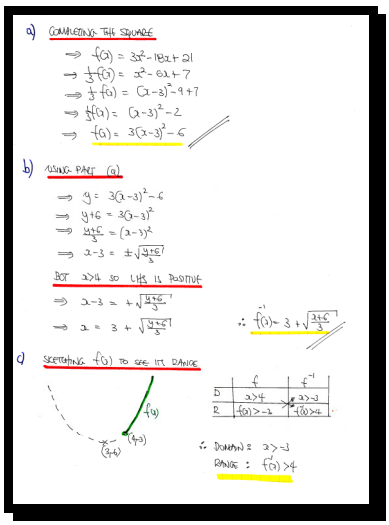
a) Express $f(x)$ in the form $A(x+B)^2+C$, where A, B and C are integers ans hence find the range of $f(x)$.

b) Find a simplified expression for $f^{-1}(x)$, the inverse of $f(x)$.

c) Determine the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{A=3,\ B=-3,\ C=-6}$, $\boxed{f(x)>-3}$, $\boxed{f^{-1}(x)=3+\sqrt{\dfrac{x+6}{3}}}$,

$\boxed{x>-3,\ f^{-1}(x)>4}$



Question 67 (****)

The piecewise continuous function f is **even** with domain $x \in \mathbb{R}$.

It is defined by

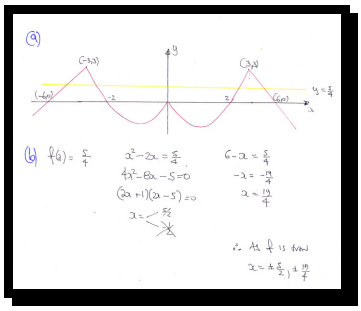
$$f(x) \equiv \begin{cases} x^2 - 2x & 0 \le x \le 3 \\ 6 - x & x > 3 \end{cases}$$

a) Sketch the graph of f for all values of x.

b) Solve the equation

$$f(x) = \frac{5}{4}$$

$\square$, $\boxed{x = \pm\frac{5}{2}, \pm\frac{19}{4}}$



Question 68 (****)

$$f(x)=x^2-4x-5,\ x\in\mathbb{R},\ x\geq 2.$$

a) Find the range of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Sketch the graph of $f^{-1}(x)$, marking clearly the coordinates of any points where the graph meets the coordinate axes.

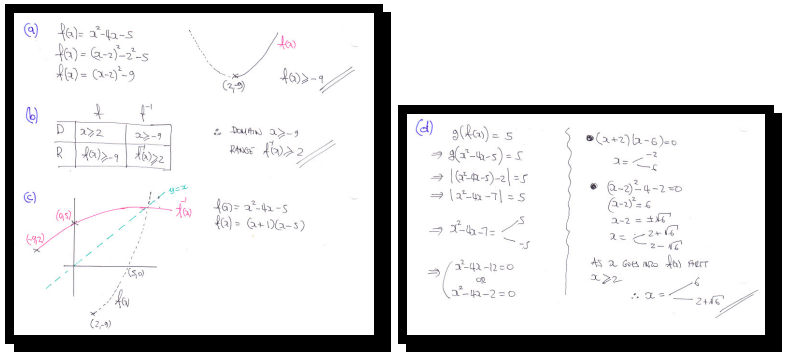
The function g is given

$$g(x)=|x-2|,\ x\in\mathbb{R}.$$

d) Find, in exact form where appropriate, the solutions of the equation

$$gf(x)=5.$$

$$\boxed{f(x)\geq -9},\quad \boxed{x\geq -9,\quad f^{-1}(x)\geq 2},\quad \boxed{x=2+\sqrt{6},\ x=6}$$



Question 68 (****)

The function f is given by

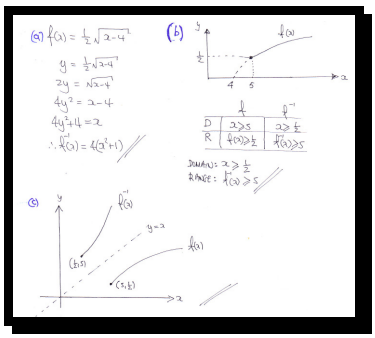
$$f(x)=\frac{1}{2}\sqrt{x-4}\,,\ x\in\mathbb{R},\ x\geq 5\,.$$

a) Determine an expression for $f^{-1}(x)$, in its simplest form.

b) Find the domain and range of $f^{-1}(x)$.

c) Sketch in the same diagram the graph of $f(x)$ and the graph of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=4\left(x^2+1\right)},\ \boxed{x\in\mathbb{R},\ x\geq\frac{1}{2}},\ \boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)\geq 5}$$



Question 70 (****)

The functions f and g are defined by

$$f(x)=\sqrt{x+4}\,,\ x\in\mathbb{R},\ x\geq -3$$

$$g(x)=2x^2-3\,,\ x\in\mathbb{R},\ x\leq 47\ .$$

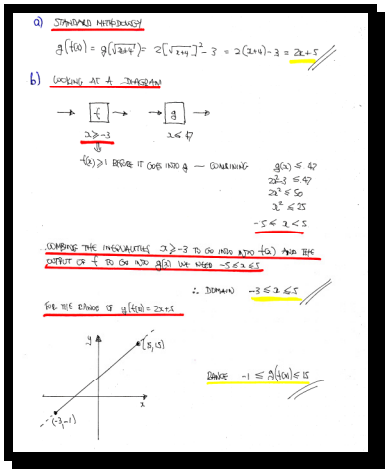
a) Find a simplified expression for $gf(x)$.

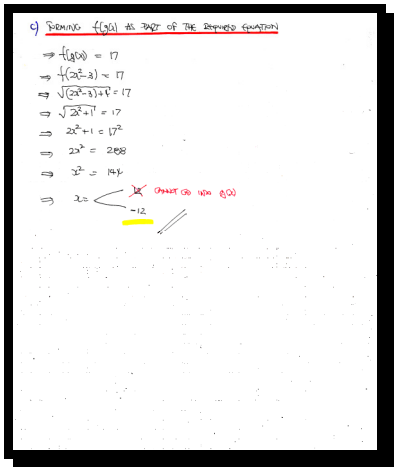
b) Determine the domain and range of $gf(x)$.

c) Solve the equation

$$fg(x)=17\,.$$

$\boxed{}$, $\boxed{gf(x)=2x+5}$, $\boxed{-3\leq x\leq 5}$, $\boxed{-1\leq gf(x)\leq 15}$, $\boxed{x=-12}$





Question 71 (****)

The function $f(x)$ is given by

$$f(x)=2x^2+3,\ x\in\mathbb{R},\ x\le 0.$$

a) Sketch the graph of $f(x)$.

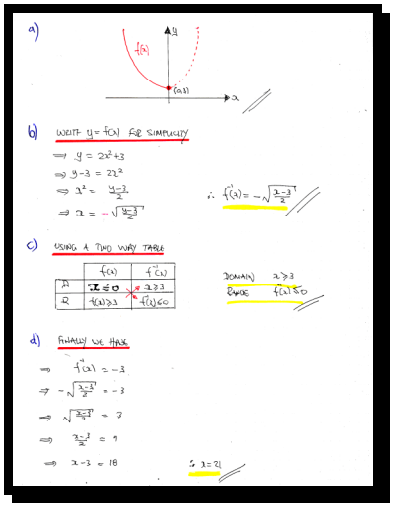
b) Find $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x)=-3.$$

$\boxed{}$, $\boxed{f^{-1}(x)=-\sqrt{\dfrac{x-3}{2}}}$, $\boxed{x\in\mathbb{R},\ x\ge 3}$, $\boxed{f(x)\in\mathbb{R},\ f(x)\le 0}$, $\boxed{x=21}$



Question 72 (****)

The function f is defined by

$$f(x)=\begin{cases}4-x, & x\in\mathbb{R},\ x\leq 2\\ 2(x-1)^2, & x\in\mathbb{R},\ x\geq 2\end{cases}$$

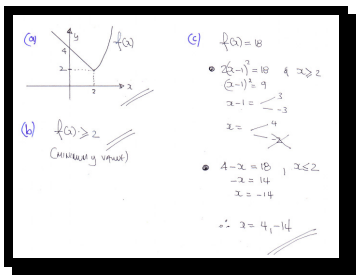
a) Sketch the graph of $f(x)$.

b) State the range of $f(x)$.

c) Solve the equation

$$f(x)=18.$$

$\boxed{}$, $\boxed{f(x)\geq 2}$, $\boxed{x=-14,4}$



Question 73 **(****)**

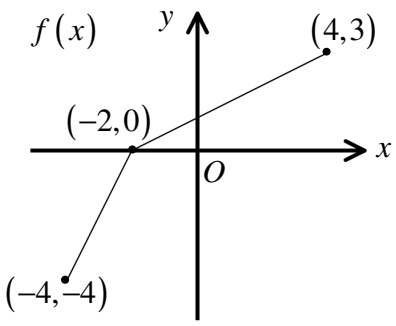


figure 1

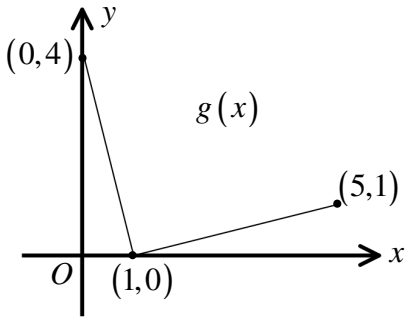


figure 2

Figure 1 and figure 2 above, show the graphs of two piecewise continuous functions $f(x)$ and $g(x)$, respectively.

Each graph consists of two straight line segments joining the points with the coordinates shown in each figure.

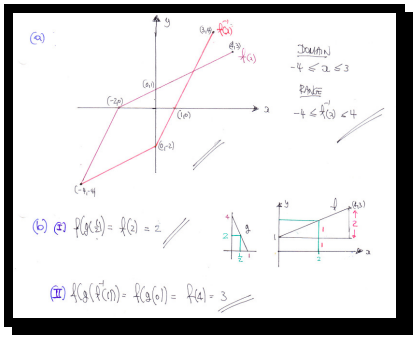
a) Sketch on the same set of axes the graphs of $f(x)$ and its inverse $f^{-1}(x)$, stating the domain and range of $f^{-1}(x)$.

b) Evaluate …

i. … $fg\left(\frac{1}{2}\right)$.

ii. … $fgf^{-1}(1)$.

$\boxed{}$, $\boxed{-4 \le x \le 3}$, $\boxed{-4 \le f^{-1}(x) \le 4}$, $\boxed{fg\left(\frac{1}{2}\right)=2}$, $\boxed{fgf^{-1}(1)=3}$



Question 74 **(****)**

$$f(x)=2-\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

a) Sketch the graph of $f(x)$.

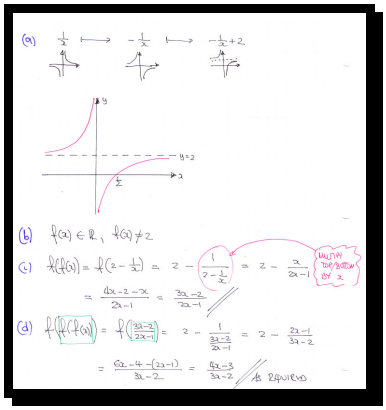
b) State the range of $f(x)$.

c) Find as a simplified fraction, an expression for $ff(x)$.

d) Hence, show that

$$fff(x)=\frac{4x-3}{3x-2}.$$

$$\boxed{f(x)\in\mathbb{R},\ f(x)\neq 2},\quad \boxed{ff(x)=\frac{3x-2}{2x-1}}$$



Question 75 (****)

The functions f and g satisfy

$$f(x)=2\mathrm{e}^{\frac{1}{2}x},\ x\in\mathbb{R}$$

$$g(x)=\ln 4x\ \ x\in\mathbb{R},\ x>\frac{1}{4}.$$

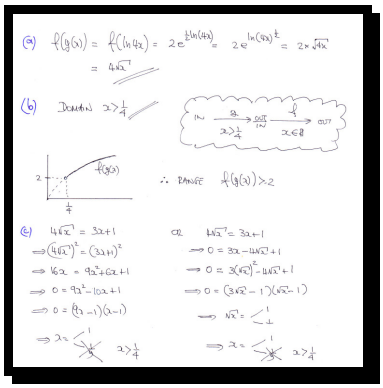
a) Find $fg(x)$ in its simplest form.

b) Find the domain and range of $fg(x)$.

c) Solve the equation

$$fg(x)=3x+1.$$

$\square$, $\boxed{fg(x)=4\sqrt{x}}$, $\boxed{x\in\mathbb{R},\ x>\frac{1}{4}}$, $\boxed{fg(x)\in\mathbb{R},\ f(x)>2}$, $\boxed{x=1,\ x\neq\frac{1}{9}}$



Question 76 (****)

The function $f(x)$ is given by

$$f(x)=\frac{4}{x-2},\ x\in\mathbb{R},\ x\neq 2.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

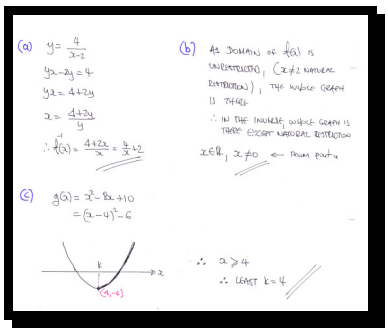
b) State the domain of $f^{-1}(x)$.

The function g is defined as

$$g(x)=x^2-8x+10,\ x\in\mathbb{R},\ x\geq k.$$

c) Given that $g^{-1}(x)$ exists, find the least value of k.

$$\boxed{f^{-1}(x)=\frac{2x+4}{x}},\ \boxed{x\in\mathbb{R},\ x\neq 0},\ \boxed{k=4}$$



Question 77 (****)

The functions f and g are given below

$$f(x)=\frac{1}{2-2x},\ x\in\mathbb{R},\quad x\neq 0, x\neq\frac{1}{2}, x\neq 1$$

$$g(x)=ff(x).$$

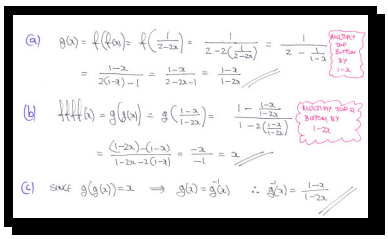
a) Find a simplified expression for $g(x)$.

b) Hence show clearly that

$$ffff(x)=x.$$

c) Find an expression for the inverse function $g^{-1}(x)$.

$$\boxed{g(x)=\frac{1-x}{1-2x}},\ \boxed{g^{-1}(x)=\frac{1-x}{1-2x}}$$



Question 78 (****)

$$f(x)=x^2-6x,\ x\in\mathbb{R},\ x\leq 3.$$

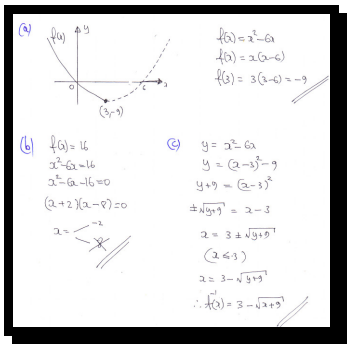
a) Find the range of $f(x)$.

b) Solve the equation

$$f(x)=16.$$

c) Find an expression for the inverse function $f^{-1}(x)$.

$\boxed{f(x)\geq -9}$, $\boxed{x=-2}$, $\boxed{f^{-1}(x)=3-\sqrt{x+9}}$



Question 79 (****)

The following functions are defined as follows

$$f(x)=3-x^2,\ x\in\mathbb{R}$$

$$g(x)=\frac{2}{x+1},\ x\in\mathbb{R},\ x\neq -1.$$

a) Find the range of $f(x)$.

b) Find $g^{-1}(x)$ in its simplest form, further stating its range.

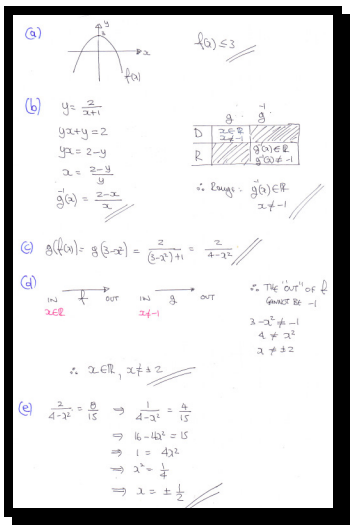
c) Determine the composite function $gf(x)$.

d) Find the domain of $gf(x)$.

e) Solve the equation

$$gf(x)=\frac{8}{15}.$$

$\boxed{f(x)\leq 3}$, $\boxed{g^{-1}(x)=\frac{2}{x}-1=\frac{2-x}{x}}$, $\boxed{g^{-1}(x)\neq -1}$, $\boxed{gf(x)=\frac{2}{4-x^2}}$, $\boxed{x\in\mathbb{R},x\neq\pm 2}$, $\boxed{x=\pm\frac{1}{2}}$



Question 80 (****)

The function f is given by

$$f:x\mapsto\frac{3-x}{1+x},\ x\in\mathbb{R},\ x\leq -2.$$

a) Show that for some constants a and b

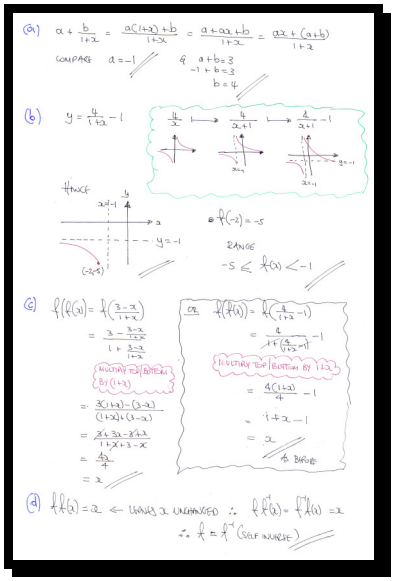
$$\frac{3-x}{1+x}\equiv a+\frac{b}{1+x}.$$

b) Sketch the graph of f and hence state its range.

c) Show that $ff(x)=x$, for all $x\leq -2$.

d) Without finding $f^{-1}(x)$ explain how part (c) can be used to deduce $f^{-1}(x)$.

$\boxed{a=-1}$, $\boxed{b=4}$, $\boxed{-5\leq f(x)<-1}$



Question 81 (****)

The function $f(x)$ is defined by

$$f(x)=(x-2)^2-1,\ x\in\mathbb{R},\ x\le 2.$$

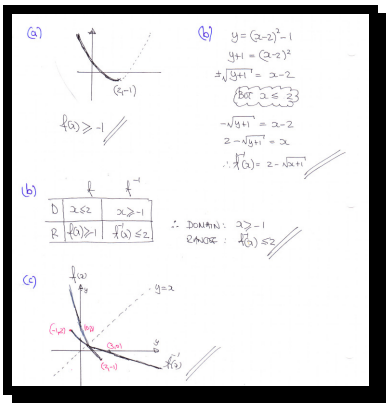
a) Find the range of $f(x)$.

b) Find $f^{-1}(x)$ in its simplest form.

c) Determine the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$$\boxed{f(x)\ge -1},\quad \boxed{f^{-1}(x)=2-\sqrt{x+1}},\quad \boxed{x\ge -1,\ \ f^{-1}(x)\le 2}$$



Question 82 (****)

The functions f and g are defined by

$$f(x)=\mathrm{e}^{x},\ x\in\mathbb{R}$$

$$g(x)=\ln\left(x^2-4\right),\ x\in\mathbb{R},\ x>2\ .$$

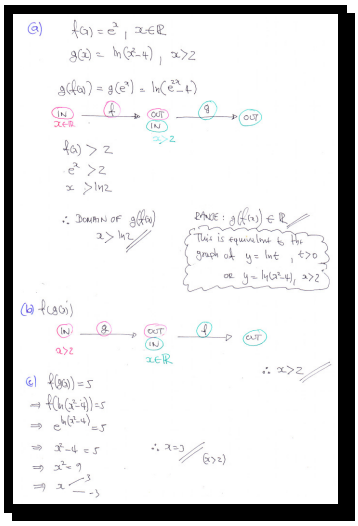
a) Find the domain and range of $gf(x)$.

b) Find the domain of $fg(x)$.

c) Solve the equation

$$fg(x)=5.$$

$\boxed{x>\ln 2,\ gf(x)\in\mathbb{R}}$, $\boxed{x>2}$, $\boxed{x=3}$



Question 83 (****)

The functions f and g are defined by

$$f(x)=\mathrm{e}^{x}-3,\ x\in\mathbb{R}$$

$$g(x)=x+1,\ x\in\mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Solve the equation

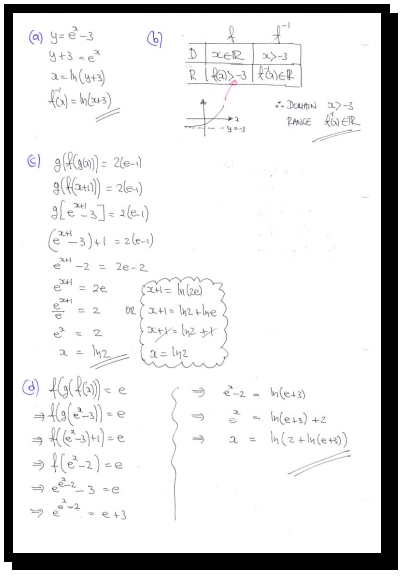
$$gfg(x)=2(\mathrm{e}-1),$$

giving the final answer in terms of logarithms in its simplest form.

d) Find an exact solution of the equation

$$fgf(x)=\mathrm{e}.$$

$\boxed{f^{-1}(x)=\ln(x+3)}$, $\boxed{x\in\mathbb{R},\ x>-3}$, $\boxed{f^{-1}(x)\in\mathbb{R}}$, $\boxed{x=\ln 2}$, $\boxed{x=\ln\left[2+\ln(3+\mathrm{e})\right]}$



Question 84 (****)

The functions f and g are defined by

$$f(x)=x^2+3x-7\,,\ x\in\mathbb{R}$$

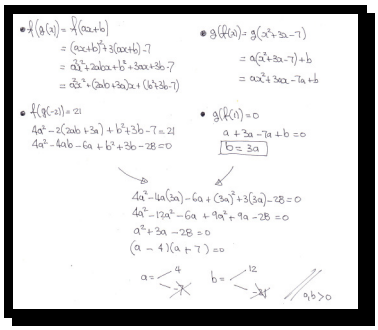
$$g(x)=ax+b\,,\ x\in\mathbb{R}\,,$$

where a and b are positive constants.

When the composition $fg(x)$ is divided by $(x+2)$ the remainder is 21, while $(x-1)$ is a factor of the composition $gf(x)$.

Determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=4}$, $\boxed{b=12}$



Question 85 (****)

The function f is defined as

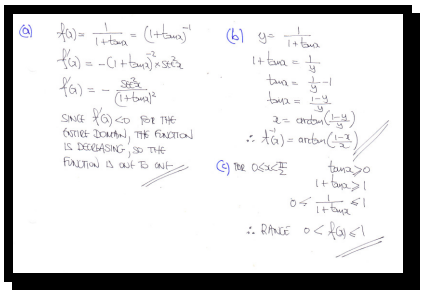
$$f(x)=\frac{1}{1+\tan x},\ 0\le x<\frac{\pi}{2}.$$

a) Use differentiation to show that f is a one to one function.

b) Find a simplified expression for the inverse of f.

c) Determine the range of f.

$\square$, $\boxed{f^{-1}(x)=\arctan\left(\frac{1-x}{x}\right)}$, $\boxed{0<f(x)\le 1}$



Question 86 (****)

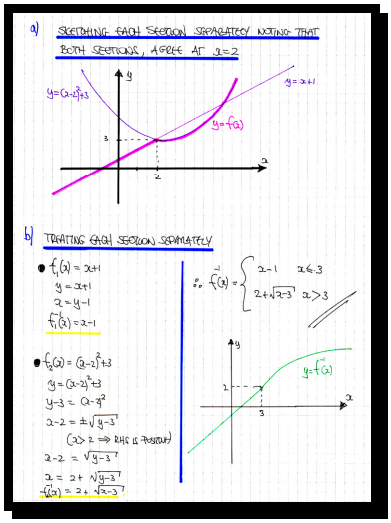
The function f is defined by

$$f(x)=\begin{cases} x+1, & x\in\mathbb{R},\ x\leq 2 \\ (x-2)^2+3, & x\in\mathbb{R},\ x>2 \end{cases}$$

a) Sketch the graph of $f(x)$.

b) Find an expression for $f^{-1}(x)$, fully specifying its domain.

$\square$, $\boxed{f(x)=\begin{cases} x-1, & x\in\mathbb{R},\ x\leq 3 \\ 2+\sqrt{x-3}, & x\in\mathbb{R},\ x\geq 3 \end{cases}}$



Question 87 (****)

The function $f(x)$ is defined by

$$f(x)=\ln(3x-2)+3,\ x\in\mathbb{R},\ x\geq 1.$$

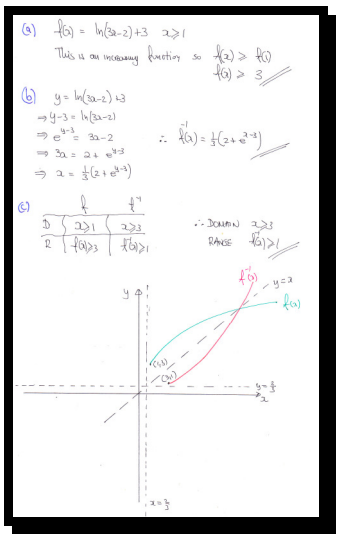
a) Find the range of $f(x)$.

b) Find $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$\boxed{f(x)\geq 3}$, $\boxed{f^{-1}(x)=\frac{1}{3}\left(2+\mathrm{e}^{x-3}\right)}$, $\boxed{x\geq 3,\ f^{-1}(x)\geq 1}$



Question 88 (****)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x)=\ln x,\ x\in\mathbb{R},\ x>0$$

$$g(x)=\mathrm{e}^{3x},\ x\in\mathbb{R},\ x>1.$$

a) Find, in its simplest form, the function compositions

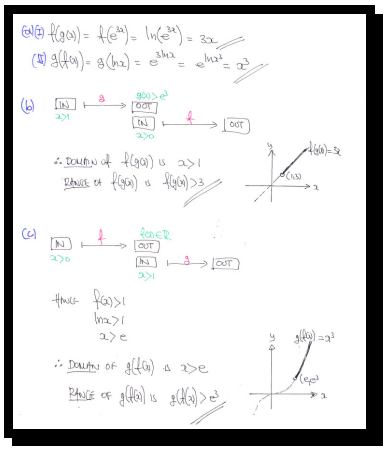
i. $fg(x)$.

ii. $gf(x)$.

b) Find the domain and range of $fg(x)$.

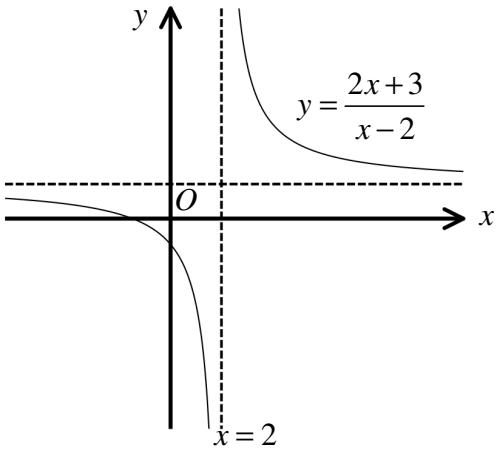
c) Find the domain and range of $gf(x)$.

$$\boxed{fg(x)=3x},\ \boxed{gf(x)=x^3},\ \boxed{x>1,\ fg(x)>3},\ \boxed{x>\mathrm{e},\ gf(x)>\mathrm{e}^3}$$



Question 89 (****)

The figure below shows the graph of the curve with equation $y=\dfrac{2x+3}{x-2}$.



a) Write down the equation of the horizontal asymptote to the curve.

The function f is defined as

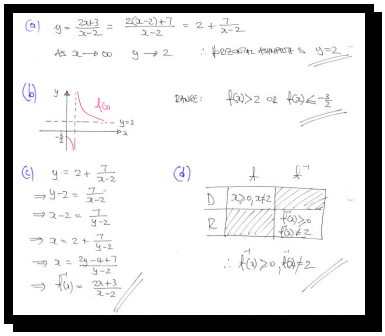
$$f(x)=\frac{2x+3}{x-2},\ x\in\mathbb{R},\ x\geq 0, x\neq 2\,.$$

b) Find the range of $f(x)$.

c) Find $f^{-1}(x)$ in its simplest form.

d) State the range of $f^{-1}(x)$.

$\boxed{y=2}$, $\boxed{f(x)\leq -\frac{3}{2}\ \text{ or }\ f(x)>2}$, $\boxed{f^{-1}(x)=\frac{2x+3}{x-2}}$, $\boxed{f^{-1}(x)\geq 0,\ f^{-1}(x)\neq 2}$



Question 90 **(****)**

The function $f(x)$ is defined by

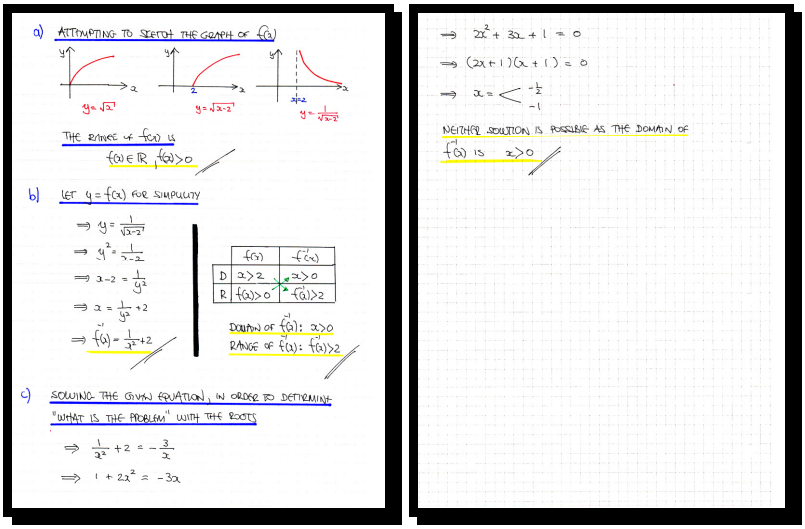
$$f(x)=\frac{1}{\sqrt{x-2}},\ x\in\mathbb{R},\ x>2.$$

a) Find the range of $f(x)$.

b) Determine a simplified expression for $f^{-1}(x)$, further stating the domain and range of $f^{-1}(x)$.

c) Show that the equation $f^{-1}(x)=-\frac{3}{x}$ has no real solutions.

$\boxed{}$, $\boxed{f(x)>0}$, $\boxed{f^{-1}(x)=\frac{1}{x^2}+2}$, $\boxed{x>0,\ f^{-1}(x)>2}$



Question 91 (****)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x)=2\mathrm{e}^{x},\ x\in\mathbb{R}$$

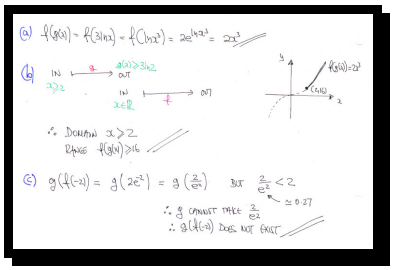
$$g(x)=3\ln x,\ x\in\mathbb{R},\ x\geq 2\,.$$

a) Find, in its simplest form, the function composition $fg(x)$.

b) Find the domain and range of $fg(x)$.

c) Show that $gf(-2)$ does not exist.

$$\boxed{fg(x)=2x^{3}},\ \boxed{x\geq 2,\ fg(x)\geq 16}$$



Question 92 **(****)**

$$f(x)=2\cos 2x,\quad x\in\mathbb{R},\ \ 0\le x\le\frac{\pi}{2}$$

$$g(x)=|x|,\quad x\in\mathbb{R}.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the curve crosses the coordinate axes.

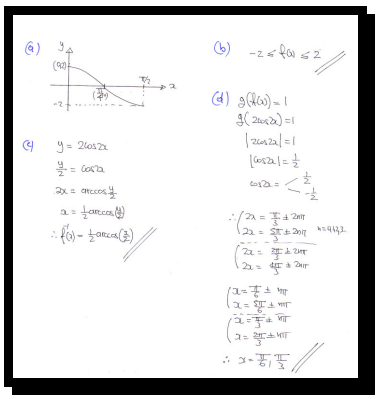
b) State the range of $f(x)$.

c) Find an expression for $f^{-1}(x)$.

d) Solve the equation

$$gf(x)=1.$$

$\boxed{\left(\frac{\pi}{4},0\right),(0,2)}$, $\boxed{-2\le f(x)\le 2}$, $\boxed{f^{-1}(x)=\frac{1}{2}\arccos\left(\frac{x}{2}\right)}$, $\boxed{x=\frac{\pi}{6},\frac{\pi}{3}}$



Question 93 **(****)**

The functions f and g are defined as

$$f(x)=3\left(2^{-x}\right)-1,\quad x\in\mathbb{R},\quad x\geq 0$$

$$g(x)=\log_2 x,\quad x\in\mathbb{R},\quad x\geq 1.$$

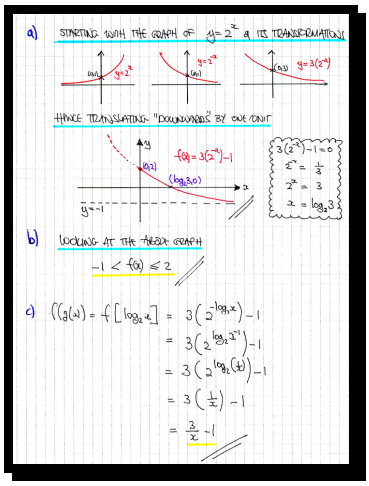
a) Sketch the graph of f.

- Mark clearly the exact coordinates of any points where the curve meets the coordinate axes. Give the answers, where appropriate, in exact form in terms of logarithms base 2.
- Mark and label the equation of the asymptote to the curve.

b) State the range of f.

c) Find $f\left(g(x)\right)$ in its simplest form.

$\boxed{}$, $\boxed{(0,2)}$, $\boxed{(\log_2 3,0)}$, $\boxed{y=-1}$, $\boxed{-1<f(x)\leq 2}$, $\boxed{f\left(g(x)\right)=\frac{3}{x}-1}$



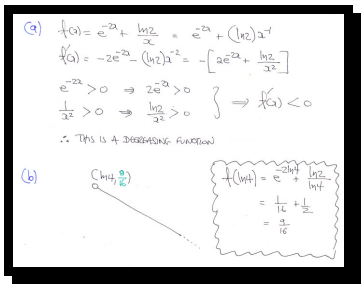
Question 94 **(****)**

$$f(x)=\mathrm{e}^{-2x}+\frac{\ln 2}{x},\ x\in\mathbb{R},\ x>\ln 4\,.$$

a) Show that $f(x)$ in a decreasing function.

b) Find the range of $f(x)$ in its simplest form.

$\square$, $\boxed{f(x)\in\mathbb{R},\ f(x)<\frac{9}{16}}$



Question 95 (****)

The function $f(x)$ satisfies

$$f(x)=\frac{2x+1}{x-1},\ x\in\mathbb{R},\ x\geq 2.$$

a) Show that

$$f(x)=A+\frac{B}{x-1},$$

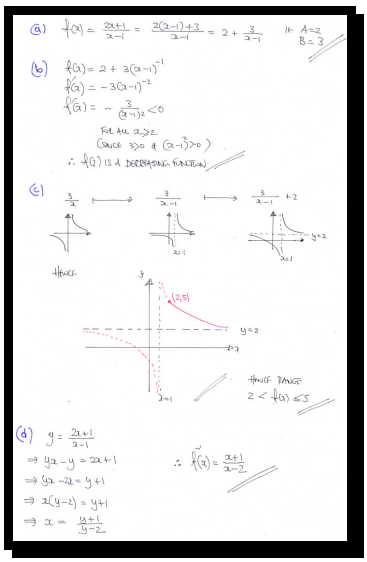
where A and B are positive constants to be found.

b) Show that $f(x)$ is s decreasing function.

c) Sketch the graph of $f(x)$ and hence find its range.

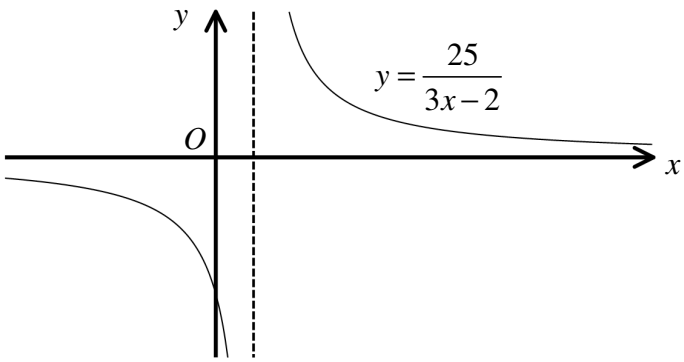
d) Find $f^{-1}(x)$, in its simplest form.

$$\boxed{A=2,B=3},\ \boxed{f(x)\in\mathbb{R},\ 2<f(x)\leq 5},\ \boxed{f^{-1}(x)=\frac{x+1}{x-2}}$$



Question 96 (****)

The figure below shows the graph of the curve C with equation $y=\dfrac{25}{3x-2}$.



a) State the equation of the vertical asymptote of the curve, marked with a dotted line in the diagram.

The function f is defined as

$$f(x)=\frac{25}{3x-2},\quad x\in\mathbb{R},\quad 1<x\leq 9.$$

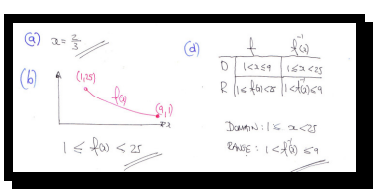
b) Write down the range of $f(x)$.

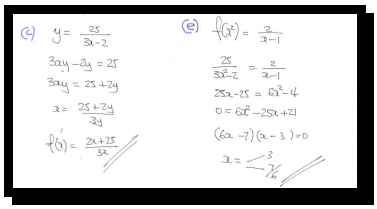
c) Find an expression for $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

e) Solve the equation $f(x^2)=\dfrac{2}{x-1}$.

$\boxed{x=\frac{2}{3}}$, $\boxed{1\leq f(x)<25}$, $\boxed{f^{-1}(x)=\frac{2x+25}{3x}}$, $\boxed{1\leq x<25}$, $\boxed{1<f^{-1}(x)\leq 9}$, $\boxed{x=\frac{7}{6},3}$





Question 97 (****)

The functions $f(x)$ and $g(x)$ are defined as

$$f(x)=\frac{2x^2-50}{x+5},\ x\in\mathbb{R},\ x\neq -6.$$

$$g(x)=x^2+1,\ x\in\mathbb{R}.$$

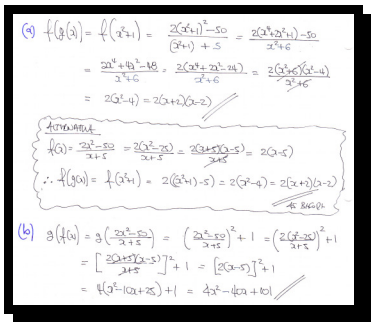
Show that …

a) … $fg(x)=k(x+k)(x-k)$,

stating the value of the constant k.

b) … $gf(x)=4x^2-40x+101$.

$\boxed{k=2}$



Question 98 **(****)**

The functions f and g are defined as

$$f(x)=x^2-16,\ x\in\mathbb{R},\ x<0$$

$$g(x)=12-\frac{1}{2}x,\ x\in\mathbb{R},\ x>8.$$

a) Find, in any order, ...

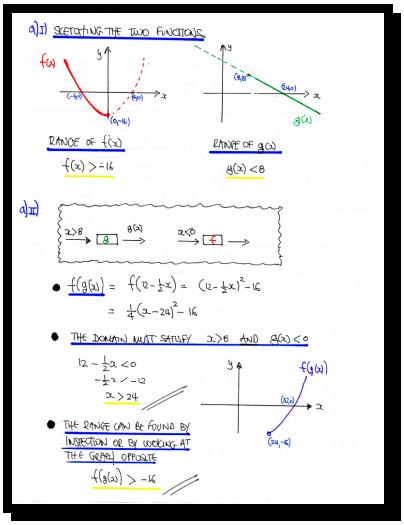
i. ... the range of $f(x)$ and the range of $g(x)$.

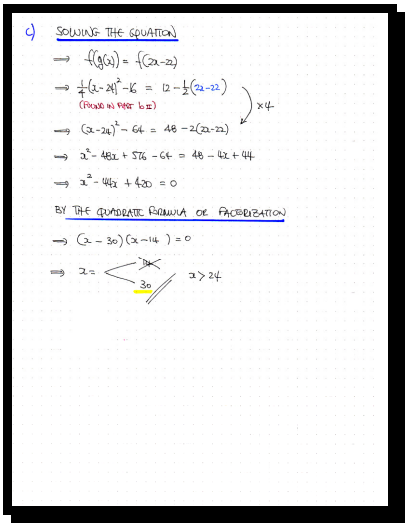
ii. ... the domain and range of $fg(x)$.

b) Solve the equation

$$fg(x)=g(2x-22).$$

$\boxed{}$, $\boxed{f(x)>-16}$, $\boxed{g(x)<8}$, $\boxed{x>24,\ fg(x)>-16}$, $\boxed{x=30}$





Question 99 (****)

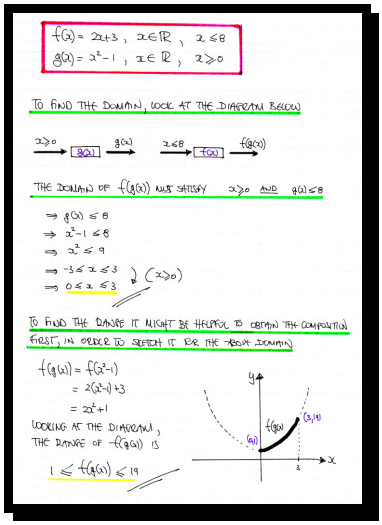
The functions f and g are defined by

$$f(x)=2x+3,\ x\in\mathbb{R},\ x\leq 8$$

$$g(x)=x^2-1,\ x\in\mathbb{R},\ x\geq 0.$$

Find the domain and range of $fg(x)$.

$\boxed{}$, $\boxed{0\leq x\leq 3}$, $\boxed{1\leq fg(x)\leq 19}$



Question 100 (****)

The functions f and g are given by

$$f:x\mapsto x^2,\ x\in\mathbb{R}.$$

$$g:x\mapsto 2x+1,\ x\in\mathbb{R}.$$

a) Solve the equation

$$fg(x)=gf(x).$$

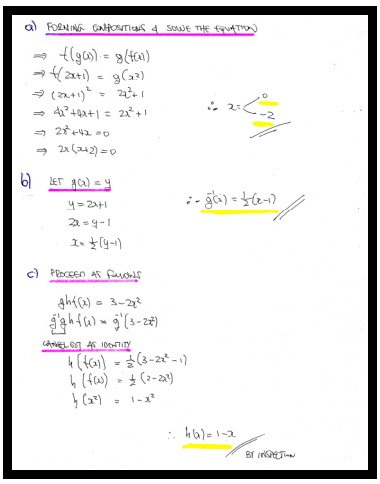
b) Find the inverse function of g.

The function h is defined on a suitable domain such so that

$$ghf(x)=3-2x^2,\ x\in\mathbb{R}.$$

c) Determine an equation of h.

$$\boxed{\ },\ \boxed{x=-2\ \cup\ x=0},\ \boxed{g^{-1}(x)=\frac{x-1}{2}},\ \boxed{h(x)=1-x}$$



Question 101 **(****)**

The piecewise continuous function f is **even** with domain $x\in\mathbb{R}$, $-6\le x\le 6$.

It is defined by

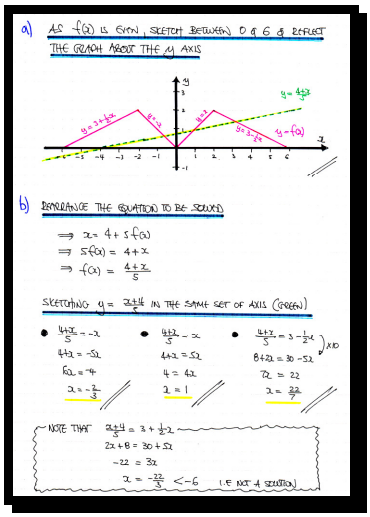
$$f(x)\equiv\begin{cases} x & 0\le x\le 2 \\ 3-\frac{1}{2}x & 2\le x\le 6 \end{cases}$$

a) Sketch the graph of f for $-6\le x\le 6$.

b) Hence, solve the equation

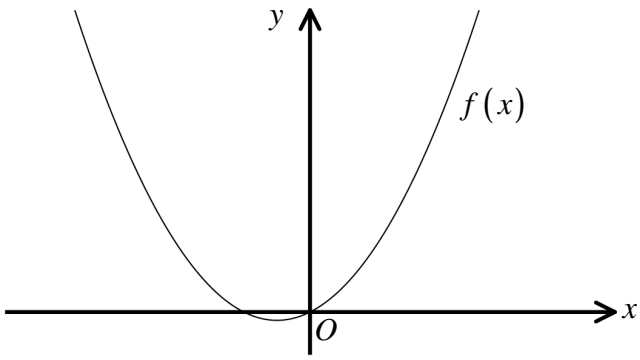
$$x=4+5f(x).$$

$\square$, $\boxed{x=-\frac{2}{3}\ \cup\ x=1\ \cup\ x=\frac{22}{7}}$



Question 102 (****)

The graph below shows the graph of a function $f(x)$.



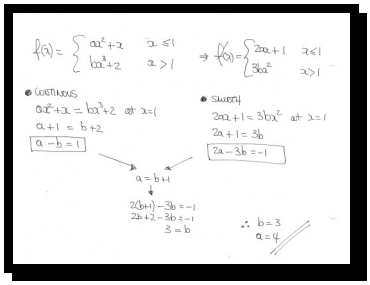
The function f is defined by

$$f(x)=\begin{cases} ax^2+x, & x\in\mathbb{R},\ x\leq 1 \\ bx^3+2, & x\in\mathbb{R},\ x>1 \end{cases}$$

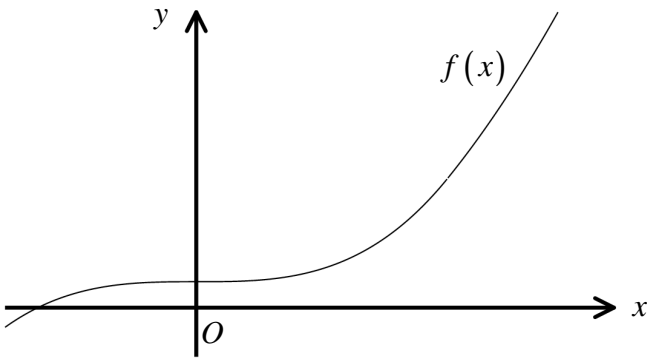
The function is **continuous** and **smooth.**

Find the value of a and the value of b.

$\boxed{}$, $\boxed{a=4,\ b=3}$



Question 103 (****)



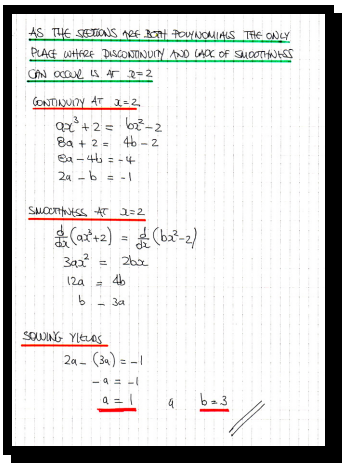
The figure above shows the graph of a function $f(x)$, defined by

$$f(x)=\begin{cases} ax^3+2, & x\in\mathbb{R},\ x\leq 2 \\ bx^2-2, & x\in\mathbb{R},\ x>2 \end{cases}$$

The function is **continuous** and **smooth.**

Find the value of a and the value of b.

$\boxed{}$, $\boxed{a=1}$, $\boxed{b=3}$


AS THE SECTIONS ARE BOTH POLYNOMIALS THE ONLY PLACE WHERE DISCONTINUITY AND LACK OF SMOOTHNESS CAN OCCUR IS AT $x=2$

CONTINUITY AT $x=2$

$ax^3+2 = bx^2-2$

$8a+2 = 4b-2$

$8a-4b = -4$

$2a-b = -1$

SMOOTHNESS AT $x=2$

$\frac{d}{dx}(ax^3+2) = \frac{d}{dx}(bx^2-2)$

$3ax^2 = 2bx$

$12a = 4b$

$b = 3a$

SOLVING YIELDS

$2a-(3a) = -1$

$-a = -1$

$a = 1$ & $b = 3$

Question 104 **(****)**

The function $f(x)$ is defined by

$$f(x)\equiv 3-2x^2,\ x\in\mathbb{R},\ x\le 0.$$

a) State the range of $f(x)$.

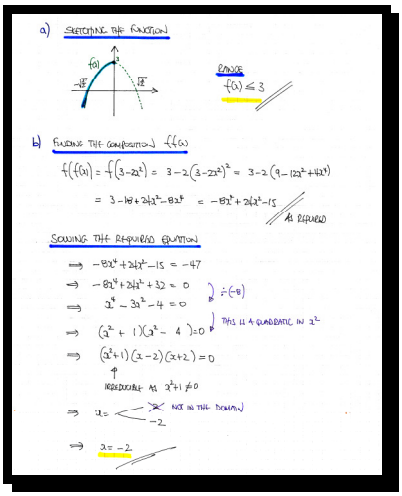
b) Show that $ff(x)=-8x^4+24x^2-15$ and hence solve the equation $ff(x)=-47$.

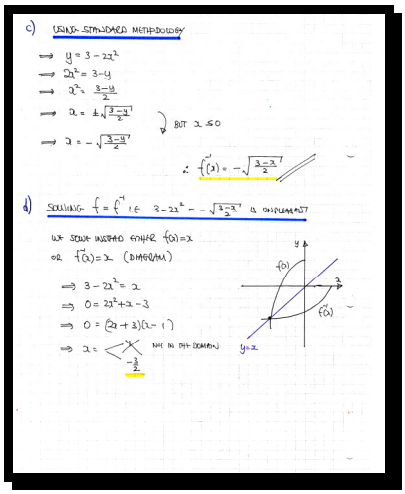
c) Find an expression for the inverse function, $f^{-1}(x)$.

d) Solve the equation

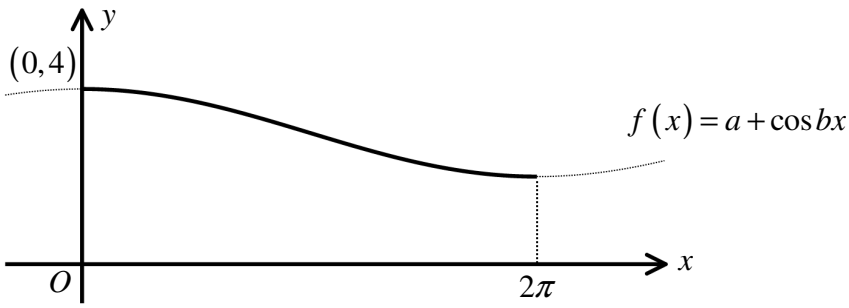
$$f(x)=f^{-1}(x).$$

$\boxed{}$, $\boxed{f(x)\le 3}$, $\boxed{x=-2}$, $\boxed{f^{-1}(x)=-\sqrt{\frac{3-x}{2}}}$, $\boxed{x=-\frac{3}{2}}$





Question 105 **(****)**



The figure above shows the graph of the function

$$f(x)=a+\cos bx,\ 0\le x\le 2\pi,$$

where a and b are non zero constants.

The stationary points $(0,4)$ and $(2\pi,2)$ are the endpoints of the graph.

a) State the range of $f(x)$ and hence find the value of a and the value of b.

b) Find an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

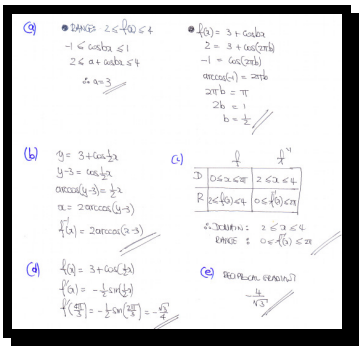
c) State the domain and range of $f^{-1}(x)$.

d) Find the gradient at the point on $f(x)$ with coordinates $\left(\frac{4\pi}{3},\frac{5}{2}\right)$.

e) State the gradient at the point on $f^{-1}(x)$ with coordinates $\left(\frac{5}{2},\frac{4\pi}{3}\right)$.

$\boxed{\ }$, $\boxed{2\le f(x)\le 4}$, $\boxed{a=3,\ b=\frac{1}{2}}$, $\boxed{f^{-1}(x)=2\arccos(x-3)}$, $\boxed{2\le x\le 4}$,

$\boxed{0\le f^{-1}(x)\le 2\pi}$, $\boxed{-\frac{\sqrt{3}}{4}}$, $\boxed{-\frac{4}{\sqrt{3}}}$



Question 106 (****)

The function f is defined in a suitable domain of real numbers and satisfies

$$f(x)=\ln\left(\frac{\mathrm{e}-x}{\mathrm{e}+x}\right).$$

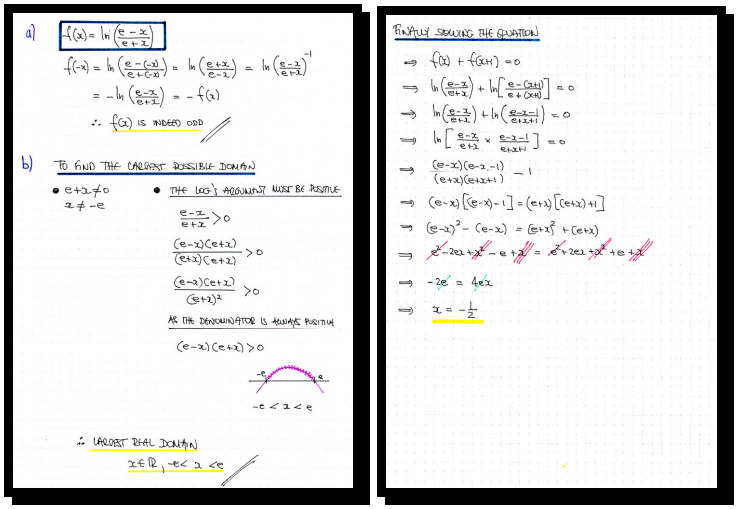
a) Show that f is odd.

b) Determine the largest possible domain of f.

c) Solve the equation

$$f(x)+f(x+1)=0.$$

$\boxed{}$, $\boxed{x\in\mathbb{R},\quad -\mathrm{e}<x<\mathrm{e}}$, $\boxed{x=-\frac{1}{2}}$



Question 107 (****)

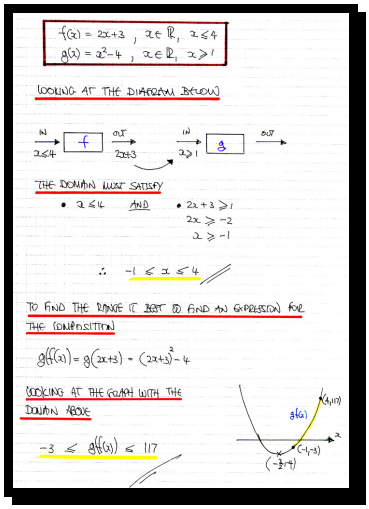
The functions f and g are defined by

$$f(x)=2x+3,\ x\in\mathbb{R},\ x\leq 4$$

$$g(x)=x^2-4,\ x\in\mathbb{R},\ x\geq 1.$$

Find the domain and range of $gf(x)$.

$\boxed{}$, $\boxed{-1\leq x\leq 4}$, $\boxed{-3\leq gf(x)\leq 117}$



Question 108 **(****)**

The function f satisfies

$$f(x)\equiv x^2-4x+1,\ x\in\mathbb{R},\ x>4.$$

a) Find an expression for $f^{-1}(x)$.

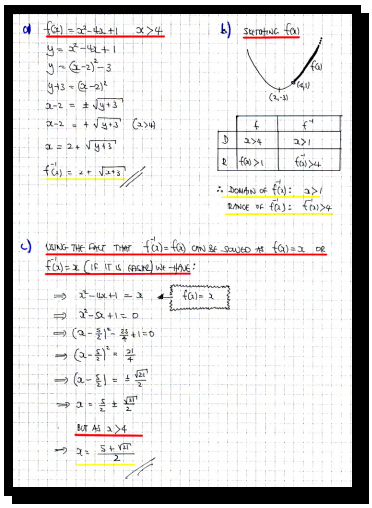
b) Determine the domain and range of $f^{-1}(x)$.

c) Solve the equation

$$f(x)=f^{-1}(x).$$

$\boxed{}$, $\boxed{f^{-1}(x)=2+\sqrt{x+3}}$, $\boxed{x\in\mathbb{R},\ x>1}$, $\boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)>4}$,

$\boxed{x=\dfrac{5+\sqrt{21}}{2}}$



Question 109 (****)

The functions $f(x)$ and $g(x)$ are given by

$$f(x)=3x-k,\ x\in\mathbb{R},\ x\geq 1,\ k\in\mathbb{R}$$

$$g(x)=2x^2+4,\ x\in\mathbb{R},\ x\geq 0$$

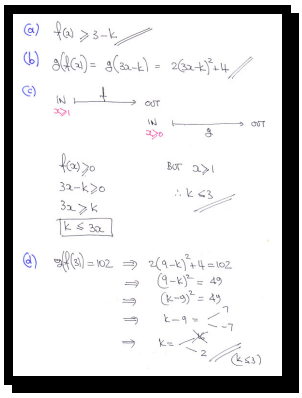
a) State the range of $f(x)$.

b) Find an expression for $gf(x)$ in terms of k.

c) Find the range of values of k which allows $gf(x)$ to be formed.

d) Find the value of k, given that $gf(3)=102$.

$\boxed{f(x)\in\mathbb{R},\ f(x)\geq 3-k}$, $\boxed{f(x)\in\mathbb{R},\ 2(3x-k)^2+4}$, $\boxed{k\leq 3}$, $\boxed{k=2,\ k\neq 16}$



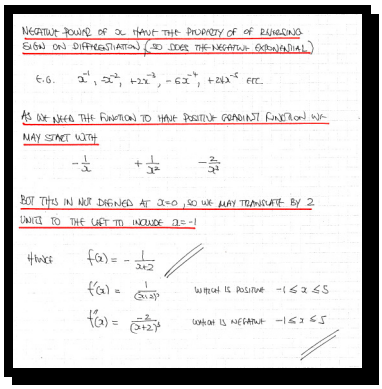
Question 110 (****)

The function $f(x)$ has domain $x \in \mathbb{R}$, $-1 \leq x \leq 5$.

It is further given that $f'(x) > 0$ and $f''(x) < 0$

Find a possible equation of $f(x)$, which ***does not*** contain exponentials.

$\square$, $\boxed{f(x) = \dfrac{1}{x+2}}$



Question 111 (**+)**

The function f is defined by

$$f(x)=\begin{cases} -x^2+8x-5, & x\in\mathbb{R},\ x\le 2 \\ x^2-2x+8, & x\in\mathbb{R},\ x>2 \end{cases}$$

a) Show that f …

i. … is not continuous.

ii. … is an increasing function.

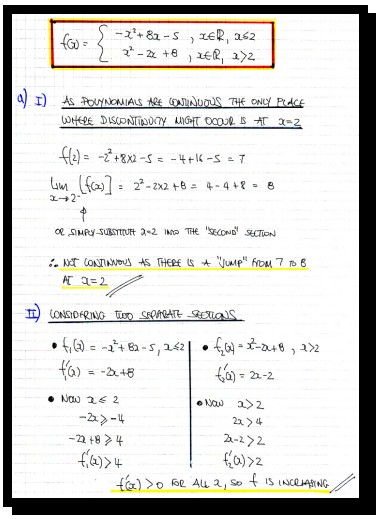
Let the set A be defined

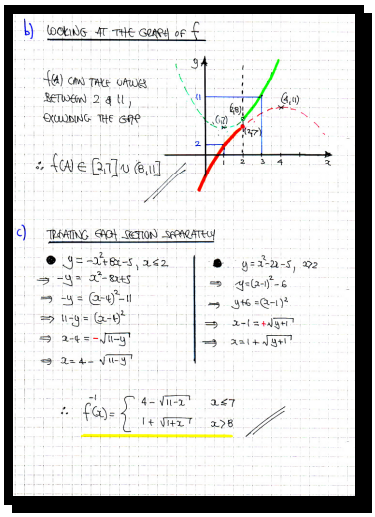
$$A=\{x\in\mathbb{R}:1\le x\le 3\}.$$

b) Determine the range of $f(A)$.

c) Find an expression for $f^{-1}(x)$, indicating clearly its domain.

$\square$, $\boxed{f(A)\in[2,7]\cup(8,11]}$, $\boxed{f^{-1}(x)=\begin{cases} 4-\sqrt{11-x} & x\le 7 \\ 1+\sqrt{1+x} & x>8 \end{cases}}$





Question 112 **(****+)**

The function $f(x)$ is defined

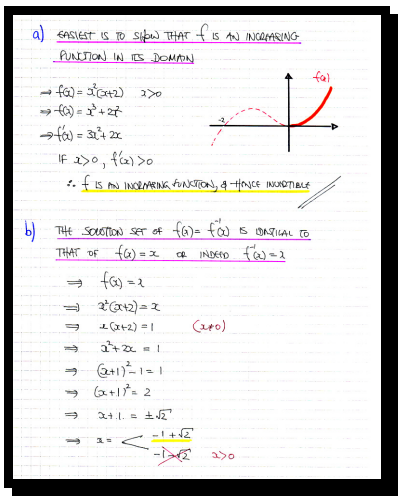
$$f(x)=x^2(x+2),\ x\in\mathbb{R},\ x>0.$$

a) Show that $f(x)$ is invertible.

b) Solve the equation

$$f(x)=f^{-1}(x).$$

$\square$, $\boxed{x=-1+\sqrt{2}}$



Question 113 **(****+)**

The function f is defined as

$$f:x\mapsto 6-\ln\left(x+3\right),\ x\in\mathbb{R},\ x\geq -2\,.$$

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x\xrightarrow{T_1}\ln\left(x+3\right)\xrightarrow{T_2}-\ln\left(x+3\right)\xrightarrow{T_3}-\ln\left(x+3\right)+6\,.$$

a) Describe geometrically T_1, T_2 and T_3, and hence sketch the graph of $f\left(x\right)$. Indicate clearly any intersections with the axes and the graph's starting point.

b) Find, in its simplest form, an expression for $f^{-1}\left(x\right)$, stating further the domain and range of $f^{-1}\left(x\right)$.

The function g satisfies

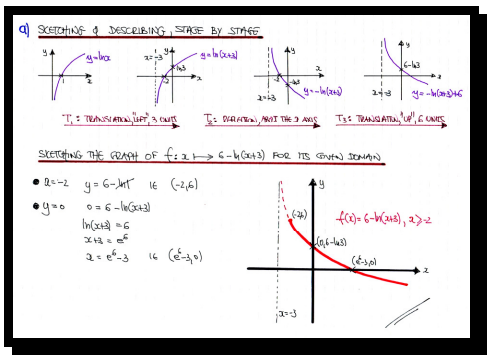
$$g:x\to \mathrm{e}^{x^2}-3,\quad x\in\mathbb{R}\,.$$

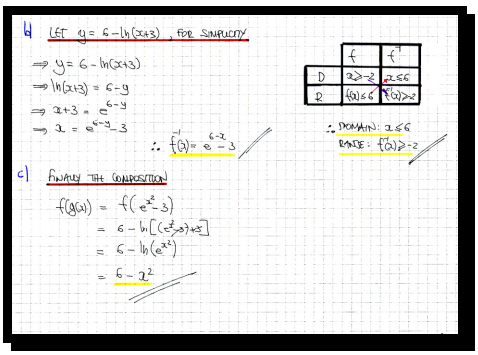
c) Find, in its simplest form, an expression for the composition $fg\left(x\right)$.

$\boxed{}$, $\boxed{T_1=\text{translation, "left", 3 units}}$, $\boxed{T_2=\text{reflection about } x\text{-axis}}$,

$\boxed{T_3=\text{translation, "up", 6 units}}$, $\boxed{\left(0,6-\ln 3\right)}$, $\boxed{\left(-3+\mathrm{e}^6,0\right)}$, $\boxed{\left(-2,6\right)}$,

$\boxed{f^{-1}:x\mapsto -3+\mathrm{e}^{6-x}}$, $\boxed{x\leq 6,\quad f^{-1}\left(x\right)\geq -2}$, $\boxed{fg:x\mapsto 6-x^2}$





Question 114 (**+)**

The functions $f(x)$ and $g(x)$ are defined by

$$f(x)=\frac{4}{x+3},\ x\in\mathbb{R},\ x>0$$

$$g(x)=9-2x^2,\ \ x\in\mathbb{R},\ \ x\geq 2\,.$$

a) Find, in its simplest form, the function $fg(x)$.

b) Find the domain of $fg(x)$.

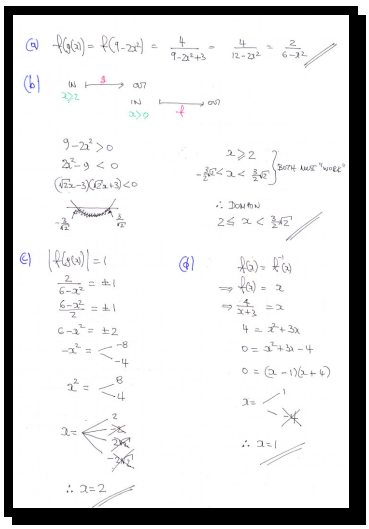
c) Find in exact form where appropriate the solutions of the equation

$$\left|fg(x)\right|=1\,.$$

d) Solve the equation

$$f(x)=f^{-1}(x)\,.$$

$$\boxed{fg(x)=\frac{2}{6-x^2}},\ \boxed{2\leq x<\frac{3\sqrt{2}}{2}},\ \boxed{x=2,\ x\neq\pm 2\sqrt{2},\ x\neq -2},\ \boxed{x=1,\ x\neq -4}$$



Question 115 **(****+)**

The functions f and g are defined by

$$f(x)=2x+1,\ x\in\mathbb{R},\ x\leq 5$$

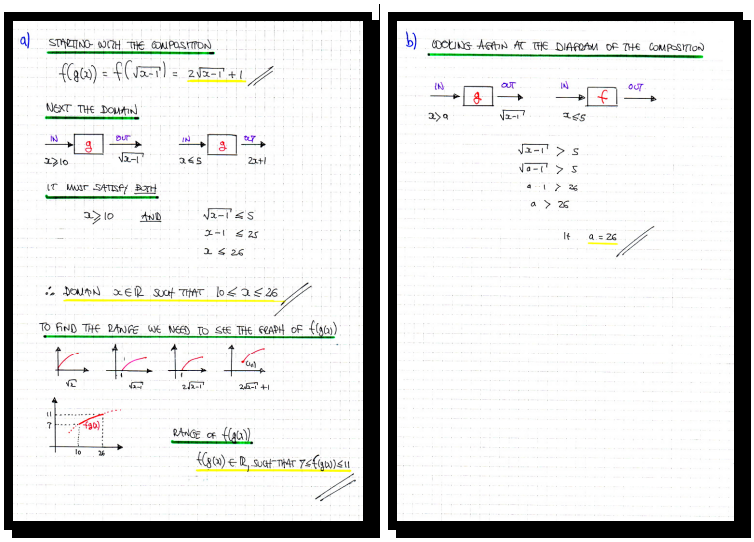
$$g(x)=\sqrt{x-1},\quad x\in\mathbb{R},\quad x\geq 10.$$

a) Find an expression for the composite function $fg(x)$, further stating its domain and range.

The domain of $g(x)$ is next changed to $x>a$.

b) Given that now $gf(x)$ **cannot** be formed, determine the smallest possible value of the constant a.

$\boxed{}$, $\boxed{fg(x)=1+2\sqrt{x-1}}$, $\boxed{x\in\mathbb{R},\ 10\leq x\leq 26}$, $\boxed{fg(x)\in\mathbb{R},\quad 7\leq fg(x)\leq 11}$, $\boxed{a=26}$



Question 116 (**+)**

The function f satisfies

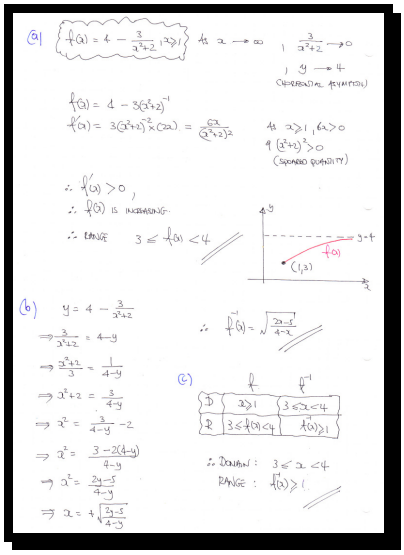
$$f(x)=4-\frac{3}{x^2+2},\ x\in\mathbb{R},\ x\geq 1.$$

a) By considering the horizontal asymptote of $f(x)$ and showing further it is an increasing function, find its range.

b) Find $f^{-1}(x)$, in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f(x)\in\mathbb{R},\ 3\leq f(x)<4}$, $\boxed{f^{-1}(x)=\sqrt{\frac{2x-5}{4-x}}}$, $\boxed{x\in\mathbb{R},\ 3\leq x<4}$, $\boxed{f(x)\in\mathbb{R},\ f(x)\geq 1}$



Question 117 (**+)**

The function f is given by

$$f(x)=1+\sqrt{x+1},\ x\in\mathbb{R},\ x\geq 0.$$

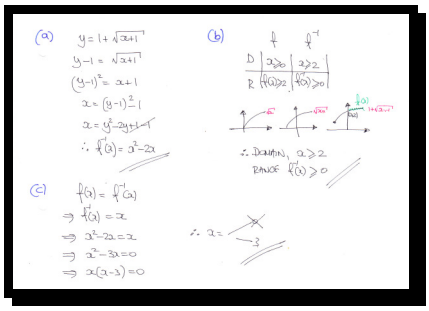
a) Find an expression for the inverse function $f^{-1}(x)$.

b) Determine the domain and range of $f^{-1}(x)$.

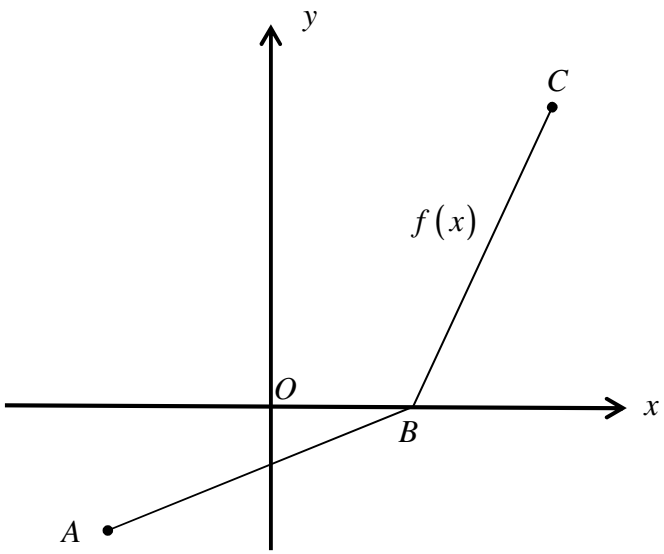
c) Solve the equation

$$f(x)=f^{-1}(x).$$

$\boxed{f^{-1}(x)=x^2-2x}$, $\boxed{x\in\mathbb{R},\ x\geq 0}$, $\boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)\geq 0}$, $\boxed{x=3}$



Question 118 **(****+)**



The above figure shows the graph of the function $f(x)$, consisting of two straight line segments starting at $A(-4,-4)$ and $B(6,8)$ meeting at the point $C(4,0)$.

a) State the range of $f(x)$.

b) Evaluate $ff(4)$.

c) Hence find $fff(5)$.

[continues overleaf]

[continued from overleaf]

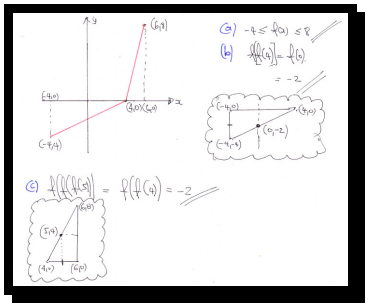
The inverse of $f(x)$ is $f^{-1}(x)$.

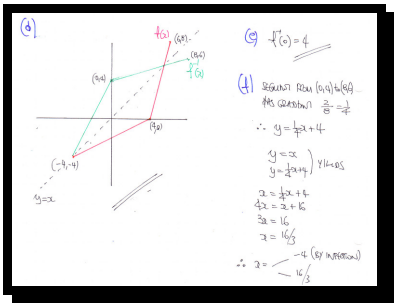
d) Sketch the graph of $f^{-1}(x)$.

e) State the value of $f^{-1}(x)$.

f) Solve the equation $f(x)=f^{-1}(x)$.

$\boxed{-4\le f(x)\le 6}$, $\boxed{ff(4)=-2}$, $\boxed{fff(5)=-2}$, $\boxed{f^{-1}(0)=4}$, $\boxed{x=-4,\frac{16}{3}}$





Question 119 **(****+)**

The functions f and g are given by

$$f(x)=5\mathrm{e}^{-x}+1,\ x\in\mathbb{R},\ x\geq 0$$

$$g(x)=2x+1,\ x\in\mathbb{R}.$$

a) Find …

i. … an expression for $gf(x)$.

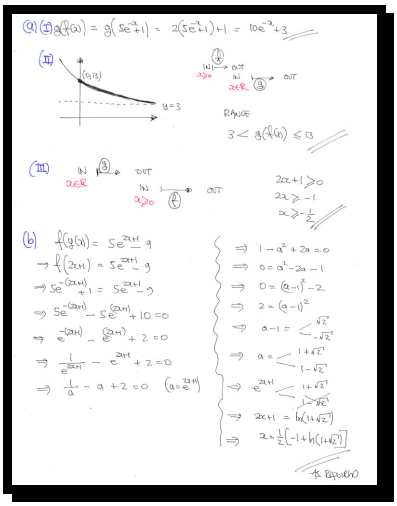
ii. … the range of $gf(x)$.

iii. … the domain of $fg(x)$.

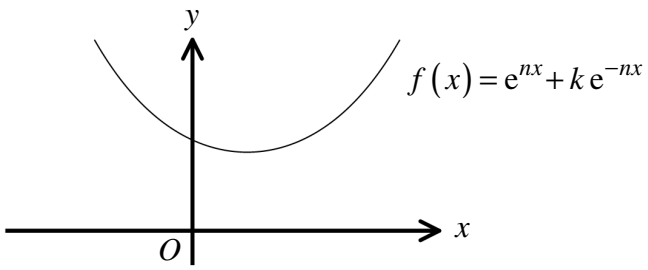
b) Show that the only solution of the equation $fg(x)=5\mathrm{e}^{2x+1}-9$ can be written as

$$x=\frac{1}{2}\left[-1+\ln\left(1+\sqrt{2}\right)\right].$$

$\boxed{gf(x)=10\mathrm{e}^{-x}+3}$, $\boxed{3<gf(x)\leq 13}$, $\boxed{x\geq -\frac{1}{2}}$



Question 120 (**+)**



The figure above shows the graph of the function with equation

$$f(x)=e^{nx}+k\,e^{-nx}\,,\ x\in\mathbb{R}\ ,\ k>1,\ n>0\,.$$

Find the range of $f(x)$ in exact form.

$\boxed{}$, $\boxed{f(x)\geq 2\sqrt{k}}$

LOCATE THE CO-ORDINATES OF THE MINIMUM BY DIFFERENTIATION

$f(x)=e^{nx}+ke^{-nx}$

$f'(x)=ne^{nx}-nke^{-nx}$

SOLVE $f'(x)=0$

$\Rightarrow ne^{nx}-nke^{-nx}=0$

$\Rightarrow e^{nx}-ke^{-nx}=0 \qquad n\neq 0$

$\Rightarrow e^{nx}=ke^{-nx}$

$\Rightarrow e^{nx}=\frac{k}{e^{nx}}$

$\Rightarrow (e^{nx})^2=k$

$\Rightarrow e^{nx}=+\sqrt{k} \qquad e^{nx}>0$

NEXT WE CAN FIND THE y CO-ORDINATE — WE DON'T REALLY x

$\Rightarrow y=e^{nx}+ke^{-nx}$

$\Rightarrow y=e^{nx}+\frac{k}{e^{nx}}$

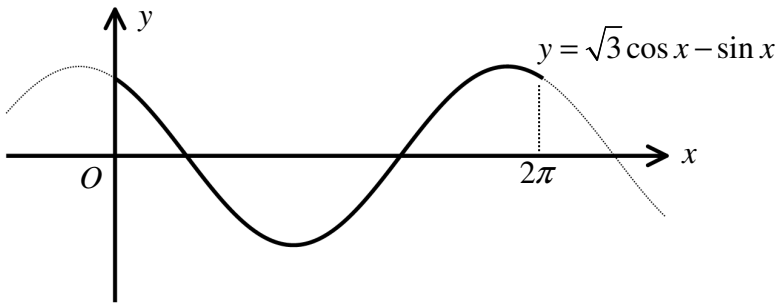
$\Rightarrow y=\sqrt{k}+\frac{k}{\sqrt{k}}$

$\Rightarrow y=\sqrt{k}+\sqrt{k}$

$\Rightarrow y=2\sqrt{k}$

$\therefore$ THE RANGE IS $f(x)\geq 2\sqrt{k}$

Question 121 (**+)**



The graph of $y=\sqrt{3}\cos x-\sin x$ for $0\le x\le 2\pi$ is shown in the figure above.

a) Express $\sqrt{3}\cos x-\sin x$ in the form $R\cos(x+\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

The function f is defined as

$$f(x)=\sqrt{3}\cos x-\sin x,\ x\in\mathbb{R},\ 0\le x\le 2\pi.$$

b) State the range of $f(x)$.

c) Explain why $f(x)$ does not have an inverse.

[continues overleaf]

[continued from overleaf]

The function $g(x)$ is defined as

$$g(x)=\sqrt{3}\cos x-\sin x,\ x\in\mathbb{R},\ 0<x_1\leq x\leq x_2<2\pi.$$

The ranges of $f(x)$ and $g(x)$ are the same and the inverse function $g^{-1}(x)$ exists.

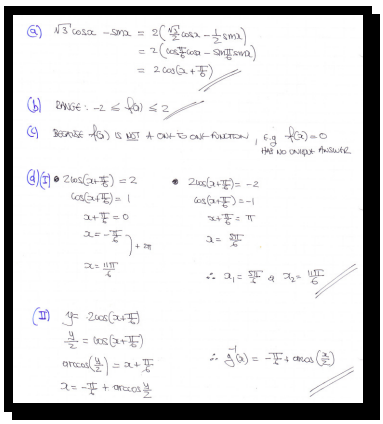
d) Find …

i. …the value of x_1 and the value of x_2.

ii. … an expression for $g^{-1}(x)$.

$\square$, $\boxed{\sqrt{3}\cos x-\sin x=2\cos\left(x+\frac{\pi}{6}\right)}$, $\boxed{2\leq f(x)\leq 2}$, $\boxed{x_1=\frac{5\pi}{6}}$, $\boxed{x_2=\frac{11\pi}{6}}$,

$$\boxed{g^{-1}(x)=-\frac{\pi}{6}+\arccos\left(\frac{1}{2}\right)}$$



Question 122 (**+)**

The function f is defined as

$$f(x)=\ln\left(\frac{1+x}{1-x}\right),\ |x|<1.$$

a) Show that $f(x)$ is an odd function.

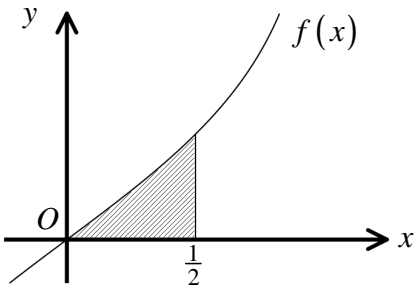
b) Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function.

c) Determine an expression for $f^{-1}(x)$.

d) Use the substitution $u=\mathrm{e}^x+1$ to find the exact value of

$$\int_0^{\ln 3} f^{-1}(x)\ dx.$$

The figure below shows part of the graph of $f(x)$.



e) Find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the straight line with equation $x=\frac{1}{2}$.

$\boxed{}$, $\boxed{f'(x)=\frac{2}{1-x^2}}$, $\boxed{f'(x)=\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}$, $\boxed{\ln\left(\frac{4}{3}\right)}$, $\boxed{\text{area}=\frac{3}{2}\ln 3-2\ln 2\approx 0.262}$

[solution overleaf]

a) USING THE STANDARD METHOD

$$f(-x)=\ln\left[\frac{1+(-x)}{1-(-x)}\right]=\ln\left(\frac{1-x}{1+x}\right)=\ln\left(\frac{1+x}{1-x}\right)^{-1}$$

$$=-\ln\left(\frac{1+x}{1-x}\right)=-f(x)$$

AS $f(-x)=-f(x)$ THE FUNCTION IS ODD

b) DIFFERENTIATING AFTER MANIPULATING

$$f(x)=\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}$$

$$f'(x)=\frac{(1-x)+(1+x)}{(1+x)(1-x)}$$

$$f'(x)=\frac{2}{1-x^2}$$

CHECKING: $f'(-x)$

$$f'(-x)=\frac{2}{1-(-x)^2}=\frac{2}{1-x^2}=f'(x)$$

AS $f'(-x)=f'(x)$, $f'(x)$ IS EVEN

c) WRITE $f(x)$ AS y & REARRANGE

$$y=\ln\left(\frac{1+x}{1-x}\right)\Rightarrow e^y=\frac{1+x}{1-x}$$

$$\Rightarrow e^y(1-x)=1+x$$

$$\Rightarrow e^y-xe^y=1+x$$

$$\Rightarrow e^y-1=x+xe^y$$

$$\Rightarrow x(1+e^y)=e^y-1$$

$$\Rightarrow x=\frac{e^y-1}{e^y+1}$$

$$\therefore f^{-1}(x)=\frac{e^x-1}{e^x+1}$$

d)

$$\int_0^{\ln 3} f^{-1}(x)\,dx=\int_0^{\ln 3}\frac{e^x-1}{e^x+1}\,dx$$

$u=e^x+1$

$\frac{du}{dx}=e^x$

$dx=\frac{du}{e^x}$

$e^x=u-1$

$x=0,\ u=2$

$x=\ln 3,\ u=4$

$$=\int_2^4\frac{e^x-1}{u}\left(\frac{du}{e^x}\right)=\int_2^4\frac{u-2}{u(u-1)}\,du$$

... PARTIAL FRACTIONS ...

$$\frac{u-2}{u(u-1)}\equiv\frac{A}{u}+\frac{B}{u-1}$$

$$u-2\equiv A(u-1)+Bu$$

• IF $u=0$: $-2=-A$, $A=2$

• IF $u=1$: $-1=B$, $B=-1$

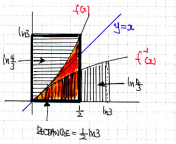
$$=\int_2^4\frac{2}{u}-\frac{1}{u-1}\,du=\left[2\ln|u|-\ln|u-1|\right]_2^4$$

$$=(2\ln 4-\ln 3)-(2\ln 2-\ln 1)=2\ln 4-\ln 3-2\ln 2$$

$$=\ln 16-\ln 3-\ln 4=\ln\left(\frac{16}{3\times 4}\right)=\ln\left(\frac{16}{12}\right)$$

$$=\ln\left(\frac{4}{3}\right)$$

c) LOOKING AT THE DIAGRAM BELOW



RECTANGLE $=\frac{1}{2}\ln 3$

$$\therefore \text{REQUIRED AREA}=\frac{1}{2}\ln 3-\ln\frac{4}{3}$$

$$=\frac{1}{2}\ln 3-(\ln 4-\ln 3)$$

$$=\frac{1}{2}\ln 3-2\ln 2+\ln 3$$

$$=\frac{3}{2}\ln 3-\ln 4$$

Question 123 (**+)**

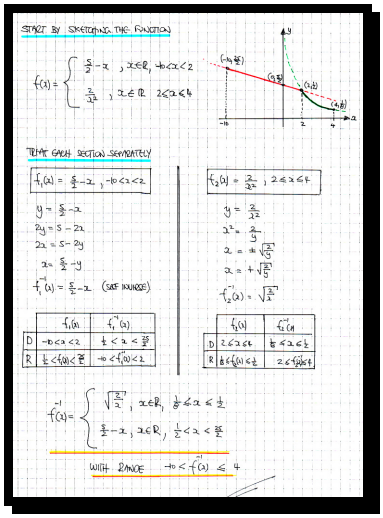
The piecewise continuous function f is defined by

$$f(x)\equiv\begin{cases}\dfrac{5}{2}-x, & x\in\mathbb{R},\ -10<x<2\\ \dfrac{2}{x^2}, & x\in\mathbb{R},\ 2\le x\le 4\end{cases}$$

Determine an expression, similar to the one above, for the inverse of f.

You must also give the range of the inverse of f.

$$\boxed{},\ \boxed{f^{-1}(x)\equiv\begin{cases}\sqrt{\dfrac{2}{x}}, & x\in\mathbb{R},\ \dfrac{1}{8}\le x\le\dfrac{1}{2}\\ \dfrac{5}{2}-x, & x\in\mathbb{R},\ \dfrac{1}{2}<x<\dfrac{25}{2}\end{cases}},\ \boxed{-10<f^{-1}(x)\le 4}$$



Question 124 **(****+)**

$$f(x)=\ln(4x-8),\ x\in\mathbb{R},\ x>2.$$

a) Find an expression for the inverse function, $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

The function g is defined as

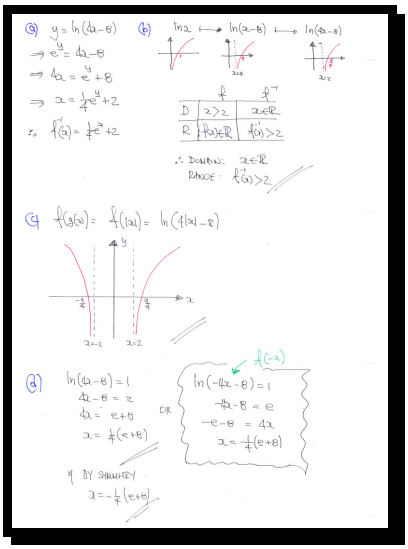
$$g(x)=|x|,\ x\in\mathbb{R}.$$

c) Sketch the graph of $fg(x)$, indicating clearly the equations of any asymptotes and the coordinates of the points where the graph meets the coordinate axes.

d) Hence solve the equation

$$fg(x)=1.$$

$$\boxed{f^{-1}(x)=2+\frac{1}{4}e^{x}},\ \boxed{x\in\mathbb{R},\ f^{-1}(x)>2},\ \boxed{\left(\frac{9}{4},0\right),\ \left(-\frac{9}{4},0\right),\ x=\pm2},\ \boxed{x=\pm\frac{1}{2}(e+8)}$$



Question 125 **(****+)**

Information about the functions f, g and h are given by

$$f(x) \equiv 1-\frac{1}{x},$$

$$g(x) \equiv ff(x),$$

$$fh(x) = \frac{x-3}{x-4}.$$

All the above functions are defined for all real numbers except for values of x for which the functions are undefined.

Find simplified expressions for …

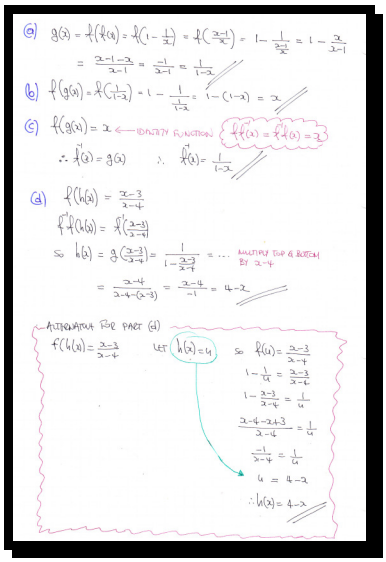
a) … $g(x)$.

b) … $fg(x)$.

c) … $f^{-1}(x)$.

d) … $h(x)$.

$$\boxed{g(x) = \frac{1}{1-x}}, \quad \boxed{fg(x) = x}, \quad \boxed{f^{-1}(x) = \frac{1}{1-x}}, \quad \boxed{h(x) = 4-x}$$



Question 126 (**+)**

The functions f and g are defined by

$$f(x) \equiv \sin x\,,\ x \in \mathbb{R}\,,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

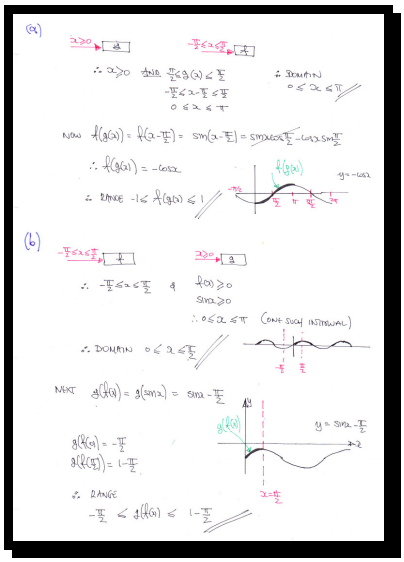
$$g(x) \equiv x - \frac{\pi}{2}\,,\ x \in \mathbb{R}\,,\ x \ge 0\,.$$

Determine, showing a clear method, the domain and range of the compositions

a) $fg(x)$.

b) $gf(x)$.

$$\boxed{0 \le x \le \pi}\,,\ \boxed{-1 \le fg(x) \le 1}\,,\ \boxed{0 \le x \le \frac{\pi}{2}}\,,\ \boxed{-\frac{\pi}{2} \le gf(x) \le 1 - \frac{\pi}{2}}$$



Question 127 (**+)**

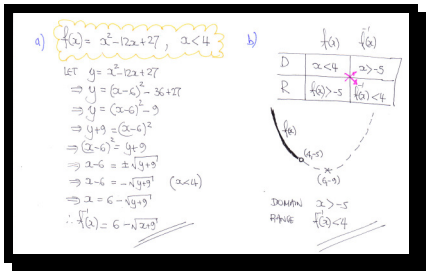
A function f is defined by

$$f(x)=x^2-12x+27,\ x\in\mathbb{R},\ x<4.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=6-\sqrt{x+9}},\ \boxed{x\geq -5},\ \boxed{f^{-1}(x)<4}$$



Question 128 **(****+)**

The functions f and g are defined by

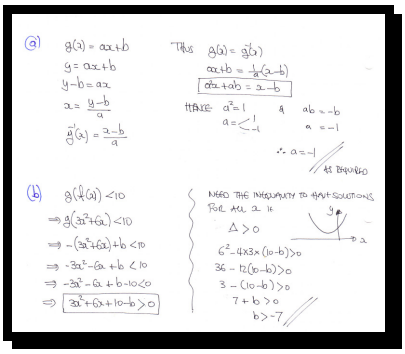
$$f(x) \equiv 3x^2 + 6x, \ x \in \mathbb{R},$$

$$g(x) \equiv ax + b, \ x \in \mathbb{R}.$$

a) Given that $g(x)$ is a self inverse function show that $a = -1$.

b) Given that $gf(x) < 10$ for all values of x, determine the range of values of b.

$\boxed{b > -7}$



Question 129 (**+)**

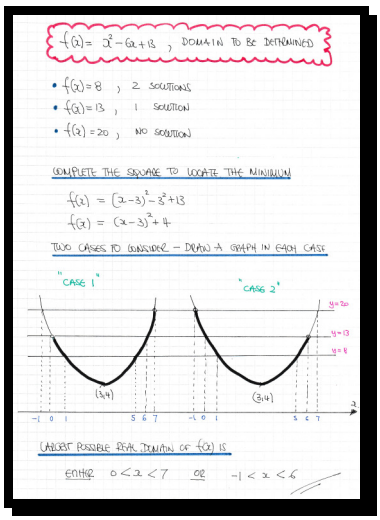
A function f is defined in a restricted real domain and has equation

$$f(x)\equiv x^2-6x+13.$$

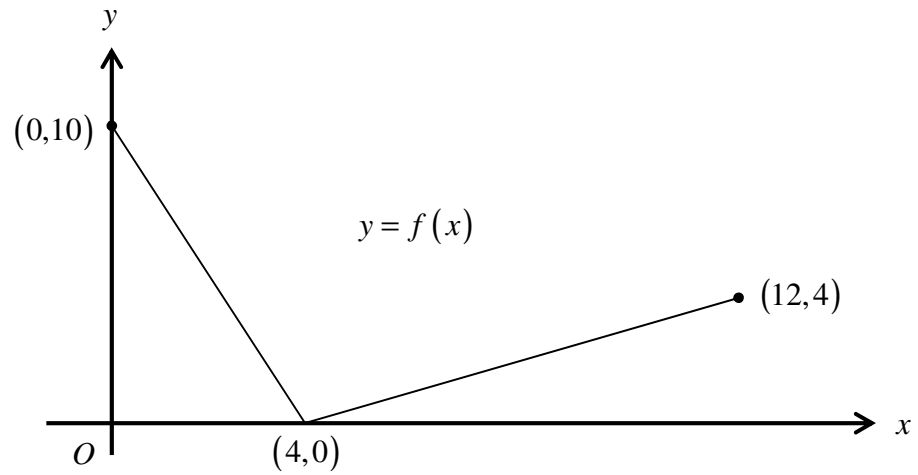
It is further given that the equations $f(x)=8$, $f(x)=13$ and $f(x)=20$ have 2 distinct solutions, 1 solution and no solutions, respectively.

Determine the possible domain of f.

$\boxed{}$, $\boxed{0<x<7 \quad \text{or} \quad -1<x<6}$



Question 130 (**+)**



The graph of the function $f(x)$ consists of two straight line segments joining the point $(0,10)$ to $(4,0)$ and the point $(12,4)$ to $(4,0)$, as shown in the figure above.

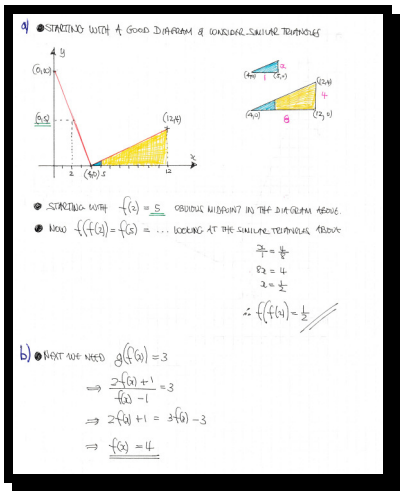
a) Find the value of $ff(2)$.

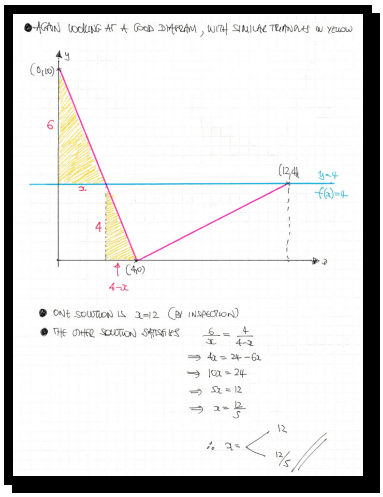
The function g is defined as

$$g(x) \equiv \frac{2x+1}{x-1}, \quad x \in \mathbb{R}, \quad x \neq 1.$$

b) Determine the solutions of the equation $gf(x) = 3$.

$\square$, $\boxed{ff(2) = \frac{1}{2}}$, $\boxed{x = \frac{12}{5}, 12}$





Question 131 **(****+)**

The function f is defined as

$$f(x)=3-\ln 4x,\ \ x\in\mathbb{R},\ x>0$$

a) Determine, in exact form, the coordinates of the point where the graph of f crosses the x axis.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln 4x \xrightarrow{T_2} -\ln 4x \xrightarrow{T_3} 3-\ln 4x$$

b) Describe geometrically each of the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.
Indicate clearly any intersections with the coordinate axes.

The function g is defined by

$$g(x)=\mathrm{e}^{5-x},\ \ x\in\mathbb{R}.$$

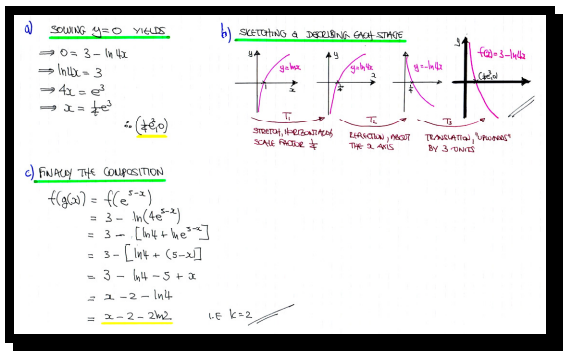
c) Show that

$$fg(x)=x-k-k\ln k\ ,$$

where k is a positive integer.

$\boxed{}$, $\boxed{\left(\frac{1}{4}\mathrm{e}^3,0\right)}$, $\boxed{T_1=\text{stretch in } x\text{, scale factor } \frac{1}{4}}$, $\boxed{T_2=\text{reflection in the } x\text{-axis}}$,

$\boxed{T_3=\text{translation, "up", 4 units}}$, $\boxed{k=2}$



Question 132 **(****+)**

The function f is defined as

$$f(x)=\ln(4-2x),\ x\in\mathbb{R},\ x<2.$$

a) Find in exact form the coordinates of the points where the graph of $f(x)$ crosses the coordinate axes.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+4) \xrightarrow{T_2} \ln(2x+4) \xrightarrow{T_3} \ln(-2x+4)$$

b) Describe geometrically the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.

Indicate clearly any asymptotes and coordinates of intersections with the axes.

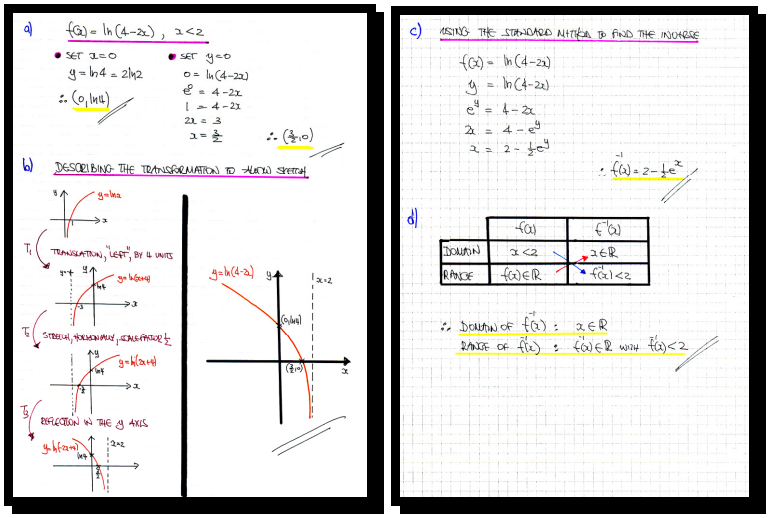
c) Find, an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

d) State the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{\left(\frac{3}{2},0\right),\ (0,\ln 4)}$, $\boxed{T_1=\text{translation, "left", 4 units}}$,

$\boxed{T_2=\text{stretch in } x\text{, scale factor } \frac{1}{2}}$, $\boxed{T_3=\text{reflection in the } y\text{-axis}}$, $\boxed{\text{asymptote } x=2}$,

$\boxed{f^{-1}(x)=2-\frac{1}{2}\mathrm{e}^x}$, $\boxed{x\in\mathbb{R}}$, $\boxed{f^{-1}(x)<2}$



Question 133 (**+)**

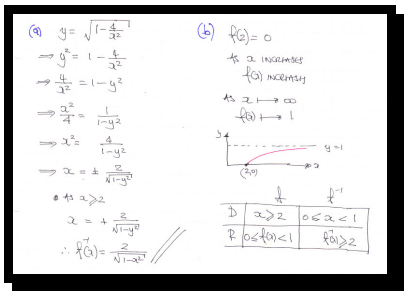
The function f is defined by

$$f(x)=\sqrt{1-\frac{4}{x^2}}\,,\ x\in\mathbb{R},\ x\geq 2\,.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Determine the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=\frac{2}{\sqrt{1-x^2}}},\quad \boxed{x\in\mathbb{R},\ 0\leq x<1},\quad \boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)\geq 2}$$



Question 134 (**+)**

The function f is defined on a suitable domain, so that the functions g and h satisfy the following relationships.

$$g(x)=\frac{1}{2}f(x)+\frac{1}{2}f(-x)$$

$$h(x)=\frac{1}{2}f(x)-\frac{1}{2}f(-x).$$

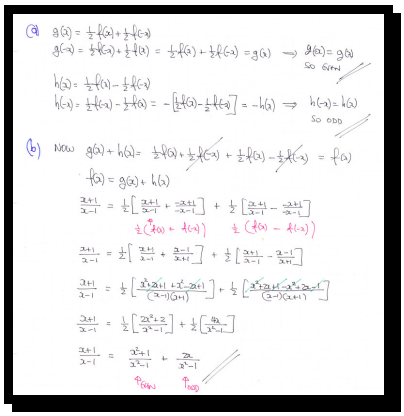
a) Show clearly that g is an even function and h is an odd function.

It is now given that

$$f(x)=\frac{x+1}{x-1},\ x\in\mathbb{R},\ x\neq\pm1.$$

b) Express $f(x)$ as the sum of an even and an odd function.

$$\boxed{f(x)=\frac{x^2+1}{x^2-1}+\frac{2x}{x^2-1}}$$



Question 135 **(****+)**

The functions f and g are defined by

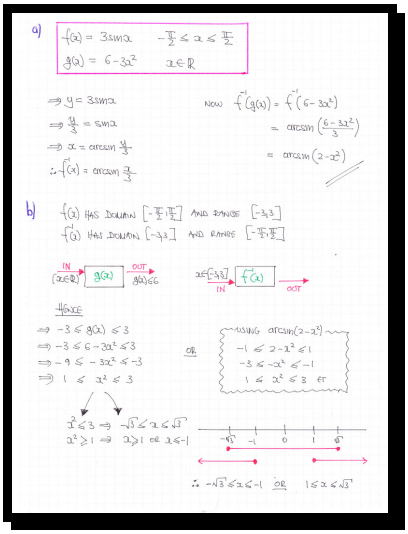
$$f(x) \equiv 3\sin x,\ x \in \mathbb{R},\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2,\ x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}g(x)$.

b) Determine the domain of $f^{-1}g(x)$.

$\square$, $\boxed{f^{-1}g(x) = \arcsin\left(2 - x^2\right)}$, $\boxed{-\sqrt{3} \le x \le -1 \ \text{ or } \ 1 \le x \le \sqrt{3}}$



Question 136 **(****+)**

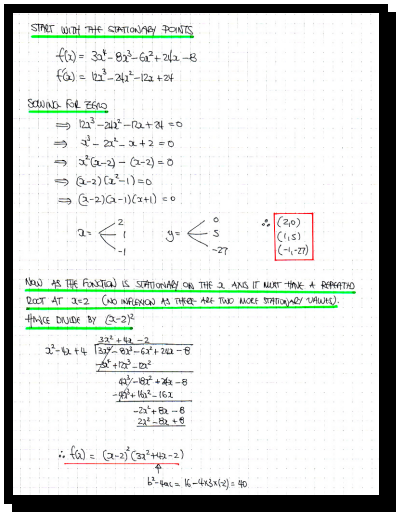
A function f is defined as

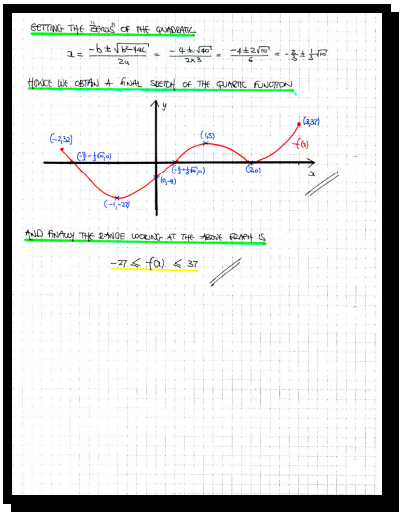
$$y=3x^4-8x^3-6x^2+24x-8\,,\ x\in\mathbb{R}\,,\ -2\le x\le 3\,.$$

Sketch the graph of f, and hence state its range.

The sketch must include the coordinates of any stationary points and any intersections with the coordinate axes.

$\boxed{}$, $\boxed{-27\le f(x)\le 37}$





Question 137 **(****+)**

The function f is defined as

$$f(x)\equiv\frac{x+1}{2x-1},\quad x\in\mathbb{R},\quad x\neq\frac{1}{2}.$$

The function g is suitably defined so that

$$f\big(g(x)\big)\equiv\frac{3x+2}{3x-5},\quad x\in\mathbb{R},\quad x\neq\frac{5}{3}.$$

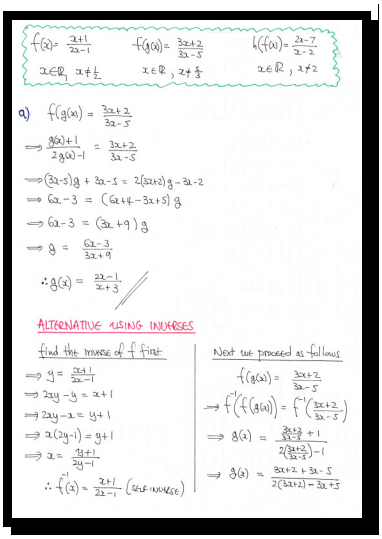
a) Determine an expression for $g(x)$.

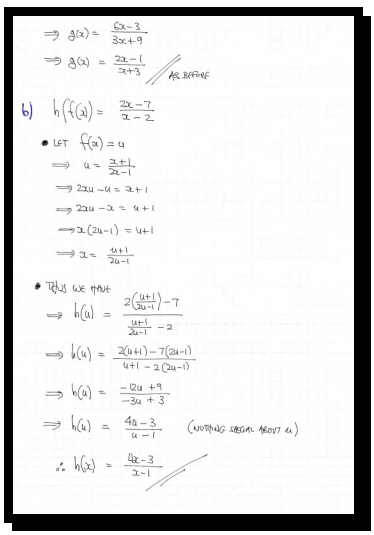
The function h is suitably defined so that

$$h\big(f(x)\big)\equiv\frac{2x-7}{x-2},\quad x\in\mathbb{R},\quad x\neq 2.$$

b) Determine an expression for $h(x)$.

$\square$, $\boxed{g(x)=\dfrac{2x-1}{x+3}}$, $\boxed{h(x)=\dfrac{4x-3}{x-1}}$





Question 138 **(*****)**

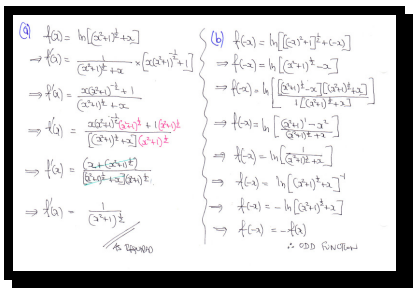
$$f(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right],\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\dfrac{1}{\sqrt{x^2+1}}$.

b) … $f(x)$ is an odd function.

proof

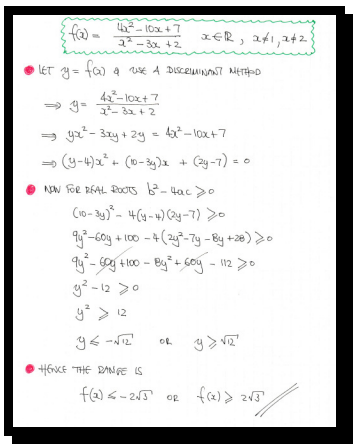


Question 139 **(*****)**

$$f(x)=\frac{4x^2-10x+7}{x^2-3x+2},\ x\in\mathbb{R},\ x\neq 1,\ x\neq 2.$$

Determine the range of $f(x)$.

$\square$, $\boxed{f(x)\leq -2\sqrt{3}\ \cup\ f(x)\geq 2\sqrt{3}}$



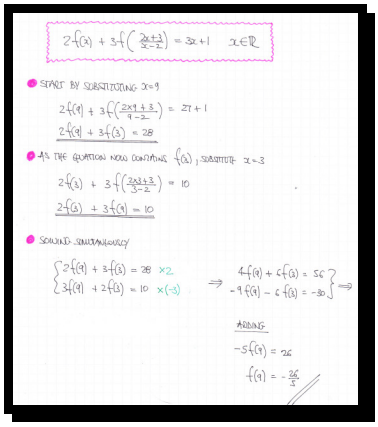
Question 140 (*****)

The function f satisfies

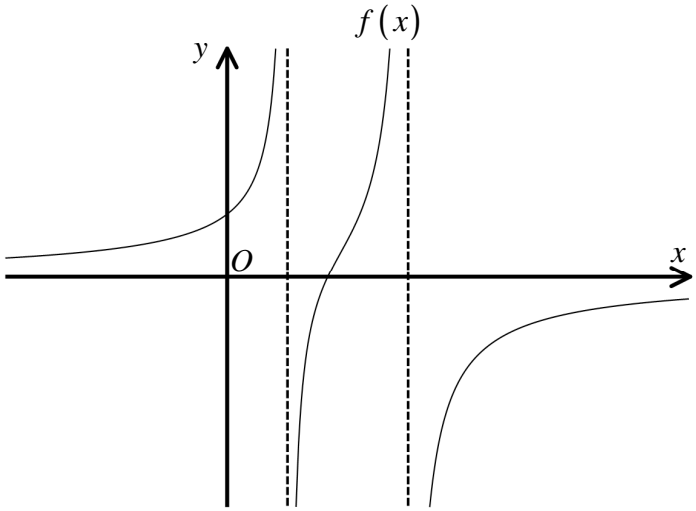
$$2f\left(x\right)+3f\left(\frac{2x+3}{x-2}\right)=3x+1,\ x\in\mathbb{R}.$$

Find the value of $f\left(9\right)$.

$\square$, $\boxed{f\left(9\right)=-\frac{26}{5}}$



Question 141 (*****)



The figure above shows the graph of

$$f(x)=\frac{5-3x}{(x-1)(x-3)},\ x\in\mathbb{R},\ x\neq 1,3.$$

a) State the equations of the vertical asymptotes of $f(x)$, which are shown as dotted lines in the figure.

[continues overleaf]

[continued from overleaf]

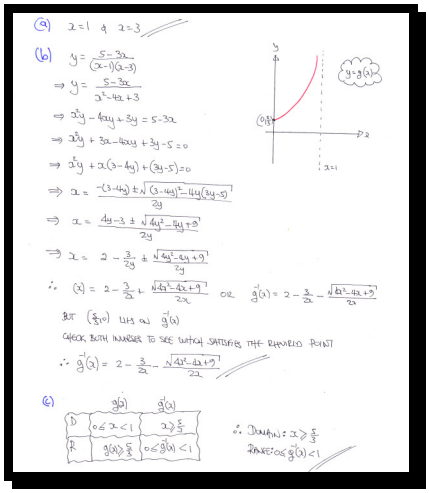
The function g is defined by as

$$g(x)=\frac{5-3x}{(x-1)(x-3)},\ x\in\mathbb{R},\ 0\leq x<1.$$

b) Find an expression for $g^{-1}(x)$.

c) State the domain and range of $g^{-1}(x)$.

$$\boxed{g^{-1}(x)=\frac{4x-3-\sqrt{4x^2-4x+9}}{2x}},\ \boxed{x\in\mathbb{R},\ \ x\geq\frac{5}{3}},\ \boxed{g^{-1}(x)\in\mathbb{R},\ \ 0\leq g^{-1}(x)\leq 1}$$



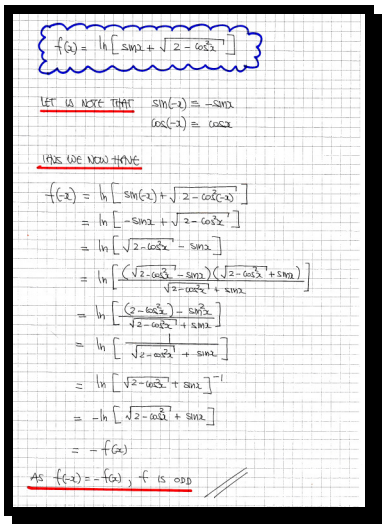
Question 142 (*****)

The function f is defined below.

$$f(x) \equiv \ln\left[\sin x + \sqrt{2-\cos^2 x}\right], \quad x \in \mathbb{R}.$$

Prove that f is odd.

[ ], [proof]



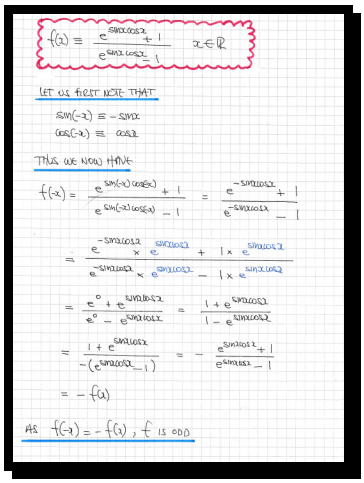
Question 143 **(*****)**

The function f is defined below.

$$f(x)\equiv\frac{e^{\sin x\cos x}+1}{e^{\sin x\cos x}-1},\ x\in\mathbb{R}.$$

Prove that f is odd.

[ ], [proof]



Question 144 **(*****)**

$$f(x)=\frac{e^x-1}{e^x+1},\ x\in\mathbb{R}.$$

a) Show clearly that …

i. … $f(-x)=-f(x)$.

ii. … $f'(x)=\dfrac{2e^x}{\left(e^x+1\right)^2}$.

b) Explain how the results of part **(a)** show that $f^{-1}(x)$ exists.

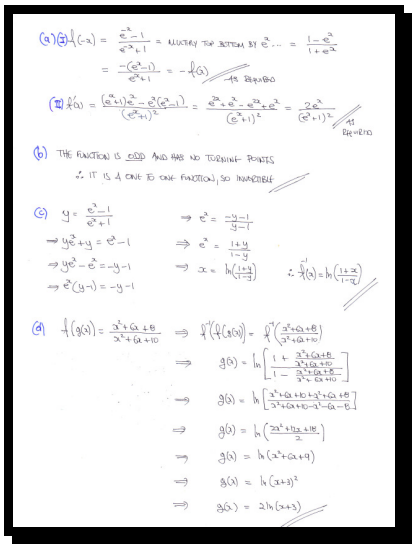
c) Find an expression for $f^{-1}(x)$.

The function $g(x)$ is defined in a suitable domain, so that

$$fg(x)=\frac{x^2+6x+8}{x^2+6x+10}.$$

d) Determine the equation of $g(x)$, in its simplest form.

$\square$, $\boxed{f^{-1}(x)=\ln\left(\dfrac{1+x}{1-x}\right)}$, $\boxed{g(x)=2\ln(x+3)}$



Question 145 (*****)

The function f satisfies the following three relationships

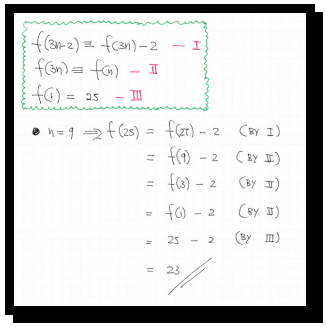
i. $f(3n-2) \equiv f(3n)-2,\ n \in \mathbb{N}$.

ii. $f(3n) \equiv f(n),\ n \in \mathbb{N}$.

iii. $f(1)=25$.

Determine the value of $f(25)$.

$\square$, $\boxed{f(25)=23}$



Question 146 (*****)

The function f is defined as

$$f(x)=-4+\sqrt{mx+12},\ x\in\mathbb{R},\ x\geq-\frac{m}{12},$$

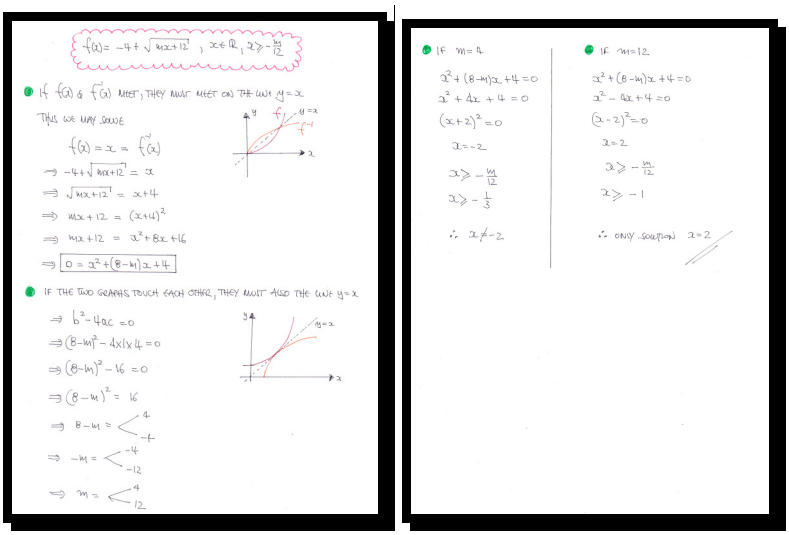
where m is a positive constant.

It is given that the graph of $f(x)$ and the graph of $f^{-1}(x)$ touch each other.

Solve the equation

$$f(x)=f^{-1}(x).$$

$\boxed{}$, $\boxed{x=\pm2}$



Question 147 **(*****)**

The functions f and g are defined by

$$f(x) \equiv \cos x, \ x \in \mathbb{R}, \ 0 \leq x \leq \pi$$

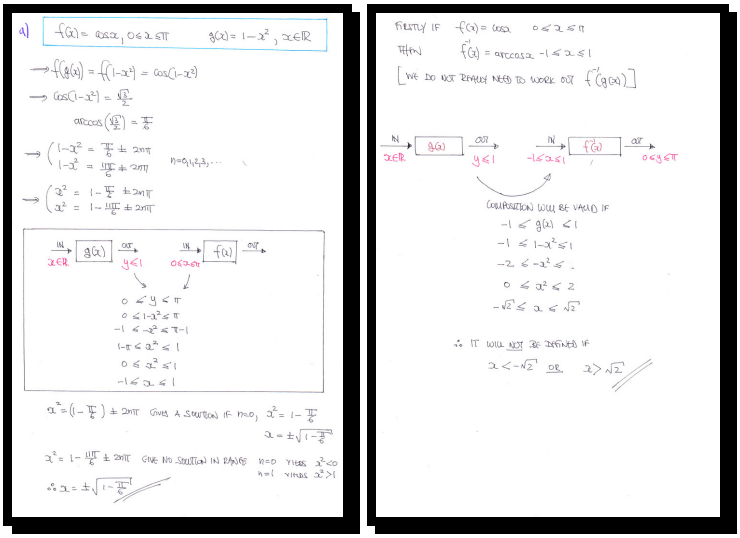
$$g(x) \equiv 1 - x^2, \ x \in \mathbb{R}.$$

a) Solve the equation

$$fg(x) = \frac{1}{2}.$$

b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

$$\boxed{}, \quad \boxed{x = \pm\sqrt{1 - \frac{\pi}{6}}}, \quad \boxed{x < -\sqrt{2} \ \text{ or } \ x > \sqrt{2}}$$



Question 148 (*****)

The function f is defined

$$f(x)=\sqrt{4-x}\,,\ x\in\mathbb{R},\ x\le 4\,.$$

It is further given that

$$fg(x)=\sqrt{4+2x}\,,\ x\in\mathbb{R},\ x\ge -2\,,$$

$$hf(x)=x-4\,,\ x\in\mathbb{R},\ x\le 4\,.$$

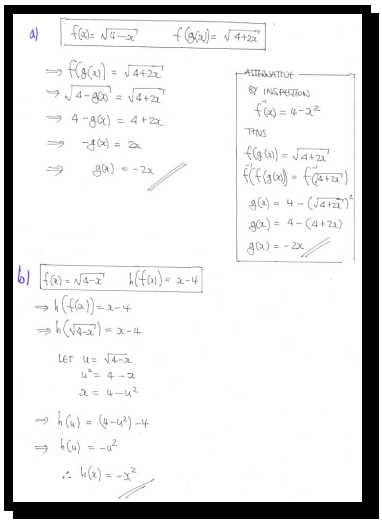
for some functions $g(x),\ x\in\mathbb{R}$ and $h(x),\ x\in\mathbb{R}$.

Find simplified expressions for …

a) … $g(x)$.

b) … $h(x)$.

$\boxed{}$, $\boxed{g(x)=-2x}$, $\boxed{h(x)=-x^2}$



Question 149 (*****)

The functions f and g are defined by

$$f(x)=\begin{cases} x^2, & x\in\mathbb{R},\ 0<x\leq 1 \\ 2-x, & x\in\mathbb{R},\ 1<x\leq 2 \end{cases}$$

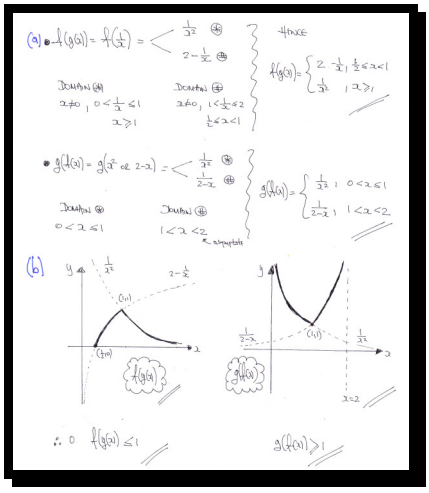
$$g(x)=\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

a) Find expressions for the function compositions $fg(x)$ and $gf(x)$, giving full descriptions of their domains.

b) Sketch the graphs of the function compositions $fg(x)$ and $gf(x)$, and hence state the ranges of $fg(x)$ and $gf(x)$.

$\square$, $fg(x)=\begin{cases} \dfrac{1}{x^2}, & x\in\mathbb{R},\ \dfrac{1}{2}\leq x<1 \\ \dfrac{1}{x^2}, & x\in\mathbb{R},\ x\geq 1 \end{cases}$, $gf(x)=\begin{cases} \dfrac{1}{x^2}, & x\in\mathbb{R},\ 0<x\leq 1 \\ \dfrac{1}{2-x}, & x\in\mathbb{R},\ 1<x<2 \end{cases}$

$fg(x)\in\mathbb{R},\ 0\leq fg(x)\leq 1$
$gf(x)\in\mathbb{R},\ gf(x)\geq 1$



Question 150 (*****)

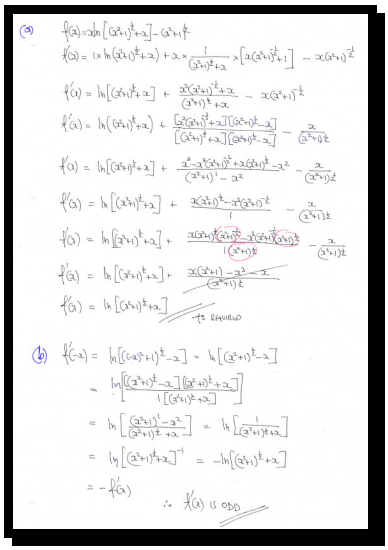
$$f(x)=x\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right]-\left(x^2+1\right)^{\frac{1}{2}},\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right]$.

b) … $f'(x)$ is an odd function.

, proof



Question 151 **(*****)**

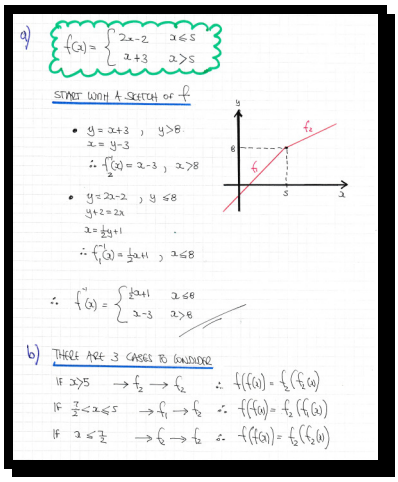
The piecewise continuous function f is given below.

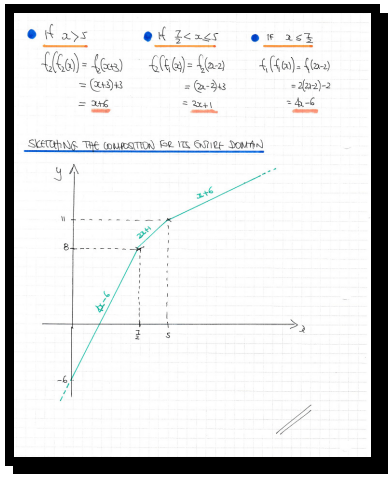
$$f(x)\equiv\begin{cases}2x-2 & x\le 5\\ x+3 & x>5\end{cases}$$

a) Determine an expression, in similar form to that of $f(x)$ above, for the inverse function, $f^{-1}(x)$.

b) Sketch a detailed graph for the composition $ff(x)$.

$\square$, $f^{-1}(x)\equiv\begin{cases}\frac{1}{2}x+1 & x\le 8\\ x-3 & x>8\end{cases}$





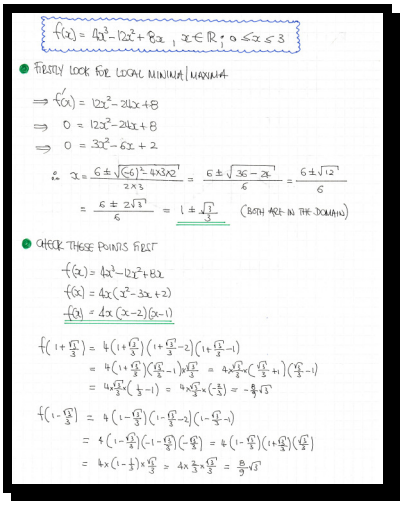
Question 152 (*****)

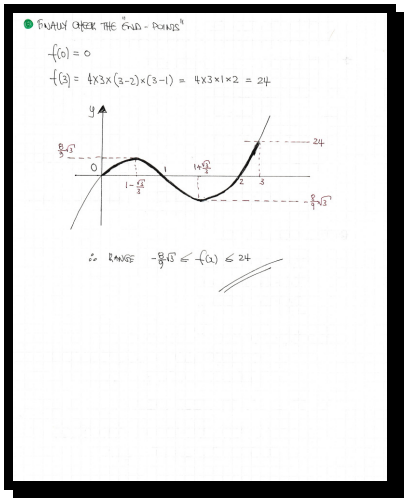
The function f is defined as

$$f(x) \equiv 4x^3 - 12x^2 + 8x, \quad x \in \mathbb{R}, \ 0 \le x \le 3.$$

Find the range of f, and hence sketch its graph, showing clearly the coordinates of any relevant points.

$\square$, $\boxed{-\frac{8}{9}\sqrt{3} \le f(x) \le 24}$





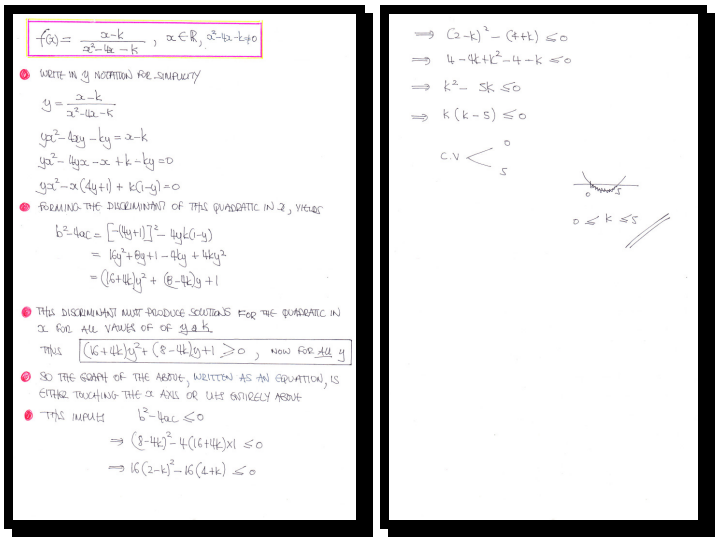
Question 153 (*****)

$$f(x)\equiv\frac{x-k}{x^2-4x-k},\quad x\in\mathbb{R},\quad x^2-4x-k\neq 0,$$

where k is a constant.

Given that the range of the function is all the real numbers determine the range of possible values of k.

$\boxed{}$, $\boxed{0\le k\le 5}$

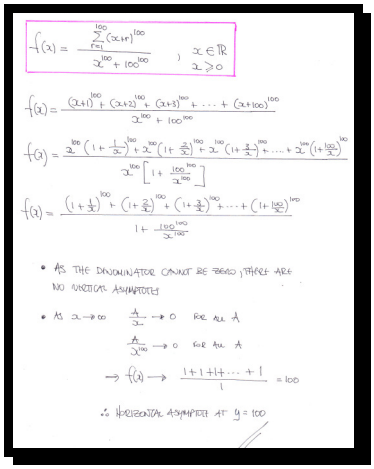


Question 154 (*****)

$$f(x) \equiv \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \ x \in \mathbb{R}, \ x \geq 0 .$$

Use a formal method to find the equations of any asymptotes of $f(x)$.

$\boxed{}$, $\boxed{y=100}$



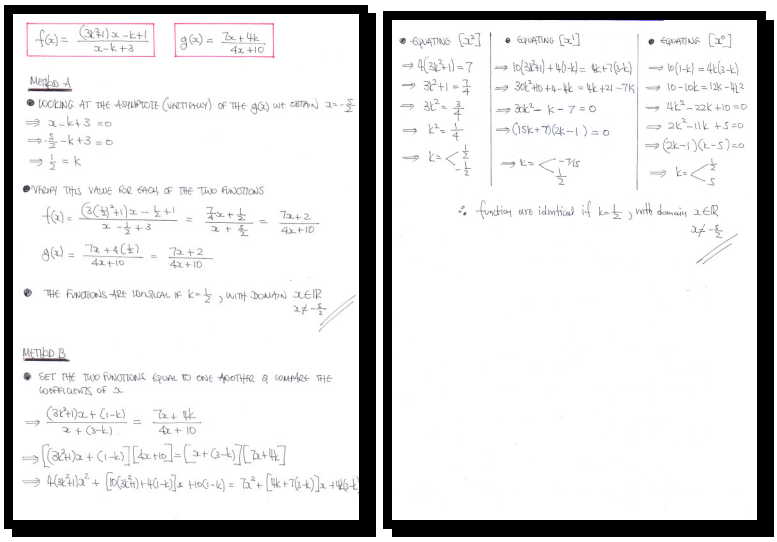
Question 155 (*****)

The functions f and g are defined in the largest possible real domain and their equations are given in terms of a constant k by

$$f(x)=\frac{\left(3k^2+1\right)x-k+1}{x-k+3} \qquad \text{and} \qquad g(x)=\frac{7x+4k}{4x+10}.$$

Given that f and g are identical, determine the possible value or values of k.

$\square$, $\boxed{k=\frac{1}{2}}$



Question 156 (*****)

The function f is defined by

$$f(x)=2-\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

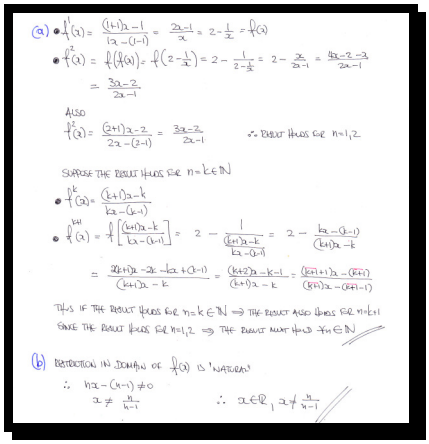
a) Prove that

$$f^n(x)=\frac{(n+1)x-n}{nx-(n-1)},\ n\geq 1,$$

where $f^n(x)$ denotes the n^{th} composition of $f(x)$ by itself.

b) State an expression for the domain of $f^n(x)$.

$\square$, $\boxed{x\in\mathbb{R},\ x\neq\frac{n-1}{n}}$



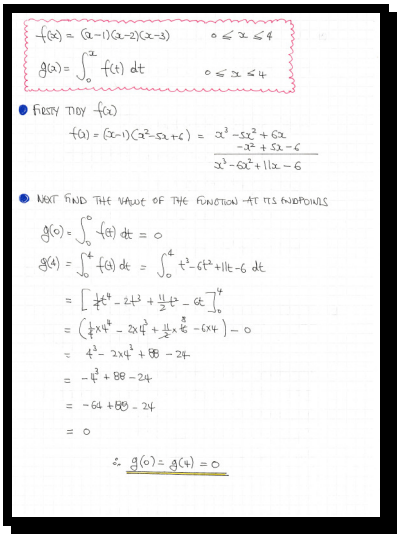
Question 157 (*****)

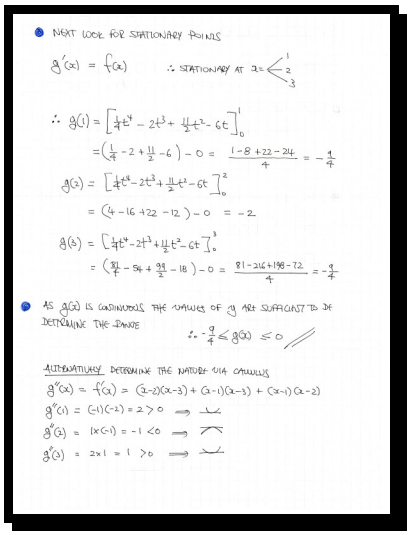
The real functions f and g have a common domain $0 \le x \le 4$, and defined as

$$f(x) \equiv (x-1)(x-2)(x-3) \qquad \text{and} \qquad g(x) \equiv \int_0^x f(t)\ dt\ .$$

Use a detailed algebraic method to determine the range of g.

$\square$, $\boxed{-\frac{9}{4} \le g(x) \le 0}$





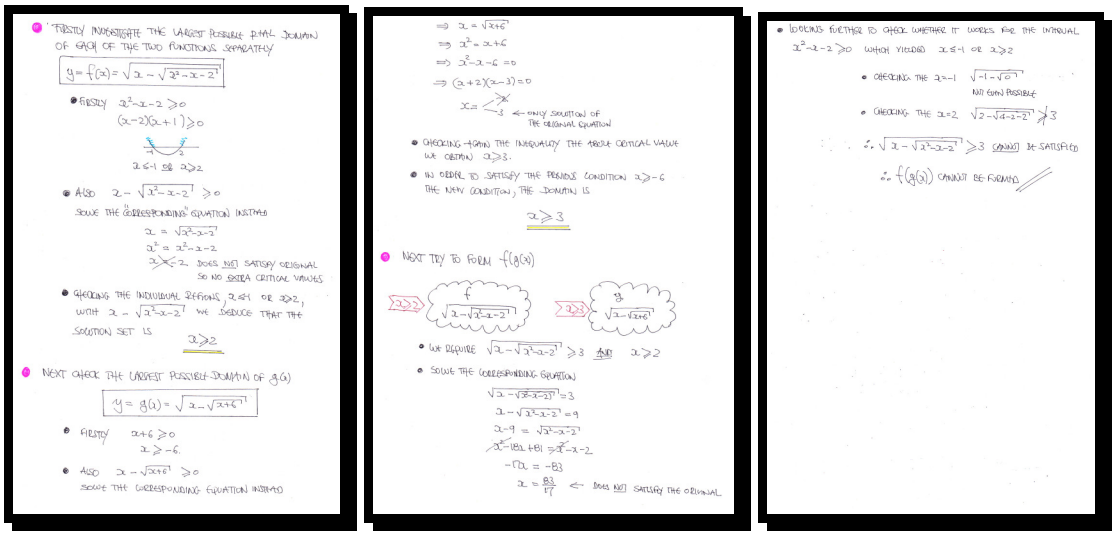
Question 158 (*****)

The functions f and g are each defined in the largest possible real number domain and given by

$$f(x)=\sqrt{x-\sqrt{x^2-x-2}} \qquad \text{and} \qquad g(x)=\sqrt{x-\sqrt{x+6}}\,.$$

By considering the domains of f and g, show that $fg(x)$ cannot be formed.

[ ], [proof]



Question 159 (*****)

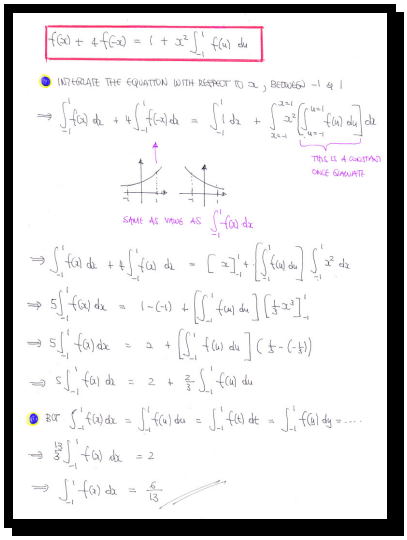
The function f, defined for all real numbers, satisfies the following relationship

$$f(x)+4f(-x)\equiv 1+x^2\int_{-1}^{1} f(u)\ du\,.$$

Determine as an exact fraction the value of

$$\int_{-1}^{1} f(x)\ dx\,.$$

$\square$, $\boxed{\frac{6}{13}}$



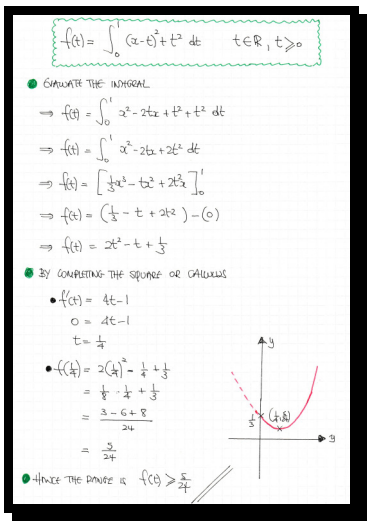
Question 160 (*****)

The function $y=f\left(t\right)$ is defined by the integral

$$f\left(t\right)\equiv\int_0^1\left(x-t\right)^2+t^2\ \ dx\,,\quad t\in\mathbb{R}\,,\ \ t\geq 0\,.$$

Determine the range of y.

$\square$, $\boxed{f\left(t\right)\geq\frac{5}{24}}$



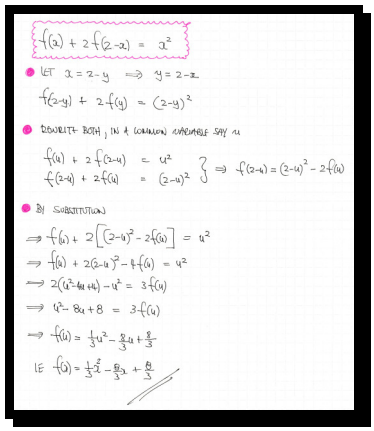
Question 161 (*****)

The function $y=f\left(x\right)$, $x\in\mathbb{R}$ satisfies

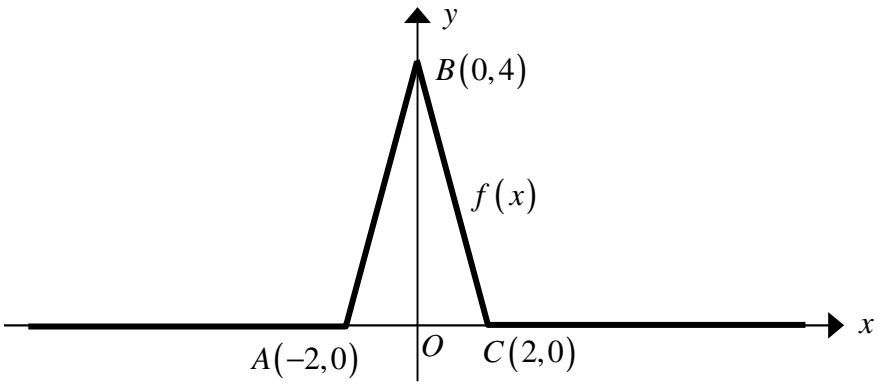
$$f\left(x\right)+2f\left(2-x\right)=x^2,\quad t\in\mathbb{R},\ t\geq 0.$$

Determine a simplified expression for $y=f\left(x\right)$.

$\square$, $\boxed{f\left(x\right)=\frac{1}{3}x^2-\frac{8}{3}x+\frac{8}{3}}$



Question 162 **(*****)**



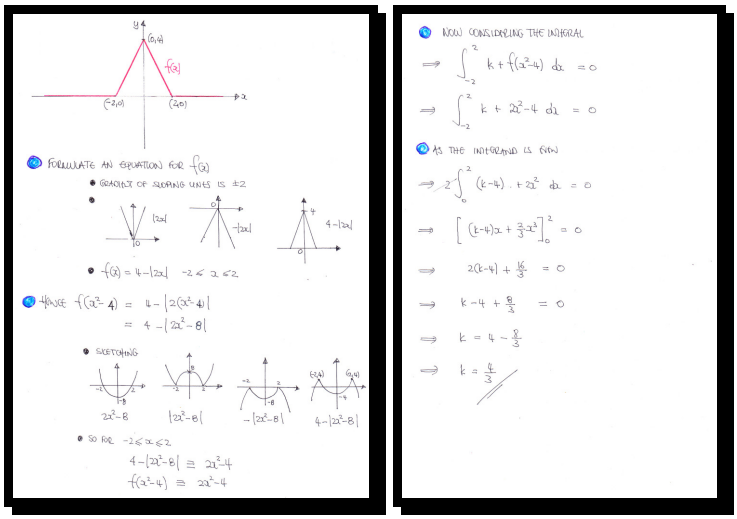
The figure above shows the graph of the function $f(x)$, consisting entirely of straight line sections. The coordinates of the joints of these straight line sections which make up the graph of $f(x)$ are also marked in the figure.

Given further that

$$\int_{-2}^{2} k + f\left(x^2-4\right)\,dx \;=\; 0\,,$$

determine as an exact fraction the value of the constant k.

$\square$, $\boxed{k=\frac{4}{3}}$



Question 163 (*****)

$$f(x)\equiv\frac{1}{k}\left(x^2-1\right)\left(x^2-9\right),\ x\in\mathbb{R},\ k\in\mathbb{N}.$$

Determine the solution interval (n,k), $n\in\mathbb{N}$, so that the equation

$$\left|f(x)\right|=n,$$

has exactly n distinct real roots.

$\boxed{}$, $\boxed{(n,k)=(8,1)=(6,2)=(4,4)=(5,2)=(6,2)=(7,2)}$

