EXPONENTIALS
&
LOGARITHMS
EXAM QUESTIONS

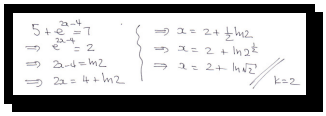
Question 1 (**)

Solve the equation

$$5+\mathrm{e}^{2x-4}=7$$

giving the answer in the form $k+\ln\sqrt{k}$, where k is an integer.

$\boxed{k=2}$



Question 2 (**)

A new antibiotic is tested by spraying it on a lab dish covered in bacteria.

Initially 12000 bacteria were placed on the dish and 24 hours later this number has fallen to 2000.

The number of bacteria N on this lab dish reduces according to the equation

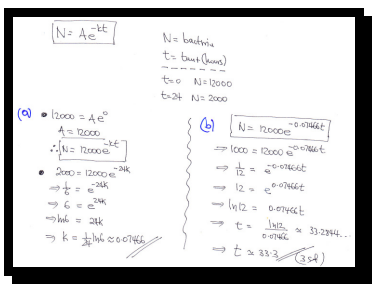
$$N=A\mathrm{e}^{-kt}\,,\ t\geq 0\,,$$

where t is the time in hours since the bacteria were first placed on the dish and, A and k are positive constants.

a) Show that $k=0.07466$, correct to 4 significant figures.

b) Find the value of t when the bacteria will reach 1000.

$\boxed{}$, $\boxed{t\approx 33.3}$



Question 3 (**)

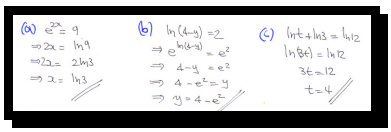
Find, in exact form where appropriate, the solution of each of the following equations.

a) $\mathrm{e}^{2x}=9$

b) $\ln\left(4-y\right)=2$

c) $\ln t+\ln 3=\ln 12$

$\boxed{}$, $\boxed{x=\ln 3}$, $\boxed{y=4-\mathrm{e}^2\approx -3.39}$, $\boxed{t=4}$



Question 4 (**)

A preservation programme, for elephants in Africa, was introduced 8 years ago. The elephants were then released to the wild. Let t be the number of years since the start of the programme.

The population of elephants P, is given by

$$P=400\,\mathrm{e}^{\frac{1}{12}(t-8)},\ t\geq 0.$$

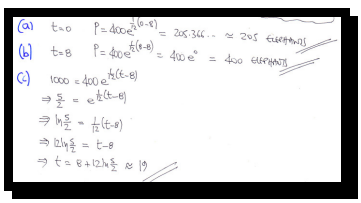
Assuming that P can be treated as a continuous variable, find …

a) … the number of elephants when the programme started.

b) … the number of elephants released to the wild.

c) … the value of t when the number of elephants will reach 1000.

$\boxed{}$, $\boxed{205}$, $\boxed{400}$, $\boxed{t=19}$



Question 5 (**)

The value $£V$ of a certain model of car, t years after it was purchased, is given by

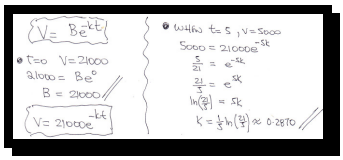
$$V = B\mathrm{e}^{-kt}\ ,\ t \geq 0$$

where B and k are positive constants.

The value of the car when new was $£21000$ and after five years it dropped to $£5000$.

Find the value of B and the value of k.

$$\boxed{B = 21000}\ ,\ \boxed{k \approx 0.2870}$$



Question 6 (**)

A curve has equation

$$y = A\mathrm{e}^{kx},$$

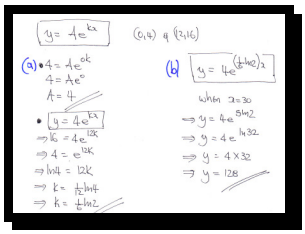
where A and k are non zero constants.

The curve passes through the points $(0,4)$ and $(12,16)$.

a) Find the value of A and the exact value of k.

b) Determine the value of y when $x = 30$

$\boxed{A = 4}$, $\boxed{k = \frac{1}{6}\ln 2}$, $\boxed{y = 128}$



Question 7 (**)

A cup of coffee is cooling down in a room.

The temperature T °C of the coffee, t minutes after it was made is modelled by

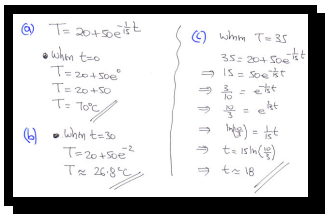
$$T = 20 + 50\mathrm{e}^{-\frac{t}{15}}, \ t \geq 0.$$

a) State the temperature of the coffee when it was first made.

b) Find the temperature of the coffee, after 30 minutes.

c) Calculate, to the nearest minute, the value of t when the temperature of the coffee has reached $35°\mathrm{C}$.

$\boxed{}$, $\boxed{T = 70}$, $\boxed{26.8°\mathrm{C}}$, $\boxed{t = 18}$



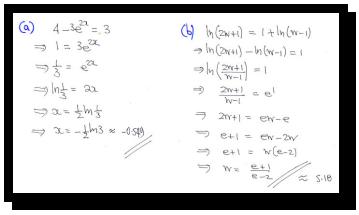
Question 8 (+)**

Find, in exact form where appropriate, the solution of each of the following equations.

a) $4 - 3\mathrm{e}^{2x} = 3$

b) $\ln(2w+1) = 1 + \ln(w-1)$

$\boxed{}$, $\boxed{x = -\frac{1}{2}\ln 3 \approx -0.549}$, $\boxed{w = \frac{\mathrm{e}+1}{\mathrm{e}-2} \approx 5.18}$



Question 9 **(**+)**

A microbiologist models the population of bacteria in culture by the equation

$$P = 1000 - 950\mathrm{e}^{-\frac{1}{2}t}\ ,\ t \geq 0$$

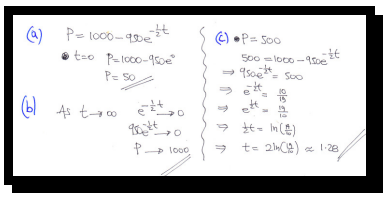
where P is the number of bacteria in time t hours.

a) Find the initial number of bacteria in the culture.

b) Show mathematically that the limiting value for P is 1000.

c) Find the value of t when $P = 500$.

$$\boxed{P_0 = 50}\ ,\ \boxed{t \approx 1.28}$$



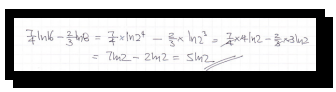
Question 10 **(**+)**

It is given that

$$\frac{7}{4}\ln 16 - \frac{2}{3}\ln 8 \equiv k\ln 2\ .$$

Determine the value of k.

$$\boxed{k = 5}$$



Question 11 (+)**

The function f is given by

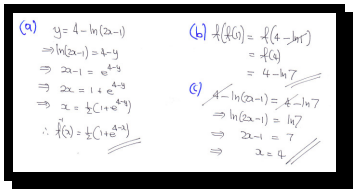
$$f(x)=4-\ln(2x-1),\ x\in\mathbb{R},\ x>\frac{1}{2}.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Find the exact value of $ff(1)$.

c) Hence, or otherwise, solve the equation $f(x)=ff(1)$.

$$\boxed{f^{-1}(x)=\frac{1}{2}\left(1+\mathrm{e}^{4-x}\right)},\ \boxed{ff(1)=4-\ln 7},\ \boxed{x=4}$$



Question 12 (***)

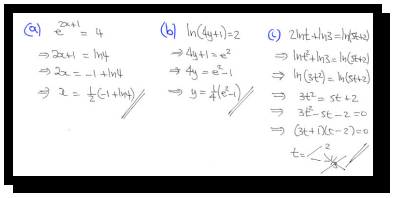
Find, in exact form where appropriate, the solution of each of the following equations.

a) $\mathrm{e}^{2x+1}=4$

b) $\ln(4y+1)=2$

c) $2\ln t+\ln 3=\ln(5t+2)$

$$\boxed{x=\frac{1}{2}(-1+\ln 4)},\ \boxed{y=\frac{1}{4}\left(\mathrm{e}^{2}-1\right)},\ \boxed{t=2}$$



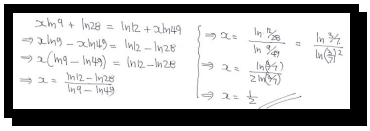
Question 13 (***)

Find, without the use of a calculating aid, the solution of the equation

$$x\ln 9+\ln 28=\ln 12+x\ln 49$$

giving the answer as an exact fraction.

$$\boxed{x=\frac{1}{2}}$$



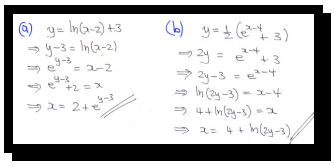
Question 14 (***)

Rearrange each of the following equations for x.

a) $y=\ln\left(x-2\right)+3, \quad x\in\mathbb{R},\ x>2$

b) $y=\frac{1}{2}\left(\mathrm{e}^{x-4}+3\right)$

$$\boxed{x=\mathrm{e}^{y-3}+2}, \quad \boxed{x=4+\ln\left(2y-3\right)}$$



Question 15 (***)

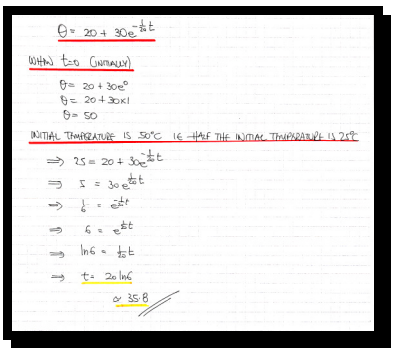
A liquid is cooling down and its temperature θ °C satisfies

$$\theta = 20 + 30\mathrm{e}^{-\frac{t}{20}}, \ t \geq 0$$

where t is the time in minutes after a given instant.

Find the value of t when the temperature of the liquid has reduced to half its initial temperature.

$\boxed{}$, $\boxed{t = 35.8}$



Question 16 (***)

A car tyre develops a puncture.

The tyre pressure P, measured in suitable units known as p.s.i., t minutes after the tyre got punctured is given by the expression

$$P = 8 + 32\mathrm{e}^{-kt}, \ t \geq 0,$$

where k is a positive constant.

a) State the tyre pressure when the tyre got punctured.

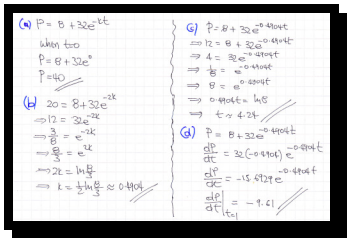
The tyre pressure halves 2 minutes after the puncture occurred.

b) Show that $k = 0.4904$, correct to 4 significant figures.

c) Calculate the time it takes for the tyre pressure to drop to 12 p.s.i.

d) Find the rate at which the pressure of the tyre is changing one minute after the puncture occurred.

$\boxed{}$, $\boxed{P = 40}$, $\boxed{t \approx 4.24}$, $\boxed{-9.61 \text{ p.s.i/min}}$



Question 17 (***)

The population P, in thousands, of a colony of rabbits in time t years after a certain instant, is given by

$$P = 5 + a\,\mathrm{e}^{-bt}\,,\ t \geq 0$$

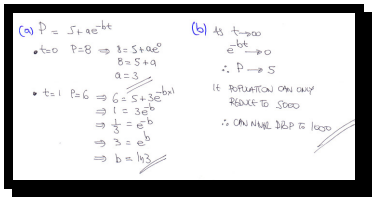
where a and b are positive constants.

It is given that the initial population is 8 thousands rabbits, and one year later this population has reduced by 2 thousands.

a) Find the value of a and the value of b.

b) Explain mathematically, why the population can never reach 1000, according to this model.

$\boxed{a = 3}$, $\boxed{b = \ln 3}$



Question 18 (***)

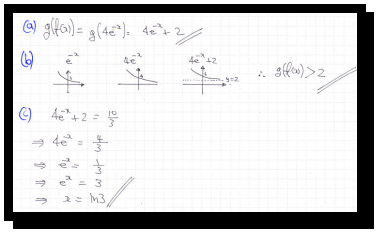
$$f(x)=4\mathrm{e}^{-x},\ x\in\mathbb{R}$$
$$g(x)=x+2,\ x\in\mathbb{R}$$

a) Find an expression for $gf(x)$.

b) State the range of $gf(x)$.

c) Solve the equation $gf(x)=\dfrac{10}{3}$.

$$\boxed{gf(x)=4\mathrm{e}^{-x}+2},\ \boxed{gf(x)>2},\ \boxed{x=\ln 3}$$

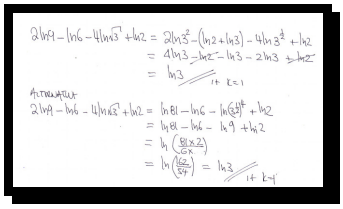


Question 19 (***)

Simplify the following expression, giving the final answer in the form $k\ln 3$, where k is an integer.

$$2\ln 9-\ln 6-4\ln\sqrt{3}+\ln 2$$

$$\boxed{k=1}$$



Question 20 (***)

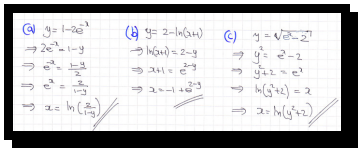
Rearrange each of the following equations for x.

a) $y=1-2\mathrm{e}^{-x}$

b) $y=2-\ln\left(x+1\right),\quad x>-1$

c) $y=\sqrt{\mathrm{e}^{x}-2},\quad x\geq \ln 2$

$$\boxed{x=-\ln\left(\frac{1-y}{2}\right)=\ln\left(\frac{2}{1-y}\right)},\quad \boxed{x=\mathrm{e}^{2-y}-1},\quad \boxed{x=\ln\left(y^2+2\right)}$$



Question 21 (***)

A hot drink is cooling down in a room.

The temperature T °C of the drink, t minutes after it was made is modelled by

$$T = 22 + 50\mathrm{e}^{-\frac{1}{8}t}, \quad t \geq 0.$$

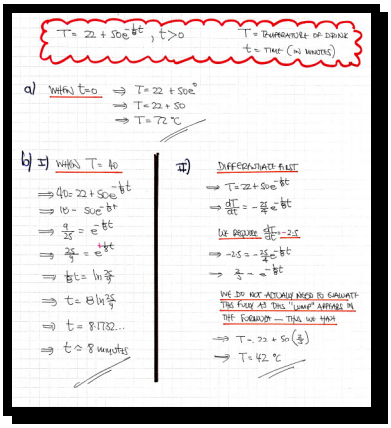
a) State the temperature of the drink when it was first made.

b) Assuming the drink is not consumed ...

i. ... calculate, to the nearest minute, the value of t when the temperature of the drink has reached $40\,°\mathrm{C}$.

ii. ... determine the value of T, when the drink is cooling at the rate of $2.5\,°\mathrm{C}$ per minute.

$\boxed{}$, $\boxed{T = 72}$, $\boxed{t \approx 8 \text{ min}}$, $\boxed{T = 42}$



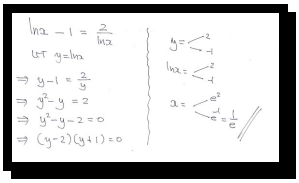
Question 22 (*+)**

Solve the equation

$$\ln x = \frac{2}{\ln x} + 1, \ x > 0,$$

giving the answers in exact form.

$$\boxed{x = \frac{1}{\mathrm{e}}, \mathrm{e}^2}$$



Question 23 (*+)**

A population P is decreasing according to the formula

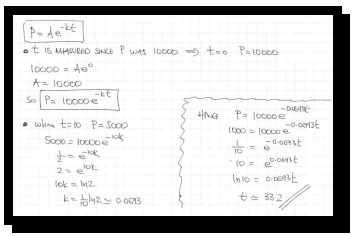
$$P = A\mathrm{e}^{-kt}, \ t \geq 0$$

where A and k are positive constants and t is the time in years since the population was 10000.

Ten years later the population has reduced to 5000.

Find the value of t when the population reaches 1000.

$$\boxed{t \approx 33.22}$$

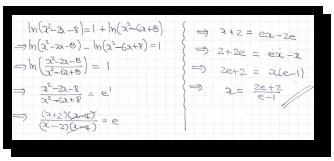


Question 24 (*+)**

Find in exact form the solution of the equation

$$\ln\left(x^2-2x-8\right)=1+\ln\left(x^2-6x+8\right).$$

$$\boxed{x=\frac{2\mathrm{e}+2}{\mathrm{e}-1}}$$



Question 25 (*+)**

The volume of water in a tank V m^3, t hours after midnight, is given by the equation

$$V=10+8\mathrm{e}^{-\frac{1}{12}t},\ t\geq 0.$$

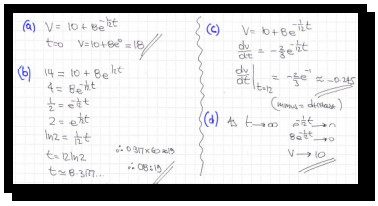
a) State the volume of water in the tank at midnight.

b) Find the time, using 24 hour clock notation, when the volume of the water in the tank is 14 m^3.

c) Determine the rate at which the volume of the water is changing at midday, explaining the significance of its sign.

d) State the limiting value of V.

$$\boxed{},\ \boxed{V=18},\ \boxed{08:19},\ \boxed{-\frac{2}{3\mathrm{e}}\approx -0.245,\ \text{decrease}},\ \boxed{t\to\infty,\ V\to 10}$$



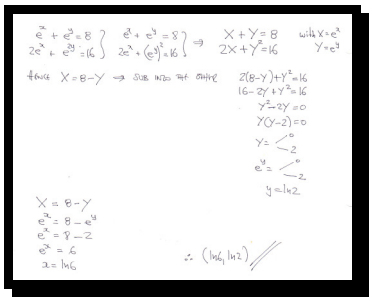
Question 26 (*+)**

Find the exact solution of the simultaneous equations

$$e^x + e^y = 8$$

$$2e^x + e^{2y} = 16 .$$

$\boxed{x = \ln 6}$, $\boxed{y = \ln 2}$



Question 27 (*+)**

A curve has equation

$$y = A\ln(Bx),\ x > 0$$

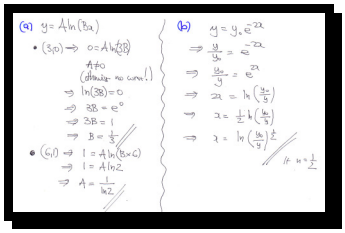
where A and B are positive constants.

a) Given the curve passes through the points with coordinates $(3,0)$ and $(6,1)$ find the exact values of A and B.

A different curve has equation $y = y_0 \mathrm{e}^{-2x}$, where y_0 is a constant.

b) Show clearly that $x = \ln\left(\dfrac{y_0}{y}\right)^n$, where n is a constant to be found.

$$\boxed{A = \frac{1}{\ln 2}},\ \boxed{B = \frac{1}{3}},\ \boxed{n = \frac{1}{2}}$$



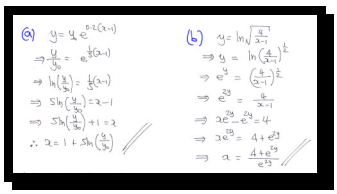
Question 28 (*+)**

Make x the subject in each of the following expressions.

a) $y = y_0 \, \mathrm{e}^{0.2(x-1)}$.

b) $y = \ln \sqrt{\dfrac{4}{x-1}}$, $x > 1$.

$$\boxed{x = 1 + 5\ln\left(\frac{y}{y_0}\right)}, \quad \boxed{x = 1 + 4e^{-2y} = \frac{\mathrm{e}^{2y} + 4}{\mathrm{e}^{2y}}}$$



Question 29 (*+)**

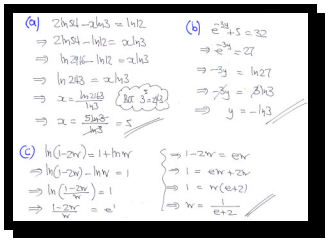
Determine, in exact simplified form where appropriate, the solution of each of the following equations.

a) $2\ln 54 - x\ln 3 = \ln 12$

b) $\mathrm{e}^{-3y} + 5 = 32$

c) $\ln(1-2w) = 1 + \ln w$

$$\boxed{x = 5}, \quad \boxed{y = -\ln 3}, \quad \boxed{w = \frac{1}{2+\mathrm{e}} \approx 0.212}$$

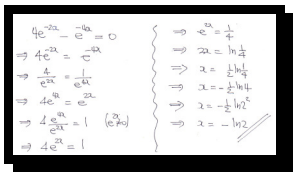


Question 30 (*+)**

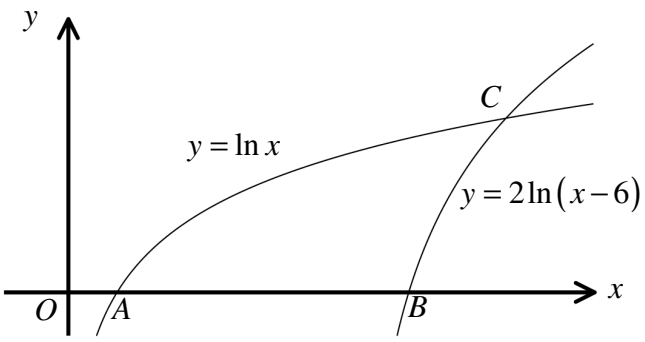
Show that $x=-\ln 2$ is the only real solution of the equation

$$4\mathrm{e}^{-2x}-\mathrm{e}^{-4x}=0\,.$$

proof



Question 31 (*+)**



The figure above shows the curves with equations

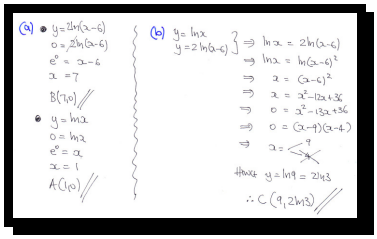
$$y=\ln x\,,\ x>0 \quad \text{and} \quad y=2\ln\left(x-6\right),\ x>6\,.$$

The curves cross the x axis at the points A and B respectively, and intersect each other at the point C.

a) Write down the coordinates of A and B.

b) Determine the exact coordinates of C.

$\boxed{A(1,0)}$, $\boxed{B(7,0)}$, $\boxed{B(9,2\ln 3)}$

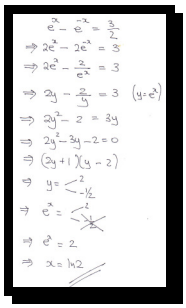


Question 32 (*+)**

Find the exact solution of the following equation

$$\mathrm{e}^{x}-\mathrm{e}^{-x}=\frac{3}{2}.$$

$\boxed{}$, $\boxed{x=\ln 2}$



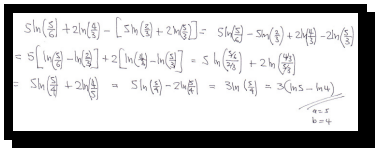
Question 33 (*+)**

Show clearly that

$$5\ln\left(\frac{5}{6}\right)+2\ln\left(\frac{4}{3}\right)-\left[5\ln\left(\frac{2}{3}\right)+2\ln\left(\frac{5}{3}\right)\right]\equiv 3\left(\ln a-\ln b\right)$$

where a and b are integers to be found.

$\boxed{a=5}$, $\boxed{b=4}$



Question 34 (*+)**

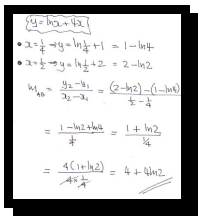
A curve has equation

$$y = \ln x + 4x, \ \ x > 0 .$$

The points A and B lie on this curve, where $x = \frac{1}{4}$ and $x = \frac{1}{2}$, respectively.

Show that the gradient of the straight line segment AB is $4 + 4\ln 2$.

proof



Question 35 (*+)**

The functions $f(x)$ and $g(x)$ are defined

$$f(x)=4+\ln x,\ x\in\mathbb{R},\ x>0$$

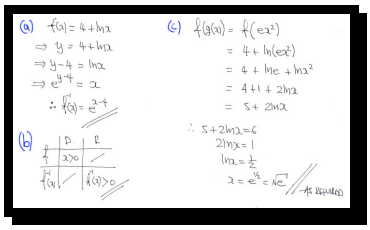
$$g(x)=\mathrm{e}x^2,\ x\in\mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Show that $x=\sqrt{\mathrm{e}}$ is the solution of the equation $fg(x)=6$.

$$\boxed{f^{-1}(x)=\mathrm{e}^{x-4}},\ \boxed{f^{-1}(x)>0}$$



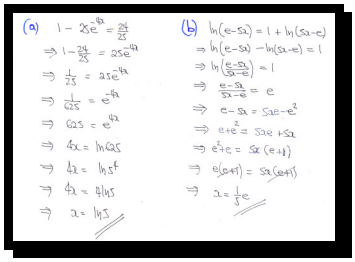
Question 36 (*+)**

Find, in exact simplified form, the solution of each of the following equations.

a) $1-25\mathrm{e}^{-4x}=\frac{24}{25}$.

b) $\ln\left(\mathrm{e}-5x\right)=1+\ln\left(5x-\mathrm{e}\right)$.

$\boxed{x=\ln 5}$, $\boxed{x=\frac{1}{5}\mathrm{e}}$



Question 37 **(***+)**

The value, £V, of a painting is modelled by the equation

$$V = 600 + 200\ln(2t+1), \ t \geq 0$$

where t is the number of years since the painting was purchased.

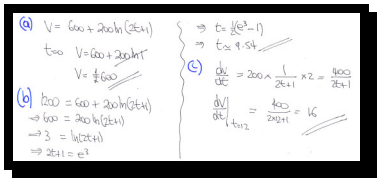
Determine…

a) … the value of the painting when it was first purchased.

b) … the value of t when the value of the painting doubles.

c) … the rate at which the value of the painting is increasing twelve years after it was purchased.

$\boxed{V = 600}$, $\boxed{t \approx 9.54}$, $\boxed{\text{£16 per year}}$

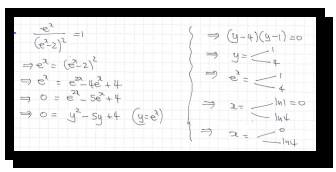


Question 38 **(***+)**

Find, in exact form where appropriate, the solutions of the equation

$$\frac{e^x}{\left(e^x-2\right)^2}=1.$$

$\boxed{x=0, \ x=\ln 4}$



Question 39 (*+)**

The function $f(x)$ is given by

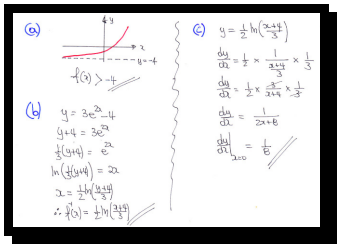
$$f(x)=3\mathrm{e}^{2x}-4,\ x\in\mathbb{R}.$$

a) State the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$.

c) Find the value of the gradient on $f^{-1}(x)$ at the point where $x=0$.

$$\boxed{f(x)>-4},\ \boxed{f^{-1}(x)=\frac{1}{2}\ln\left(\frac{x+4}{3}\right)},\ \boxed{\frac{1}{8}}$$



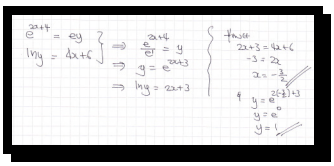
Question 40 (*+)**

Find the solution of the following simultaneous equations

$$\mathrm{e}^{2x+4}=\mathrm{e}\,y$$

$$\ln y=4x+6.$$

$$\boxed{\ },\ \boxed{x=-\frac{3}{2},\ y=1}$$



Question 41 (*+)**

The functions $f(x)$ and $g(x)$ are defined

$$f(x)=x^2-10x,\ \ x\in\mathbb{R}$$

$$g(x)=\mathrm{e}^x+5,\ \ x\in\mathbb{R}.$$

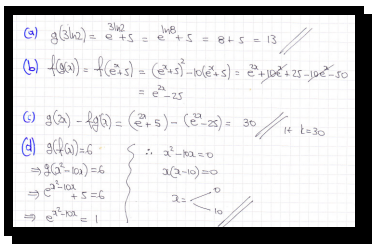
a) Find, showing all the steps of the calculation, the value of $g(3\ln 2)$.

b) Find, in its simplest form, an expression for $fg(x)$.

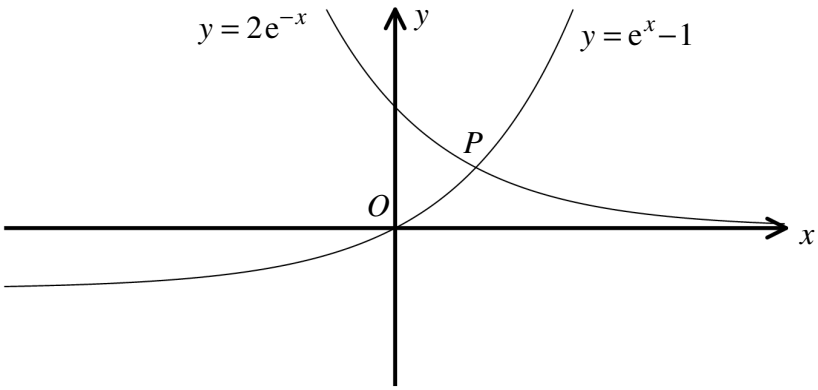
c) Show that $g(2x)-fg(x)=k$, stating the value of the constant k.

d) Solve the equation $gf(x)=6$.

$\boxed{g(3\ln 2)=13}$, $\boxed{fg(x)=\mathrm{e}^{2x}-25}$, $\boxed{k=30}$, $\boxed{x=0,10}$



Question 42 **(***+)**



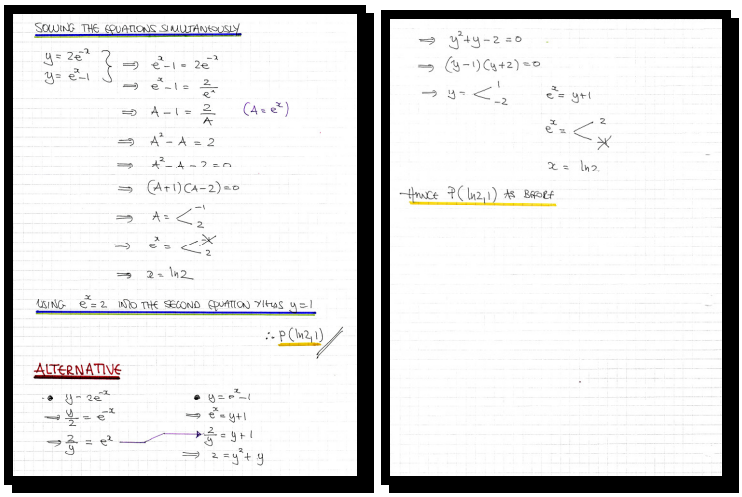
The figure above shows the graphs of the curves with equations

$$y = 2\mathrm{e}^{-x} \quad \text{and} \quad y = \mathrm{e}^{x} - 1 .$$

The two graphs intersect at the point P.

Determine the exact coordinates of P.

$\boxed{}$, $\boxed{P(\ln 2, 1)}$



Question 43 (*+)**

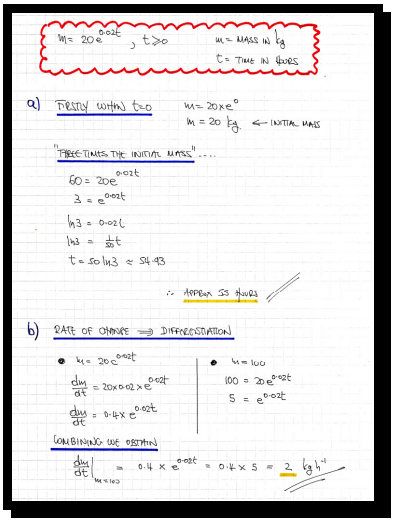
During a chemical process the mass of a substance, m kg, at time t hours grows exponentially according to the formula

$$m = 20\mathrm{e}^{0.02t}, \ t \geq 0.$$

a) Find the time taken for the substance to increase to three times its initial mass.

b) Calculate the rate of change of m, when $m = 100$.

$\boxed{}$, $\boxed{t = 50\ln 3 \approx 54.93 \text{ hours}}$, $\boxed{\left.\dfrac{dm}{dt}\right|_{m=100} = 2}$



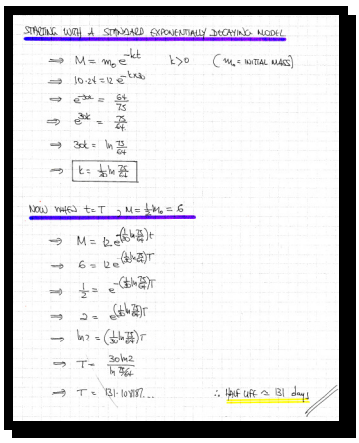
Question 44 (*+)**

The half life of a radioactive isotope is the time it takes for a given mass to reduce to half the size of that given mass.

A radioactive substance reduces from 12 g to exactly 10.24 g in 30 days.

Assuming that the isotope decays exponentially, determine the half life of the isotope, correct to the nearest day.

$\boxed{}$, $\boxed{\approx 131 \text{ days}}$



Question 45 (****)

A curve has equation

$$y=\ln\left(Ax+B\right),\ x>0,$$

where A and B are non zero constants.

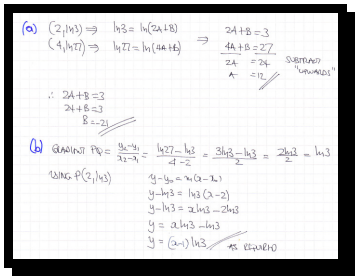
The curve passes through the points $P\left(2,\ln 3\right)$ and $Q\left(4,\ln 27\right)$.

a) Find the value of A and the value of B.

b) Show that the equation of the straight line through P and Q can be written as

$$y=\left(x-1\right)\ln 3.$$

$\boxed{A=12}$, $\boxed{B=-21}$



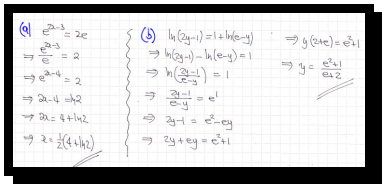
Question 46 (****)

Find, in exact simplified form, the solution of each of the following equations.

a) $\mathrm{e}^{2x-3}=2\mathrm{e}$.

b) $\ln(2y-1)=1+\ln(\mathrm{e}-y)$.

$\boxed{x=\frac{1}{2}(4+\ln 2)}$, $\boxed{y=\frac{\mathrm{e}^2+1}{\mathrm{e}+2}}$



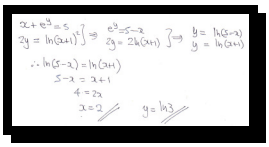
Question 47 (****)

Find, in exact form if appropriate, the solution of the following simultaneous equations

$$x+\mathrm{e}^{y}=5$$

$$\ln(x+1)^2=2y.$$

$\boxed{}$, $\boxed{x=2}$, $\boxed{y=\ln 3}$



Question 48 (****)

The function f is defined as

$$f(x)=a\ln(bx),\ x\in\mathbb{R},\ x>0$$

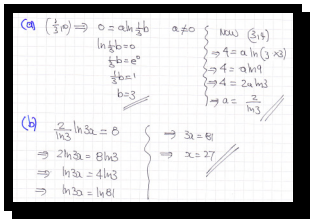
where a and b are positive constants.

a) Given that the graph of $f(x)$ passes through the points $\left(\frac{1}{3},0\right)$ and $(3,4)$, find the exact value of a and the value of b.

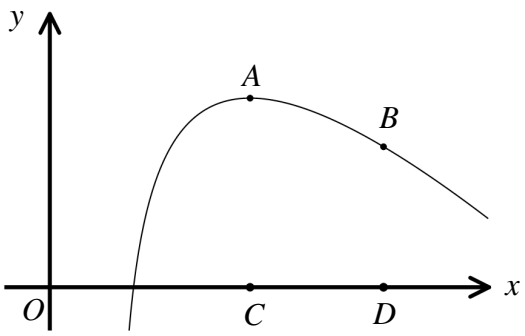
b) Solve the equation

$$f(x)=8.$$

$$\boxed{a=\frac{2}{\ln 3}},\ \boxed{b=3},\ \boxed{x=27}$$



Question 49 **(****)**



The figure above shows the graph of the curve with equation

$$y=4-x+2\ln(x-1),\ x>1.$$

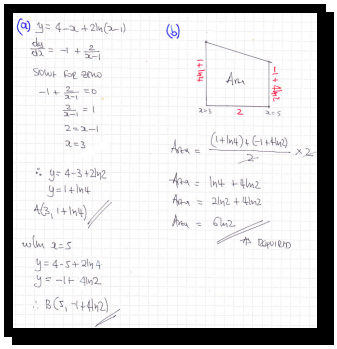
The point A is the maximum point of the curve and the point B lies on the curve where $x=5$.

a) Find the coordinates of A and B.

The points C and D lie on the x axis directly below the points A and B, respectively.

b) Show that the area of the trapezium $ABDC$ is $6\ln 2$ square units.

$$\boxed{A(3,1+2\ln 2),B(5,-1+4\ln 2)}$$



Question 50 (****)

An area is to be replanted with eucalyptus trees after a large fire.

The height, H m, of one such tree is given by the formula

$$H = 25 - 24\mathrm{e}^{-0.1t}, \ t \geq 0,$$

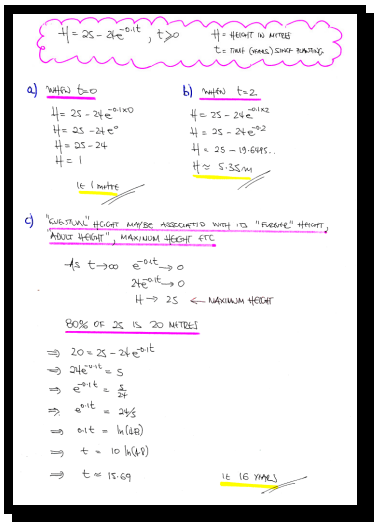
where t is the time in years since the tree was planted.

a) State the height of a newly planted tree.

b) Find the height of a tree, after 2 years.

c) Calculate, to the nearest integer, the value of t when the height of a tree has reached 80% of its eventual height.

$\boxed{}$, $\boxed{H_0 = 1 \text{ m}}$, $\boxed{H \approx 5.35 \text{ m}}$, $\boxed{t \approx 16}$



Question 51 (****)

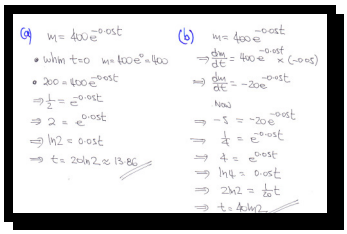
A radioactive substance decays so that its mass, m kg, at time t years from now, satisfies the exponential equation

$$m = 400\,\mathrm{e}^{-0.05t}\,,\ t \geq 0\,.$$

a) Find the time it takes for the substance to halve its mass.

b) Determine the **exact** value of t when the radioactive substance is decaying at the rate of 5 kg per year, giving the answer in terms of $\ln 2$.

$\boxed{t = 20\ln 2 \approx 13.86 \text{ years}}$, $\boxed{t = 40\ln 2}$



Question 52 (****)

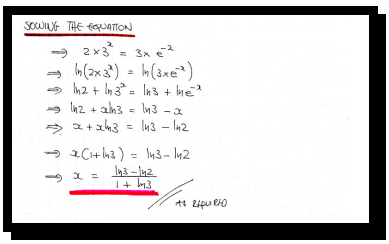
It is given that

$$x = \frac{\ln 3 - \ln 2}{1 + \ln 3}\,.$$

Show clearly that x is the solution of the equation

$$2 \times 3^x = 3 \times \mathrm{e}^{-x}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 53 (****)

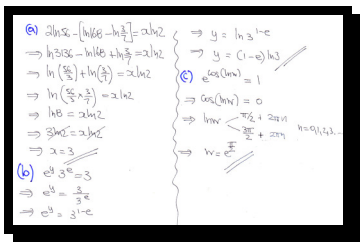
Find, in exact form where appropriate, the solutions of each of the following equations.

a) $2\ln 56-\left[\ln 168-\ln\left(\frac{3}{7}\right)\right]=x\ln 2$.

b) $\mathrm{e}^{y}3^{\mathrm{e}}=3$.

c) $\mathrm{e}^{\cos(\ln w)}=1$, $1\le w<5$.

$\boxed{}$, $\boxed{x=3}$, $\boxed{y=(1-\mathrm{e})\ln 3}$, $\boxed{w=\mathrm{e}^{\frac{\pi}{2}}}$



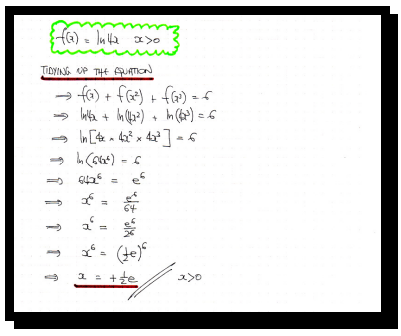
Question 54 (****)

$$f(x)=\ln(4x),\ x\in\mathbb{R},\ x>0.$$

Find, in exact simplified form, the solution of the equation

$$f(x)+f\left(x^2\right)+f\left(x^3\right)=6.$$

$\boxed{}$, $\boxed{x=\frac{1}{2}\mathrm{e}}$



Question 55 (****)

Water is heated in a kettle which is kept in a kitchen. The kitchen is kept at a constant temperature, T_0.

The temperature, T °C, of the water in the kettle satisfies

$$T = 95 - 75\mathrm{e}^{-t}, \ t \geq 0,$$

where t is the time in minutes since the kettle was switched on.

a) Find the time it takes for the water in the kettle to reach a temperature of 85 °C.

b) Determine the initial rate of the temperature rise of the water in the kettle.

Once the water has reached a temperature of 85 °C the kettle is switched off and is allowed to cool. Its temperature is now given by

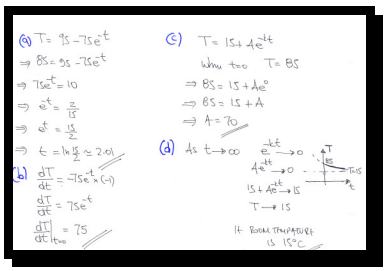
$$T = 15 + A\mathrm{e}^{-kt}, \ t \geq 0,$$

where A and k are positive constants, and t now represents the time in minutes since the kettle was switched off.

c) Find the value of A.

d) State, with a reason, the constant temperature of the kitchen, T_0.

$\boxed{}$, $\boxed{t \approx 2.01 \text{ minutes}}$, $\boxed{75\,^\circ\text{C/min}}$, $\boxed{A = 70}$, $\boxed{T_0 = 15}$



Question 56 (****)

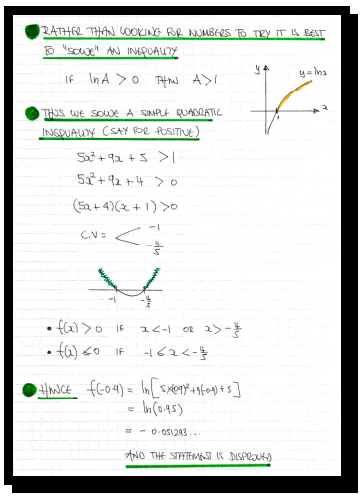
$$f(x)=\ln\left(5x^2+9x+5\right),\ x\in\mathbb{R}.$$

Show that the statement

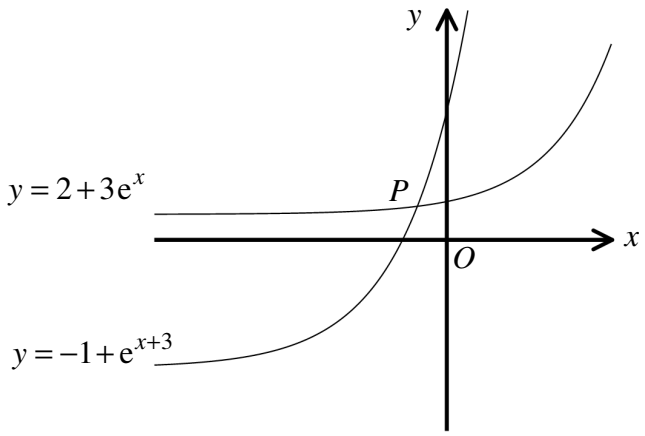
"$f(x)$ *is positive for all real values of* x"

is in fact false.

[ ], [proof]



Question 57 (****)



The figure above shows the graphs of

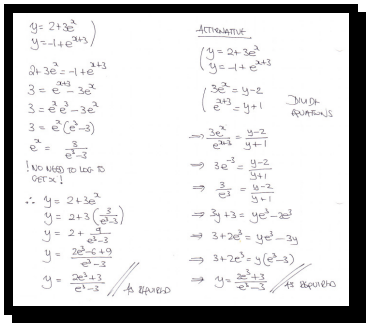
$$y = 2 + 3\mathrm{e}^{x} \quad \text{and} \quad y = -1 + \mathrm{e}^{x+3}.$$

The graphs meet at the point P.

Show that the y coordinate of P is

$$\frac{2\mathrm{e}^3 + 3}{\mathrm{e}^3 - 3}.$$

[ ], [proof]



Question 58 (****)

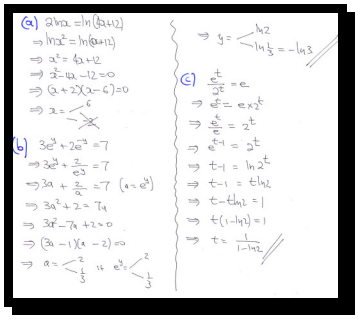
Find, in exact form where appropriate, the solution for each of the following equations.

a) $2\ln x = \ln\left(4x+12\right)$.

b) $3\mathrm{e}^{y}+2\mathrm{e}^{-y}=7$.

c) $\dfrac{\mathrm{e}^{t}}{2^{t}}=\mathrm{e}$.

$\boxed{x=6,\ x\neq -2}$, $\boxed{y=\ln 2,-\ln 3}$, $\boxed{t=\dfrac{1}{1-\ln 2}}$



Question 59 (****)

When hot cooking oil cools down, its temperature, T °C, is related to the time, t minutes, for which it has been cooling, by the formula

$$T = 20 + 160\mathrm{e}^{-0.1t}, \ t \geq 0.$$

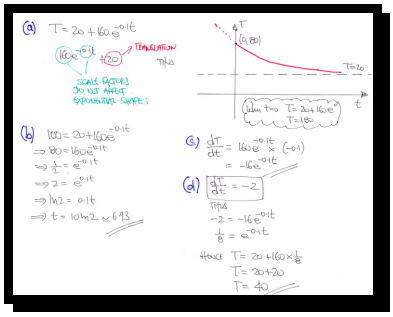
a) Sketch the graph of T against t, clearly marking its asymptote and the coordinates of the starting point of the curve.

b) Determine the value of t, when $T = 100$.

c) Find an expression for $\dfrac{dT}{dt}$.

d) Calculate the value of T, at the instant when the oil is cooling down at the rate of 2 °C per minute.

$\boxed{}$, $\boxed{t = 10\ln 2 \approx 6.93}$, $\boxed{\dfrac{dT}{dt} = -16\mathrm{e}^{-0.1t}}$, $\boxed{T = 40}$



Question 60 (****)

The point $P\left(\ln 2, 5b-3a\right)$ lies on the curve with equation

$$y=a+b\mathrm{e}^{x},$$

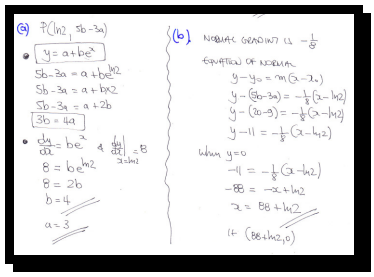
where a and b are non zero constants.

The gradient at P is 8.

a) Find the value of a and the value of b.

b) Determine the exact coordinates of the point where the normal to the curve at P crosses the x axis.

$\boxed{a=3,\ b=4}$, $\boxed{\left(88+\ln 2, 0\right)}$

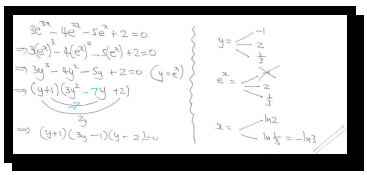


Question 61 (****)

Find the exact solutions of the equation

$$3\mathrm{e}^{3x}-4\mathrm{e}^{2x}-5\mathrm{e}^{x}+2=0.$$

$\boxed{x=\ln 2, -\ln 3}$



Question 62 (****)

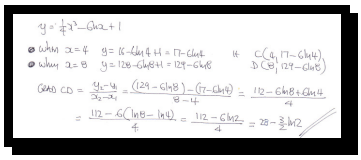
A curve has equation

$$y=\frac{1}{4}x^3-6\ln x+1,\ x>0.$$

The points C and D lie on this curve, where $x=4$ and $x=8$, respectively.

Show that the gradient of the straight line segment CD is $28-\frac{1}{2}\ln 2$.

proof

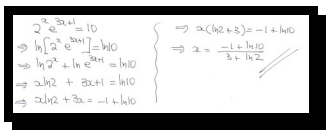


Question 63 (****)

Find in exact simplified form the solution to the equation

$$2^x\,\mathrm{e}^{3x+1}=10.$$

$$\boxed{x=\frac{-1+\ln 10}{3+\ln 2}}$$



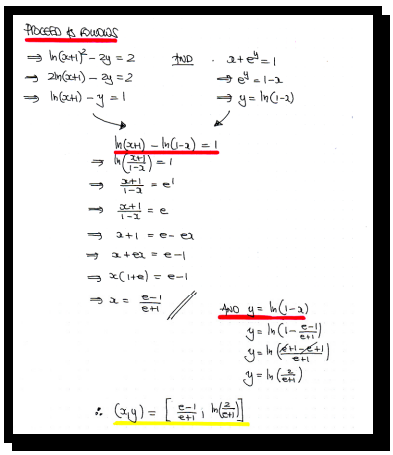
Question 64 (****)

Find the exact solution of the following simultaneous equations

$$x+\mathrm{e}^{y}=1$$

$$\ln\left(x+1\right)^{2}-2y=2\,.$$

$$\boxed{}\,,\quad \boxed{x=\frac{\mathrm{e}-1}{\mathrm{e}+1}}\,,\quad \boxed{y=\ln\left(\frac{2}{\mathrm{e}+1}\right)}$$

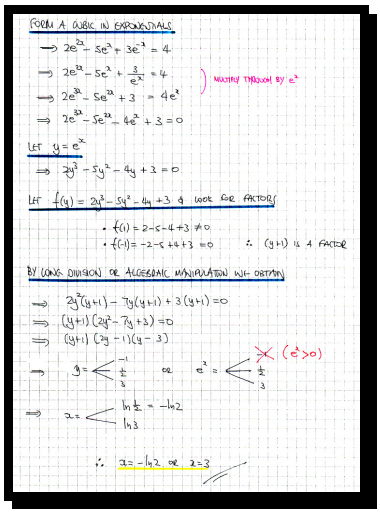


Question 65 (****)

Find the exact solutions of the equation

$$2\mathrm{e}^{2x}-5\mathrm{e}^{x}+3\mathrm{e}^{-x}=4\,.$$

$\boxed{}$, $\boxed{x=\ln 3,-\ln 2}$



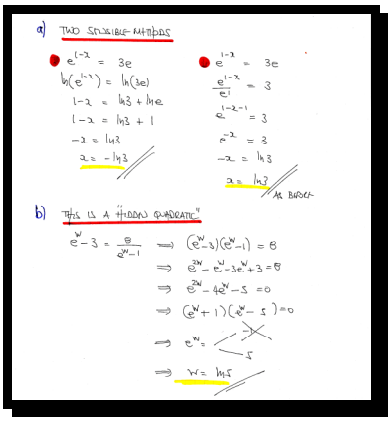
Question 66 (****)

Find, in exact form where appropriate, the solutions for each of the following equations.

a) $e^{1-x}=3e$.

b) $e^{w}-3=\dfrac{8}{e^{w}-1}$.

$\boxed{}$, $\boxed{x=-\ln 3}$, $\boxed{w=\ln 5}$



Question 67 (****)

Show that the following simultaneous equations

$$e^{2y}+4=x$$

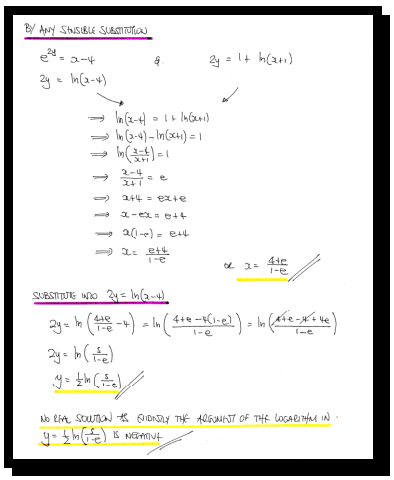
$$\ln(x+1)=2y-1$$

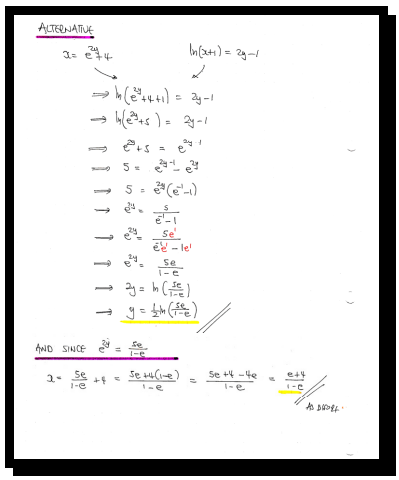
are satisfied by the solution pair

$$x=\frac{4+e}{1-e},\quad y=\frac{1}{2}\ln\left(\frac{5e}{1-e}\right)$$

and hence explain why the equations have no real solutions.

[ ], [proof]



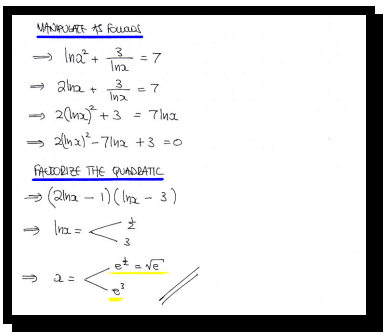


Question 68 (****)

Solve the following logarithmic equation

$$\ln x^2 + \frac{3}{\ln x} = 7 .$$

$\square$, $\boxed{x = \mathrm{e}^3, \sqrt{\mathrm{e}}}$



Question 69 (****)

The value V, in £, of a computer system t years after it was purchased is modelled by the following expression

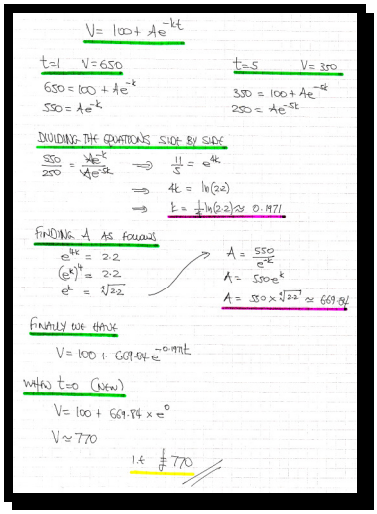
$$V = 100 + A\mathrm{e}^{-kt}, \ t \geq 0,$$

where A and k are positive constants.

Its value after one year was £650 and after a further period of four years £350.

Find, correct to the nearest £, the value of the system when new.

$\boxed{}$, $\boxed{£770}$



Question 70 (****)

The curve C has equation

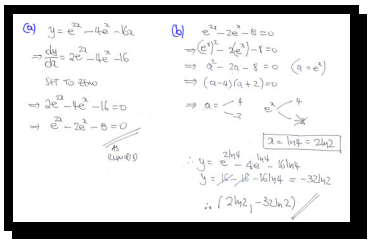
$$y = \mathrm{e}^{2x} - 4\mathrm{e}^{x} - 16x .$$

a) Show that the x coordinates of the stationary points of C satisfy the equation

$$\mathrm{e}^{2x} - 2\mathrm{e}^{x} - 8 = 0 .$$

b) Hence find the exact coordinates of the stationary point of C, giving the answer in terms of $\ln 2$.

$\boxed{}$, $\boxed{(2\ln 2, -32\ln 2)}$

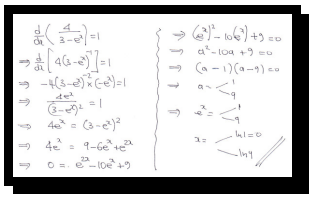


Question 71 (****)

Find, in exact form where appropriate, the solutions of the following equation

$$\frac{d}{dx}\left(\frac{4}{3-\mathrm{e}^{x}}\right) = 1 .$$

$\boxed{x = 0, \ \ln 9}$



Question 72 (****)

The temperature of an oven is modelled by

$$\theta = 225 - 200\mathrm{e}^{-0.1t}\,,\ t \geq 0$$

where θ °C is the temperature of the oven, t minutes after it was switched on.

a) State the highest temperature of the oven according to this model.

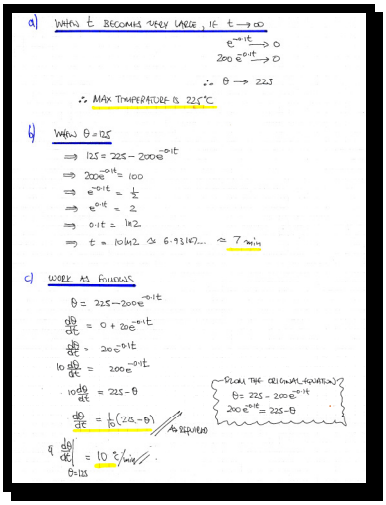
b) Determine the value of t when the oven temperature reaches $125°\mathrm{C}$.

c) Show clearly that

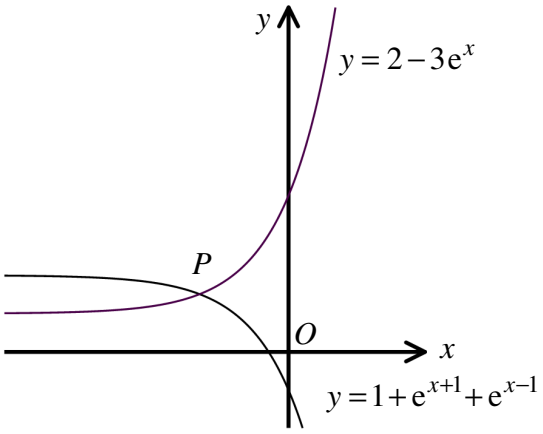
$$\frac{d\theta}{dt} = \frac{1}{10}(225 - \theta)\,,$$

and hence find the rate at which the temperature of the oven is increasing when its temperature has reached $125°\mathrm{C}$.

$\boxed{}$, $\boxed{\theta_{\max} = 225}$, $\boxed{t \approx 6.93}$, $\boxed{10°\mathrm{C/min}}$



Question 73 (****)



The figure above shows the graphs of

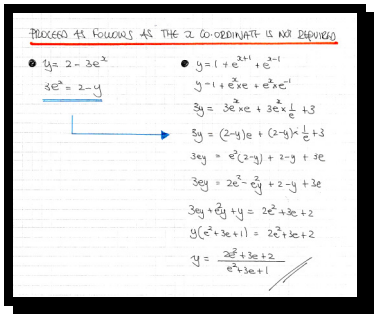
$$y = 2 - 3\mathrm{e}^{x} \qquad \text{and} \qquad y = 1 + \mathrm{e}^{x+1} + \mathrm{e}^{x-1}.$$

The graphs meet at the point P.

Show that the y coordinate of P is

$$\frac{2\mathrm{e}^2 + 3\mathrm{e} + 2}{\mathrm{e}^2 + 3\mathrm{e} + 1}.$$

$\boxed{}$, proof



Question 74 (****)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x)=\mathrm{e}^{x}-3,\ x\in\mathbb{R}$$

$$g(x)=x+1,\quad x\in\mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Solve the equation

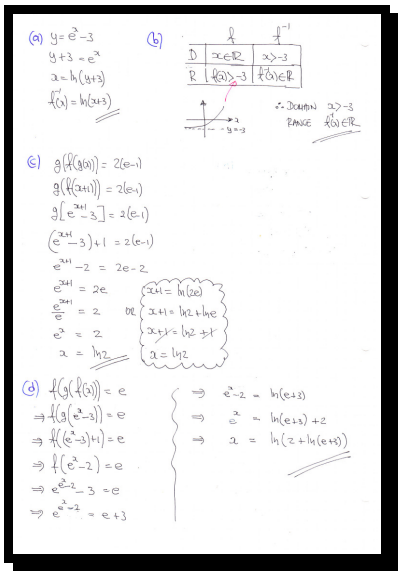
$$gfg(x)=2(\mathrm{e}-1),$$

giving the final answer in terms of logarithms in its simplest form.

d) Solve the equation

$$fgf(x)=\mathrm{e}.$$

$\boxed{f^{-1}(x)=\ln(x+3)}$, $\boxed{x\in\mathbb{R},\ x>-3}$, $\boxed{f^{-1}(x)\in\mathbb{R}}$, $\boxed{x=\ln 2}$, $\boxed{x=\ln\left[2+\ln(3+\mathrm{e})\right]}$



Question 75 (****)

The number of bacteria N present in a culture at time t hours, is modelled by the equation

$$N = A\mathrm{e}^{kt}\,,\ t \geq 0\,.$$

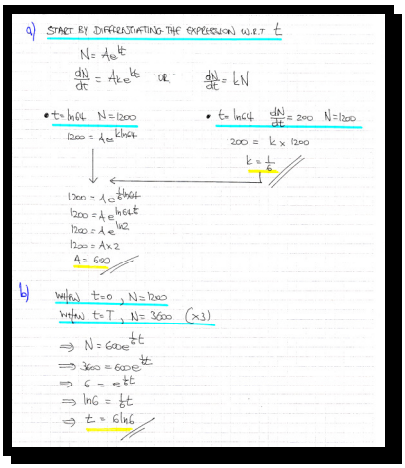
At the instant when $t = \ln 64$, there are 1200 bacteria present in the culture and bacteria are increasing at the rate of 200 bacteria per hour.

a) Find the value of A and the value of k.

The time T it takes for the bacteria to triple in number is constant.

b) Find the exact value of T.

$\boxed{}$, $\boxed{A = 600}$, $\boxed{k = \frac{1}{6}}$, $\boxed{T = 6\ln 3}$



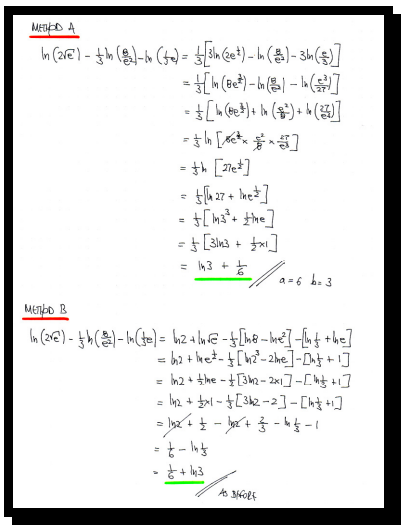
Question 76 (****)

Simplify the following logarithmic expression

$$\ln\left(2\sqrt{\mathrm{e}}\right)-\frac{1}{3}\ln\left(\frac{8}{\mathrm{e}^2}\right)-\ln\left(\frac{1}{3}\mathrm{e}\right),$$

giving the answer in the form $\frac{1}{a}+\ln b$, where a and b are positive integers.

$\boxed{}$, $\boxed{\frac{1}{2}+\ln 3}$



Question 77 (****)

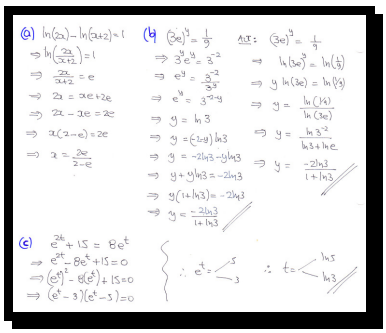
Find, in exact form where appropriate, the solutions for each of the following equations.

a) $\ln(2x)-\ln(x+2)=1$.

b) $(3\mathrm{e})^{y}=\dfrac{1}{9}$.

c) $\mathrm{e}^{2t}+15=8\mathrm{e}^{t}$.

$$\boxed{x=\frac{2\mathrm{e}}{2-\mathrm{e}}\approx -7.57}\ ,\quad \boxed{y=-\frac{2\ln 3}{1+\ln 3}}\ ,\quad \boxed{t=\ln 3,\ \ln 5}$$



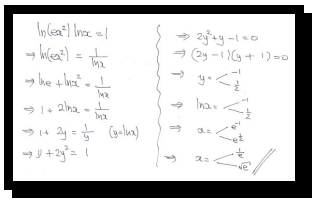
Question 78 (****)

Solve the equation

$$\ln\left(\mathrm{e}\,x^2\right)\ln x = 1,\ x > 0,$$

giving the answers in exact form.

$\square$, $\boxed{x = \frac{1}{\mathrm{e}}, \sqrt{\mathrm{e}}}$



Question 79 (****)

A curve has equation

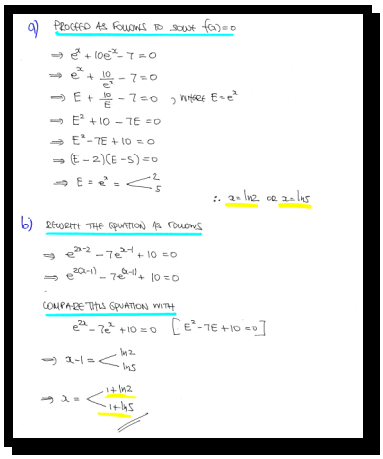
$$f(x)=\mathrm{e}^{x}+10\mathrm{e}^{-x}-7,\quad x\in\mathbb{R}.$$

a) Solve the equation $f(x)=0$.

b) Hence, or otherwise, solve the equation

$$\mathrm{e}^{2x-2}-7\mathrm{e}^{x-1}+10=0.$$

$\boxed{}$, $\boxed{x=\ln 2,\ x=\ln 5}$, $\boxed{x=\ln(2\mathrm{e}),\ x=\ln(5\mathrm{e})}$



Question 80 (****)

A hot metal rod is cooled down by dipping it into a large pool of water which is maintained at constant temperature.

The temperature of the metal rod, T °C, is given by

$$T = 20 + 480\mathrm{e}^{-0.1t}, \ t \geq 0,$$

where t is time in minutes since the rod was dipped in the water.

a) State the temperature …

i. … of the rod before it enters the water.

ii. … of the water.

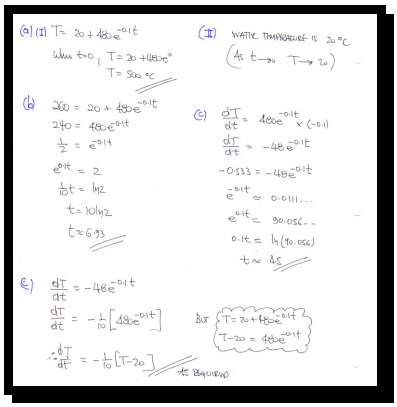
b) Determine the value of t when the rod reaches a temperature of 260 °C.

c) Find the value of t when the rod is cooling at the rate of 0.533 °C per minute.

d) Show clearly that

$$\frac{dT}{dt} = -\frac{1}{10}(T - 20).$$

$\boxed{}$, $\boxed{T = 500}$, $\boxed{T_{\text{water}} = 20}$, $\boxed{t \approx 6.93}$, $\boxed{t \approx 45}$



Question 81 (****)

The mass, M grams, of a leaf t days after it was picked from a tree is given by

$$M = A\mathrm{e}^{-kt}, \ t \geq 0$$

where A and k are positive constants.

When the leaf is picked its mass is 10 grams and 5 days later its mass is 5 grams.

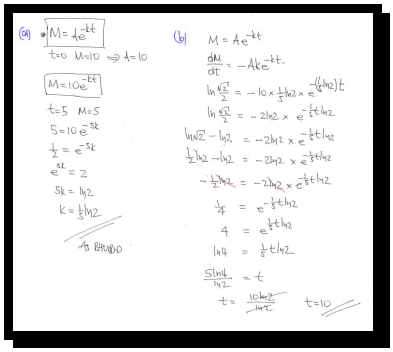
a) Show clearly that

$$k = \frac{1}{5}\ln 2 .$$

b) Find the value of t that satisfies the equation

$$\frac{dM}{dt} = \ln\left(\frac{\sqrt{2}}{2}\right).$$

$\boxed{}$, $\boxed{t = 10}$



Question 82 (**+)**

A scientist investigating the growth of a certain species of mushroom observes that this mushroom grows to a height of 41 mm in 5 hours.

He decides to model the height, H mm, t hours after the mushroom started forming, by the equation

$$H = k\left(1-\mathrm{e}^{-\frac{1}{12}t}\right),\ t \geq 0$$

where k is a positive constant.

a) Show that $k = 120$, correct to three significant figures.

The equation

$$H = k\left(1-\mathrm{e}^{-\frac{1}{12}t}\right),\ t \geq 0,$$

is to be used for the rest of this question.

b) Determine the value of t when $H = 90$, giving the answer in the form $a\ln 2$, where a is an integer.

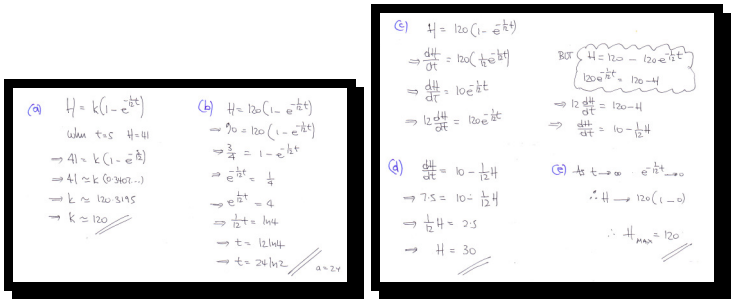
c) Show clearly that

$$\frac{dH}{dt} = 10 - \frac{1}{12}H\ .$$

d) Hence find the value of H when the height is the mushroom is growing at the rate of 7.5 mm per hour.

e) State the maximum height of the mushroom according to this model.

$\boxed{}$, $\boxed{a = 24}$, $\boxed{H = 30}$, $\boxed{H_{\max} = 120}$



Question 83 (**+)**

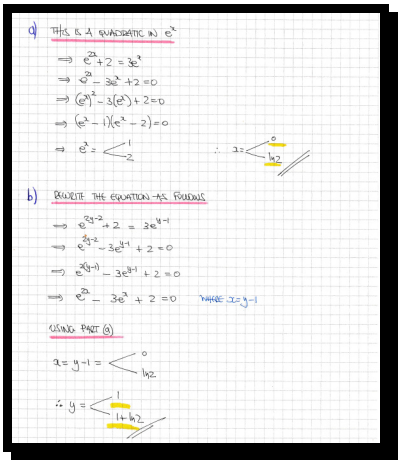
Determine, in exact simplified form where appropriate, the solutions for each of the following equations.

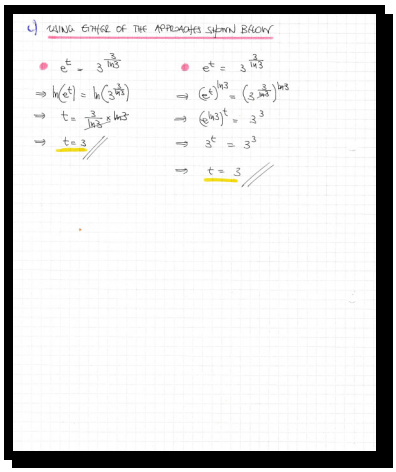
a) $\mathrm{e}^{2x}+2=3\mathrm{e}^{x}$.

b) $\mathrm{e}^{2y-2}+2=3\mathrm{e}^{y-1}$.

c) $\mathrm{e}^{t}=3^{\frac{3}{\ln 3}}$.

$\boxed{}$, $\boxed{x=0,\quad x=\ln 2}$, $\boxed{y=1,\quad y=1+\ln 2}$, $\boxed{t=3}$



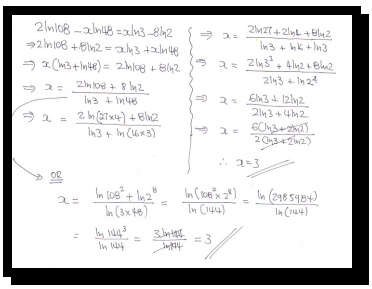


Question 84 (**+)**

Find the solution of the following logarithmic equation

$$2\ln 108 - x\ln 48 = x\ln 3 - 8\ln 2 .$$

$\boxed{x = 3}$



Question 85 (**+)**

The function f is defined as

$$f(x)=3-\ln 4x,\ x\in\mathbb{R},\ x>0$$

a) Determine, in exact form, the coordinates of the point where the graph of f crosses the x axis.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln 4x \xrightarrow{T_2} -\ln 4x \xrightarrow{T_3} 3-\ln 4x$$

b) Describe geometrically each of the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.
Indicate clearly any intersections with the coordinate axes.

The function g is defined by

$$g(x)=\mathrm{e}^{5-x},\ x\in\mathbb{R}.$$

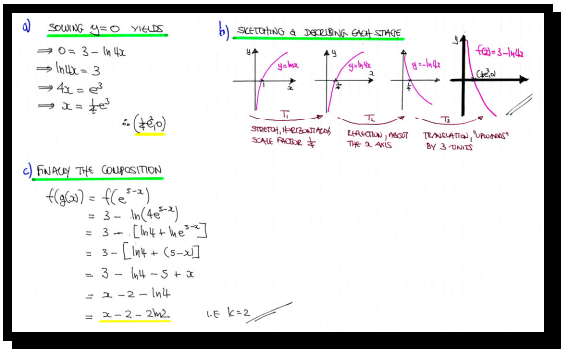
c) Show that

$$fg(x)=x-k-k\ln k\ ,$$

where k is a positive integer.

$\boxed{}$, $\boxed{\left(\frac{1}{4}\mathrm{e}^3,0\right)}$, $\boxed{T_1=\text{stretch in } x\text{, scale factor } \frac{1}{4}}$, $\boxed{T_2=\text{reflection in the } x\text{-axis}}$,

$\boxed{T_3=\text{translation, "up", 4 units}}$, $\boxed{k=2}$



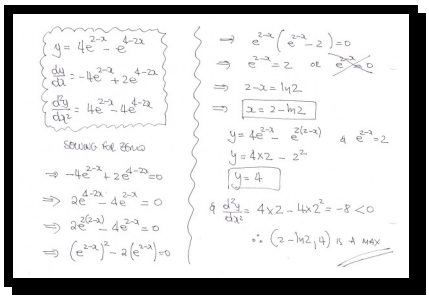
Question 86 (**+)**

A curve has equation

$$y = 4\mathrm{e}^{2-x} - \mathrm{e}^{4-2x}, \quad x \in \mathbb{R}.$$

Use differentiation to find the exact coordinates of the stationary point of the curve, and determine its nature.

$\square$, $\boxed{\max\left(2-\ln 2, 4\right)}$

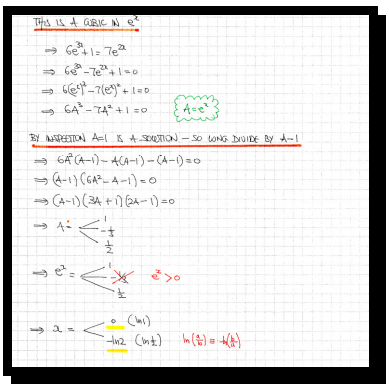


Question 87 (**+)**

Find, in exact form where appropriate, the solutions of the following equation

$$6\mathrm{e}^{3x} + 1 = 7\mathrm{e}^{2x}.$$

$\square$, $\boxed{x = 0, -\ln 2}$



Question 88 (**+)**

When a tree of a certain species was planted it was 2 metres in height and after 2 years its height was measured at 3.81 metres.

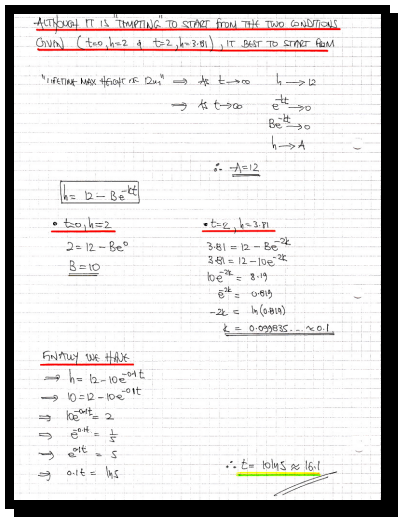
The height, h metres, of this tree, t years after it was planted, is modelled by the equation

$$h = A - B\mathrm{e}^{-kt},$$

where A, B and k are positive constants.

Given that this species of tree will reach in its lifetime a maximum height of 12 metres, find the value of t when $h = 10$.

$\boxed{}$, $\boxed{t \approx 16}$

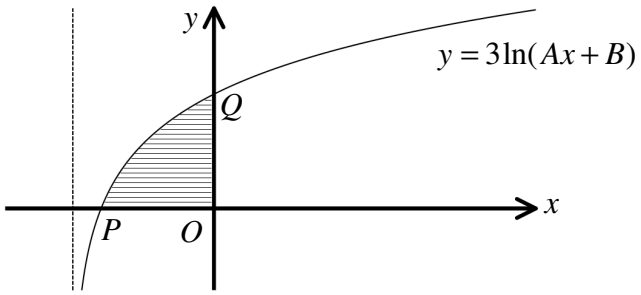


Question 89 (**+)**

The figure below shows the curve with equation

$$y = 3\ln(Ax + B),\ Ax + B > 0,$$

where A and B are positive constants.



The curve passes through the points $P(-1,0)$ and $Q(0,\ln 27)$.

a) Find the values of A and B.

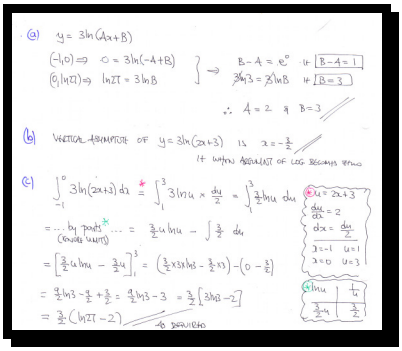
b) State the equation of the vertical asymptote to the curve.

The shaded region R is bounded by the curve and the coordinate axes.

c) Show that the area of R is

$$\frac{3}{2}\left(\ln 27 - 2\right).$$

$\boxed{A = 2}$, $\boxed{B = 3}$, $\boxed{x = -\frac{3}{2}}$



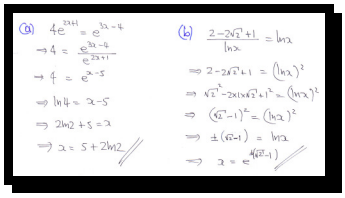
Question 90 (**+)**

Find, in exact form where appropriate, the solutions for each of the following equations.

a) $4\mathrm{e}^{2x+1}=\mathrm{e}^{3x-4}$.

b) $\dfrac{2-2\sqrt{2}+1}{\ln x}=\ln x$.

$\boxed{}$, $\boxed{x=5+2\ln 2}$, $\boxed{y=\mathrm{e}^{\pm\left(\sqrt{2}-1\right)}}$



Question 91 **(****+)**

The function f is defined as

$$f:x\mapsto 6-\ln\left(x+3\right),\ \ x\in\mathbb{R},\ x\geq -2\,.$$

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x\xrightarrow{T_1}\ln\left(x+3\right)\xrightarrow{T_2}-\ln\left(x+3\right)\xrightarrow{T_3}-\ln\left(x+3\right)+6\,.$$

a) Describe geometrically T_1, T_2 and T_3, and hence sketch the graph of $f\left(x\right)$. Indicate clearly any intersections with the axes and the graph's starting point.

b) Find, in its simplest form, an expression for $f^{-1}\left(x\right)$, stating further the domain and range of $f^{-1}\left(x\right)$.

The function g satisfies

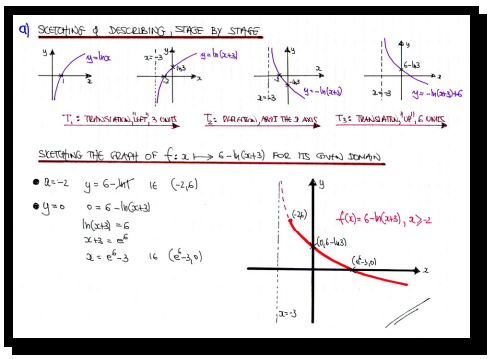
$$g:x\to \mathrm{e}^{x^2}-3,\quad x\in\mathbb{R}\,.$$

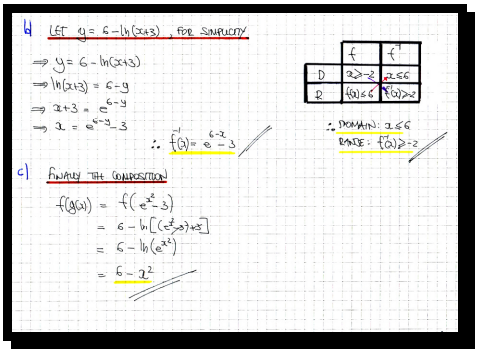
c) Find, in its simplest form, an expression for the composition $fg\left(x\right)$.

$\boxed{}$, $\boxed{T_1=\text{translation, "left", 3 units}}$, $\boxed{T_2=\text{reflection about } x\text{-axis}}$,

$\boxed{T_3=\text{translation, "up", 6 units}}$, $\boxed{\left(0,6-\ln 3\right)}$, $\boxed{\left(-3+\mathrm{e}^6,0\right)}$, $\boxed{\left(-2,6\right)}$,

$\boxed{f^{-1}:x\mapsto -3+\mathrm{e}^{6-x}}$, $\boxed{x\leq 6,\quad f^{-1}\left(x\right)\geq -2}$, $\boxed{fg:x\mapsto 6-x^2}$





Question 92 **(****+)**

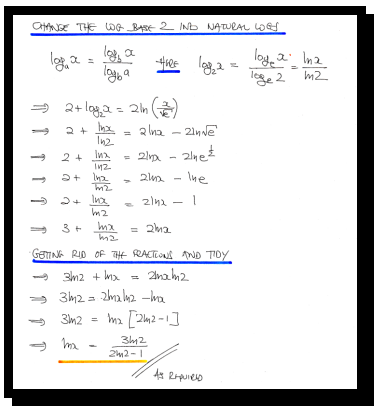
Show that the value of x in the following expression

$$\ln x = \frac{3\ln 2}{2\ln 2 - 1}$$

satisfies the logarithmic equation

$$2 + \log_2 x = 2\ln\left(\frac{x}{\sqrt{\mathrm{e}}}\right), \quad x > 0 .$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 93 (**+) non calculator**

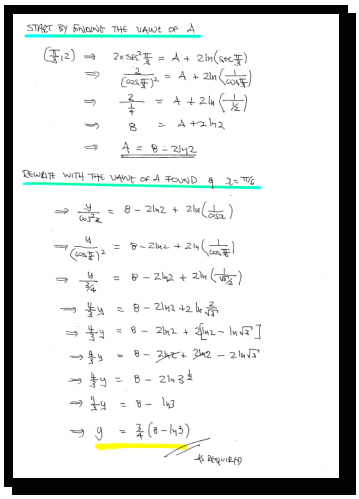
An implicit relationship between x and y is given below, in terms of a constant A.

$$y\sec^2 x = A + 2\ln(\sec x),\ 0 \le x < \frac{\pi}{2}.$$

Given that $y = 2$ at $x = \frac{\pi}{3}$, **show clearly** that when $x = \frac{1}{6}\pi$

$$y = \frac{3}{4}(8 - \ln 3).$$

$\boxed{}$, $\boxed{\text{proof}}$



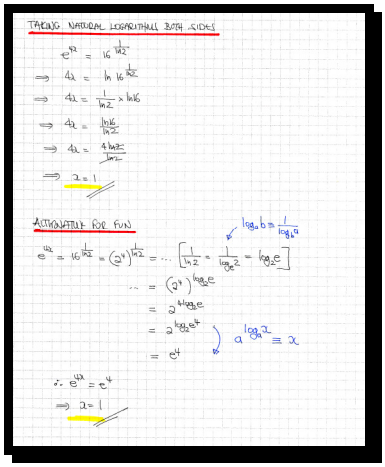
Question 94 (**+)**

Find, in its simplest form, the solution of the following equation

$$e^{4x} = 16^{\frac{1}{\ln 2}}.$$

The answer must be supported by detailed workings.

$\boxed{}$, $\boxed{x=1}$



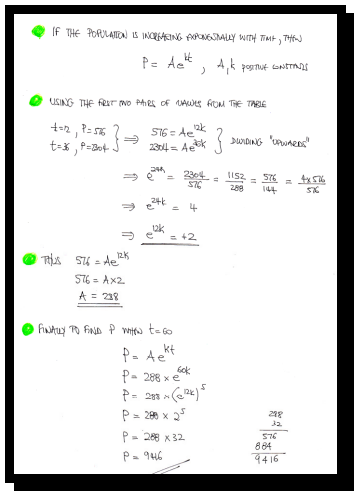
Question 95 (**+)**

A population of bacteria P is growing exponentially with time t and the table below shows some of these values.

t	12	36	60
P	576	2304	a

Show clearly that $a = 9216$.

[ ], [proof]



Question 96 (**+)**

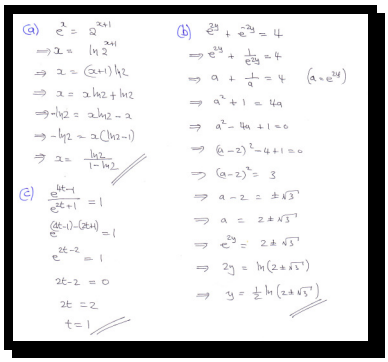
Find, in exact form where appropriate, the solutions for each of the following equations.

a) $\mathrm{e}^{x}=2^{x+1}$.

b) $\mathrm{e}^{2y}+\mathrm{e}^{-2y}=4$.

c) $\dfrac{\mathrm{e}^{4t-1}}{\mathrm{e}^{2t+1}}=1$.

$$\boxed{x=\frac{\ln 2}{1-\ln 2}},\quad \boxed{y=\frac{1}{2}\ln\left(2\pm\sqrt{3}\right)=\pm\frac{1}{2}\ln\left(2+\sqrt{3}\right)},\quad \boxed{t=1}$$



(a) $\mathrm{e}^{x}=2^{x+1}$
$\Rightarrow x=\ln 2^{x+1}$
$\Rightarrow x=(x+1)\ln 2$
$\Rightarrow x=x\ln 2+\ln 2$
$\Rightarrow \ln 2=x\ln 2-x$
$\Rightarrow \ln 2=x(\ln 2-1)$
$\Rightarrow x=\dfrac{\ln 2}{1-\ln 2}$

(b) $\mathrm{e}^{2y}+\mathrm{e}^{-2y}=4$
$\Rightarrow \mathrm{e}^{2y}+\dfrac{1}{\mathrm{e}^{2y}}=4$
$\Rightarrow a+\dfrac{1}{a}=4 \quad (a=\mathrm{e}^{2y})$
$\Rightarrow a^2+1=4a$
$\Rightarrow a^2-4a+1=0$
$\Rightarrow (a-2)^2-4+1=0$
$\Rightarrow (a-2)^2=3$
$\Rightarrow a-2=\pm\sqrt{3}$
$\Rightarrow a=2\pm\sqrt{3}$
$\Rightarrow \mathrm{e}^{2y}=2\pm\sqrt{3}$
$\Rightarrow 2y=\ln\left(2\pm\sqrt{3}\right)$
$\Rightarrow y=\frac{1}{2}\ln\left(2\pm\sqrt{3}\right)$

(c) $\dfrac{\mathrm{e}^{4t-1}}{\mathrm{e}^{2t+1}}=1$
$\mathrm{e}^{(4t-1)-(2t+1)}=1$
$\mathrm{e}^{2t-2}=1$
$2t-2=0$
$2t=2$
$t=1$

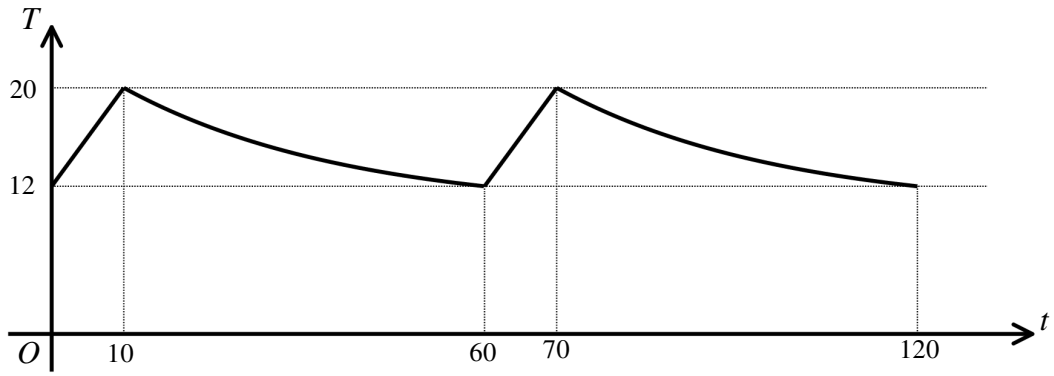
Question 97 (**+)**

The temperature T °C in a warehouse is regulated by a thermostat so that when the temperature drops to 12 °C the heating system is switched on.

It takes 10 minutes for the temperature to rise to 20 °C and at that point the thermostat switches the heating system off. It takes 50 minutes for the temperature to drop back down to 12 °C.

The cycle then repeats.

This is shown in the graph below, where the time t is in minutes.



The section of the graph for which $10 \le t \le 60$ is modelled by the equation

$$T = 10 + A\mathrm{e}^{-kt}, \ 10 \le t \le 60$$

where A and k are positive constants.

a) Find, correct to 5 significant figures, the value of A and the value of k.

[continues overleaf]

[continued from overleaf]

The section of the graph for which $70 \le t \le 120$ is a horizontal translation of the section of the graph for which $10 \le t \le 60$.

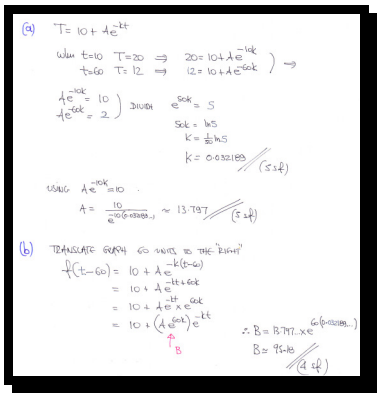
This section is modelled by the equation

$$T = 10 + B\mathrm{e}^{-kt}, \quad 70 \le t \le 120$$

where B is a positive constant.

b) Find, to 4 significant figures, the value of B.

$\boxed{A = 13.797}$, $\boxed{k = 0.032189}$, $\boxed{B = 95.18}$

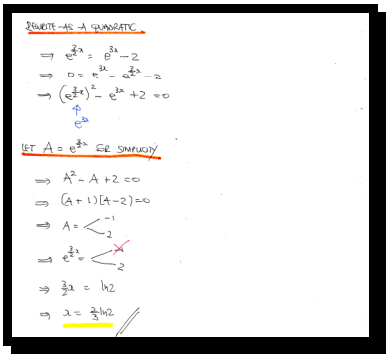


Question 98 (**+)**

Find, in exact form if appropriate, the solutions of the following equation.

$$e^{\frac{3}{2}x} = e^{3x} - 2 .$$

$\square$, $\boxed{x = \frac{2}{3}\ln 2}$



Question 99 (**+)**

A scientist is investigating the population growth of farm mice.

The number of farm mice N, t months since the start of the investigation, is modelled by the equation

$$N=\frac{600}{1+5\mathrm{e}^{-0.25t}},\ t\geq 0.$$

a) State the number of farm mice at the start of the investigation.

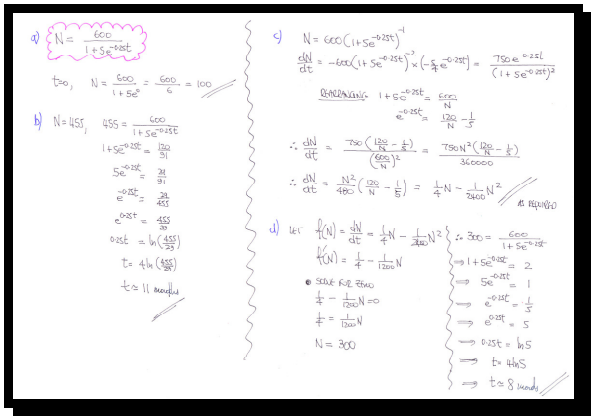
b) Calculate the number of months that it will take the population of farm mice to reach 455.

c) Show clearly that

$$\frac{dN}{dt}=\frac{1}{4}N-\frac{1}{2400}N^2.$$

d) Find the value of t when the rate of growth of the population of these farm mice is largest.

$\boxed{}$, $\boxed{100}$, $\boxed{t\approx 11}$, $\boxed{t=4\ln 5\approx 8}$



Question 100 (**+)**

The populations P_1 and P_2 of two bacterial cultures, t hours after a certain instant, are modelled by the following equations

$$P_1 = 1600\mathrm{e}^{\frac{1}{4}t}\ ,\ P_2 = 100\mathrm{e}^{\frac{1}{2}t}\ ,\ t \in \mathbb{R}\ ,\ t \geq 0\ .$$

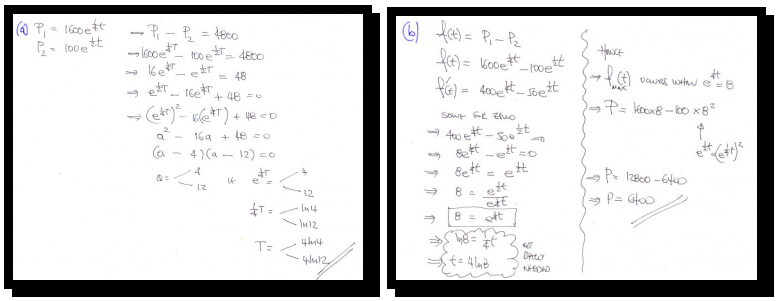
When $t = T$, P_1 contains 4800 more bacteria than P_2.

a) Find, in terms of natural logarithms, the possible values of T.

At a certain time there are P extra bacteria in P_1 compared with P_2.

b) Determine the greatest value of P.

$\boxed{}$, $\boxed{T = 8\ln 2 \ \text{ or } \ 4\ln 12}$, $\boxed{P = 6400}$



Question 101 **(****+)**

The function f is defined as

$$f(x)=\ln(4-2x),\ x\in\mathbb{R},\ x<2.$$

a) Find in exact form the coordinates of the points where the graph of $f(x)$ crosses the coordinate axes.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+4) \xrightarrow{T_2} \ln(2x+4) \xrightarrow{T_3} \ln(-2x+4)$$

b) Describe geometrically the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.

Indicate clearly any asymptotes and coordinates of intersections with the axes.

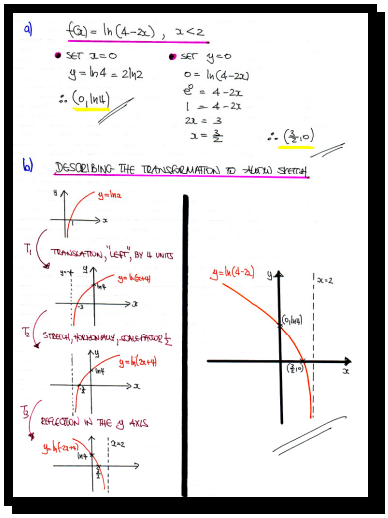
c) Find, an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

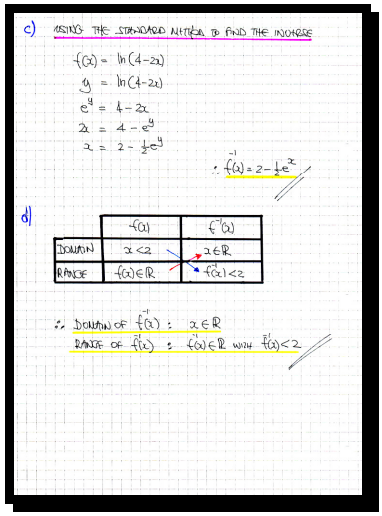
d) State the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{\left(\frac{3}{2},0\right),\ (0,\ln 4)}$, $\boxed{T_1=\text{translation, "left", 4 units}}$,

$\boxed{T_2=\text{stretch in } x\text{, scale factor } \frac{1}{2}}$, $\boxed{T_3=\text{reflection in the } y\text{-axis}}$, $\boxed{\text{asymptote } x=2}$,

$\boxed{f^{-1}(x)=2-\frac{1}{2}\mathrm{e}^x}$, $\boxed{x\in\mathbb{R}}$, $\boxed{f^{-1}(x)<2}$





Question 102 **(****+)**

The amount X milligrams, of an anaesthetic drug in the bloodstream of a patient, is given by

$$X = D\,\mathrm{e}^{-0.2t}\,,\ t \geq 0$$

where D is the dose, in milligrams, of the anaesthetic administered and t is the time in hours since the dose was administered.

A patient undergoing an operation is given an initial dose of 20 milligrams.

This patient will remain asleep if there are more than 12 milligrams of anaesthetic in his bloodstream.

a) Show that one hour later $X = 16.37$, correct to two decimal places.

b) Show, by calculation, that two hours after the initial dose was administered, the patient should still be asleep.

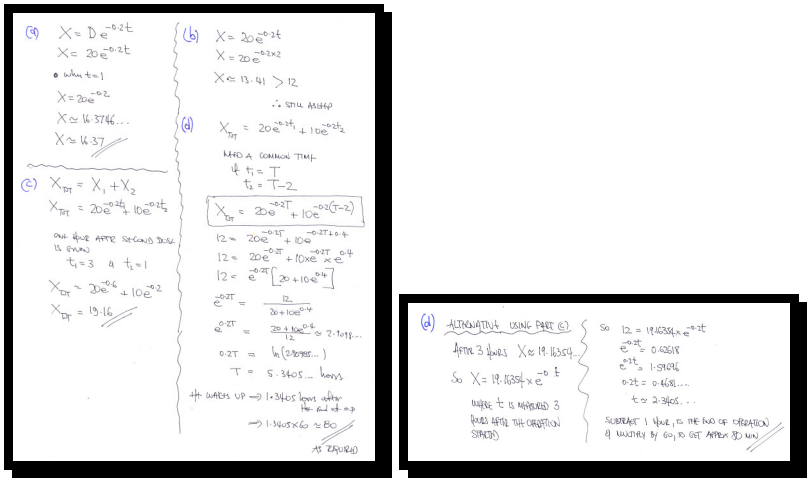
Two hours after the initial dose was administered a further dose of **10** milligrams is given to the patient.

c) Find the amount of the anaesthetic in the patient's bloodstream one hour after the second dose is given.

No more anaesthetic is given and the operation lasts for 4 hours.

d) Show by solving a relevant equation that the patient should "wake up" approximately 80 minutes after the end of his operation.

$\boxed{}$, $\boxed{X \approx 19.16}$



Question 103 (**+)**

The temperature, θ °C, of an oven t minutes after it was switched on is given by

$$\theta = 300 - 280\,\mathrm{e}^{-0.05t}\,,\ t \geq 0\,.$$

a) State the initial temperature of the oven.

b) Find the value of t when the temperature of the oven …

i. … reaches 160 °C.

ii. … is increasing at the rate of 4 °C per minute.

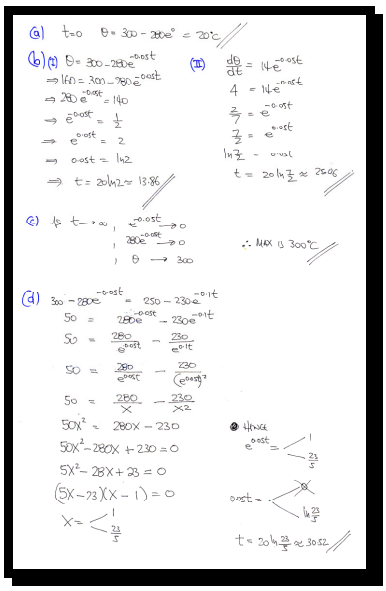
c) Determine, with justification, the maximum temperature this oven can reach.

The temperature θ °C of a **different** oven t minutes after it was switched on is modelled by a similar equation

$$\theta = 250 - 230\,\mathrm{e}^{-0.1t}\,,\ t \geq 0\,.$$

d) Assuming that both ovens are switched on at the same time find the time when both ovens will have the same temperature since they were switched on.

$\boxed{}$, $\boxed{20°\mathrm{C}}$, $\boxed{20\ln 2 \approx 13.86}$, $\boxed{20\ln\left(\frac{7}{2}\right) \approx 25.06}$, $\boxed{300°\mathrm{C}}$, $\boxed{20\ln\left(\frac{23}{5}\right) \approx 30.52}$



Question 104 (*****)

The population P of seals on an island obeys the equation

$$P=\frac{800k\,\mathrm{e}^{0.25t}}{1+k\,\mathrm{e}^{0.25t}},\quad t\geq 0,$$

where k is a positive constant and t is the time, in years, measured from a certain instant.

Initially there were 175 seals on the island.

a) Find, showing a detailed method, …

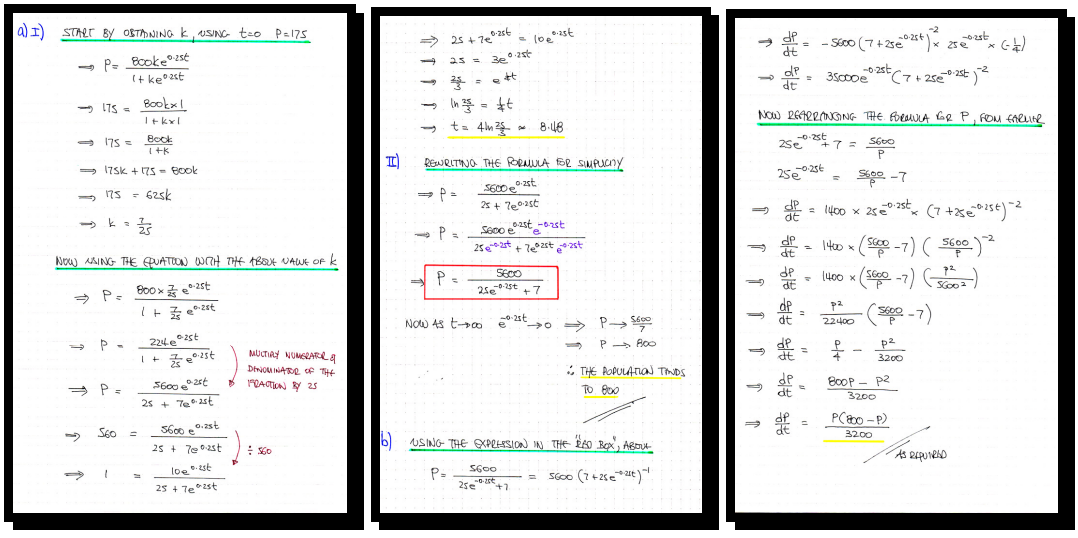
i. … the value of t when the number of seals reaches 560.

ii. … the long term prospects of the population of these seals.

b) Show further that

$$\frac{dP}{dt}=\frac{P(800-P)}{3200}.$$

$\boxed{}$, $\boxed{t\approx 8.48}$, $\boxed{P_{\max}=800}$



Question 105 **(*****)**

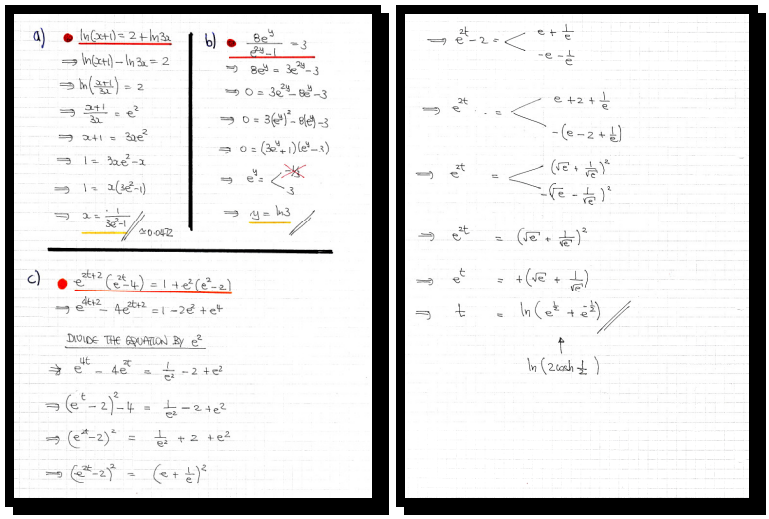
Find, in exact form where appropriate, the solutions for each of the following equations.

a) $\ln(x+1)=2+\ln(3x)$.

b) $\dfrac{8\mathrm{e}^{y}}{\mathrm{e}^{2y}-1}=3$.

c) $\mathrm{e}^{2t+2}\left(\mathrm{e}^{2t}-4\right)=1+\mathrm{e}^{2}\left(\mathrm{e}^{2}-2\right)$.

$\square$, $\boxed{x=\dfrac{1}{3\mathrm{e}^{2}-1}\approx 0.0472}$, $\boxed{y=\ln 3}$, $\boxed{t=\ln\left[\mathrm{e}^{\frac{1}{2}}+\mathrm{e}^{-\frac{1}{2}}\right]}$



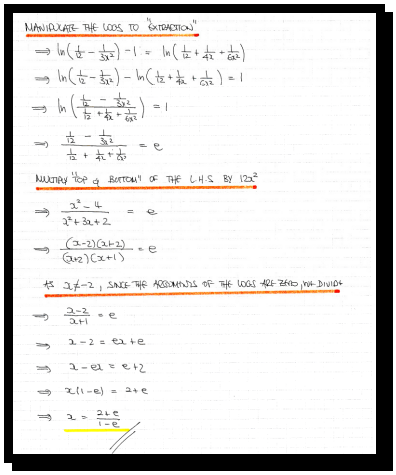
Question 106 (*****)

Solve the following logarithmic equation.

$$\ln\left(\frac{1}{12}-\frac{1}{3x^2}\right)-1=\ln\left(\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}\right),$$

giving the value of x in exact form.

$\square$, $\boxed{x=\dfrac{2+\mathrm{e}}{1-\mathrm{e}}}$



Question 107 (*****)

On the 1st January 2000 a rare stamp was purchased at an auction for £16000 and by the 1st January 2010 its value was four times as large as its purchase price.

The future value of this stamp, £V, t years after the 1st January 2000 is modelled by the equation

$$V = A\mathrm{e}^{pt}, \ t \geq 0,$$

where A and p are positive constants.

On the 1st January 1990 a different stamp was purchased for £2.

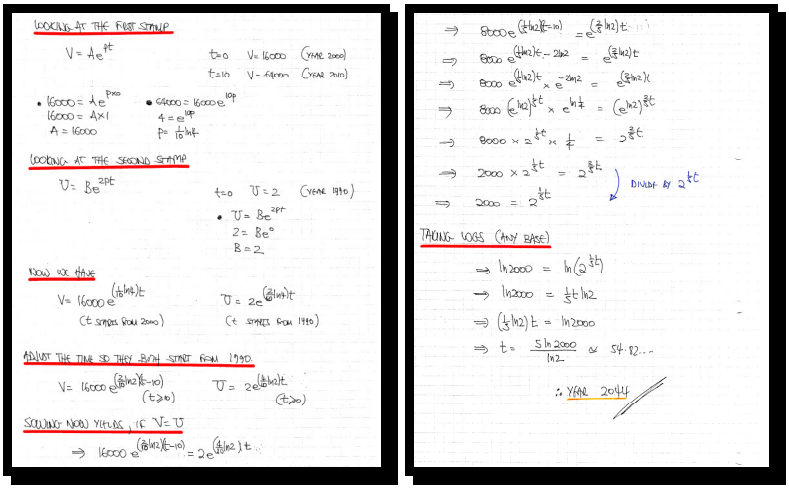
The future value of this stamp, £U, t years after the 1st January 1990 is modelled by the equation

$$U = B\mathrm{e}^{2pt}, \ t \geq 0,$$

where B is a positive constant.

Determine the year, during which the two stamps will achieve the same value, according to their modelling equations.

[ ], [2044]

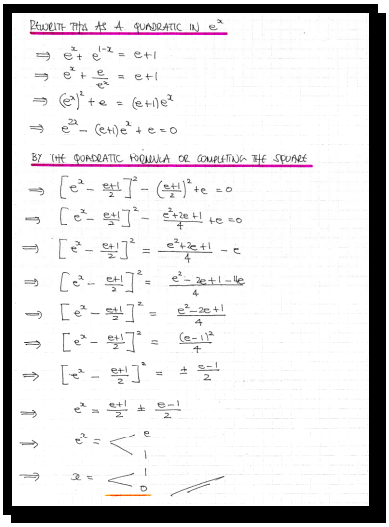


Question 108 (*****)

Use algebra, to solve the following equation.

$$e^{x}+e^{1-x}=e+1.$$

$\boxed{}$, $\boxed{x=0,\ \ x=1}$



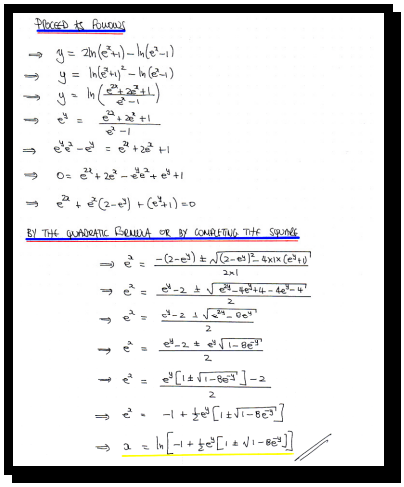
Question 109 **(*****)**

It is given that

$$y=2\ln\left(\mathrm{e}^{x}+1\right)-\ln\left(\mathrm{e}^{x}-1\right),\ x\in\mathbb{R}.$$

Express x in terms of y.

$\boxed{}$, $\boxed{x=\ln\left[\frac{1}{2}\mathrm{e}^{y}\left(1\pm\sqrt{1-8\mathrm{e}^{-y}}\right)-1\right]}$

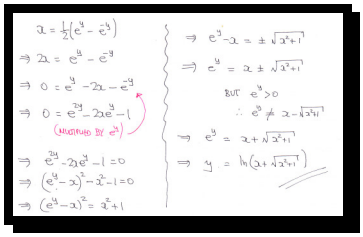


Question 110 **(*****)**

Given that $x=\frac{1}{2}\left(\mathrm{e}^{y}-\mathrm{e}^{-y}\right)$ show that

$$y=\ln\left(x+\sqrt{x^{2}+1}\right).$$

proof

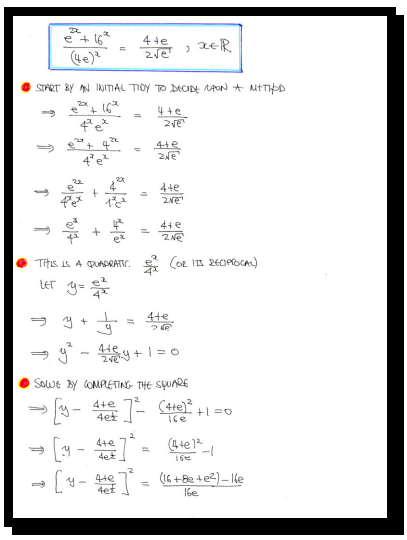


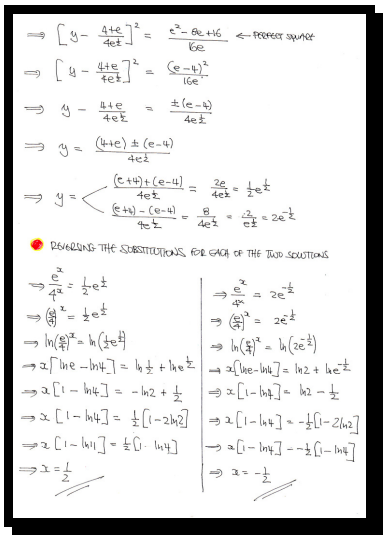
Question 111 (*****)

Solve the following equation

$$\frac{e^{2x}+16^x}{(4e)^x}=\frac{4+e}{2\sqrt{e}},\ x\in\mathbb{R}.$$

$\square$, $\boxed{x=\pm\frac{1}{2}}$





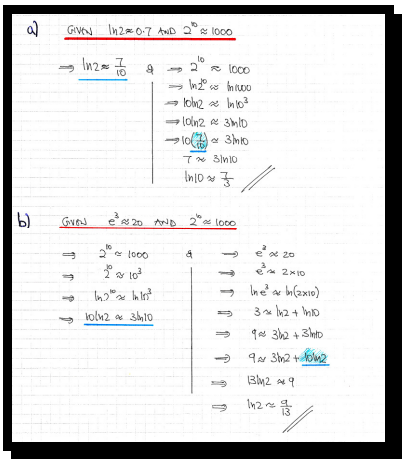
Question 112 (*****)

It is given that 2^{10} is approximately 1000.

a) Given further that $\ln 2$ is approximately 0.7, find the approximate value of $\ln 10$, giving the answer in the form $\frac{a}{b}$, where a and b are positive integers.

b) Given further that e^3 is approximately 20, show that the approximate value of $\ln 2$ is $\frac{9}{13}$.

$\square$, $\boxed{\ln 10 \approx \frac{7}{3}}$



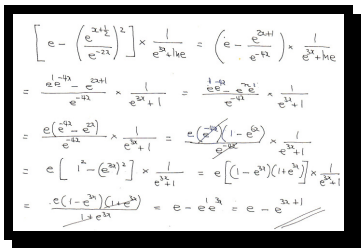
Question 113 (*****)

Show that the expression

$$\left[\mathrm{e}-\left(\frac{\mathrm{e}^{x+\frac{1}{2}}}{\mathrm{e}^{-2x}}\right)^2\right]\times\frac{1}{\mathrm{e}^{3x}+\ln\mathrm{e}}$$

simplifies to $\mathrm{e}-\mathrm{e}^{3x+1}$.

$\square$, proof

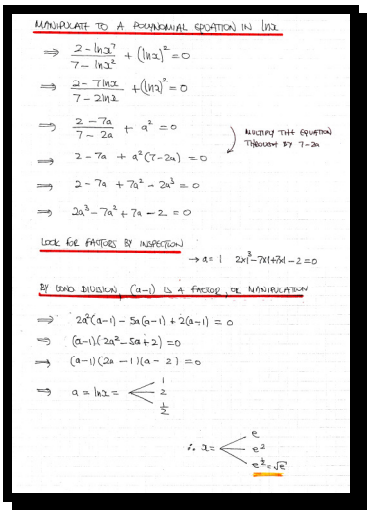


Question 114 (*****)

Find, in exact form, the solutions of the following equation

$$\frac{2-\ln x^7}{7-\ln x^2}+(\ln x)^2=0\,.$$

$\square$, $x=\mathrm{e},\ \mathrm{e}^2,\ \sqrt{\mathrm{e}}$

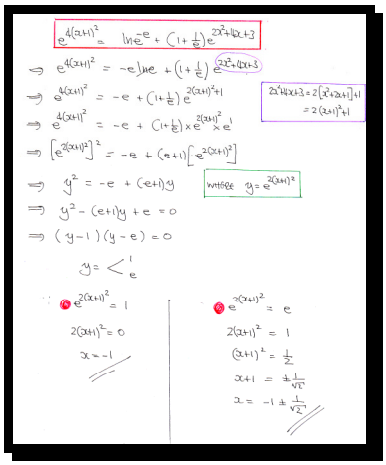


Question 115 (*****)

Solve the following equation

$$\mathrm{e}^{4(x+1)^2}=\ln \mathrm{e}^{-\mathrm{e}}+\left(1+\frac{1}{\mathrm{e}}\right)\mathrm{e}^{2x^2+4x+3}.$$

$\square$, $\boxed{x=-1,\ \frac{1}{2}\left(-2\pm\sqrt{2}\right)}$

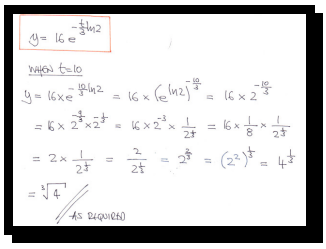


Question 116 (*****)

$$y=16\mathrm{e}^{-\frac{t}{3}\ln 2},\ t\in\mathbb{R}.$$

Show clearly that when $t=10$, $y=\sqrt[3]{4}$

$\square$, $\boxed{\text{proof}}$

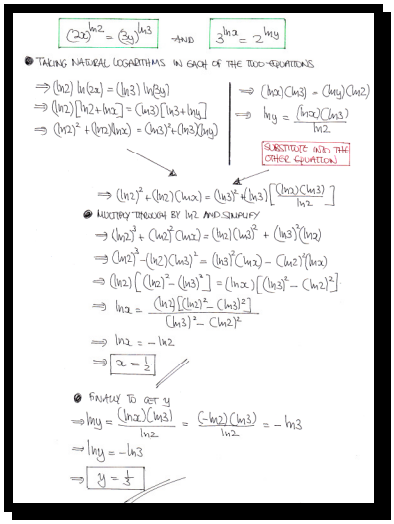


Question 117 (*****)

Solve the following simultaneous equations.

$$(2x)^{\ln 2} = (3y)^{\ln 3} \quad \text{and} \quad 3^{\ln x} = 2^{\ln y}.$$

$\square$, $\boxed{x = \frac{1}{2},\ \ y = \frac{1}{3}}$



Question 118 (*****)

A function f is defined as

$$f(x)=\mathrm{e}^{ax}+b,\quad -\ln 2\le x\le \ln 2,$$

where a and b are positive constants.

It is given that $f\left(\ln\left(\frac{3}{2}\right)\right)=\frac{13}{4}$.

a) Show clearly that

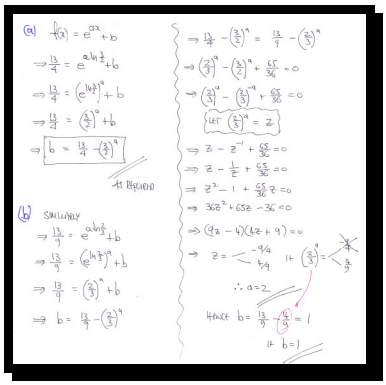
$$b=\frac{13}{4}-\left(\frac{3}{2}\right)^{a}.$$

It is further given that

$$f\left(\ln\left(\frac{2}{3}\right)\right)=\frac{13}{9}.$$

b) Find the value of a and the value of b.

$\square$, $\boxed{a=2,\ b=1}$



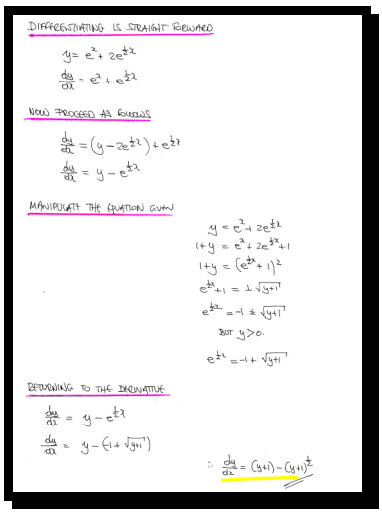
Question 119 (*****)

A curve has equation

$$y = \mathrm{e}^{x} + 2\mathrm{e}^{\frac{1}{2}x}.$$

Find a simplified expression for $\dfrac{dy}{dx}$ in terms of y.

$\square$, $\boxed{\dfrac{dy}{dx} = 1 + y - \sqrt{y+1} = (y+1) - (y+1)^{\frac{1}{2}}}$

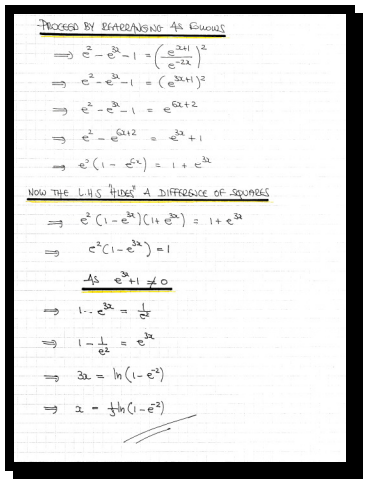


Question 120 (*****)

Find in exact form the solution of the following equation.

$$e^2-e^{3x}-1=\left(\frac{e^{x+1}}{e^{-2x}}\right)^2, \ x\in\mathbb{R}.$$

$\square$, $\boxed{x=\frac{1}{2}\ln\left(1+e^{-2}\right)}$



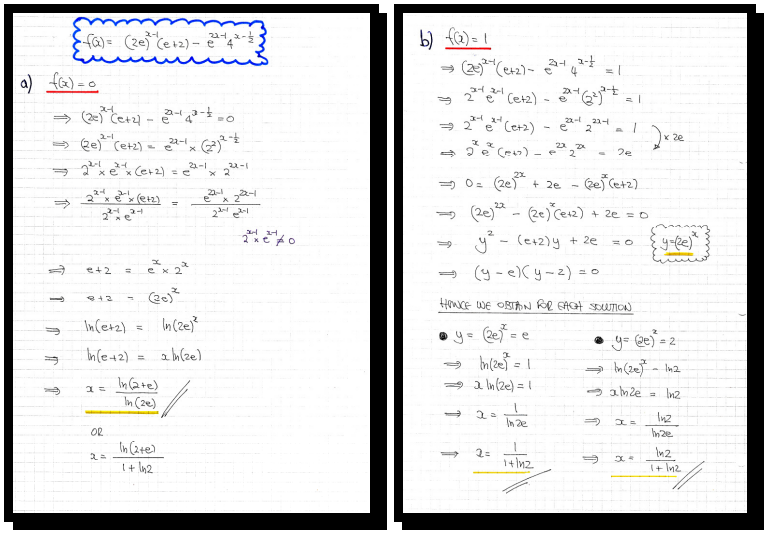
Question 121 **(*****)**

$$f(x)\equiv(2+\mathrm{e})(2\mathrm{e})^{x-1}-\mathrm{e}^{2x-1}4^{x-\frac{1}{2}},\quad x\in\mathbb{R}.$$

a) Find in exact simplified form the solution of the equation $f(x)=0$.

b) Determine, in terms of $\ln 2$, the two solutions of the equation $f(x)=1$.

$$\square,\quad \boxed{x=\frac{\ln(2+\mathrm{e})}{1+\ln 2}},\quad \boxed{x=\frac{1}{1+\ln 2}\cup x=\frac{\ln 2}{1+\ln 2}}$$



Question 122 (*****)

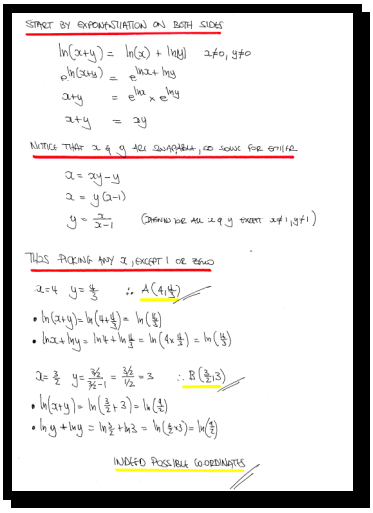
The distinct points A and B lie on the curve with equation

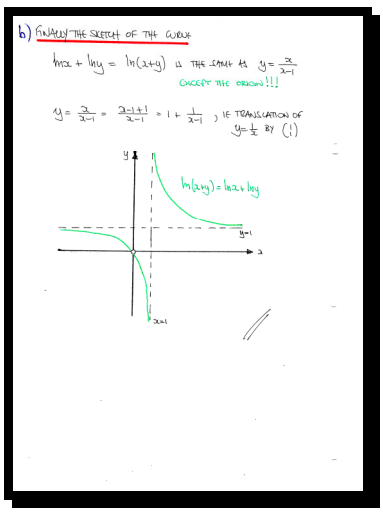
$$\ln(x+y) = \ln x + \ln y, \quad x \in (0,\infty), \quad y \in (0,\infty).$$

a) Determine possible coordinates for A and B, further verifying that these coordinates indeed satisfy the above given equation.

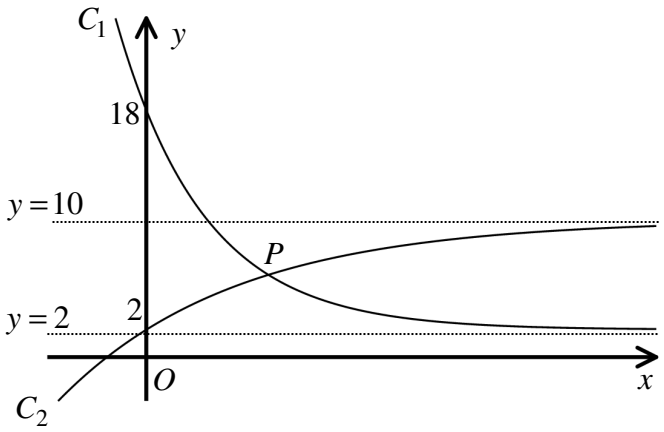
b) Sketch the curve, showing clearly all the relevant details.

$\boxed{}$, $\boxed{A\left(4,\frac{4}{3}\right)}$, $\boxed{B\left(\frac{3}{2},3\right)}$





Question 123 (*****)



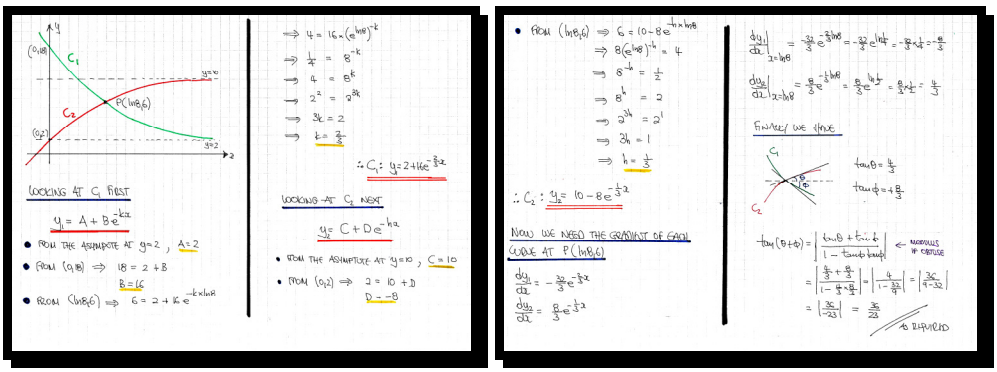
Two exponential curves, C_1 and C_2, intersect at the point $P(\ln 8, 6)$.

- C_1 meets the y axis at $(0,18)$ and the straight line with equation $y=2$ is an asymptote to C_1.

- C_2 meets the y axis at $(0,2)$ and the straight line with equation $y=10$ is an asymptote to C_2.

Show that at P, C_1 and C_1 cross each other at an acute angle of $\arctan\left(\frac{36}{23}\right)$.

[ ], [proof]



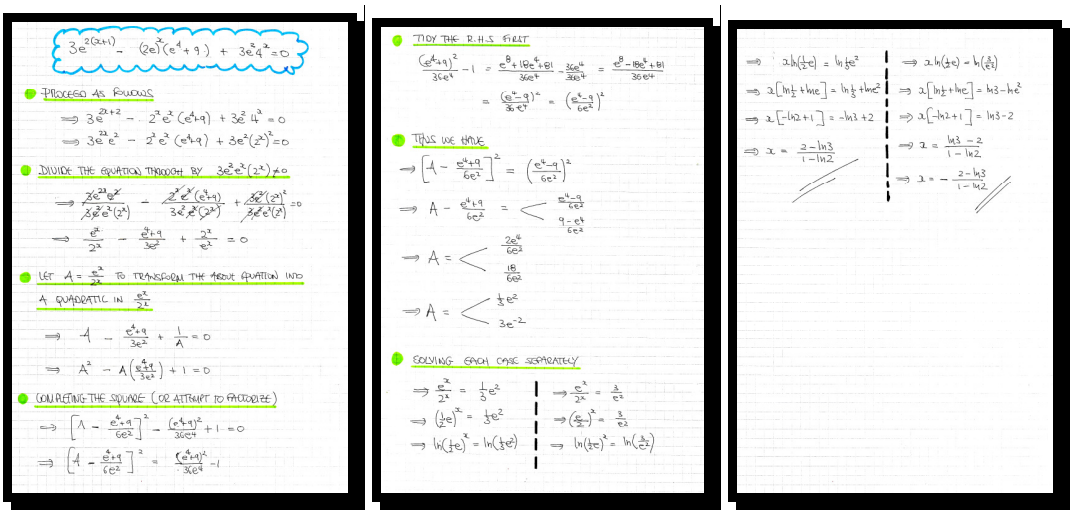
Question 124 (*****)

Solve the following equation.

$$3\mathrm{e}^{2(x+1)}-(2\mathrm{e})^{x}\left(\mathrm{e}^{4}+9\right)+3\mathrm{e}^{2}\times 4^{x}\,,\quad x\in\mathbb{R}\,.$$

Give the two solutions of the equation in the form $x=\pm A$, where A is in the form $\dfrac{a-\ln 3}{b-\ln 2}$, where a and b are positive integers.

$\square$, $\boxed{x=\pm\dfrac{2-\ln 3}{1-\ln 2}}$

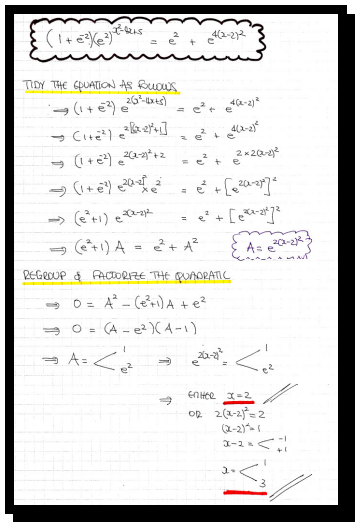


Question 125 (*****)

Solve the following equation.

$$\left(1+\mathrm{e}^{-2}\right)\left(\mathrm{e}^{2}\right)^{x^2-4x+5} = \mathrm{e}^{2}+\mathrm{e}^{4(x-2)^2}, \quad x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=1, \quad x=2, \quad x=3}$



Question 126 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

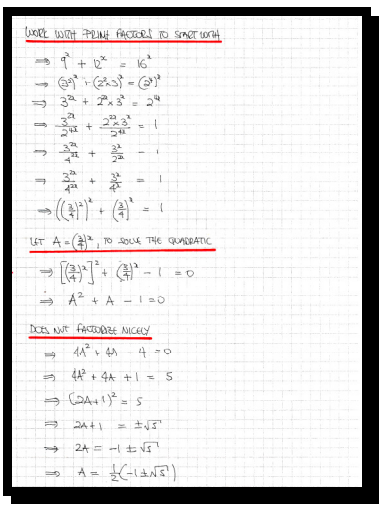
Show, with a detailed method, that the real solution of the following exponential equation

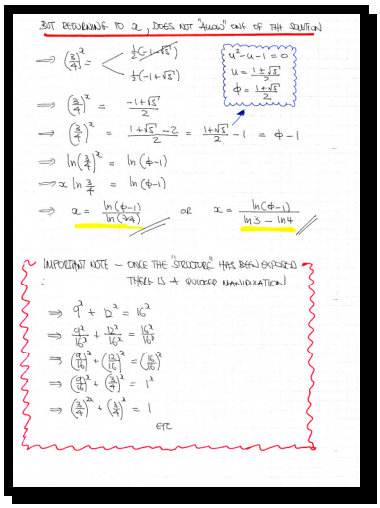
$$9^x + 12^x = 16^x,$$

can be written in the form

$$\frac{\ln(\varphi - 1)}{\ln 3 - \ln 4},$$

, proof





Question 127 (*****)

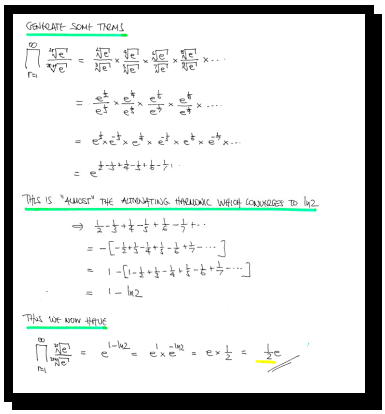
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1\times u_2\times u_3\times u_4\times ...\times u_{k-1}\times u_k\,.$$

Given that e is Euler's number, use a detailed method to find the exact value of

$$\prod_{r=1}^{\infty}\left[\frac{\sqrt[(2r)]{\mathrm{e}}}{\sqrt[(2r+1)]{\mathrm{e}}}\right].$$

$\square$, $\boxed{\frac{1}{2}\mathrm{e}}$



Question 128 (*****)

The distinct points A and B lie on the curve with equation

$$\ln(x-y) = \ln x + \ln y, \quad x \in (0,\infty), \quad y \in (0,\infty).$$

a) Determine possible coordinates for A and B, further verifying that these coordinates indeed satisfy the above given equation.

b) Sketch the curve, showing clearly all the relevant details.

$\boxed{}$, $\boxed{A(4,2)}$, $\boxed{B\left(\frac{9}{2},\frac{3}{2}\right)}$

