DIFFERENTIATION II
EXAM QUESTIONS

Question 1 (**)

The curve C has equation

$$y=\frac{x^2}{2x+1},\ x\neq -\frac{1}{2}$$

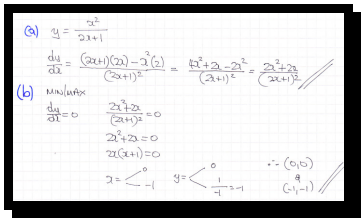
a) Show clearly that

$$\frac{dy}{dx}=\frac{2x^2+2x}{(2x+1)^2}.$$

b) Find the coordinates of the stationary points of C.

[*the nature of these stationary points need not be determined*]

$$\boxed{(0,0)\ \&\ (-1,-1)}$$



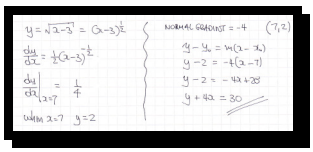
Question 2 (**)

A curve C has equation

$$y=\sqrt{x-3},\ x>3.$$

Find an equation of the normal to C at the point where $x=7$

$$\boxed{},\ \boxed{4x+y=30}$$



Question 3 **(**)**

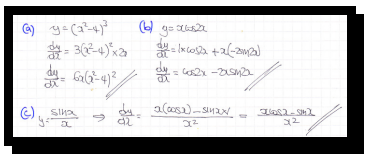
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y=\left(x^2-4\right)^3$

b) $y=x\cos 2x$

c) $y=\dfrac{\sin x}{x}$

$\square$, $\boxed{\dfrac{dy}{dx}=6x\left(x^2-4\right)^2}$, $\boxed{\dfrac{dy}{dx}=\cos 2x-2x\sin 2x}$, $\boxed{\dfrac{dy}{dx}=\dfrac{x\cos x-\sin x}{x^2}}$

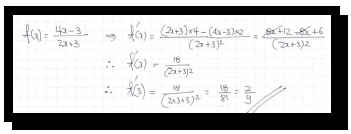


Question 4 **(**)**

$$f\left(x\right)=\frac{4x-3}{2x+3},\ x\neq -\frac{3}{2}.$$

Evaluate $f'\left(3\right)$.

$\boxed{f'(3)=\dfrac{2}{9}}$



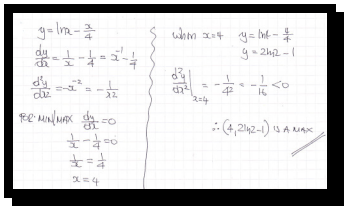
Question 5 (**)

The curve C has equation

$$y = \ln x - \frac{x}{4}, \ x > 0 .$$

Find the exact coordinates of the turning point of C, determining by calculation whether it is a maximum or minimum.

$$\boxed{\max\left(4, 2\ln 2 - 1\right)}$$



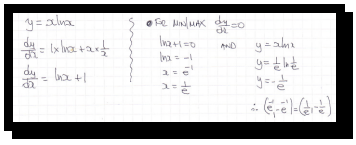
Question 6 (**)

The curve C has equation

$$y = x\ln x, \ x > 0 .$$

Find the exact coordinates of the turning point of C.

$$\boxed{\left(\frac{1}{\mathrm{e}}, -\frac{1}{\mathrm{e}}\right)}$$



Question 7 (**)

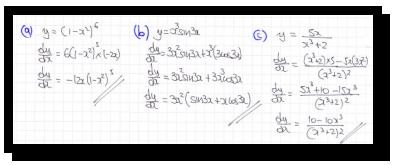
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible.

a) $y=\left(1-x^2\right)^6$

b) $y=x^3\sin 3x$

c) $y=\dfrac{5x}{x^3+2}$

$\square$, $\boxed{\dfrac{dy}{dx}=-12x\left(1-x^2\right)^5}$, $\boxed{\dfrac{dy}{dx}=3x^2\left(\sin 3x+x\cos 3x\right)}$, $\boxed{\dfrac{dy}{dx}=\dfrac{10\left(1-x^3\right)}{\left(x^3+2\right)^2}}$



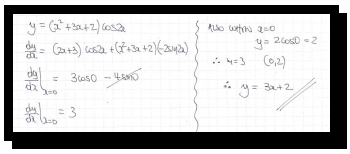
Question 8 (+)**

A curve has equation

$$y=\left(x^2+3x+2\right)\cos 2x\,.$$

Determine an equation of the tangent to the curve at the point where the curve crosses the y axis.

$\square$, $\boxed{y=3x+2}$



Question 9 (+)**

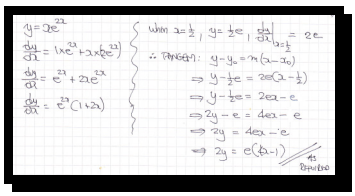
A curve C has equation

$$y = x\mathrm{e}^{2x},\ x \in \mathbb{R}.$$

Show that an equation of the tangent to C at the point where $x = \frac{1}{2}$ is

$$2y = \mathrm{e}(4x - 1).$$

$\boxed{}$, proof



Question 10 (+)**

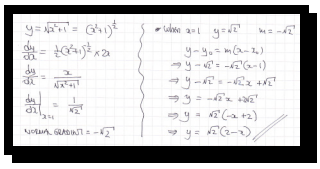
A curve C has equation

$$y = \sqrt{x^2 + 1},\ x \in \mathbb{R}.$$

Show that an equation of the normal to C at the point where $x = 1$ is given by

$$y = \sqrt{2}(2 - x).$$

$\boxed{}$, proof



Question 11 (+)**

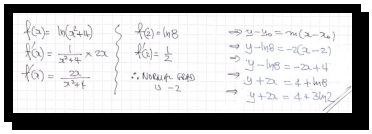
The point P, where $x=2$, lies on the curve with equation

$$f(x)=\ln\left(x^2+4\right).$$

Show that an equation of the normal to the curve at P, is given by

$$y+2x=4+3\ln 2.$$

proof



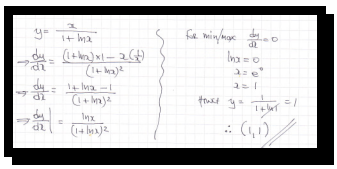
Question 12 (+)**

The curve C has equation

$$y=\frac{x}{1+\ln x},\ x>0,\ x\neq \mathrm{e}^{-1}.$$

Show that C has a single stationary point and find its coordinates.

$\square$, $(1,1)$



Question 13 (+)**

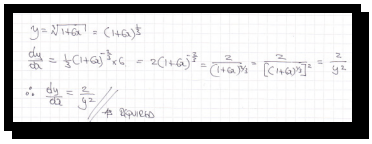
The curve C has equation

$$y=\sqrt[3]{1+6x}\ ,\ x\geq -\frac{1}{6}.$$

Show clearly that

$$\frac{dy}{dx}=\frac{2}{y^2}\ .$$

proof



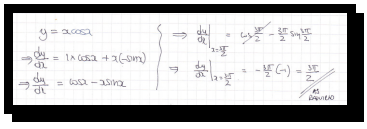
Question 14 (+)**

Given that

$$y=x\cos x\ ,\ x\in\mathbb{R}$$

Show clearly that the value of $\frac{dy}{dx}$ at $x=\frac{3\pi}{2}$ is $\frac{3\pi}{2}$.

proof



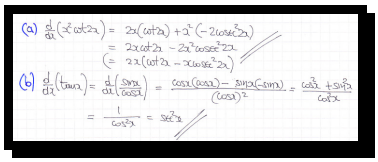
Question 15 (***)

a) Find $\dfrac{d}{dx}\left(x^2\cot 2x\right)$

b) Show clearly that

$$\frac{d}{dx}\left(\tan x\right)=\sec^2 x$$

$$\boxed{\frac{dy}{dx}=2x\left(\cot 2x-x\operatorname{cosec}^2 2x\right)}$$



Question 16 (***)

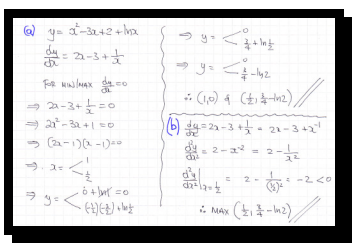
The curve C has equation

$$y=(x-1)(x-2)+\ln x,\ x>0.$$

a) Show that one of the turning points of C has coordinates $\left(\frac{1}{2},\frac{3}{4}-\ln 2\right)$ and find the coordinates of the other.

b) Determine the nature of the turning point with coordinates $\left(\frac{1}{2},\frac{3}{4}-\ln 2\right)$.

$$\boxed{(1,0)},\ \boxed{\max\left(\frac{1}{2},\frac{3}{4}-\ln 2\right)}$$



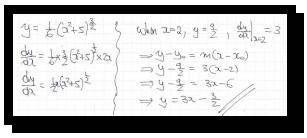
Question 17 (***)

The point P, where $x=2$, lies on the curve with equation

$$y=\frac{1}{6}\left(x^2+5\right)^{\frac{3}{2}},\ x\in\mathbb{R}.$$

Find an equation of the tangent to the curve at P.

$\square$, $\boxed{y=3x-\frac{3}{2}}$



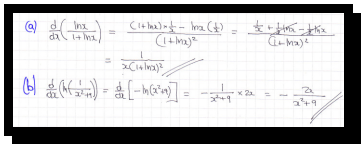
Question 18 (***)

Differentiate each of the following expressions with respect to x, writing the final answers as simplified fractions.

a) $y=\dfrac{\ln x}{1+\ln x}$.

b) $y=\ln\left(\dfrac{1}{x^2+9}\right)$.

$\square$, $\boxed{\dfrac{dy}{dx}=\dfrac{1}{x\left(1+\ln x\right)^2}}$, $\boxed{\dfrac{dy}{dx}=-\dfrac{2x}{x^2+9}}$



Question 19 (***)

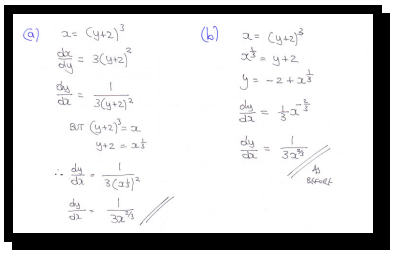
A curve has equation

$$x=\left(y+2\right)^3.$$

a) Find $\dfrac{dy}{dx}$ in **terms of** x, by **first** finding $\dfrac{dx}{dy}$.

b) By making y the subject of the equation and differentiating the resulting equation, verify the result of part (a).

$$\boxed{\frac{dy}{dx}=\frac{1}{3x^{\frac{2}{3}}}}$$



Question 20 (***)

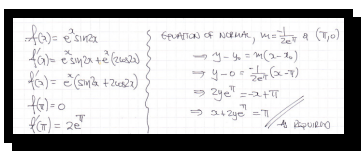
The point P, where $x=\pi$, lies on the curve with equation

$$f\left(x\right)=\mathrm{e}^x\sin 2x,\ 0\le x<2\pi.$$

Show that an equation of the normal to the curve at P, is given by

$$x+2y\mathrm{e}^{\pi}=\pi.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 21 (***)

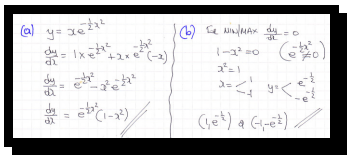
The curve C has equation

$$y = x\mathrm{e}^{-\frac{1}{2}x^2}, \ x \in \mathbb{R}.$$

a) Find an expression for $\dfrac{dy}{dx}$.

b) Find the exact coordinates of the turning points of C.

$$\square, \ \boxed{\frac{dy}{dx} = \left(1-x^2\right)\mathrm{e}^{-\frac{1}{2}x^2}}, \ \boxed{\left(1, \frac{1}{\sqrt{\mathrm{e}}}\right), \left(-1, -\frac{1}{\sqrt{\mathrm{e}}}\right)}$$



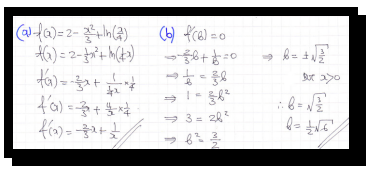
Question 22 (***)

$$f(x) = 2 - \frac{x^2}{3} + \ln\left(\frac{x}{4}\right), \ x > 0.$$

a) Find an expression for $f'(x)$.

b) Find β in exact surd form, such that $f'(\beta) = 0$.

$$\boxed{f'(x) = \frac{1}{x} - \frac{2}{3}x}, \ \boxed{\beta = \frac{1}{2}\sqrt{6}}$$



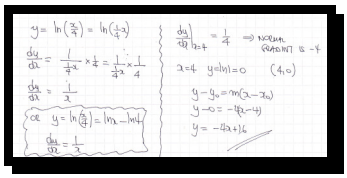
Question 23 (***)

A curve C has equation

$$y=\ln\left(\frac{x}{4}\right),\ x>0\,.$$

Find an equation of the normal to C at the point where $x=4$

$\boxed{}$, $\boxed{y=16-4x}$



Question 24 (***)

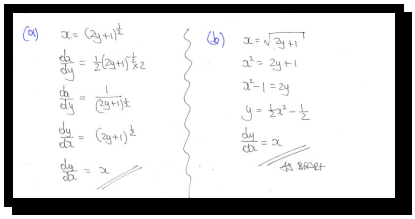
A curve has equation

$$x=\sqrt{2y+1}\,,\ y\geq -\frac{1}{2}\,.$$

a) Find $\dfrac{dy}{dx}$ in **terms of** x, by **first** finding $\dfrac{dx}{dy}$.

b) By making y the subject of the above equation and differentiating the resulting equation, verify the result of part **(a)**.

$\boxed{}$, $\boxed{\dfrac{dy}{dx}=x}$



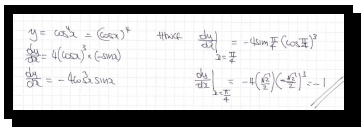
Question 25 (***)

Given that

$$y=\cos^4 x$$

find the value of $\dfrac{dy}{dx}$ at $x=\dfrac{\pi}{4}$.

$\boxed{\dfrac{dy}{dx}=-1}$



Question 26 (***)

The point P with $x=\dfrac{\pi}{4}$ lies on the curve with equation

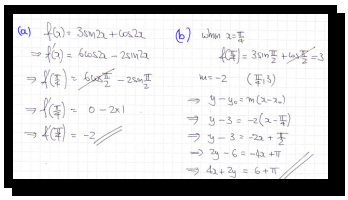
$$f\left(x\right)=3\sin 2x+\cos 2x\,,\ 0\le x<2\pi\,.$$

a) Find the gradient at P.

b) Show that an equation of the tangent to the curve at P, is given by

$$4x+2y=6+\pi\,.$$

$\boxed{-2}$



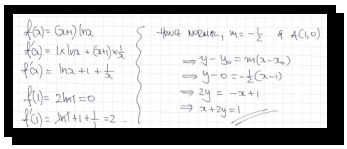
Question 27 (***)

The point A, where $x=1$, lies on the curve with equation

$$f(x)=(x+1)\ln x,\ x>0.$$

Find an equation of the normal to the curve at A.

$$\boxed{2y+x=1}$$



Question 28 (***)

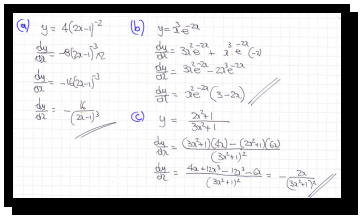
Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y=\dfrac{4}{(2x-1)^2}$.

b) $y=x^3\mathrm{e}^{-2x}$.

c) $y=\dfrac{2x^2+1}{3x^2+1}$.

$$\square,\ \boxed{\frac{dy}{dx}=-\frac{16}{(2x-1)^3}},\ \boxed{\frac{dy}{dx}=x^2(3-2x)\mathrm{e}^{-2x}},\ \boxed{\frac{dy}{dx}=-\frac{2x}{(3x^2+1)^2}}$$



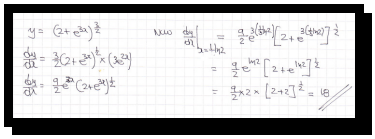
Question 29 (***)

Given that

$$y=\left(2+\mathrm{e}^{3x}\right)^{\frac{3}{2}}$$

find the value of $\dfrac{dy}{dx}$ at $x=\dfrac{1}{3}\ln 2$.

$\square$, $\boxed{\dfrac{dy}{dx}=18}$



Question 30 (***)

A curve has equation

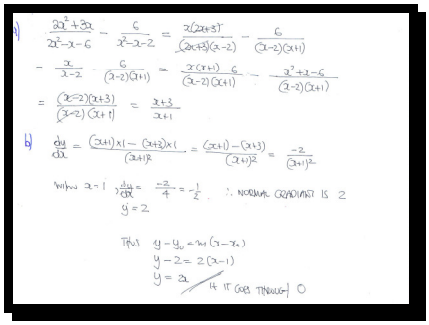
$$y=\frac{2x^2+3x}{2x^2-x-6}-\frac{6}{x^2-x-2},\ x\in\mathbb{R},\ 0<x<2.$$

a) Show clearly that

$$y=\frac{x+3}{x+1},\ x\in\mathbb{R},\ 0<x<2.$$

b) Show further that the equation of the normal to the curve at the point where $x=1$ passes through the origin.

[ ], [proof]



Question 31 (***)

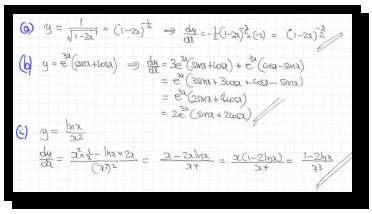
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y=\dfrac{1}{\sqrt{1-2x}}$.

b) $y=\mathrm{e}^{3x}\left(\sin x+\cos x\right)$.

c) $y=\dfrac{\ln x}{x^2}$.

$\square$, $\boxed{\dfrac{dy}{dx}=\left(1-2x\right)^{-\frac{3}{2}}}$, $\boxed{\dfrac{dy}{dx}=2\mathrm{e}^{3x}\left(\sin x+2\cos x\right)}$, $\boxed{\dfrac{dy}{dx}=\dfrac{1-2\ln x}{x^3}}$



Question 32 (***)

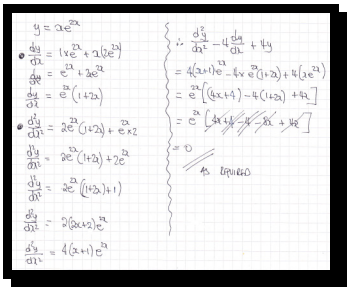
A curve has equation

$$y = x\mathrm{e}^{2x},\ x \in \mathbb{R}.$$

Show clearly that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

proof



Question 33 (***)

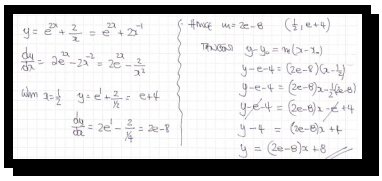
The point A, where $x = \frac{1}{2}$, lies on the curve with equation

$$y = \mathrm{e}^{2x} + \frac{2}{x},\ x \neq 0.$$

Show that an equation of the tangent to the curve at A is given by

$$y = (2\mathrm{e} - 8)x + 8.$$

proof



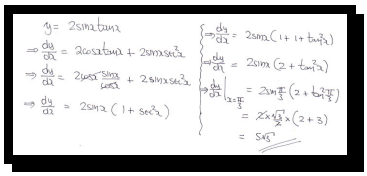
Question 34 (***)

Given that

$$y = 2\sin x \tan x$$

find the exact value of $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{3}$.

$$\boxed{\frac{dy}{dx} = 5\sqrt{3}}$$



Question 35 (***)

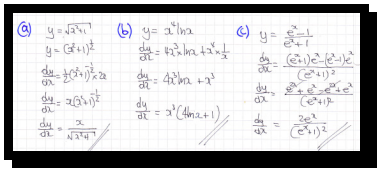
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y = \sqrt{x^2 - 1}$

b) $y = x^4 \ln x$

c) $y = \dfrac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}$

$$\square\,, \quad \boxed{\frac{dy}{dx} = x\left(x^2-1\right)^{-\frac{1}{2}}}\,, \quad \boxed{\frac{dy}{dx} = 4x^3 \ln x + x^3}\,, \quad \boxed{\frac{dy}{dx} = \frac{2\mathrm{e}^x}{\left(\mathrm{e}^x+1\right)^2}}$$



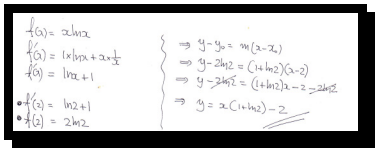
Question 36 (***)

The point A, where $x=2$, lies on the curve with equation

$$f(x)=x\ln x,\ x>0.$$

Find an equation of the tangent to the curve at A, giving the answer in the form $y=mx+c$, where m and c are exact constants.

$$\boxed{y=x(1+\ln 2)-2}$$



Question 37 (***)

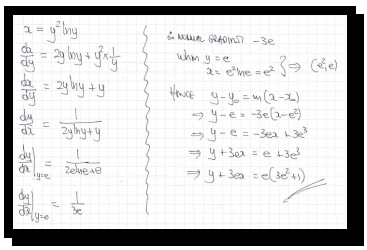
A curve C has equation

$$x=y^2\ln y,\ y>0.$$

Show that an equation of the normal to C at the point where $y=\mathrm{e}$ is

$$y+3\mathrm{e}x=\mathrm{e}\left(3\mathrm{e}^2+1\right).$$

$$\boxed{},\ \boxed{\text{proof}}$$



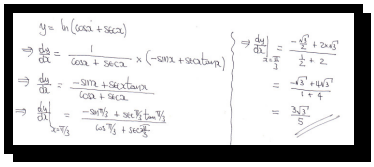
Question 38 (***)

Given that

$$y=\ln\left(\cos x+\sec x\right)$$

find the exact value of $\dfrac{dy}{dx}$ at $x=\dfrac{\pi}{3}$.

$$\boxed{\frac{dy}{dx}=\frac{3}{5}\sqrt{3}}$$



Question 39 (***)

The curve C has equation

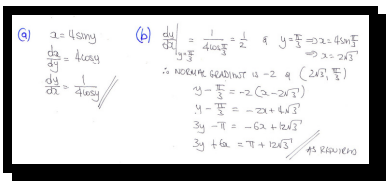
$$x=4\sin y\,,\ x>0\,.$$

a) Find $\dfrac{dy}{dx}$ in terms of y.

b) Show that an equation of the normal to the curve at the point where $y=\dfrac{\pi}{3}$ is

$$3y+6x=\pi+12\sqrt{3}\,.$$

$$\boxed{\frac{dy}{dx}=\frac{1}{4\cos y}}$$



Question 40 **(***)**

$$f(x)=5\ln x+\frac{1}{x},\ x>0.$$

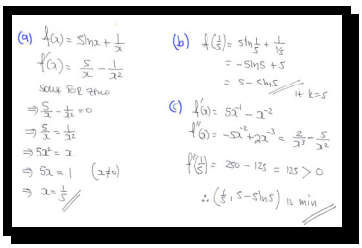
a) Solve the equation

$$f'(x)=0.$$

b) Hence write down the y coordinate of the turning point of $f(x)$ in the form $k-k\ln k$, where k is an integer.

c) Find $f''(x)$ and use it to determine the nature of the turning point of $f(x)$.

$$\boxed{x=\frac{1}{5}},\ \boxed{y=5-5\ln 5}\ \boxed{f''(x)=\frac{2}{x^3}-\frac{5}{x^2},\ f''\left(\frac{1}{5}\right)=125>0\ \text{so minimum}}$$



Question 41 (***)

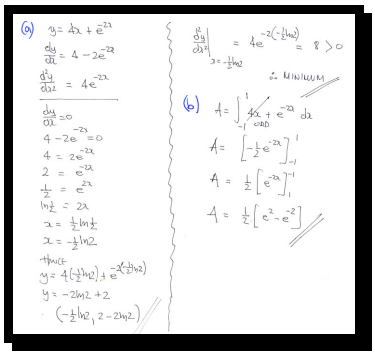
The curve C with equation $y=4x+\mathrm{e}^{-2x}$ has a turning point at A.

a) Find the exact coordinates of A and determine whether it is a local maximum or a local minimum.

The curve C lies entirely above the x axis.

b) Calculate the exact value of area bounded by the curve C, the x axis and the lines $x=1$ and $x=-1$.

$$\boxed{\min\left(-\tfrac{1}{2}\ln 2, 2-2\ln 2\right)}, \quad \boxed{\text{area}=\tfrac{1}{2}\left(\mathrm{e}^{2}-\mathrm{e}^{-2}\right)}$$



Question 42 (***)

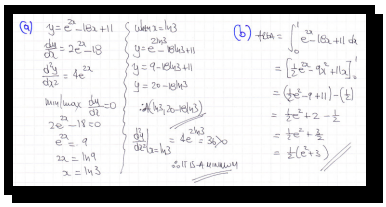
The curve C with equation $y=\mathrm{e}^{2x}-18x+11$ has a turning point at A.

a) Find the exact coordinates of A and determine whether it is a local maximum or a local minimum.

The curve C lies entirely above the x axis.

b) Calculate the exact value of area bounded by the curve C, the coordinate axes and the line $x=1$.

$\boxed{\min\left(\ln 3, 20-18\ln 3\right)}$, $\boxed{\text{area}=\frac{1}{2}\left(\mathrm{e}^2+3\right)}$



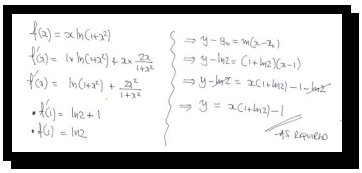
Question 43 (***)

$$f\left(x\right)=x\ln\left(1+x^2\right),\ x\in\mathbb{R}.$$

Show that an equation of the tangent to the curve with equation $y=f\left(x\right)$, at the point where $x=1$, is given by

$$y=x\left(1+\ln 2\right)-1.$$

$\boxed{\phantom{\text{proof}}}$, $\boxed{\text{proof}}$



Question 44 (***)

The equation of the curve C is given by

$$y = \mathrm{e}^{2x}\left(\cos x + \sin x\right).$$

a) Find an expression for $\dfrac{dy}{dx}$.

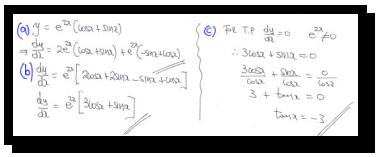
b) Show further that $\dfrac{dy}{dx}$ can be simplified to

$$\frac{dy}{dx} = \mathrm{e}^{2x}\left(\sin x + 3\cos x\right).$$

c) Hence show that the x coordinates of the turning points of C satisfy

$$\tan x = -3$$

$\square$, $\boxed{\dfrac{dy}{dx} = 2\mathrm{e}^{2x}\left(\cos x + \sin x\right) + \mathrm{e}^{2x}\left(\cos x - \sin x\right)}$



Question 45 (***)

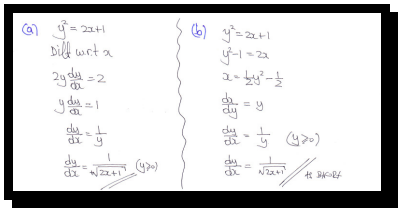
A curve has equation

$$y^2 = 2x+1,\ x \geq -\frac{1}{2},\ y \geq 0.$$

a) Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of x.

b) By making x the subject of the equation and differentiating the resulting equation, verify the result of part **(a)**.

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{2x+1}}}$$



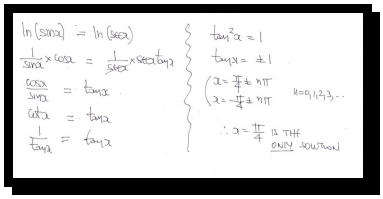
Question 46 (***)

By differentiating both sides of the equation

$$\ln(\sin x) = \ln(\sec x),\ 0 \leq x < \frac{\pi}{2},$$

show that the only solution is $x = \dfrac{\pi}{4}$.

$$\boxed{\text{proof}}$$



Question 47 (***)

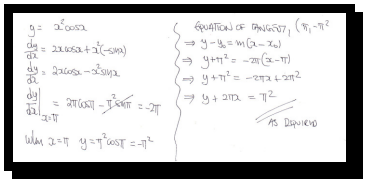
A curve has equation

$$y = x^2 \cos x, \ x \in \mathbb{R}.$$

Show that the tangent to the curve at the point where $x = \pi$ is given by

$$y + 2\pi x = \pi^2.$$

proof



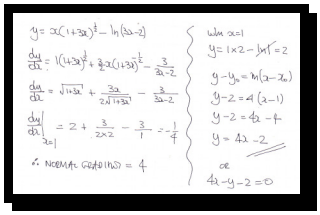
Question 48 (***)

Find an equation of the normal to the curve with equation

$$y = x\sqrt{1+3x} - \ln(3x-2), \ x \in \mathbb{R}, \ x > \frac{2}{3},$$

at the point on the curve where $x = 1$.

$y = 4x - 2$



Question 49 (*+)**

A function is defined by

$$f(x)=\sqrt{\mathrm{e}^{x}-1}\,,\ x\geq 0\,.$$

a) Find the values of …

i. … $f(\ln 5)$.

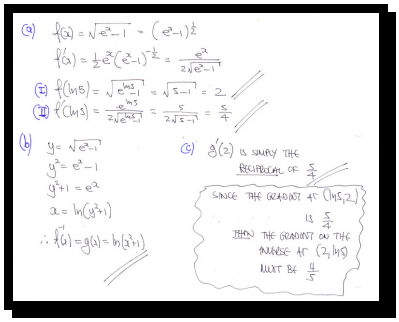
ii. … $f'(\ln 5)$.

The inverse function of $f(x)$ is $g(x)$.

b) Determine an expression for $g(x)$.

c) State the value of $g'(2)$.

$$\boxed{f(\ln 5)=2}\,,\ \boxed{f'(\ln 5)=\frac{5}{4}}\,,\ \boxed{g(x)=\ln\left(x^2+1\right)}\,,\ \boxed{g'(2)=\frac{4}{5}}$$



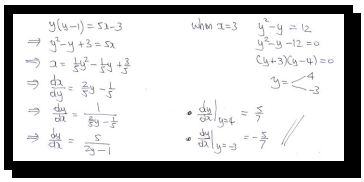
Question 50 (*+)**

A curve has equation

$$y(y-1)=5x-3.$$

Find the gradient at each of the points on the curve where $x=3$.

$\square$, $\boxed{\pm\frac{5}{7}}$



Question 51 (*+)**

The curve C has equation

$$f(x)=(2x-1)\mathrm{e}^{-2x},\ x\in\mathbb{R}.$$

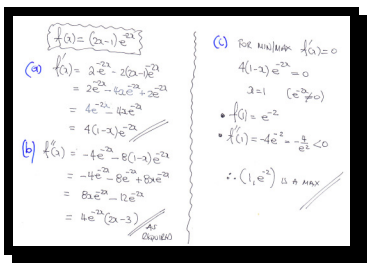
a) Find an expression for $f'(x)$.

b) Show clearly that

$$f''(x)=4(2x-3)\mathrm{e}^{-2x}.$$

c) Hence find the exact coordinates of the stationary point of C and determine its nature.

$\boxed{f'(x)=4(1-x)\mathrm{e}^{-2x}}$ $\boxed{\max\left(1,\mathrm{e}^{-2}\right)}$



Question 52 (*+)**

A curve has equation

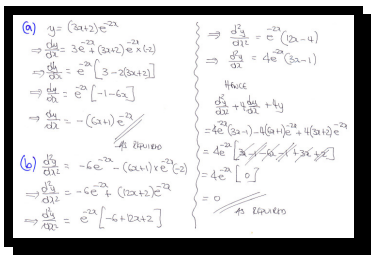
$$y=(3x+2)\mathrm{e}^{-2x}.$$

Show clearly that …

a) … $\dfrac{dy}{dx}=-(6x+1)\mathrm{e}^{-2x}$.

b) … $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=0$.

proof



Question 53 (*+)**

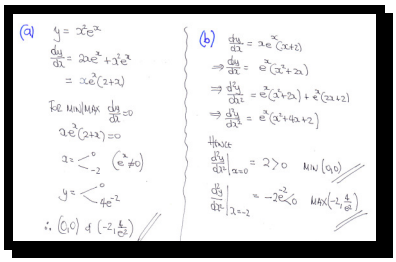
The curve C has equation

$$y = x^2 \mathrm{e}^x, \ x \in \mathbb{R}.$$

a) Find the exact coordinates of the stationary points of C.

b) By considering the sign of $\dfrac{d^2y}{dx^2}$ at each of these points determine their nature.

$\square$, $\boxed{\min(0,0)}$, $\boxed{\max\left(-2, \dfrac{4}{\mathrm{e}^2}\right)}$



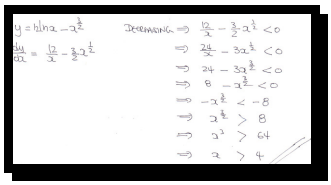
Question 54 (*+)**

The curve C has equation

$$y = 12\ln x - x^{\frac{3}{2}}, \ x > 0.$$

Determine the range of values of x for which y is decreasing.

$\boxed{x > 4}$



Question 55 (*+)**

The curve C has equation

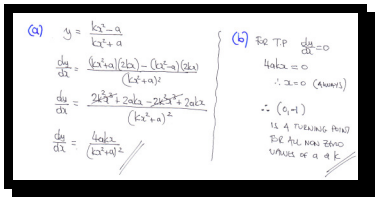
$$y=\frac{kx^2-a}{kx^2+a},$$

where k and a are non zero constants.

a) Find a simplified expression for $\dfrac{dy}{dx}$ in terms of a and k.

b) Hence show that C has a single turning point for all values of a and k, and state its coordinates.

$\square$, $\boxed{\dfrac{dy}{dx}=\dfrac{4akx}{\left(kx^2+a\right)^2}}$, $\boxed{(0,-1)}$



Question 56 (*+)**

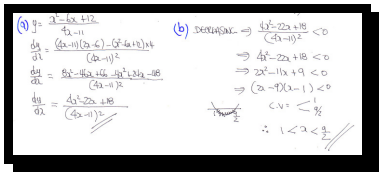
The curve C has equation

$$y=\frac{x^2-6x+12}{4x-11},\ x\in\mathbb{R},\ x\neq\frac{11}{4}\ .$$

a) Find a simplified expression for $\dfrac{dy}{dx}$.

b) Determine the range of values of x, for which y is decreasing.

$\square$, $\boxed{\dfrac{dy}{dx}=\dfrac{4x^2-22x+18}{(4x-11)^2}}$, $\boxed{1<x<\dfrac{9}{2}}$



Question 57 (*+)**

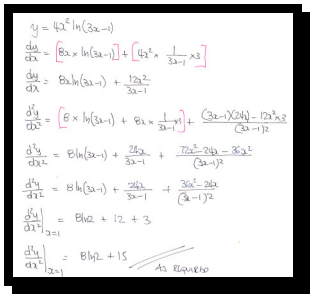
A curve C has equation

$$y=4x^2\ln(3x-1),\quad x\in\mathbb{R},\quad x>\frac{1}{3}.$$

Show that the value of $\dfrac{d^2y}{dx^2}$ at the point where $x=1$ is

$$15+8\ln 2.$$

proof



Question 58 (*+)**

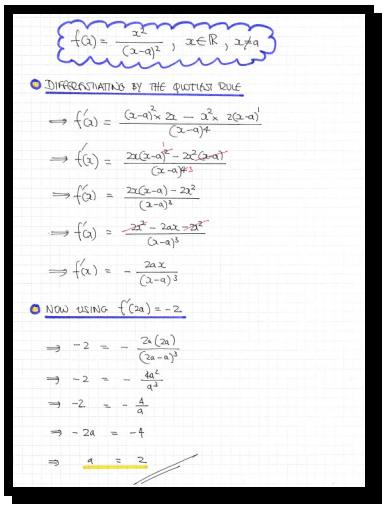
The curve C has equation

$$f(x)=\frac{x^2}{(x-a)^2},\ x\in\mathbb{R},\ x\neq a\ ,$$

where a is a non zero constant.

Given that $f'(2a)=-2$, determine the value of a.

$\boxed{}$, $\boxed{a=2}$



Question 59 (*+)**

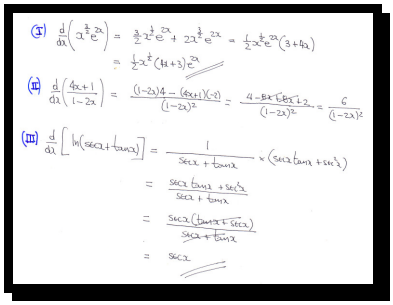
Show clearly that …

i. … $\dfrac{d}{dx}\left(x^{\frac{3}{2}}\mathrm{e}^{2x}\right)=\dfrac{1}{2}(4x+3)x^{\frac{1}{2}}\mathrm{e}^{2x}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{4x+1}{1-2x}\right)=\dfrac{6}{(1-2x)^2}$.

iii. … $\dfrac{d}{dx}\big(\ln(\sec x+\tan x)\big)=\sec x$.

proof



(i) $\frac{d}{dx}\left(2^{\frac{3}{2}}e^{2x}\right) = \frac{3}{2}2^{\frac{1}{2}}e^{2x} + 2^{\frac{3}{2}}e^{2x} = \frac{1}{2}2^{\frac{1}{2}}e^{2x}(3+4x)$

$= \frac{1}{2}2^{\frac{1}{2}}(4x+3)e^{2x}$

(ii) $\frac{d}{dx}\left(\frac{4x+1}{1-2x}\right) = \frac{(1-2x)4-(4x+1)(-2)}{(1-2x)^2} = \frac{4-8x+8x+2}{(1-2x)^2} = \frac{6}{(1-2x)^2}$

(iii) $\frac{d}{dx}\left[\ln(\sec x+\tan x)\right] = \frac{1}{\sec x+\tan x}\times(\sec x\tan x+\sec^2 x)$

$= \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$

$= \frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}$

$= \sec x$

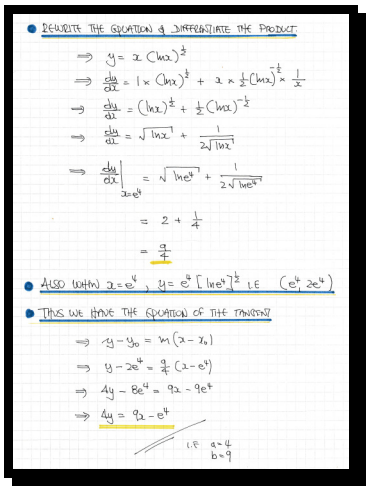
Question 60 (*+)**

The curve C has equation

$$y = x\sqrt{\ln x}\,, \quad x > 1\,.$$

Find an equation of the tangent to the curve at the point where $x = \mathrm{e}^4$ giving the answer in the form $ay = bx - \mathrm{e}^4$, where a and b are integers.

$\boxed{}$, $\boxed{4y = 9x - \mathrm{e}^4}$



Question 61 (*+)**

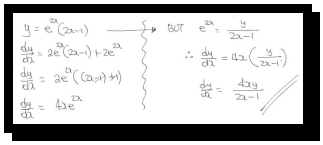
The curve C has equation

$$y=\mathrm{e}^{2x}\left(2x-1\right),\ x\in\mathbb{R}.$$

Show clearly that

$$\frac{dy}{dx}=\frac{4xy}{2x-1}\ .$$

proof



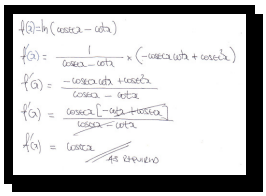
Question 62 (*+)**

$$f\left(x\right)=\ln\left(\operatorname{cosec}x-\cot x\right),\ 0<x<\pi\ .$$

Show clearly that

$$f'\left(x\right)=\operatorname{cosec}x\ .$$

proof

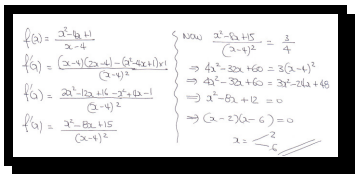


Question 63 (*+)**

$$f(x)=\frac{x^2-4x+1}{x-4},\quad x\in\mathbb{R},\ x\neq 4.$$

Solve the equation $f'(x)=\frac{3}{4}$.

$\boxed{x=2,\ 6}$



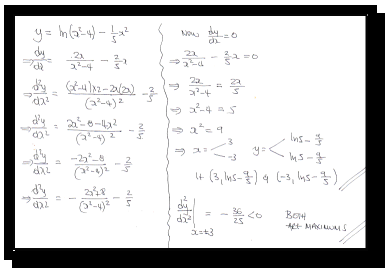
Question 64 (*+)**

The curve C has equation

$$y=\ln\left(x^2-4\right)-\frac{1}{5}x^2\ ,\ |x|>2.$$

Find the exact coordinates of the turning points of C and determine their nature.

$\boxed{\max\left(-3,\ln 5-\frac{9}{5}\right)}$, $\boxed{\max\left(3,\ln 5-\frac{9}{5}\right)}$



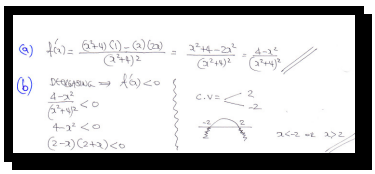
Question 65 **(***+)**

$$f(x)=\frac{x}{x^2+4},\ x\in\mathbb{R}.$$

a) Find an expression for $f'(x)$.

b) Determine the range of values of x, for which $f(x)$ is decreasing.

$\square$, $\boxed{f'(x)=\dfrac{4-x^2}{\left(x^2+4\right)^2}}$, $\boxed{x<-2 \text{ or } x>2}$



Question 66 (*+)**

A curve C is defined by the function

$$f(x)=\frac{1+\sin 2x}{4+\cos 2x},\ 0\le x<2\pi .$$

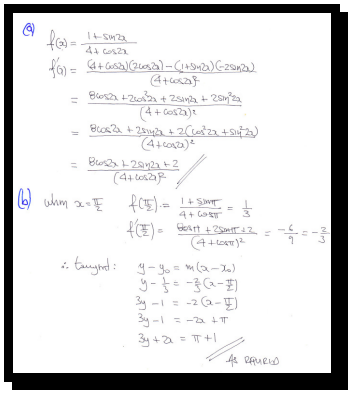
a) Show clearly that

$$f'(x)=\frac{2+2\sin 2x+8\cos 2x}{(4+\cos 2x)^2}.$$

b) Show further that the equation of the tangent to C at the point with $x=\frac{\pi}{2}$ is given by

$$2x+3y=\pi+1.$$

proof



Question 67 (*+)**

A curve C has equation

$$y=\frac{4x+k}{4x-k},\ x\neq\frac{k}{4},$$

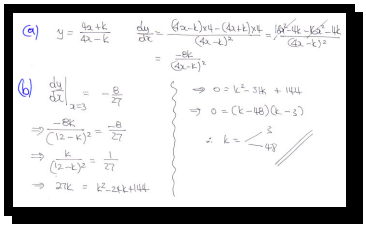
where k is a non zero constant.

a) Find a simplified expression for $\dfrac{dy}{dx}$, in terms k.

The point P lies on C, where $x=3$.

b) Given that the gradient at P is $-\dfrac{8}{27}$, show that one possible value of k is 48 and find the other.

$\square$, $\boxed{\dfrac{dy}{dx}=-\dfrac{8k}{(4x-k)^2}}$, $\boxed{k=3}$



Question 68 (*+)**

A curve C has equation

$$y=\mathrm{e}^{2x}\left(x^2-4x-2\right),\ x\in\mathbb{R}.$$

a) Show clearly that

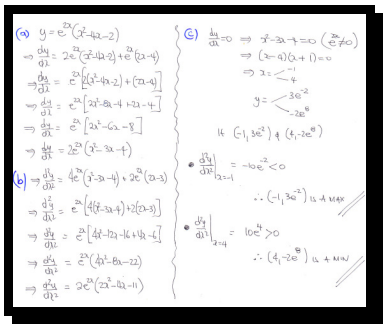
$$\frac{dy}{dx}=2\mathrm{e}^{2x}\left(x^2-3x-4\right).$$

b) Show further that

$$\frac{d^2y}{dx^2}=2\mathrm{e}^{2x}\left(2x^2-4x-11\right).$$

c) Hence find the exact coordinates of the stationary points of C and use $\dfrac{d^2y}{dx^2}$ to determine their nature.

$$\boxed{\phantom{\min\left(4,-2\mathrm{e}^8\right)}},\ \boxed{\min\left(4,-2\mathrm{e}^8\right)},\ \boxed{\max\left(-1,3\mathrm{e}^{-2}\right)}$$



Question 69 (*+)**

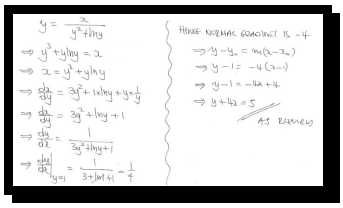
The curve C has equation given by

$$y=\frac{x}{y^2+\ln y},\ y>0.$$

Show that an equation of the normal to C at the point $(1,1)$ is

$$4x+y=5.$$

$\boxed{}$, $\boxed{\text{proof}}$

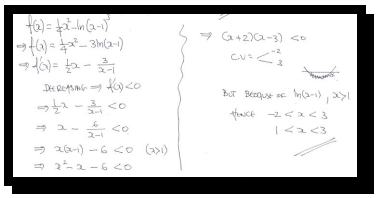


Question 70 (*+)**

$$f(x)=\frac{1}{4}x^2-\ln(x-1)^3,\ x\in\mathbb{R},\ x>1.$$

Find the range of values of x for which $f(x)$ is a decreasing function.

$\boxed{1<x<3}$



Question 71 (*+)**

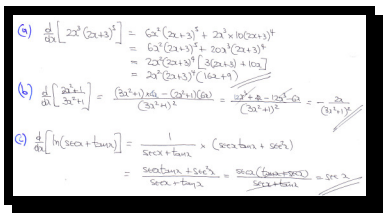
Show clearly that …

i. … $\dfrac{d}{dx}\left[2x^3\left(2x+3\right)^5\right]=2x^2\left(16x+9\right)\left(2x+3\right)^4$.

ii. … $\dfrac{d}{dx}\left[\dfrac{2x^2+1}{3x^2+1}\right]=-\dfrac{2x}{\left(3x^2+1\right)^2}$.

iii. … $\dfrac{d}{dx}\left[\ln\left(\sec x+\tan x\right)\right]=\sec x$.

proof



Question 72 **(***+)**

The curve C has equation

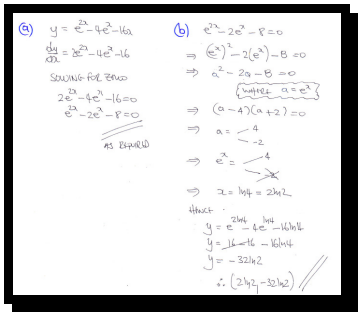
$$y=\mathrm{e}^{2x}-4\mathrm{e}^{x}-16x\,.$$

a) Show that the x coordinates of the stationary points of C satisfy the equation

$$\mathrm{e}^{2x}-2\mathrm{e}^{x}-8=0\,.$$

b) Hence determine the exact coordinates of the stationary point of C, giving the answer in terms of $\ln 2$.

$\square$, $\boxed{(2\ln 2,-32\ln 2)}$



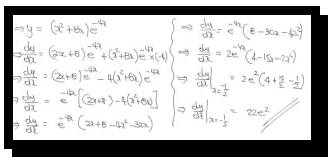
Question 73 **(***+)**

Given that

$$y=\left(x^2+8x\right)\mathrm{e}^{-4x},$$

find the exact value of $\dfrac{dy}{dx}$ at $x=-\dfrac{1}{2}$.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=-\frac{1}{2}}=22\mathrm{e}^{2}}$$



Question 74 **(***+)**

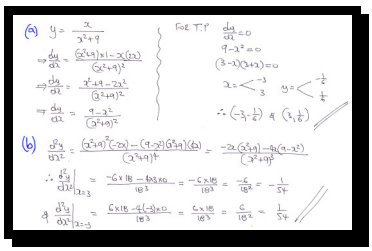
The equation of a curve C is

$$y=\frac{x}{x^2+9},\ x\in\mathbb{R}\ .$$

a) Find the coordinates of the stationary points of C.

b) Evaluate the exact value of $\dfrac{d^2y}{dx^2}$ at each of these stationary points.

$$\square,\ \boxed{\left(3,\frac{1}{6}\right),\ \left(-3,-\frac{1}{6}\right)},\ \boxed{\left.\frac{d^2y}{dx^2}\right|_{x=\pm3}=\mp\frac{1}{54}}$$



Question 75 **(***+)**

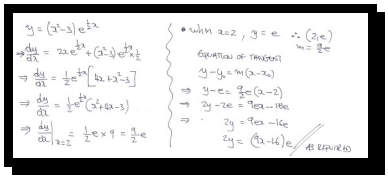
The point A, where $x=2$, lies on the curve with equation

$$y=\left(x^2-3\right)\mathrm{e}^{\frac{1}{2}x}\ .$$

Show that an equation of the tangent to the curve at A is given by

$$2y=\left(9x-16\right)\mathrm{e}\ .$$

proof



Question 76 (*+)**

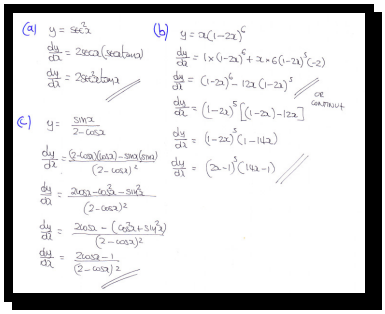
Differentiate each of the following expressions with respect to x, simplifying the final answer as far as possible.

a) $y=\sec^2 x$.

b) $y=x(1-2x)^6$.

c) $y=\dfrac{\sin x}{2-\cos x}$.

$$\boxed{\frac{dy}{dx}=2\sec^2 x\tan x},\quad \boxed{\frac{dy}{dx}=(14x-1)(2x-1)^5=(1-14x)(1-2x)^5},\quad \boxed{\frac{dy}{dx}=\frac{2\cos x-1}{(2-\cos x)^2}}$$



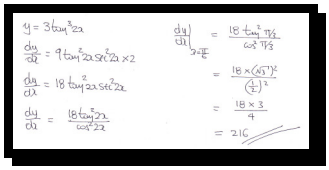
Question 77 (*+)**

Given that

$$y=3\tan^3 2x$$

find the value of $\dfrac{dy}{dx}$ at $x=\dfrac{\pi}{6}$.

$$\boxed{\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}}=216}$$



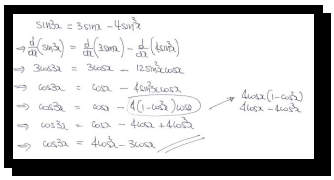
Question 78 (*+)**

The following trigonometric identity is given

$$\sin 3x \equiv 3\sin x - 4\sin^3 x .$$

By differentiating both sides of the above trigonometric identity with respect to x, find the corresponding identity for $\cos 3x$ in terms of $\cos x$.

$\boxed{}$, $\boxed{\cos 3x = 4\cos^3 x - 3\cos x}$



Question 79 (*+)**

$$f(x) = 8x^2 + 8x + \ln x , \ x > 0 .$$

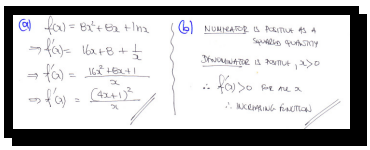
a) Show clearly that

$$f'(x) = \frac{(ax+b)^2}{x},$$

where a and b are integers.

b) Hence show that $f(x)$ is an increasing function.

$\boxed{a = 4, \ b = 1}$



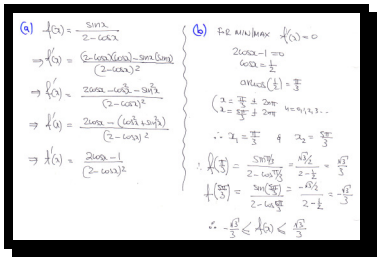
Question 80 **(***+)**

$$f(x)=\frac{\sin x}{2-\cos x},\ 0\le x<2\pi\ .$$

a) Find a simplified expression for $f'(x)$.

b) Hence find the minimum and maximum value of $f(x)$.

$$\boxed{f(x)=\frac{2\cos x-1}{(2-\cos x)^2}},\ \boxed{-\frac{\sqrt{3}}{3}\le f(x)\le\frac{\sqrt{3}}{3}}$$



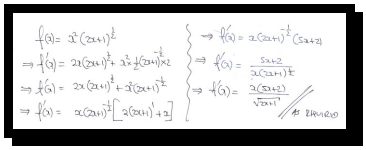
Question 81 **(***+)**

$$f(x)=x^2\sqrt{2x+1},\ x\ge-\frac{1}{2}\ .$$

Show clearly that

$$f'(x)=\frac{x(5x+2)}{\sqrt{2x+1}}.$$

$$\boxed{\text{proof}}$$



Question 82 (*+)**

The curve C has equation

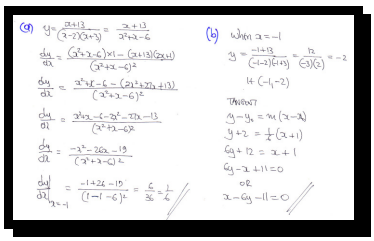
$$y=\frac{x+13}{(x-2)(x+3)},\ x\neq -3,2\,.$$

The point A lies on C and has $x=-1$.

a) Find the value of $\dfrac{dy}{dx}$ at A.

b) Find an equation of the tangent to C at A, giving the final answer in the form $ax+by+c=0$, where a, b and c are integers.

$$\boxed{\frac{1}{6}},\ \boxed{x-6y-11=0}$$



Question 83 (*+)**

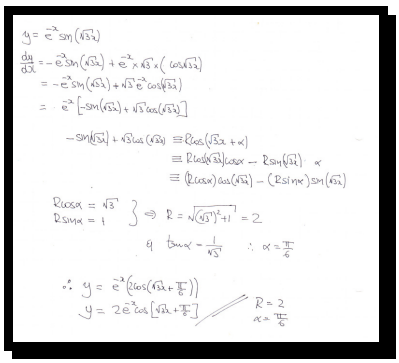
$$y = \mathrm{e}^{-x}\sin\left(\sqrt{3}x\right),\ x \in \mathbb{R}.$$

Find the exact value of each of the constants R and α so that

$$\frac{dy}{dx} = R\,\mathrm{e}^{-x}\cos\left(\sqrt{3}x+\alpha\right),$$

where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$.

$\square$, $\boxed{R=2}$, $\boxed{\alpha=\dfrac{\pi}{6}}$



Question 84 **(***+)**

A curve has equation

$$y=\frac{1}{2}\ln\left(\frac{x}{3}\right),\ x>0\,.$$

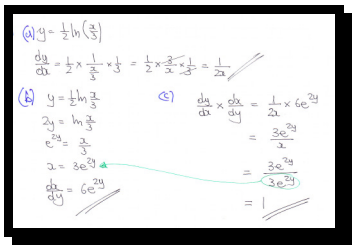
a) Find an expression for $\dfrac{dy}{dx}$ in terms of x.

b) By making x the subject of the equation and differentiating the resulting equation, find $\dfrac{dx}{dy}$.

c) Use the results of parts (a) and (b), to deduce that

$$\frac{dy}{dx}\times\frac{dx}{dy}=1\,.$$

$\square$, $\boxed{\dfrac{dy}{dx}=\dfrac{1}{2x}}$, $\boxed{\dfrac{dx}{dy}=6\mathrm{e}^{2y}=2x}$



Question 85 (*+)**

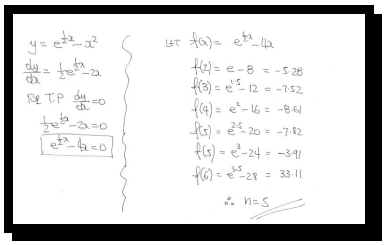
A curve C has equation

$$y = \mathrm{e}^{\frac{1}{2}x} - x^2, \ x \in \mathbb{R}.$$

The curve has a single stationary point at $x = x_0$, such that $n < x_0 < n+1, \ n \in \mathbb{N}$.

Determine the value of n.

$\boxed{n = 5}$



Question 86 (*+)**

A curve has equation

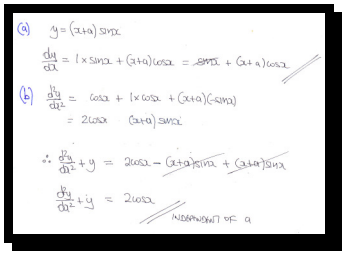
$$y=(x+a)\sin x,$$

where a is a non zero constant.

a) Find an expression for $\dfrac{dy}{dx}$.

b) Show that $\dfrac{d^2y}{dx^2}+y$ is independent of a.

$$\boxed{\frac{dy}{dx}=(a+x)\cos x+\sin x}\,,\qquad \boxed{\frac{d^2y}{dx^2}+y=2\cos x}$$



Question 87 **(***+)**

The curve C has equation

$$y=\frac{(x+3)(x-1)}{x+2}-3\ln(x+2),\ x>-2.$$

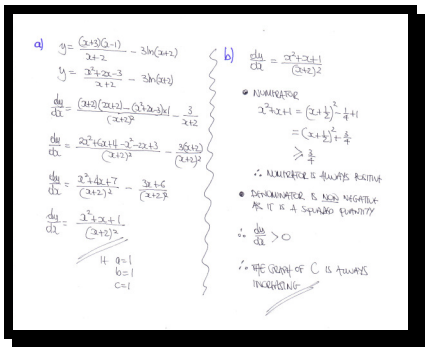
a) Show clearly that

$$\frac{dy}{dx}=\frac{ax^2+bx+c}{(x+2)^2},$$

where a, b and c are constants to be found.

b) Deduce that the graph of C is increasing for all allowable values of x.

$$\boxed{a=1},\ \boxed{b=1},\ \boxed{c=1}$$



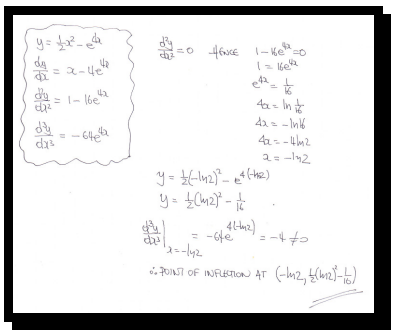
Question 88 (*+)**

The curve C has equation

$$y=\frac{1}{2}x^2-\mathrm{e}^{4x}.$$

Show clearly that C has a point of inflection, determining its exact coordinates.

$$\left[-\ln 2,\frac{1}{2}(\ln 2)^2-\frac{1}{16}\right]$$

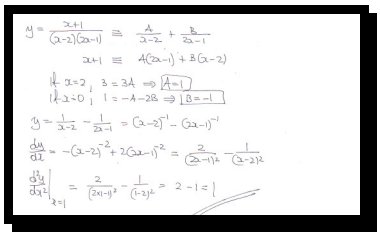


Question 89 (*+)**

$$y=\frac{x+1}{(x-2)(2x-1)},\ x\neq 2,\ x\neq\frac{1}{2}.$$

Find the value of $\dfrac{dy}{dx}$ at $x=1$.

$$\left.\frac{dy}{dx}\right|_{x=1}=1$$



Question 90 **(***+)**

A curve C has equation

$$y=\frac{1}{4}\mathrm{e}^{2x}+3\,,\ x\in\mathbb{R}\,.$$

The point P lies on C where $x=\ln 2$.

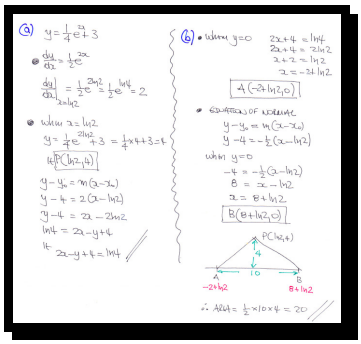
a) Show that the equation of the tangent to the curve at the point P is

$$2x-y+4=\ln 4\,.$$

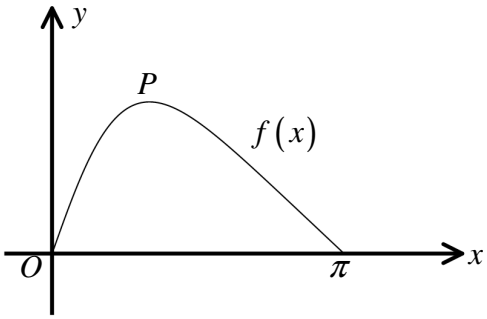
This tangent meets the x axis at the point A, and the normal to the curve at the point P meets the x axis at the point B.

b) Show that the area of the triangle APB is 20 square units.

proof



Question 91 (*+)**



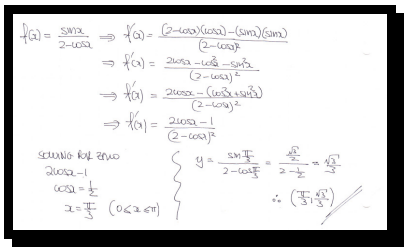
The figure above shows the graph of the curve with equation

$$f(x)=\frac{\sin x}{2-\cos x},\ 0\le x\le \pi .$$

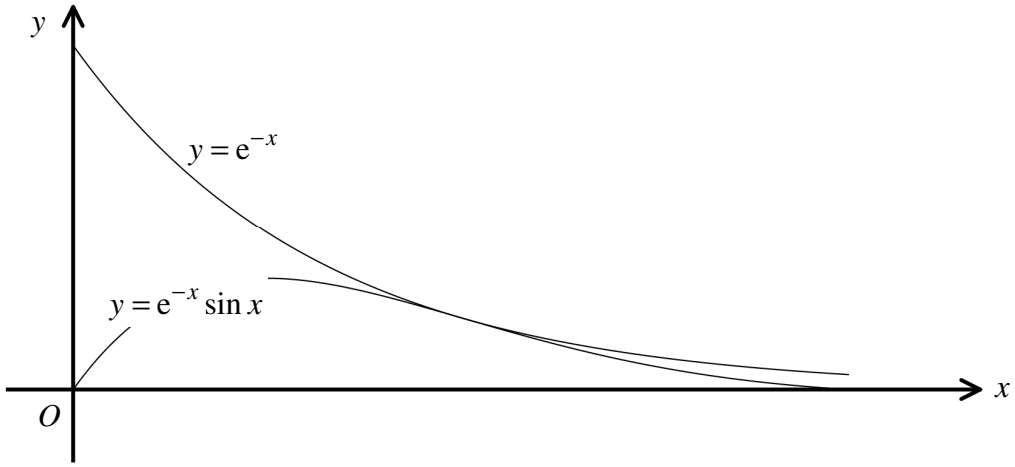
The curve has a stationary point at P.

Determine the exact coordinates of P.

$\square$, $\boxed{P\left(\frac{1}{3}\pi,\frac{1}{3}\sqrt{3}\right)}$



Question 92 (*+)**



The figure above shows the graph of

$$y = \mathrm{e}^{-x} \qquad \text{and} \qquad y = \mathrm{e}^{-x}\sin x, \ \ 0 \le x \le \pi .$$

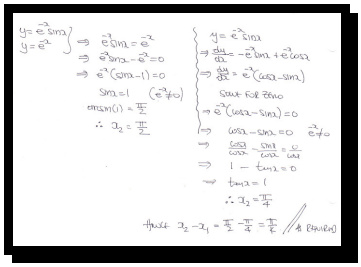
The curve with equation $y = \mathrm{e}^{-x}\sin x$ has a local maximum at the point where $x = x_1$.

The curves touch each other at the point where $x = x_2$.

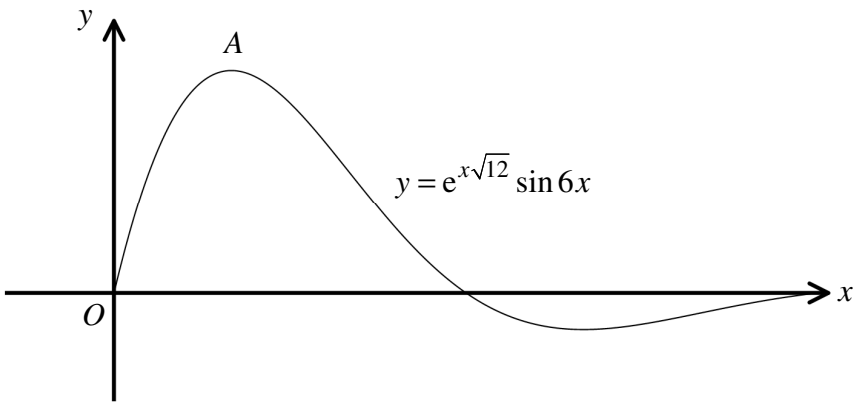
Show clearly that

$$x_2 - x_1 = \frac{\pi}{4}.$$

, proof



Question 93 (*+)**



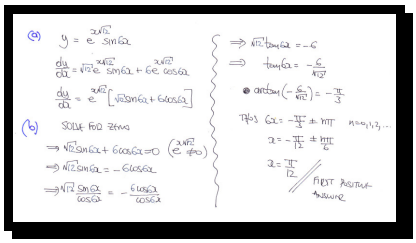
The figure above shows the graph of the curve with equation

$$y = \mathrm{e}^{x\sqrt{12}} \sin 6x\,,\ 0 \le x \le \frac{\pi}{3}\,.$$

The curve has a local maximum at the point A.

Find, in terms of π, the x coordinate of A.

$\square$, $\boxed{\dfrac{\pi}{9}}$



Question 94 **(***+)**

$$f(x)=\frac{6x-13}{(x+2)(x-3)},\ x\in\mathbb{R},\ x\neq -2,3.$$

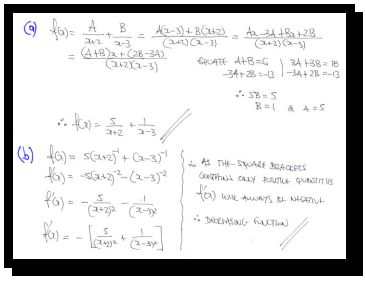
a) Show clearly that

$$f(x)=\frac{A}{(x+2)}+\frac{B}{(x-3)},$$

where A and B are integers.

b) Hence show further that $f(x)$ is a decreasing function.

$\boxed{A=5,\ B=1}$



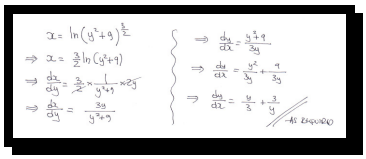
Question 95 **(***+)**

$$x=\ln\left(y^2+9\right)^{\frac{3}{2}}.$$

Show clearly that

$$\frac{dy}{dx}=\frac{y}{3}+\frac{3}{y}.$$

$\boxed{}$, $\boxed{\text{proof}}$



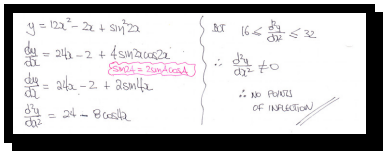
Question 96 **(****)**

The curve C has equation

$$y=12x^2-2x+\sin^2 2x\,.$$

Show clearly that C has no points of inflection.

proof



Question 97 **(****)**

$$y=a^x,\ a>0,\ a\neq 1\,.$$

a) Show clearly that

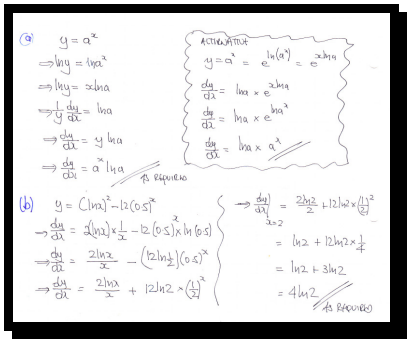
$$\frac{dy}{dx}=a^x\ln a\,.$$

A curve has equation

$$y=(\ln x)^2-12(0.5)^x,\ x>0\,.$$

b) Show that at the point of the curve where $x=2$, the gradient is $4\ln 2$.

proof



Question 98 **(****)**

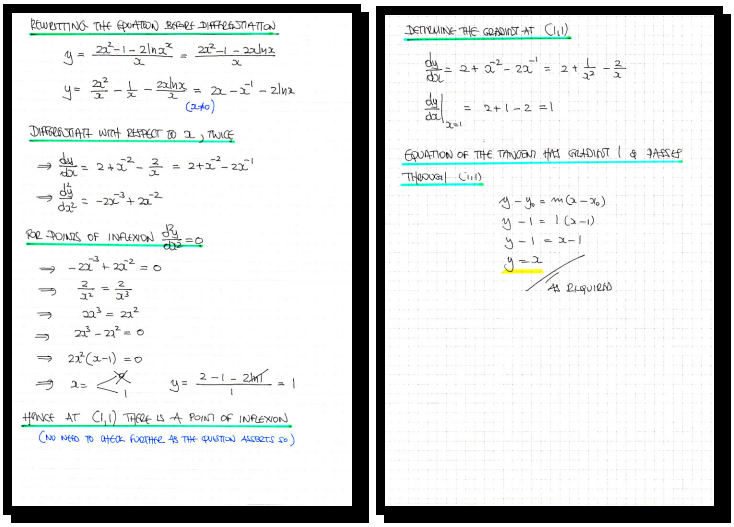
The curve C has equation

$$y=\frac{2x^2-1-2\ln x^x}{x},\ x>0.$$

The curve has a point of inflection at P.

Show that the straight line with equation $y=x$ is a tangent to C at P.

[ ], [proof]

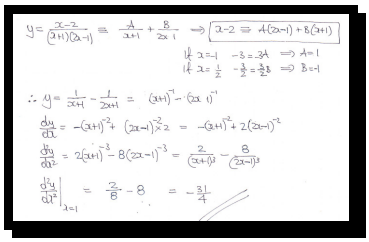


Question 99 (****)

$$y=\frac{x-2}{(x+1)(2x-1)},\ x\neq -1,\ x\neq \frac{1}{2}.$$

Find the value of $\dfrac{d^2y}{dx^2}$ at $x=1$.

$$\left.\frac{d^2y}{dx^2}\right|_{x=1}=-\frac{31}{4}$$



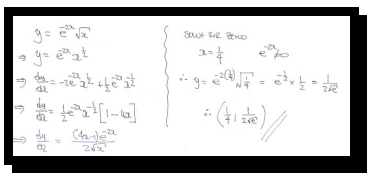
Question 100 (****)

The equation of a curve C is

$$y=\mathrm{e}^{-2x}\sqrt{x},\ x\geq 0\ .$$

Determine the exact coordinates of the stationary point of C.

$$\left(\frac{1}{4},\frac{1}{2\sqrt{\mathrm{e}}}\right)$$



Question 101 (****)

The curve C has equation

$$x = y\sqrt{1-4y}\,,\ y \le \frac{1}{4}.$$

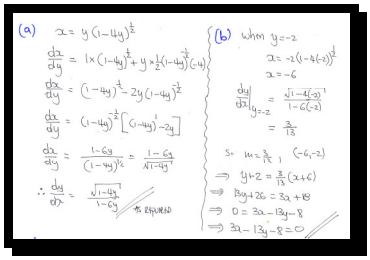
a) Show clearly that

$$\frac{dy}{dx} = \frac{\sqrt{1-4y}}{1-6y}\ .$$

b) Show further that an equation of the tangent to C at the point where $y = -2$ is

$$3x - 13y - 8 = 0\,.$$

proof



Question 102 **(****)**

Differentiate each the following expressions with respect to x, simplifying the final answers as far as possible.

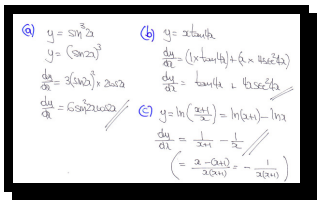
(Fractional answers must not involve double fractions)

a) $y=\sin^3 2x$.

b) $y=x\tan 4x$.

c) $y=\ln\left(\dfrac{x+1}{x}\right)$.

$$\boxed{\frac{dy}{dx}=6\sin^2 2x\cos 2x}\,,\quad \boxed{\frac{dy}{dx}=\tan 4x+4x\sec^2 4x}\,,\quad \boxed{\frac{dy}{dx}=-\frac{1}{x^2+x}}$$



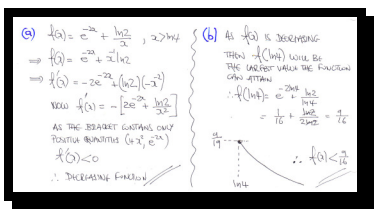
Question 103 **(****)**

$$f(x)=\mathrm{e}^{-2x}+\frac{\ln 2}{x}\,,\quad x\in\mathbb{R}\,,\quad x>\ln 4\,.$$

a) Show that $f(x)$ in a decreasing function.

b) Hence find the range of $f(x)$.

$$\boxed{}\,,\quad \boxed{f(x)\in\mathbb{R},\ f(x)<\frac{9}{16}}$$



Question 104 (****)

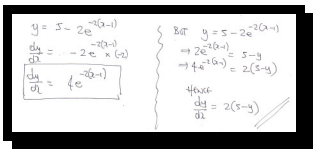
Given that

$$y = 5 - 2\mathrm{e}^{-2(x-1)}$$

show clearly that

$$\frac{dy}{dx} = 2(5 - y).$$

proof



Question 105 **(****)**

$$f(x)=x\sqrt{9-4x^2}\ ,\ 0\le x\le\frac{3}{2}\ .$$

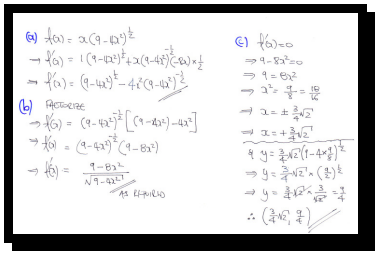
a) Find an expression for $f'(x)$.

b) Show further that

$$f'(x)=\frac{9-8x^2}{\sqrt{9-4x^2}}\ .$$

c) Hence calculate that the exact coordinates of the stationary point of C.

$$\boxed{f'(x)=\left(9-4x^2\right)^{\frac{1}{2}}-4x^2\left(9-4x^2\right)^{-\frac{1}{2}}}\ ,\ \boxed{\left(\frac{3\sqrt{2}}{4},\frac{9}{4}\right)}$$



Question 106 (****)

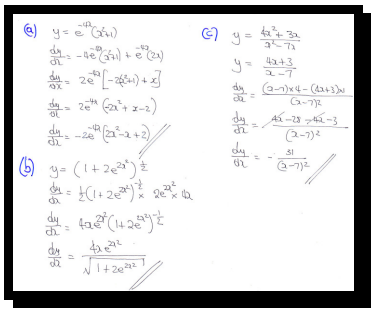
Differentiate each of the following expressions with respect to x, simplifying the answers as far as possible.

a) $y=\mathrm{e}^{-4x}\left(x^2+1\right)$.

b) $y=\sqrt{1+2\mathrm{e}^{2x^2}}$.

c) $y=\dfrac{4x^2+3x}{x^2-7x}$.

$$\boxed{\frac{dy}{dx}=-2\mathrm{e}^{-4x}\left(2x^2-x+2\right)},\quad \boxed{\frac{dy}{dx}=\frac{4x\mathrm{e}^{2x^2}}{\sqrt{1+2\mathrm{e}^{2x^2}}}},\quad \boxed{\frac{dy}{dx}=-\frac{31}{(x-7)^2}}$$



Question 107 (****)

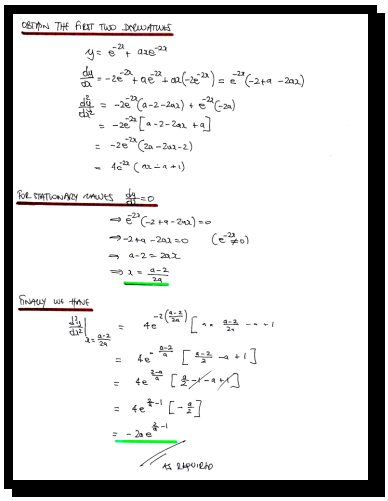
A curve has equation

$$y = \mathrm{e}^{-2x} + ax\mathrm{e}^{-2x},$$

where a is a non zero constant.

Show that the value of $\dfrac{d^2y}{dx^2}$ at the stationary point of the curve is $-2a\mathrm{e}^{\frac{2}{a}-1}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 108 **(****)**

The curve C has equation given by

$$y=\frac{x}{\sqrt{x-2}},\ x\geq 2.$$

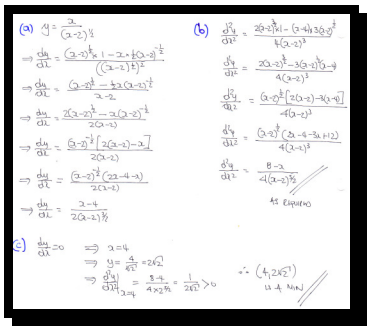
a) Show clearly that …

i. … $\dfrac{dy}{dx}=\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$.

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{8-x}{4(x-2)^{\frac{5}{2}}}$.

b) Hence find the exact coordinates of the stationary point of C, and determine its nature.

$$\boxed{\text{min at } \left(4,2\sqrt{2}\right)}$$



Question 109 **(****)**

The curve C has equation

$$y=\frac{2\ln x-1}{2\ln x+1}\ ,\ x>0\,.$$

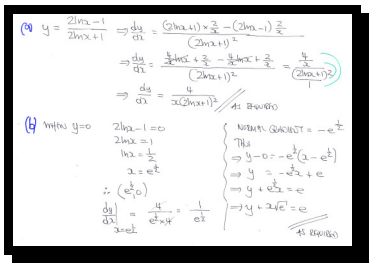
a) Show clearly that

$$\frac{dy}{dx}=\frac{4}{x\left(2\ln x+1\right)^2}\,.$$

b) Show that the equation of the normal at the point where the curve crosses the x axis is given by

$$y+x\sqrt{\mathrm{e}}=\mathrm{e}\,.$$

proof



Question 110 (****)

$$f(x)=\frac{2x}{\sqrt{1-x}},\ x<1\ .$$

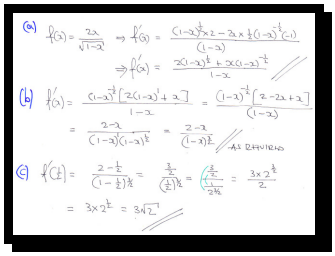
a) Find an expression for $f'(x)$.

b) Show further that

$$f'(x)=\frac{2-x}{(1-x)^{\frac{3}{2}}}\ .$$

c) Hence calculate that the exact value of $f'\left(\frac{1}{2}\right)$.

$$\boxed{f'(x)=\frac{2(1-x)^{\frac{1}{2}}+x(1-x)^{-\frac{1}{2}}}{1-x}},\quad \boxed{f'\left(\frac{1}{2}\right)=3\sqrt{2}}$$



Question 111 **(****)**

A curve C has equation

$$y = x - 2\ln\left(x^2+4\right),\ x \in \mathbb{R}.$$

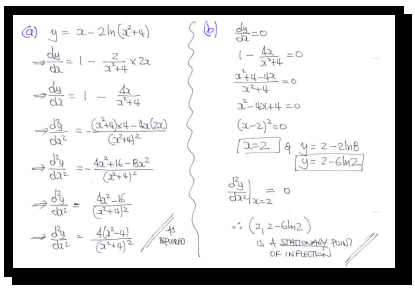
a) Show clearly that

$$\frac{d^2y}{dx^2} = \frac{4\left(x^2-4\right)}{\left(x^2+4\right)^2}.$$

The curve has a single stationary point.

b) Find its exact coordinates and determine its nature.

$\boxed{}$, $\boxed{\text{point of inflection at}\left(2, 2-6\ln 2\right)}$

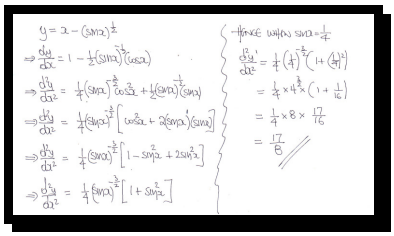


Question 112 (****)

$$y = x - \sqrt{\sin x}\,,\ 0 < x < \frac{\pi}{2}\,.$$

Show clearly that when $\sin x = \frac{1}{4}$, the value of $\frac{d^2y}{dx^2}$ is $\frac{17}{8}$.

proof



Question 113 **(****)**

$$f(x)=\mathrm{e}^{3x}-4\mathrm{e}^{-3x},\ x\in\mathbb{R}\ .$$

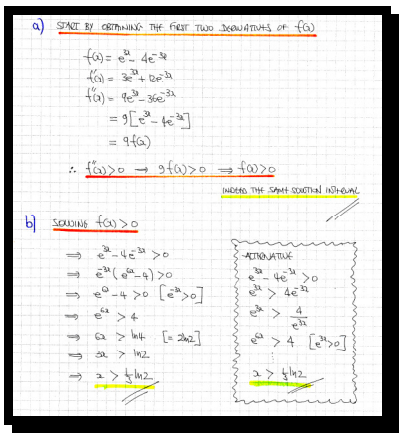
a) Show that the inequalities

$$f(x)>0 \text{ and } f''(x)>0$$

have the same solution interval.

b) Determine, in exact form, the common solution interval.

$\boxed{}$, $\boxed{x>\frac{1}{3}\ln 2}$



Question 114 **(****)**

The equation of the curve C is

$$y=(x+2)^2\,\mathrm{e}^{1-x},\ x\in\mathbb{R}.$$

a) Show clearly that …

i. … $\dfrac{dy}{dx}=-x(x+2)\mathrm{e}^{1-x}$.

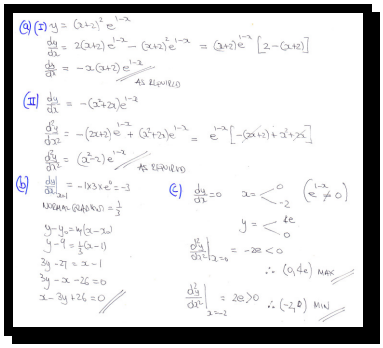
ii. … $\dfrac{d^2y}{dx^2}=(x^2-2)\mathrm{e}^{1-x}$.

b) Find an equation of the normal to C at the point $(1,9)$.

The curve has two stationary points at P and Q.

c) Find the exact coordinates of P and Q, further determining the nature of these stationary points.

$\boxed{x-3y+26=0}$, $\boxed{\text{max at } (0,4\mathrm{e})}$, $\boxed{\text{min at } (-2,0)}$



Question 115 (****)

The equation of the curve C is given by

$$y = x\sqrt{x^3+1}\,,\ x \geq -1\,.$$

a) Find an expression for $\dfrac{dy}{dx}$.

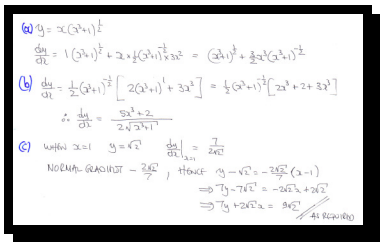
b) Show further that $\dfrac{dy}{dx}$ can be simplified to

$$\frac{dy}{dx} = \frac{5x^3+2}{2\sqrt{x^3+1}}\,.$$

c) Hence show that an equation of the normal to C at the point where $x=1$ is

$$7y + 2x\sqrt{2} = 9\sqrt{2}\,.$$

$$\boxed{\frac{dy}{dx} = \left(x^3+1\right)^{\frac{1}{2}} + \frac{3}{2}x^3\left(x^3+1\right)^{-\frac{1}{2}}}$$



Question 116 **(****)**

It is given that

$$\frac{d}{dx}(\sec x)=\sec x\tan x\,.$$

a) Prove the validity of the above result by writing $\sec x$ as $\dfrac{1}{\cos x}$.

The curve C has equation

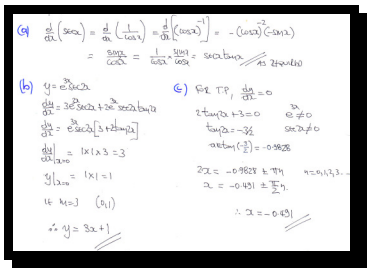
$$y=\mathrm{e}^{3x}\sec 2x\,,\quad -\frac{\pi}{4}<x<\frac{\pi}{4}\,.$$

b) Find an equation of the tangent to C at the point where C crosses the y axis.

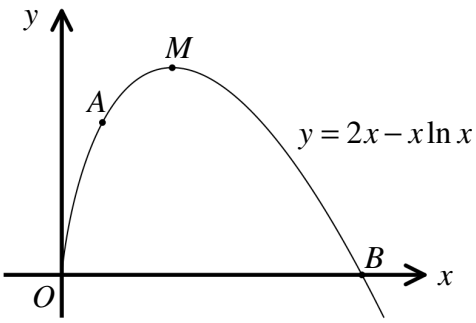
The curve has a single stationary point at P.

c) Find the x coordinate of P, correct to 3 significant figures.

$$\boxed{y=3x+1}\,,\quad \boxed{x\approx -0.491}$$



Question 117 **(****)**



The diagram above shows the graph of a curve C with equation

$$y = 2x - x\ln x,\ x > 0.$$

The curve has a maximum at M.

a) Find the exact coordinates of M.

The point A lies on C where $x = 1$.

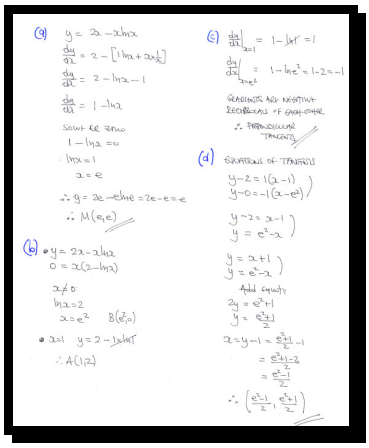
The curve crosses the x axis at the point B.

b) Determine the coordinates of A and the exact coordinates B.

c) Show that the tangents to the curve at A and B are perpendicular to each other.

d) Show that these tangents intersect at the point $Q\left(\frac{e^2-1}{2}, \frac{e^2+1}{2}\right)$.

$\boxed{M(e,e)}$, $\boxed{A(1,2),\ B(e^2,0)}$

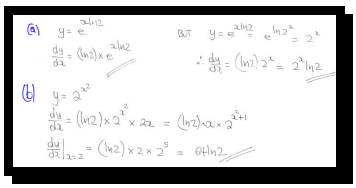


Question 118 (****)

a) By differentiating $y=\mathrm{e}^{x\ln 2}$, find the derivative of $y=2^x$.

b) Hence find the exact value of the gradient on the curve with equation $y=2^{x^2}$ at the point where $x=2$.

$\boxed{\dfrac{d}{dx}\left(2^x\right)=2^x\ln 2}$, $\boxed{64\ln 2}$



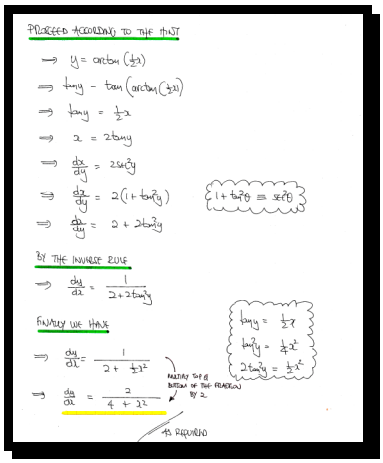
Question 119 (****)

$$y=\arctan\left(\frac{1}{2}x\right),\ x\in\mathbb{R}.$$

By writing $y=\arctan\left(\frac{1}{2}x\right)$ as $x=f\left(y\right)$, show that

$$\frac{dy}{dx}=\frac{2}{x^2+4}.$$

$\boxed{}$, $\boxed{\text{proof}}$



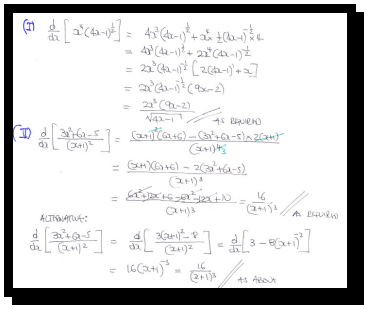
Question 120 **(****)**

Prove that …

i. … $\dfrac{d}{dx}\left(x^4\sqrt{4x-1}\right)=\dfrac{2x^3(9x-2)}{\sqrt{4x-1}}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{3x^2+6x-5}{(x+1)^2}\right)=\dfrac{16}{(x+1)^3}$.

proof



Question 121 **(****)**

The curve C has equation

$$f(x)=\mathrm{e}^{2x}\sin 2x\,,\quad 0\le x\le \pi\,.$$

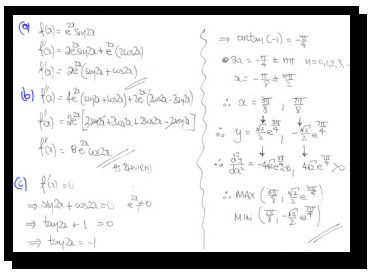
a) Find an expression for $f'(x)$.

b) Show clearly that

$$f''(x)=8\mathrm{e}^{2x}\cos 2x\,.$$

c) Hence find the exact coordinates of the stationary points of C and determine their nature.

$$\boxed{f'(x)=2\mathrm{e}^{2x}\left(\sin 2x+\cos 2x\right)}\,,\quad \boxed{\max\left(\frac{3}{8}\pi,\frac{\sqrt{2}}{2}\mathrm{e}^{\frac{3\pi}{4}}\right)},\quad \boxed{\min\left(\frac{7}{8}\pi,-\frac{\sqrt{2}}{2}\mathrm{e}^{\frac{7\pi}{4}}\right)}$$



Question 122 (****)

The curve C has equation given by

$$y = x\sqrt{1-2x}\ ,\ x \le \frac{1}{2}\ .$$

a) Show clearly that

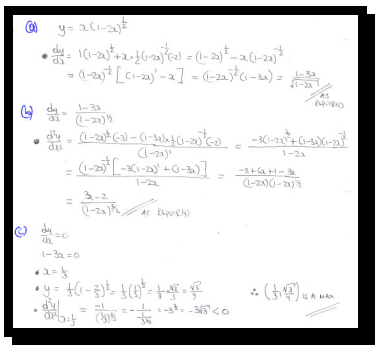
$$\frac{dy}{dx} = \frac{1-3x}{\sqrt{1-2x}}\ .$$

b) Show further that

$$\frac{d^2y}{dx^2} = \frac{3x-2}{(1-2x)^{\frac{3}{2}}}\ .$$

c) Hence find the exact coordinates of the stationary point of C, and determine its nature.

$$\boxed{\text{max at } \left(\frac{1}{3},\frac{1}{9}\sqrt{3}\right)}$$



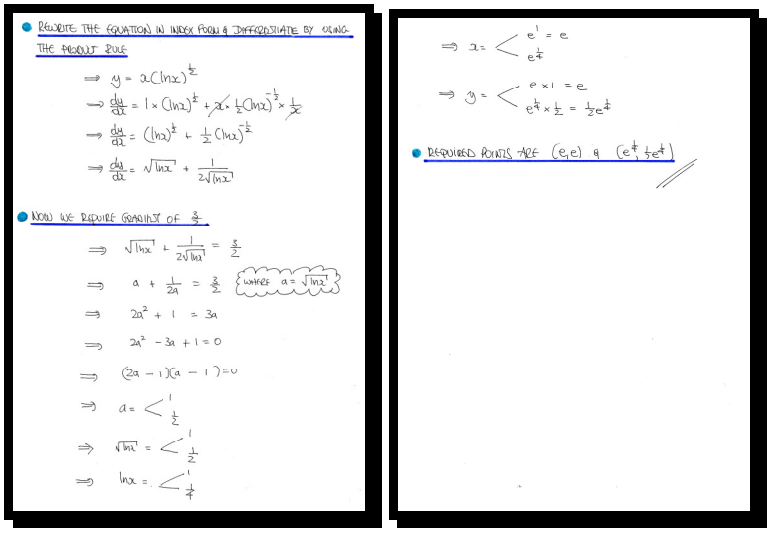
Question 123 **(****)**

A curve has equation

$$y = x\sqrt{\ln x}\,, \qquad x \in \mathbb{R}\,, \qquad x > 0\,.$$

Find the exact coordinates of the two points on the curve which have gradient $\frac{3}{2}$.

$\square$, $\boxed{(e,e),\ \left(e^{\frac{1}{4}}, \frac{1}{2}e^{\frac{1}{4}}\right)}$



Question 124 (****)

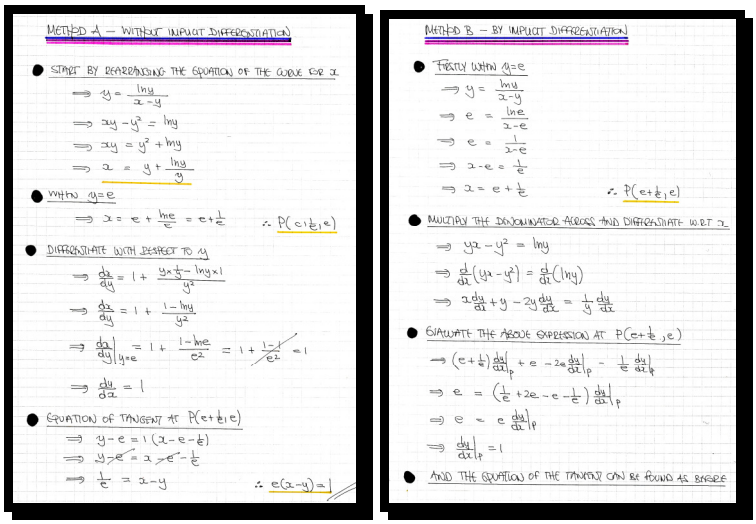
The curve C has equation

$$y=\frac{\ln y}{x-y},\ \ y>0.$$

Show that the equation of the tangent to C at the point where $y=\mathrm{e}$ can be written as

$$\mathrm{e}\left(x-y\right)=1.$$

[ ], proof



Question 125 (****)

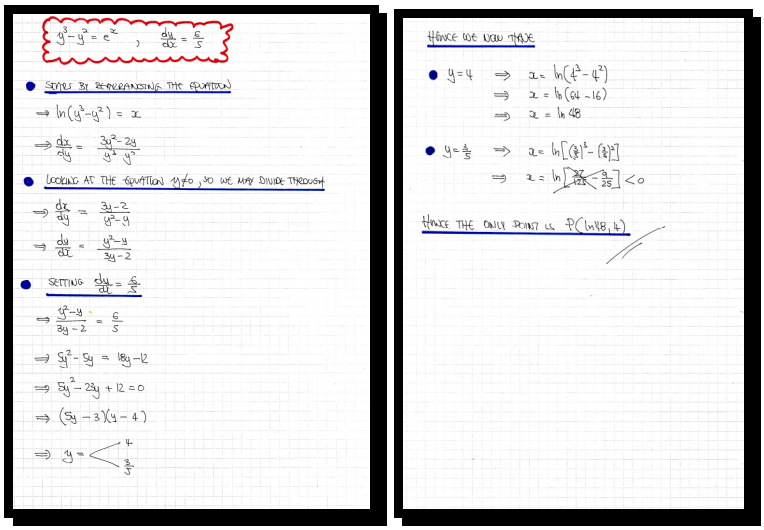
At the point P, which lies on the curve with equation

$$y^3 - y^2 = \mathrm{e}^x,$$

the gradient is $\frac{6}{5}$.

Determine the possible coordinates of P.

$\boxed{}$, $\boxed{P\left(\ln 48,\, 4\right)}$



Question 126 (****)

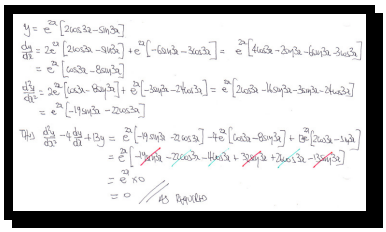
A curve has equation

$$y=\mathrm{e}^{2x}\left(2\cos 3x-\sin 3x\right).$$

Show that

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+13y=0.$$

$\boxed{}$, $\boxed{\text{proof}}$

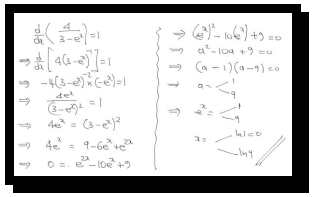


Question 127 (****)

Find, in exact form where appropriate, the solutions of the equation

$$\frac{d}{dx}\left(\frac{4}{3-\mathrm{e}^x}\right)=1.$$

$\boxed{x=0,\ x=\ln 9}$



Question 128 (****)

The point $P\left(\ln 2, 5b-3a\right)$ lies on the curve with equation

$$y=a+b\mathrm{e}^{x},$$

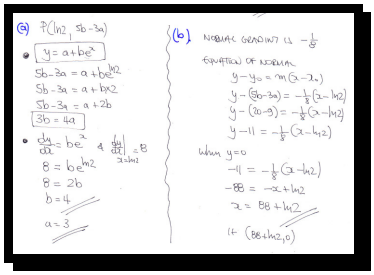
where a and b are non zero constants.

The gradient at P is 8.

a) Find the value of a and the value of b.

b) Find the exact coordinates of the point where the normal to the curve at P crosses the x axis.

$\boxed{a=3,\ b=4}$, $\boxed{\left(88+\ln 2, 0\right)}$

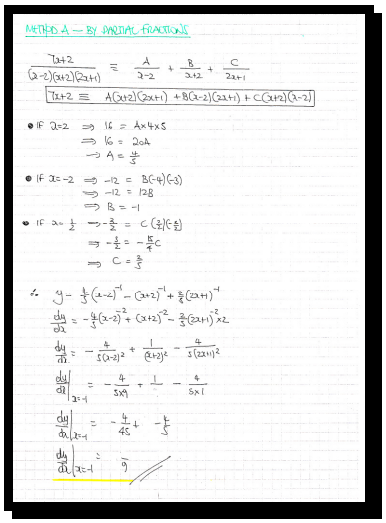


Question 129 (****)

$$y=\frac{7x+2}{(x-2)(x+2)(2x+1)},\ x\neq\pm2,\ x\neq-\frac{1}{2}.$$

Find the exact value of $\dfrac{dy}{dx}$ at $x=-1$.

$\square$, $\left.\dfrac{dy}{dx}\right|_{x=-1}=\dfrac{1}{9}$



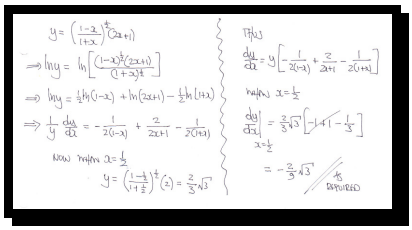
Question 130 (****)

The curve C has equation

$$y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}(2x+1),\ -1<x\leq 1.$$

By taking logarithms on both sides of this equation, or otherwise, show that at the point on C where $x=\frac{1}{2}$, the gradient is $-\frac{2}{9}\sqrt{3}$.

$\boxed{}$, $\boxed{\text{proof}}$



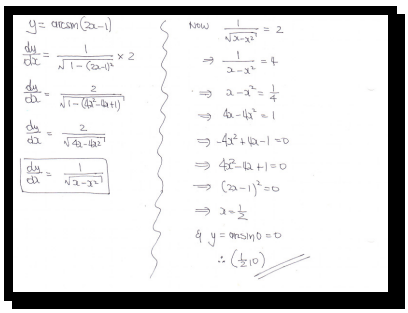
Question 131 (****)

The curve C has equation

$$y=\arcsin(2x-1),\ 0\leq x\leq 1.$$

Find the coordinates of the point on C, whose gradient is 2.

$\boxed{\left(\frac{1}{2},0\right)}$



Question 132 (****)

A curve C has equation

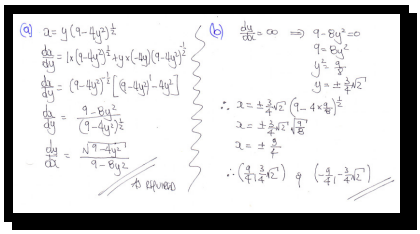
$$x = y\sqrt{9-4y^2}\ ,\ |y| \le \frac{3}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{\sqrt{9-4y^2}}{9-8y^2}.$$

b) Find the exact coordinates of the two points on C, with infinite gradient.

$\square$, $\boxed{\left(\frac{9}{4},\frac{3}{4}\sqrt{2}\right),\left(-\frac{9}{4},-\frac{3}{4}\sqrt{2}\right)}$



Question 133 **(****)**

Given that

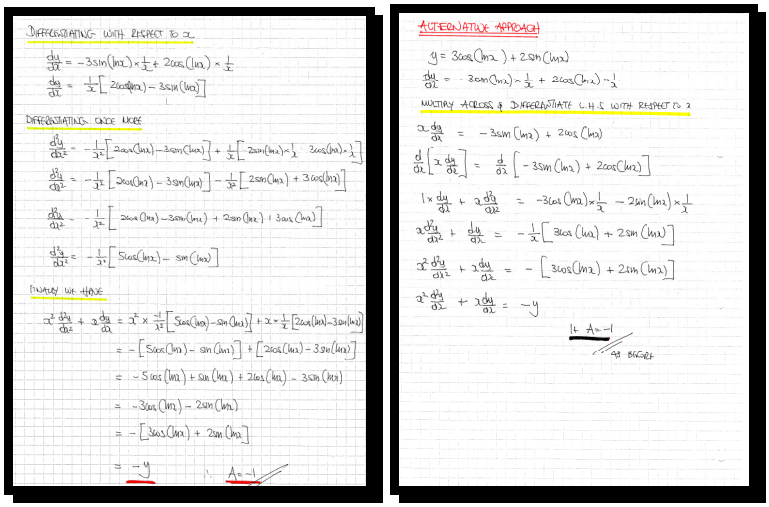
$$y=3\cos\left(\ln x\right)+2\sin\left(\ln x\right),\ x>0,$$

show clearly that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=Ay,$$

stating the value of the constant A.

$\boxed{}$, $\boxed{A=-1}$



Question 134 (****)

The equation of a curve C is

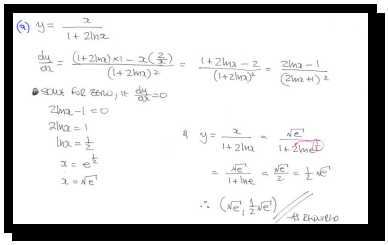
$$y=\frac{x}{1+2\ln x},\ x\in\mathbb{R},\ x>0\ .$$

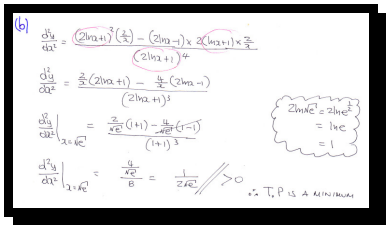
The curve has a single turning point at P.

a) Show that the coordinates of P are $\left(\sqrt{\mathrm{e}},\frac{1}{2}\sqrt{\mathrm{e}}\right)$.

b) Evaluate the exact value of $\dfrac{d^2y}{dx^2}$ at P and hence determine its nature.

$\square$, $\boxed{\dfrac{d^2y}{dx^2}=\dfrac{1}{2\sqrt{\mathrm{e}}}>0,\ \text{so minimum}}$



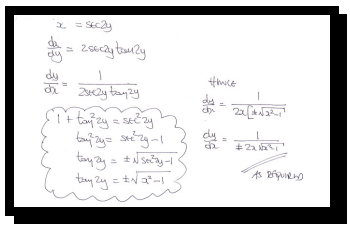


Question 135 (****)

Show that if $x=\sec 2y$, then

$$\frac{dy}{dx}=\pm\frac{1}{2x\sqrt{x^2-1}}.$$

$\boxed{\text{proof}}$



Question 136 (****)

A curve has equation given by

$$y = x\sqrt{x+1}\,,\ x \geq -1\,.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)(x+1)^{-\frac{1}{2}}\,.$$

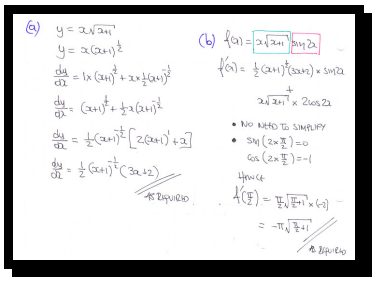
The function f is defined as

$$f(x) = x\sqrt{x+1}\sin 2x\,,\ x \geq 1\,.$$

b) Show further that

$$f'\left(\frac{\pi}{2}\right) = -\pi\sqrt{\frac{\pi}{2}+1}\ .$$

[ ], [proof]



Question 137 **(****)**

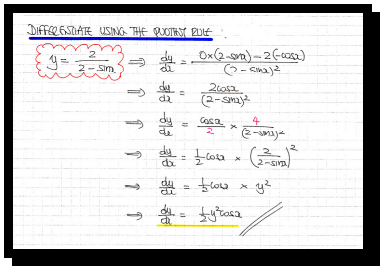
The curve C has equation

$$y=\frac{2}{2-\sin x}.$$

Show clearly that

$$\frac{dy}{dx}=\frac{1}{2}y^2\cos x\ .$$

$\boxed{}$, $\boxed{\text{proof}}$



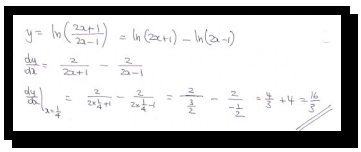
Question 138 **(****)**

Given that

$$y=\ln\left(\frac{2x+1}{2x-1}\right)$$

find the exact value of $\dfrac{dy}{dx}$ at $x=\dfrac{1}{4}$.

$$\boxed{\frac{dy}{dx}=\frac{16}{3}}$$



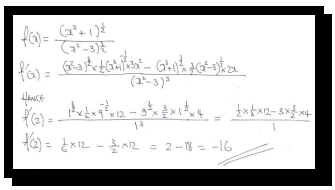
Question 139 (****)

The function f is defined by

$$f(x)=\frac{\left(x^3+1\right)^{\frac{1}{2}}}{\left(x^2-3\right)^{\frac{3}{2}}},\quad x\in\mathbb{R},\quad x>-1.$$

Show clearly that $f'(2)=-16$.

proof



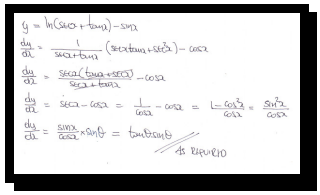
Question 140 (****)

$$y=\ln\left(\sec x+\tan x\right)-\sin x.$$

Show clearly that

$$\frac{dy}{dx}=\sin x\tan x\ .$$

proof



Question 141 (****)

It is given that

$$\frac{d}{dx}\left(\tan 2x\right)=2\sec^2 2x\,.$$

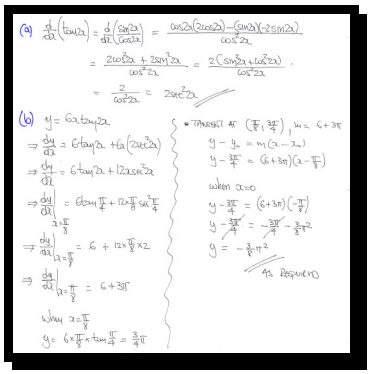
a) Prove the validity of the above result by considering the derivatives of $\sin 2x$ and $\cos 2x$.

A curve has equation

$$y=6x\tan 2x\,,\ \ x\in\mathbb{R}\,.$$

b) Show that the tangent to the curve at the point where $x=\frac{1}{8}\pi$ meets the y axis at the point with coordinates $\left(0,-\frac{3}{8}\pi^2\right)$.

$\boxed{}$, proof



Question 142 (****)

$$y=x-\frac{12}{x^2+2x-3}+\frac{3}{x-1},\ x\in\mathbb{R},\ x\neq -3,\ x\neq 1.$$

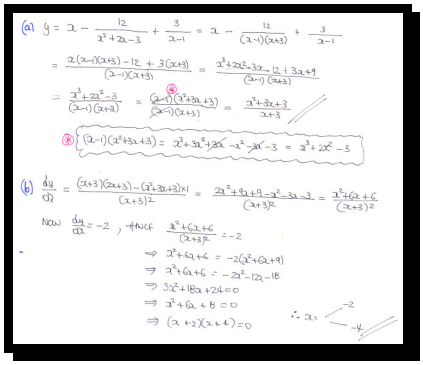
a) Show clearly that

$$y=\frac{x^2+3x+3}{x+3}.$$

b) Solve the equation

$$\frac{dy}{dx}=-2.$$

$$\boxed{x=-2,\ x=4}$$



Question 143 **(****)**

A curve has equation

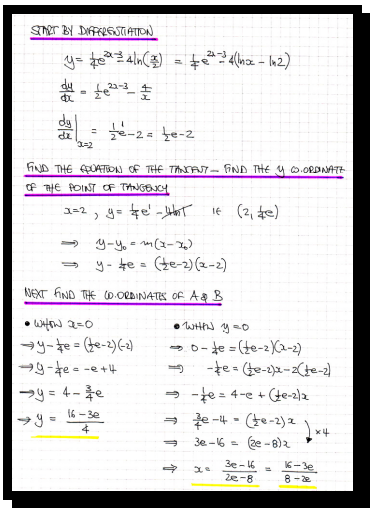
$$y=\frac{1}{4}\mathrm{e}^{2x-3}-4\ln\left(\frac{1}{2}x\right),\quad x>0.$$

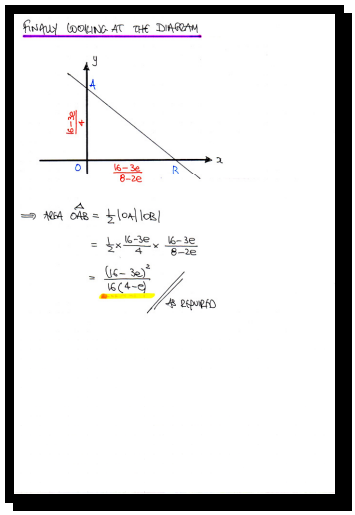
The tangent to the curve, at the point where $x=2$, crosses the coordinate axes at the points A and B.

Show that the area of the triangle OAB, where O is the origin, is given by

$$\frac{(16-3\mathrm{e})^2}{16(4-\mathrm{e})}.$$

, proof



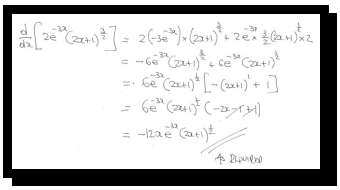


Question 144 **(****)**

Show clearly that

$$\frac{d}{dx}\left(2\mathrm{e}^{-3x}\left(2x+1\right)^{\frac{3}{2}}\right)=-12x\mathrm{e}^{-3x}\left(2x+1\right)^{\frac{1}{2}}.$$

proof



Question 145 **(****)**

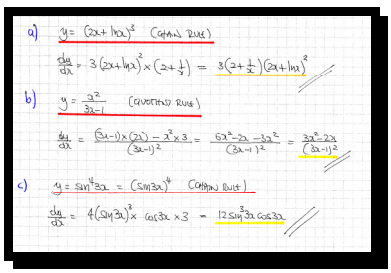
Differentiate each of the following expressions with respect to x.

a) $y=\left(2x+\ln x\right)^3$.

b) $y=\dfrac{x^2}{3x-1}$.

c) $y=\sin^4 3x$.

$\square$, $\dfrac{dy}{dx}=3\left(2x+\ln x\right)^2\left(2+\dfrac{1}{x}\right)$, $\dfrac{dy}{dx}=\dfrac{3x^2-2x}{\left(3x-1\right)^2}$, $\dfrac{dy}{dx}=12\sin^3 3x\cos 3x$



Question 146 (****)

The functions f and g are defined by

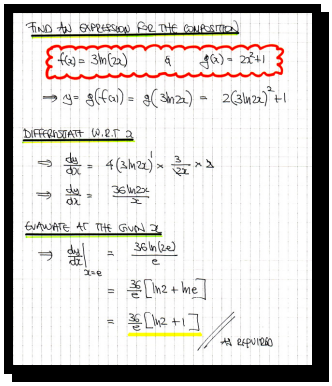
$$f(x)=3\ln 2x,\ \ x\in\mathbb{R},\ x>0$$

$$g(x)=2x^2+1,\ \ x\in\mathbb{R}.$$

Show that the value of the gradient on the curve $y=gf(x)$ at the point where $x=\mathrm{e}$ is

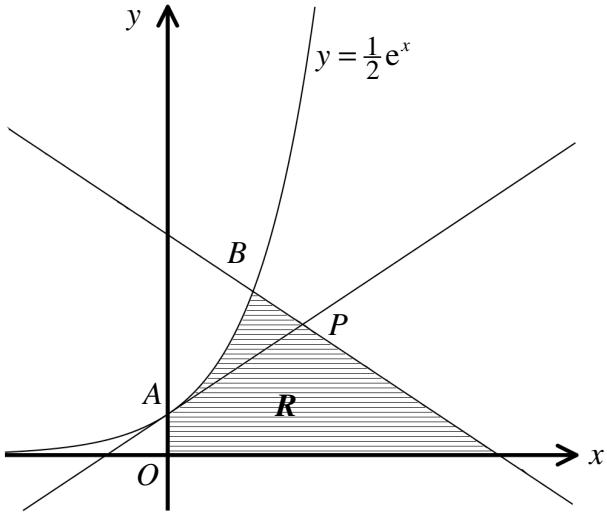
$$\frac{36}{\mathrm{e}}(1+\ln 2).$$

, proof



Question 147 (****)

The graph of the curve with equation $y=\frac{1}{2}\mathrm{e}^{x}$ is shown below.



The points A and B lie on the curve, where $x=0$ and $x=\ln 4$, respectively.

a) Show that the equation of the tangent to the curve at the point A is

$$2y-x-1=0\,.$$

This tangent meets the normal to the curve at the point B at the point P.

b) Show that the coordinates of P are given by

$$\left(\frac{3}{2}+\ln 2,\frac{5}{4}+\ln\sqrt{2}\right).$$

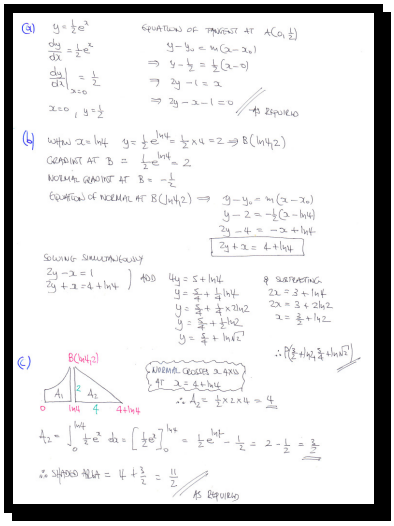
[continues overleaf]

[continued from overleaf]

The region R bounded by the curve, the normal to the curve at B and coordinate axes is shown shaded in the diagram above.

c) Show that the area of R is $\dfrac{11}{2}$ square units.

proof



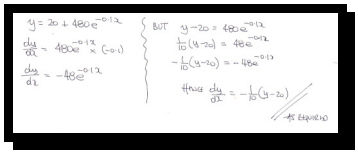
Question 148 (****)

$$y = 20 + 480\mathrm{e}^{-0.1x}\,,\ x \in \mathbb{R}\,.$$

Show clearly that

$$\frac{dy}{dx} = -\frac{1}{10}\left(y - 20\right).$$

proof



Question 149 (****)

A curve has equation

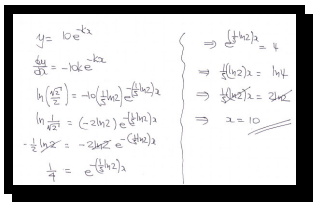
$$y = 10\mathrm{e}^{-kx}\,,\ x \in \mathbb{R}$$

where $k = \frac{1}{5}\ln 2$.

Find the value of x that satisfies the equation

$$\frac{dy}{dx} = \ln\left(\frac{\sqrt{2}}{2}\right).$$

$\square$, $x = 10$



Question 150 (****)

A curve C has equation

$$y=\frac{1}{2}\mathrm{e}^{2x}-4x+1,\ x\in\mathbb{R}.$$

The point P lies on C where $x=\ln 4$.

a) Show that the equation of the tangent to the curve at the point P is

$$y=12x+9-32\ln 2.$$

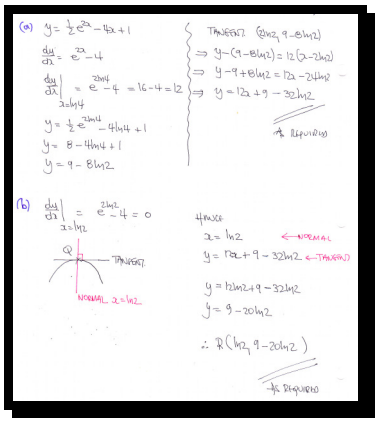
The point Q lies on C where $x=\ln 2$.

The normal to the curve at the point Q meets the tangent to the curve at the point P, at the point R.

b) Show that the coordinates of R are

$$\left(\ln 2, 9-20\ln 2\right).$$

proof



Question 151 (****)

A curve C has equation

$$y=(x+1)^2\mathrm{e}^{2x},\ x\in\mathbb{R}.$$

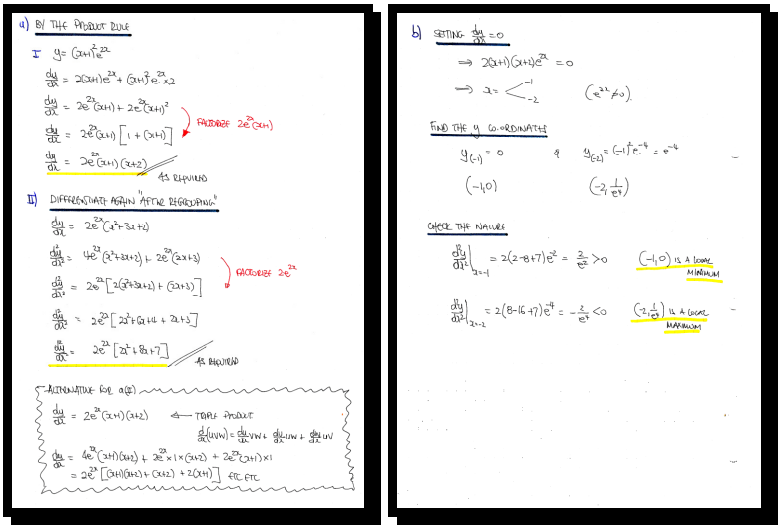
a) Show that

i. $\dfrac{dy}{dx}=2(x+1)(x+2)\mathrm{e}^{2x}$.

ii. $\dfrac{d^2y}{dx^2}=2\left(2x^2+8x+7\right)\mathrm{e}^{2x}$.

b) Hence, or otherwise, find the exact coordinates of the stationary points of C and determine their nature.

$\boxed{}$, $\boxed{\min(-1,0)}$, $\boxed{\max\left(-2,\mathrm{e}^{-4}\right)}$



Question 152 (****)

A curve C is defined by the equation

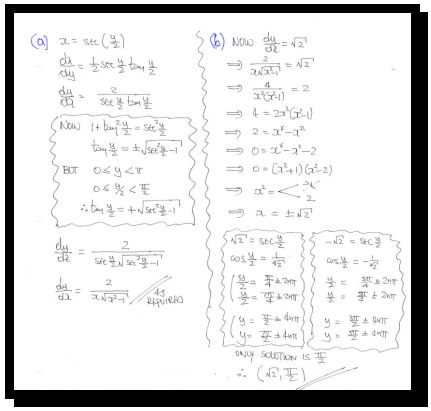
$$x=\sec\left(\frac{y}{2}\right),\ 0\le y<\pi .$$

a) Show clearly that

$$\frac{dy}{dx}=\frac{2}{x\sqrt{x^2-1}}.$$

b) Hence find the exact coordinates of the point of C, where $\frac{dy}{dx}=\sqrt{2}$.

$\square$, $\boxed{\left(\sqrt{2},\frac{\pi}{2}\right)}$

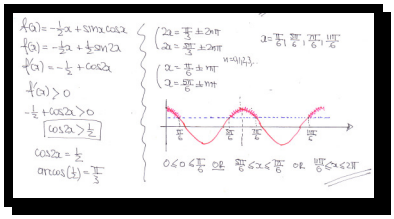


Question 153 (****)

$$f(x)=-\frac{1}{2}x+\sin x\cos x,\ 0\le x<2\pi .$$

Use differentiation to find the range of values of x for which $f(x)$ is increasing.

$$\boxed{0\le x<\frac{\pi}{6}\ \cup\ \frac{5\pi}{6}<x<\frac{7\pi}{6}\ \cup\ \frac{11\pi}{6}<x<2\pi}$$



Question 154 **(****)**

A curve has equation

$$y=\arcsin 2x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

a) By finding $\dfrac{dx}{dy}$ and using an appropriate trigonometric identity show that

$$\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}\,.$$

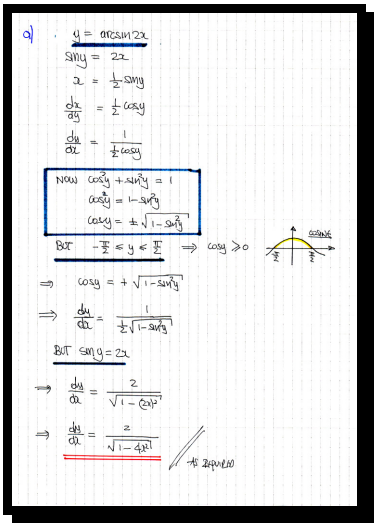
b) Show further that …

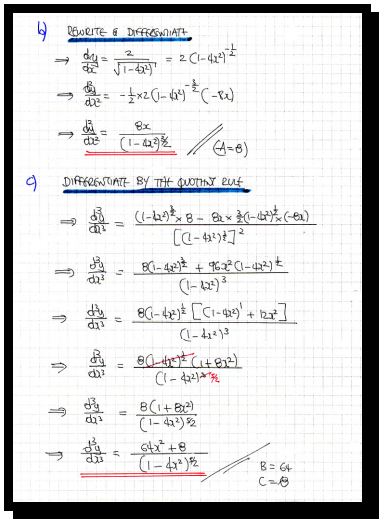
i. $\ldots\dfrac{d^2y}{dx^2}=\dfrac{Ax}{\left(1-4x^2\right)^{\frac{3}{2}}}$,

ii. $\ldots\dfrac{d^3y}{dx^3}=\dfrac{Bx^2+C}{\left(1-4x^2\right)^{\frac{5}{2}}}$,

where A, B and C are constants to be found.

$A=2$, proof





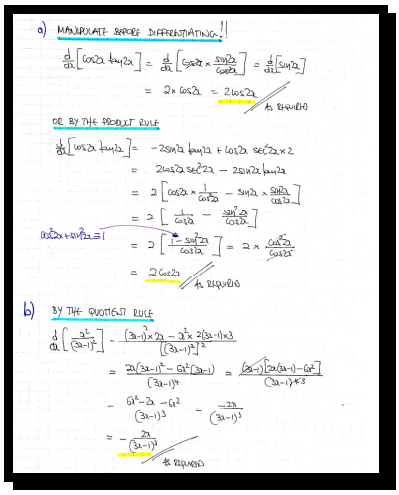
Question 155 **(****)**

Show, with detailed workings, that

a) $\dfrac{d}{dx}\left(\cos 2x \tan 2x\right) = 2\cos 2x$.

b) $\dfrac{d}{dx}\left(\dfrac{x^2}{(3x-1)^2}\right) = -\dfrac{2x}{(3x-1)^3}$.

$\boxed{\phantom{\text{proof}}}$, $\boxed{\text{proof}}$



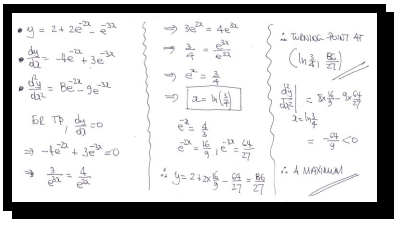
Question 156 **(****)**

A curve C has equation

$$y = 2 + 2\mathrm{e}^{-2x} - \mathrm{e}^{-3x}, \quad x \in \mathbb{R}.$$

Find the exact coordinates of the stationary point of C and determine its nature.

$\boxed{\max\left(\ln\left(\frac{3}{4}\right), \frac{86}{27}\right)}$

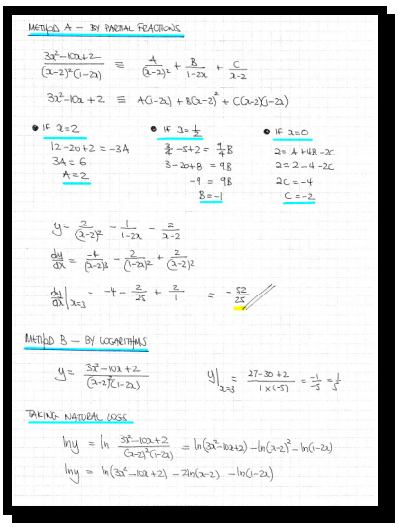


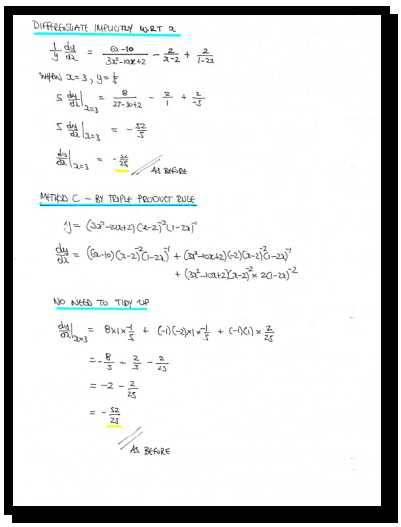
Question 157 (****)

$$y=\frac{3x^2-10x+2}{(1-2x)(x-2)^2},\quad x\in\mathbb{R},\quad x>2.$$

Find the exact value of $\dfrac{dy}{dx}$ at $x=3$.

$\square$, $\left.\dfrac{dy}{dx}\right|_{x=3}=-\dfrac{52}{25}$





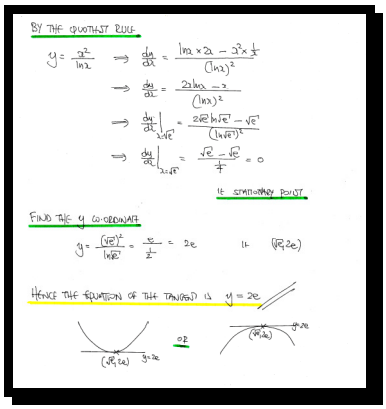
Question 158 (****)

A curve C has equation

$$y=\frac{x^2}{\ln x},\quad x\in\mathbb{R},\quad x>0.$$

Find, in exact form, the equation of the tangent to C at the point where $x=\sqrt{\mathrm{e}}$.

$\boxed{\phantom{y=2\mathrm{e}}}$, $\boxed{y=2\mathrm{e}}$



Question 159 (****)

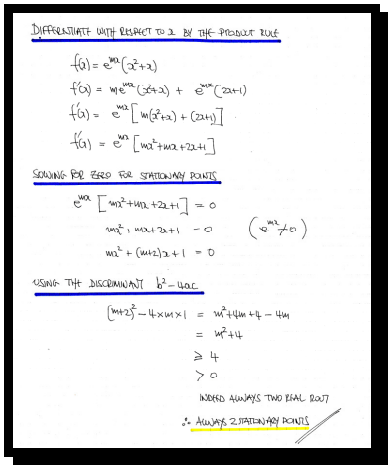
The function f is given by

$$f(x) \equiv \mathrm{e}^{mx}\left(x^2+x\right),\ x \in \mathbb{R},$$

where m is a non zero constant.

Show that f has two stationary points, for all non zero values of m.

[ ], proof



Question 160 (****)

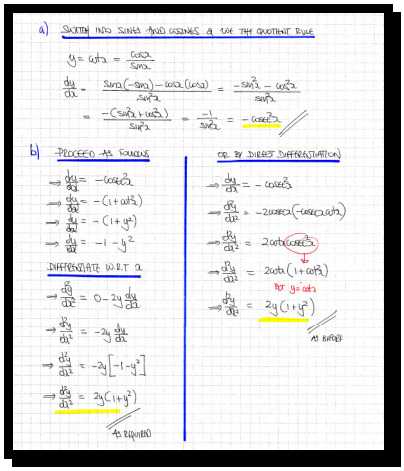
$$y = \cot x\,,\ 0 < x < \frac{\pi}{2}\,.$$

Show, with detailed workings, that

a) $\dfrac{dy}{dx} = -\text{cosec}^2 x$.

b) $\dfrac{d^2y}{dx^2} = 2y\left(y^2+1\right)$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 161 **(****)**

The curve C has equation

$$x=\sec^2 y+\tan y\,,\ 0\le y<\frac{\pi}{2}\,.$$

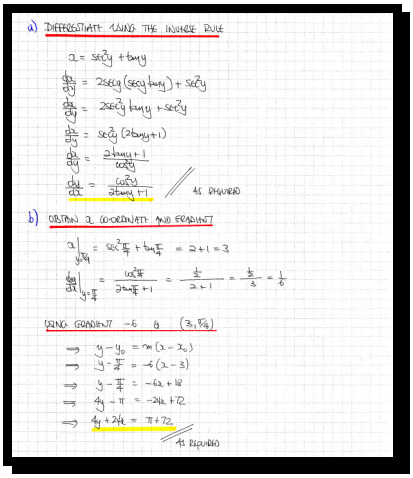
a) Show that

$$\frac{dy}{dx}=\frac{\cos^2 y}{2\tan y+1}\,.$$

b) Hence show that the equation of the normal to C at the point where $y=\frac{\pi}{4}$ is

$$4y+24x=\pi+72\,.$$

, proof



Question 162 **(****)**

$$y=\arcsin x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

a) By finding $\dfrac{dx}{dy}$ and using an appropriate trigonometric identity show that

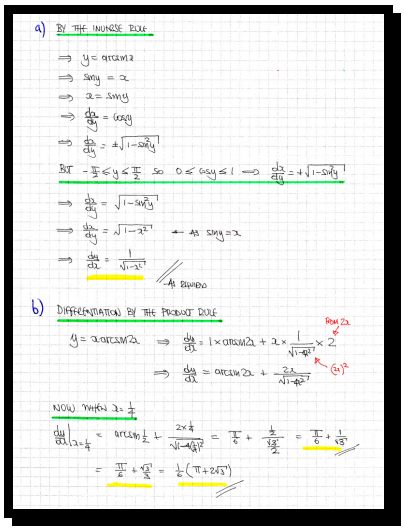
$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}.$$

A curve C has equation

$$y=x\arcsin 2x\,,\ -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{\pi}{2}\le y\le\frac{\pi}{2}.$$

b) Find the exact value of $\dfrac{dy}{dx}$ at the point on C where $x=\dfrac{1}{4}$.

$\square$, $\boxed{\dfrac{1}{6}\left(\pi+2\sqrt{3}\right)}$



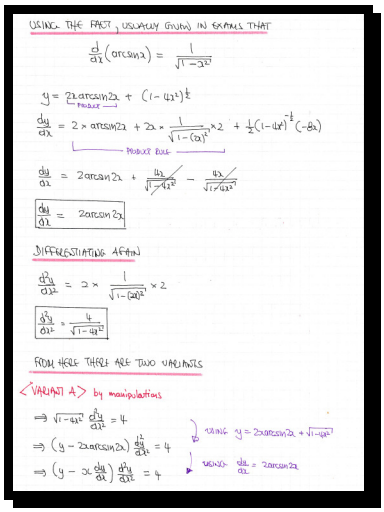
Question 163 (****)

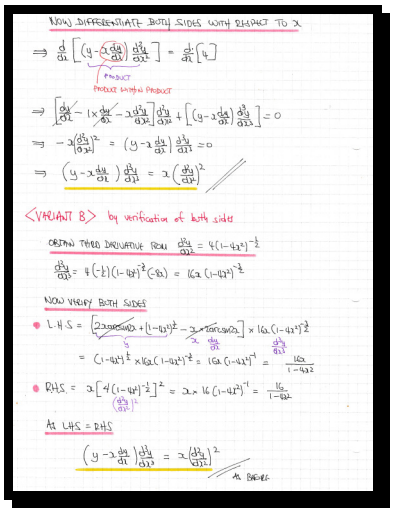
$$y=2x\arcsin 2x+\sqrt{1-4x^2}\ ,\ -\frac{1}{2}\le x\le\frac{1}{2}.$$

Show clearly that

$$\frac{d^3y}{dx^3}\left(y-x\frac{dy}{dx}\right)=x\left(\frac{d^2y}{dx^2}\right)^2.$$

, proof





Question 164 (**+)**

The curve C has equation

$$y = x\sqrt{\ln x}\,,\ x>0\,.$$

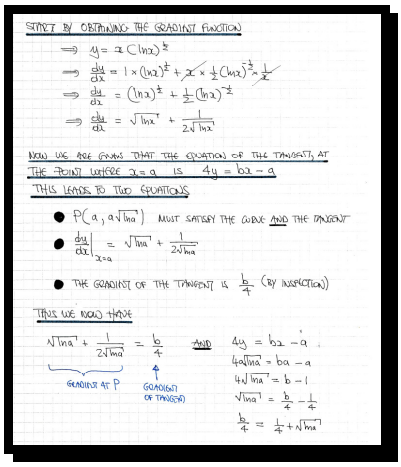
The equation of the tangent to C at the point where $x=a$ is

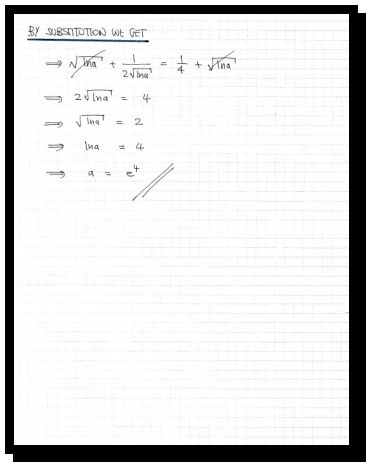
$$4y = bx - a\,,$$

where a and b are non zero constants.

Determine the exact value of a.

$\boxed{}$, $\boxed{a=\mathrm{e}^4}$





Question 165 **(****+)**

$$y=\arccos x\,,\quad -1\le x\le 1\,,\ 0\le y\le \pi\,.$$

a) Prove that

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}\,.$$

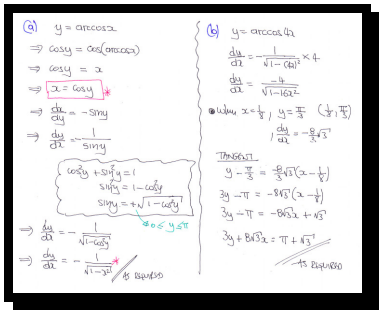
The curve C has equation given by

$$y=\arccos 4x\,,\ -\frac{1}{4}\le x\le \frac{1}{4}\,.$$

b) Show that an equation of the tangent to C where $x=\frac{1}{8}$, is given by

$$3y+8\sqrt{3}x=\pi+\sqrt{3}\,.$$

proof

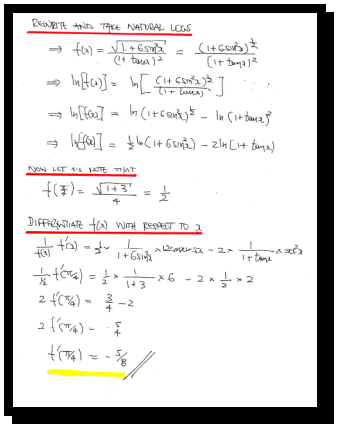


Question 166 **(****+)**

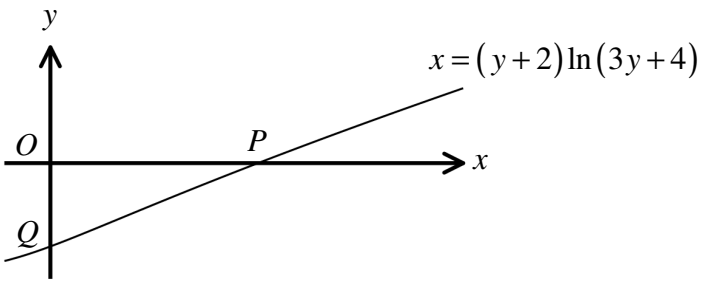
$$f(x)\equiv\frac{\sqrt{1+6\sin^2 x}}{(1+\tan x)^2},\ \tan x\neq -1.$$

By using logarithmic differentiation, or otherwise, determine the value of $f'\left(\frac{\pi}{4}\right)$.

$\square$, $\boxed{f'\left(\frac{\pi}{4}\right)=-\frac{5}{8}}$



Question 167 (**+)**



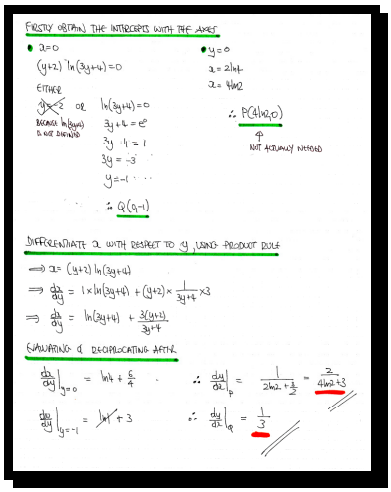
The figure above shows the graph of the curve with equation

$$x=(y+2)\ln(3y+4).$$

The curve meets the coordinate axes at the point P and at the point Q.

Determine the gradient, in exact from where appropriate, at P and at Q.

$$\boxed{},\ \boxed{\left.\frac{dy}{dx}\right|_P=\frac{2}{3+4\ln 2}},\ \boxed{\left.\frac{dy}{dx}\right|_Q=\frac{1}{3}}$$



Question 168 **(****+)**

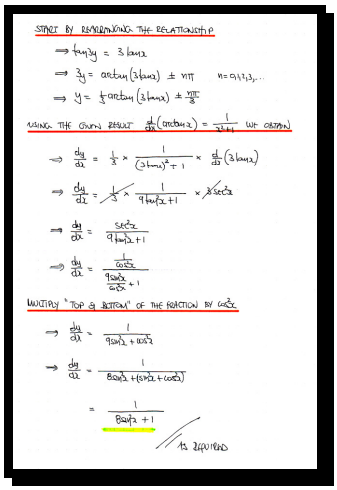
It is given that

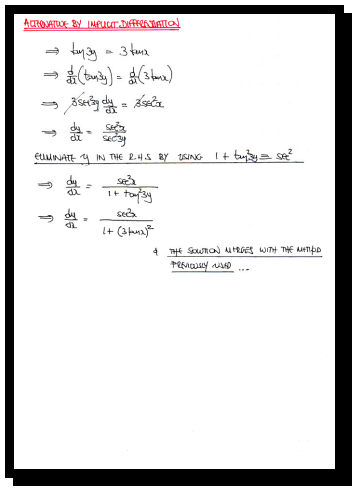
$$\frac{d}{dx}\left(\arctan x\right)=\frac{1}{1+x^2}\,.$$

If $\tan 3y=3\tan x$ show that

$$\frac{dy}{dx}=\frac{1}{1+8\sin^2 x}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 169 (**+)**

The curve C has equation

$$y=4\mathrm{e}^{-x}-2\mathrm{e}^{-2x}-\mathrm{e}^{-3x},\ x\in\mathbb{R}.$$

a) Show clearly that …

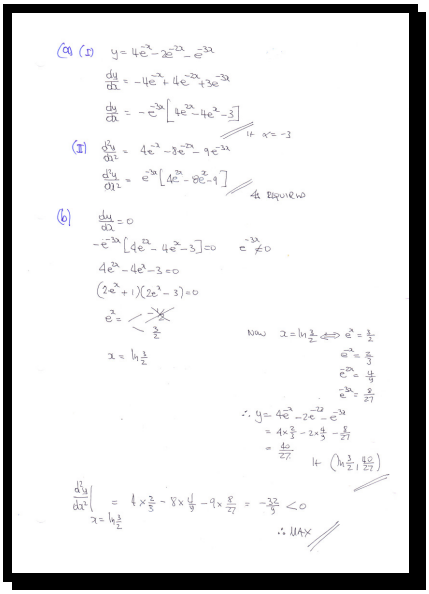
i. … $\dfrac{dy}{dx}=-\mathrm{e}^{\alpha x}\left(4\mathrm{e}^{2x}-4\mathrm{e}^{x}-3\right)$,

ii. … $\dfrac{d^2y}{dx^2}=\mathrm{e}^{\alpha x}\left(4\mathrm{e}^{2x}-8\mathrm{e}^{x}-9\right)$,

where α is a constant to be found.

b) Hence find the exact coordinates of the stationary point of C and determine its nature.

$$\boxed{\max\left(\ln\left(\frac{3}{2}\right),\frac{40}{27}\right)}$$



Question 170 (**+)**

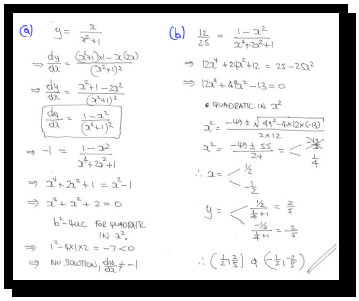
The curve C has equation

$$y=\frac{x}{x^2+1},\ x\in\mathbb{R}\ .$$

a) Show that there is no point on C where the gradient is -1.

b) Find the coordinates of the points on C where the gradient is $\frac{12}{25}$.

$\square$, $\boxed{\left(\frac{1}{2},\frac{2}{5}\right),\left(-\frac{1}{2},-\frac{2}{5}\right)}$



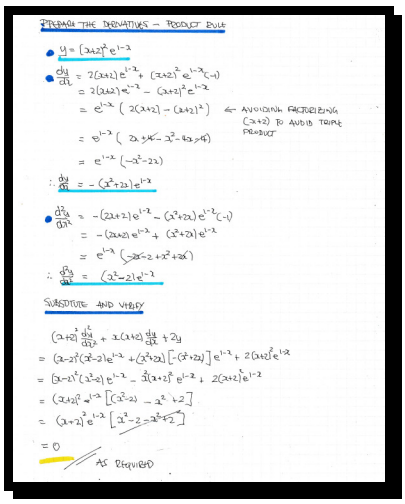
Question 171 (**+)**

$$y=(x+2)^2\,\mathrm{e}^{1-x},\ x\in\mathbb{R}$$

Show clearly that

$$(x+2)^2\frac{d^2y}{dx^2}+x(x+2)\frac{dy}{dx}+2y=0\,.$$

[ ], [proof]



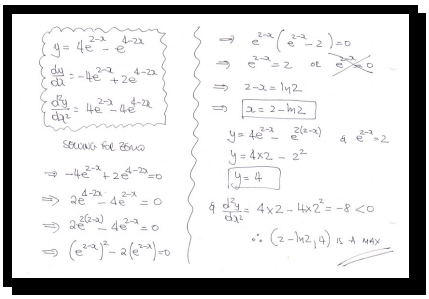
Question 172 (**+)**

A curve has equation

$$y = 4\mathrm{e}^{2-x} - \mathrm{e}^{4-2x}, \ x \in \mathbb{R}.$$

Use differentiation to find the exact coordinates of the stationary point of the curve, and further determine its nature.

$\square$, $\max\left(2-\ln 2, 4\right)$



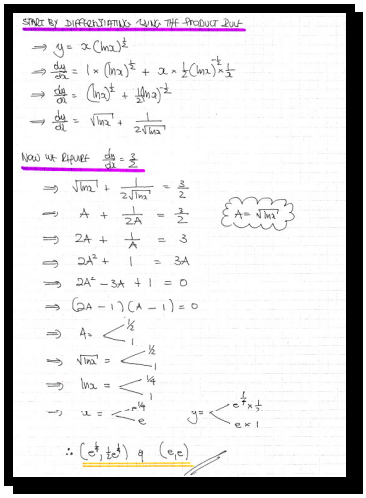
Question 173 (**+)**

The point P lies on the curve with equation

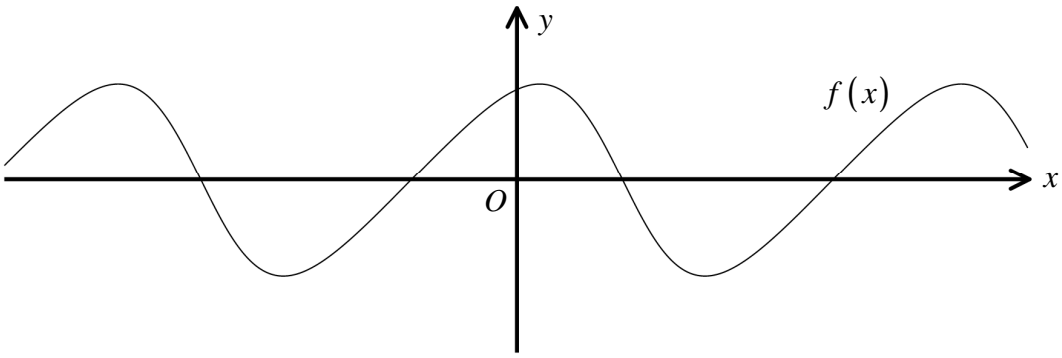
$$y = x\sqrt{\ln x}, \ x > 1.$$

Determine the two possible sets of coordinates of P given further that the gradient of the curve at P is $\frac{3}{2}$.

$\square$, $P\left(\mathrm{e}^{\frac{1}{4}}, \frac{1}{2}\mathrm{e}^{\frac{1}{4}}\right) \cup P\left(\mathrm{e}, \mathrm{e}\right)$



Question 174 (**+)**



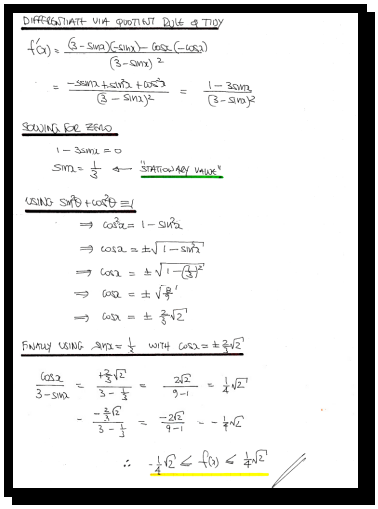
The figure above shows part of the graph of the curve with equation

$$f(x)=\frac{\cos x}{3-\sin x},\ x\in\mathbb{R}.$$

Use differentiation to show that

$$-\tfrac{1}{4}\sqrt{2}\le f(x)\le\tfrac{1}{4}\sqrt{2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 175 **(****+)**

The curve C has equation given by

$$y=\frac{\mathrm{e}^x}{\sin x},\quad 0<x<\pi .$$

a) Show clearly that

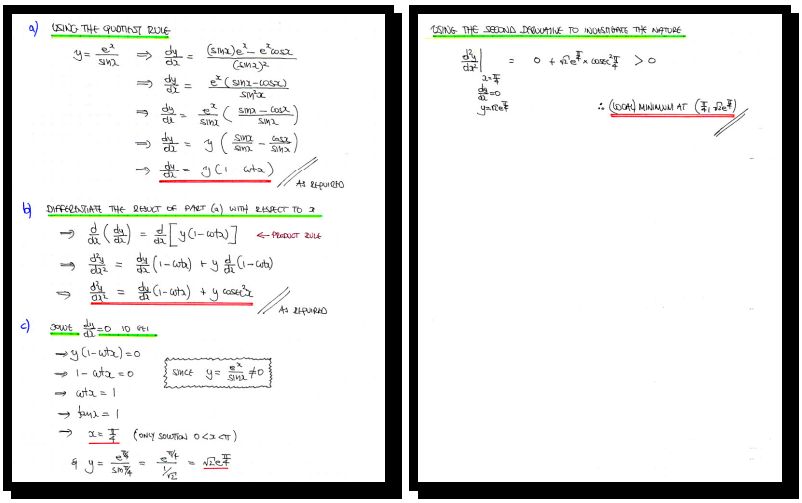
$$\frac{dy}{dx}=y\left(1-\cot x\right).$$

b) Show further that

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}\left(1-\cot x\right)+y\operatorname{cosec}^2 x.$$

c) Use the above results to find the exact coordinates of the turning point of C, and determine its nature.

$\boxed{}$, $\boxed{\text{min at } \left(\frac{\pi}{4},\sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}\right)}$



Question 176 (**+)**

The curve C has equation

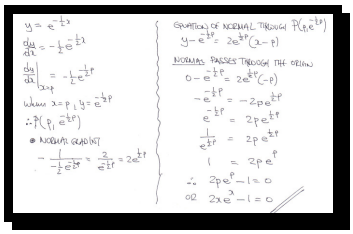
$$y=\mathrm{e}^{-\frac{1}{2}x},\ x\in\mathbb{R}.$$

The normal to the curve at the point P where $x=p$ passes through the origin.

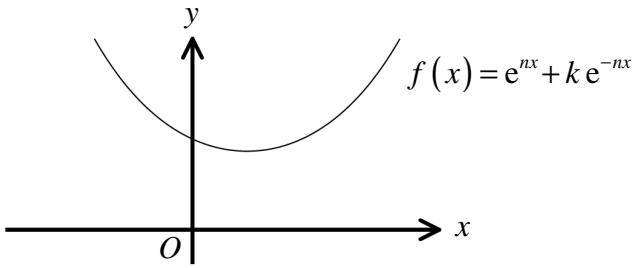
Show that $x=p$ is a solution of the equation

$$2x\mathrm{e}^{x}-1=0.$$

$\square$, proof



Question 177 **(****+)**

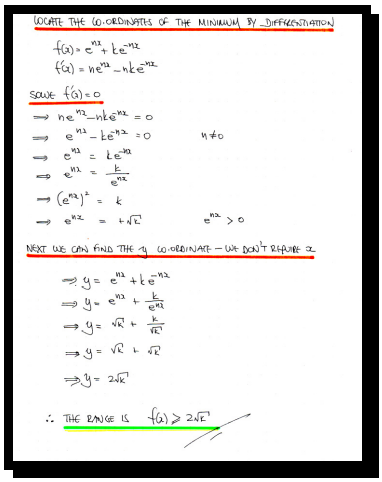


The figure above shows the graph of the curve with equation

$$f(x)=e^{nx}+k\,e^{-nx},\ x\in\mathbb{R}\ ,\ k>1,\ n>0\,.$$

Find the range of $f(x)$ in exact form.

$\boxed{\qquad}$, $\boxed{f(x)\geq 2\sqrt{k}}$


LOCATE THE CO-ORDINATES OF THE MINIMUM BY DIFFERENTIATION

$f(x)=e^{nx}+ke^{-nx}$

$f'(x)=ne^{nx}-nke^{-nx}$

SOLVE $f'(x)=0$

$\Rightarrow ne^{nx}-nke^{-nx}=0$

$\Rightarrow e^{nx}-ke^{-nx}=0$ $\quad n\neq 0$

$\Rightarrow e^{nx}=ke^{-nx}$

$\Leftrightarrow e^{nx}=\frac{k}{e^{nx}}$

$\Rightarrow (e^{nx})^2=k$

$\Rightarrow e^{nx}=+\sqrt{k}$ $\quad e^{nx}>0$

NEXT WE CAN FIND THE y CO-ORDINATE — WE DON'T REQUIRE x

$\Rightarrow y=e^{nx}+ke^{-nx}$

$\Rightarrow y=e^{nx}+\frac{k}{e^{nx}}$

$\Rightarrow y=\sqrt{k}+\frac{k}{\sqrt{k}}$

$\Rightarrow y=\sqrt{k}+\sqrt{k}$

$\Rightarrow y=2\sqrt{k}$

$\therefore$ THE RANGE IS $f(x)\geqslant 2\sqrt{k}$

Question 178 (**+)**

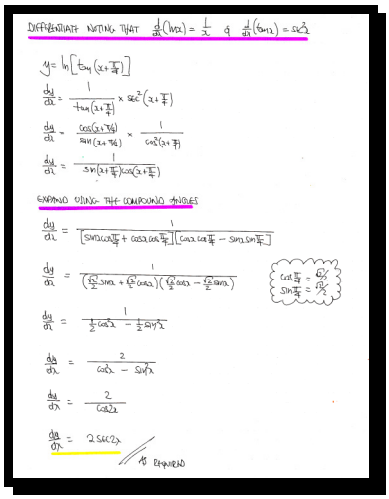
A curve has equation

$$y=\ln\left[\tan\left(x+\frac{\pi}{4}\right)\right], \text{ where } \tan\left(x+\frac{\pi}{4}\right)>0.$$

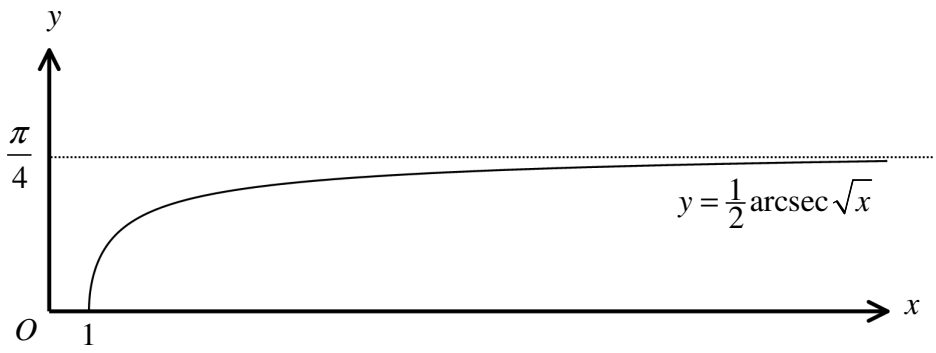
Show that

$$\frac{dy}{dx}=2\sec 2x \ .$$

[ ], [proof]



Question 179 **(****+)**



The figure above shows the graph of the curve with equation

$$y=\frac{1}{2}\text{arcsec}\sqrt{x}\,,\ x\geq 1,\ 0\leq y<\frac{1}{4}\pi\,,$$

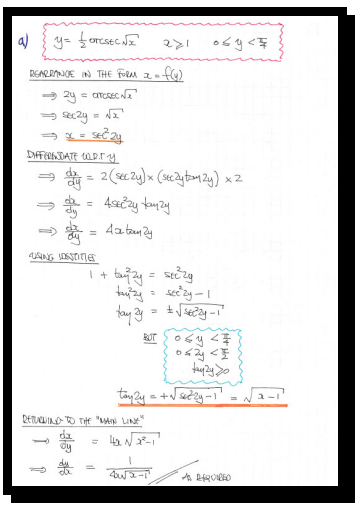
where $\text{arcsec}(u)$ is the inverse function of $\sec(u)$.

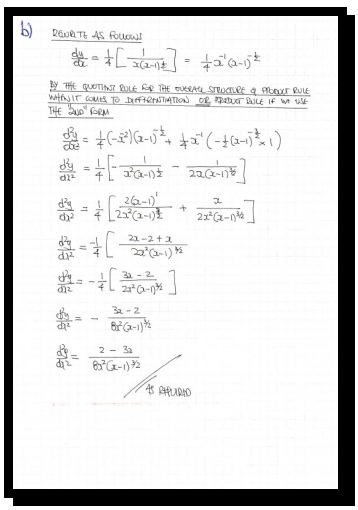
Show clearly that …

a) … $\dfrac{dy}{dx}=\dfrac{1}{4x\sqrt{x-1}}$.

b) … $\dfrac{d^2y}{dx^2}=\dfrac{2-3x}{8x^2(x-1)^{\frac{3}{2}}}$.

[ ], [proof]





Question 180 (**+)**

a) Differentiate the following expressions with respect to x, fully simplifying the answers.

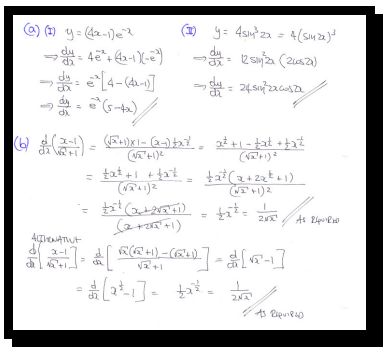
i. $y=(4x-1)\mathrm{e}^{-x}$.

ii. $y=4\sin^3 2x$.

b) Prove that

$$\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}+1}\right)=\frac{1}{2\sqrt{x}}.$$

$$\boxed{\frac{dy}{dx}=\mathrm{e}^{-x}(5-4x)}, \quad \boxed{\frac{dy}{dx}=24\sin^2 2x\cos 2x}$$



Question 181 **(****+)**

The curve C has equation

$$y=\sqrt{\mathrm{e}^{2x}-2x}\,,\ \mathrm{e}^{2x}>2x\,.$$

The tangent to the curve at the point P where $x=p$ passes through the origin.

a) Show that $x=p$ is a solution of the equation

$$(1-x)\mathrm{e}^{2x}=x\,.$$

b) Show that the equation $(1-x)\mathrm{e}^{2x}=x$ has root between 0.8 and 1.

The iterative formula

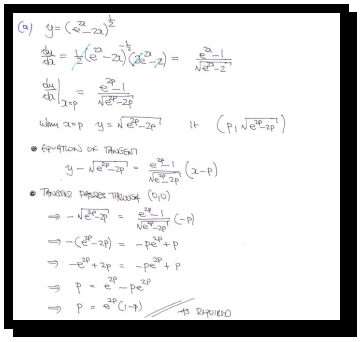
$$x_{n+1}=1-x_n\,\mathrm{e}^{-2x_n}$$

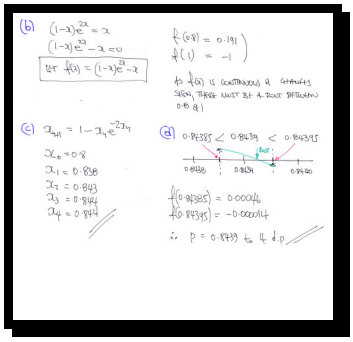
with $x_0=0.8$ is used to find this root.

c) Find, to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Hence show that the value of p is 0.8439, correct to 4 decimal places.

$\boxed{}$, $\boxed{x_1=0.838,\quad x_2=0.843,\quad x_3=0.844,\quad x_4=0.844}$





Question 182 (**+)**

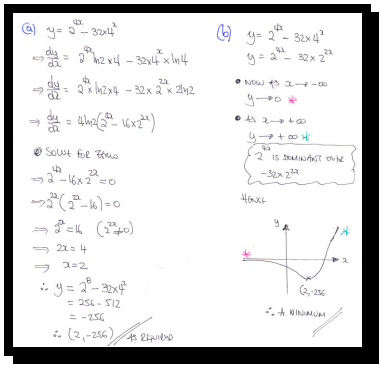
The curve C has equation

$$y = 2^{4x} - 32 \times 4^x , \ x \geq 0 .$$

a) Show that C has a turning point at $(2, -256)$.

b) Without further differentiation explain why this turning point must be a minimum.

[ ], [proof]



Question 183 (**+)**

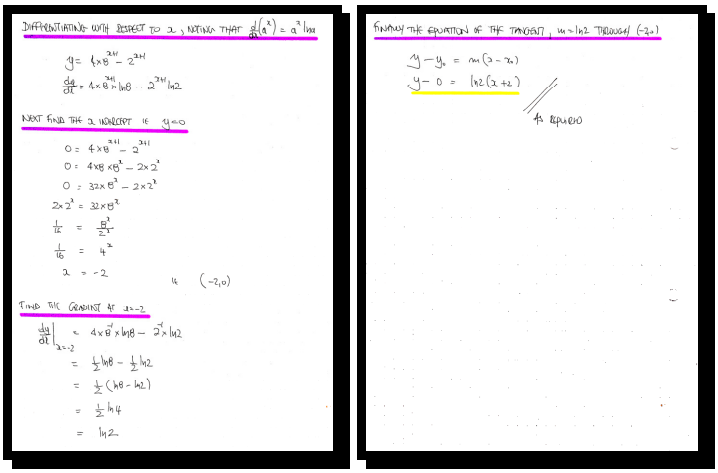
The curve C has equation

$$y = 4\times 8^{x+1} - 2^{x+1}.$$

Show that an equation of the tangent to the curve, at the point where C crosses the x axis is given by

$$y = (x+2)\ln 2.$$

[ ], proof



Question 184 **(****+)**

$$y=\arccos x\,,\ -1\le x\le 1\,,\ 0\le y\le \pi\,.$$

a) By writing $y=\arccos x$ as $x=\cos y$, show that

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}\,.$$

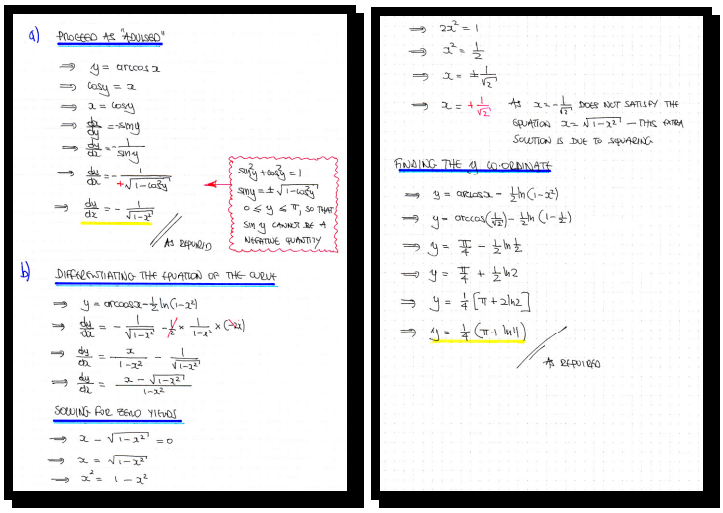
The curve C has equation

$$y=\arccos x-\frac{1}{2}\ln\left(1-x^2\right),\ x>0\,.$$

b) Show that the y coordinate of the stationary point of C is

$$\frac{1}{4}\left(\pi+\ln 4\right).$$

[ ], [proof]

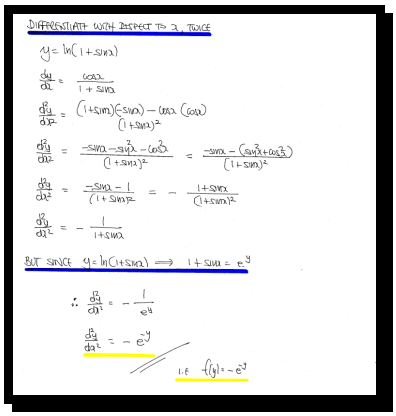


Question 185 (**+)**

$$y=\ln\left(1+\sin x\right),\ \sin x\neq -1.$$

Show clearly that $\dfrac{d^2y}{dx^2}=f\left(y\right)$, where $f\left(y\right)$ is a function to be found.

$\boxed{}$, $\boxed{y=-\mathrm{e}^{-y}}$



Question 186 (**+)**

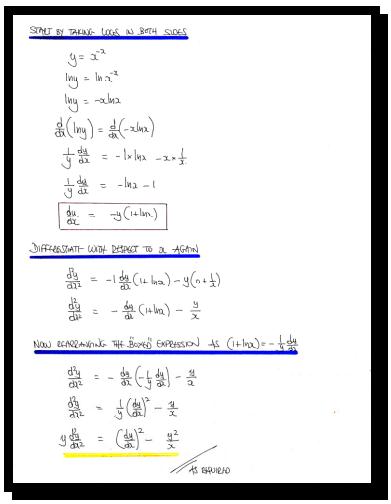
A curve C has equation

$$y = x^{-x}, \quad x \in \mathbb{R}, \quad x > 0 .$$

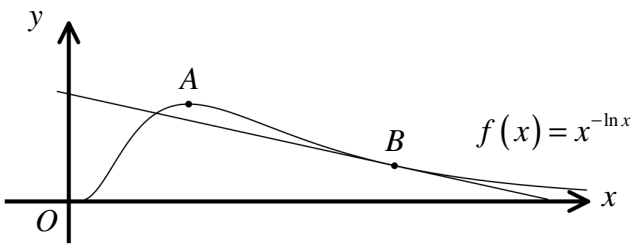
Show that y is a solution of the equation

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x} .$$

[ ], proof



Question 187 **(****+)**



The figure above shows that the graph of

$$f(x)=x^{-\ln x}, \quad x\in\mathbb{R}, \quad x>0.$$

The curve has a turning point at A.

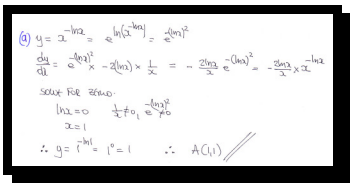
a) Find the coordinates of A.

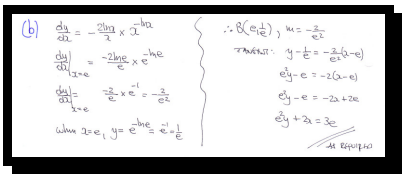
The point B lies on the curve where $x=\mathrm{e}$.

b) Show that the equation of the tangent to the curve at B is given by

$$\mathrm{e}^2 y+2x=3\mathrm{e}.$$

$\boxed{}$, $\boxed{A(1,1)}$



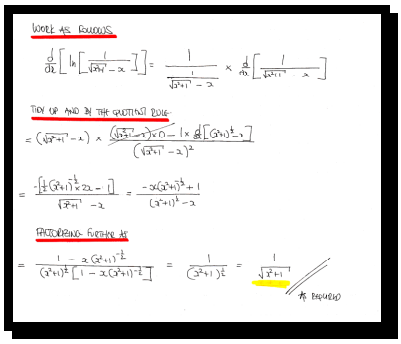


Question 188 **(****+)**

Show, with a detailed method, that

$$\frac{d}{dx}\left[\ln\left(\frac{1}{\sqrt{x^2+1}-x}\right)\right]=\frac{1}{\sqrt{x^2+1}}.$$

[ ], proof



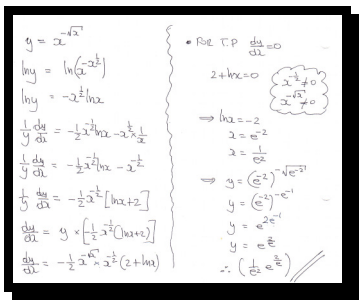
Question 189 **(****+)**

A curve C has equation

$$y=x^{-\sqrt{x}}, \quad x\in\mathbb{R}, \quad x>0.$$

Show that the coordinates of turning point of C are $\left(\frac{1}{e^2},e^{\frac{2}{e}}\right)$.

proof



Question 190 (**+)**

A curve C has equation

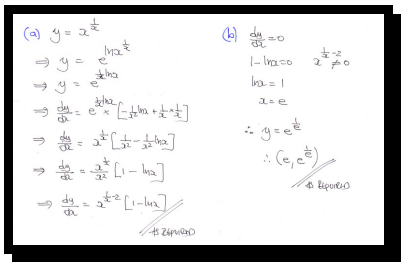
$$y = x^{\frac{1}{x}}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Show clearly that …

a) … $\dfrac{dy}{dx} = x^{\frac{1}{x}-2}\left(1-\ln x\right)$.

b) … the coordinates of turning point of C are $\left(\mathrm{e}, \mathrm{e}^{\frac{1}{\mathrm{e}}}\right)$.

proof



Question 191 (**+)**

A curve C has equation

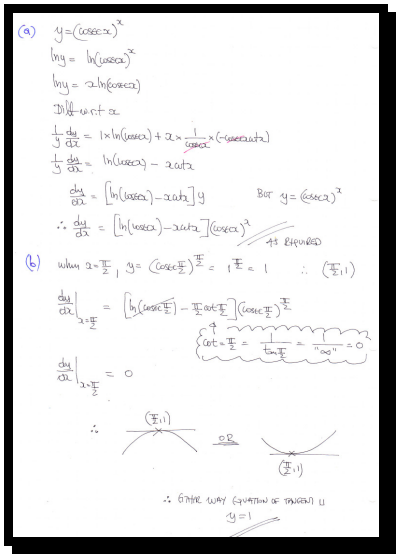
$$y=(\operatorname{cosec} x)^x,\quad x\in\mathbb{R},\quad x>0.$$

a) Show that

$$\frac{dy}{dx}=\left[\ln(\operatorname{cosec} x)-x\cot x\right](\operatorname{cosec} x)^x.$$

b) Find the equation of the tangent to C at the point where $x=\dfrac{\pi}{2}$.

$$\boxed{y=1}$$



Question 192 **(****+)**

$$y=\arctan x\,,\ x\in\mathbb{R}\,.$$

a) By writing the above equation in the from $x=g\left(y\right)$, show that

$$\frac{d}{dx}\left(\arctan x\right)=\frac{1}{1+x^2}\,.$$

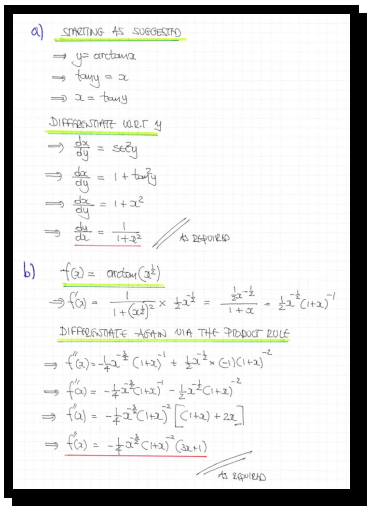
The function f is defined as

$$f\left(x\right)=\arctan\sqrt{x}\,,\quad x\in\mathbb{R}\,,\ x\geq 0\,.$$

b) Show further that

$$f''\left(x\right)=-\frac{1}{4}x^{-\frac{3}{2}}\left(3x+1\right)\left(x+1\right)^{-2}\,.$$

[ ], [proof]



Question 193 (**+)**

The function f is defined as

$$f(x)=x^{x}\,\mathrm{e}^{-2x},\quad x\in\mathbb{R},\quad x>0.$$

a) Find an expression for $f'(x)$.

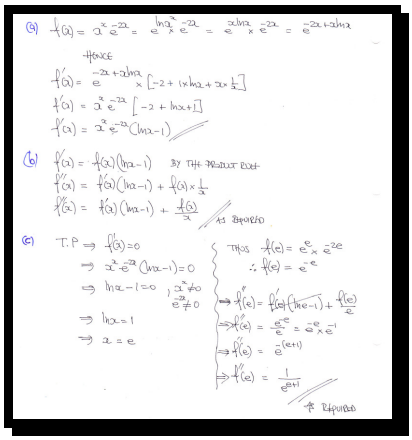
b) Show clearly that

$$f''(x)=f'(x)(\ln x-1)+\frac{f(x)}{x}.$$

c) Show that the value of $f''(x)$ at the turning point of the function is

$$\frac{1}{\mathrm{e}^{\mathrm{e}+1}}.$$

$$\boxed{f'(x)=x^{x}\,\mathrm{e}^{-2x}(\ln x-1)}$$



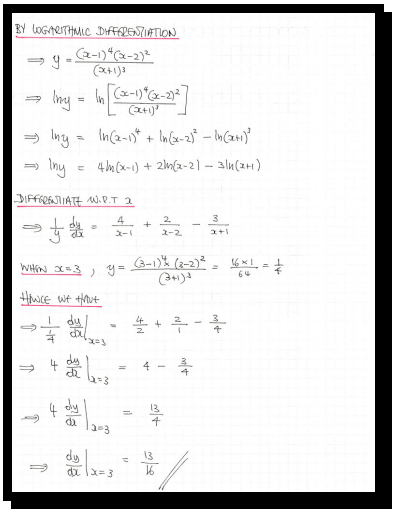
Question 194 **(****+)**

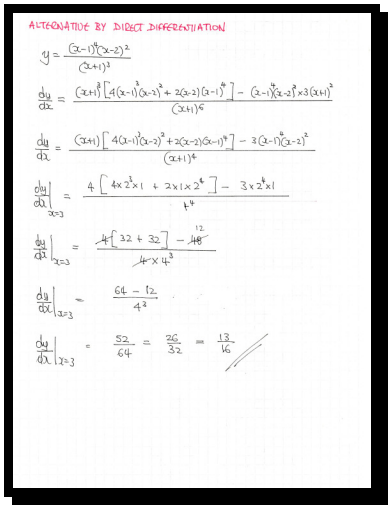
Given that

$$y=\frac{(x-1)^4(x-2)^2}{(x+1)^3},\ x\in\mathbb{R},\ x\neq -1,$$

find the value of $\dfrac{dy}{dx}$ at $x=3$.

$\square$, $\left.\dfrac{dy}{dx}\right|_{x=3}=\dfrac{13}{16}$





Question 195 **(****+)**

$$y = x^x, \ x > 0 .$$

a) Show by using logarithms that

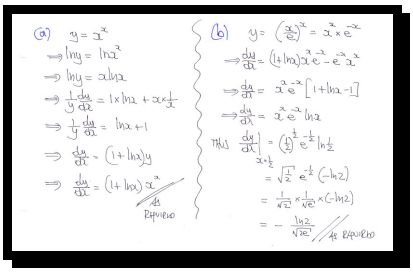
$$\frac{dy}{dx} = (1 + \ln x) x^x .$$

A curve C has equation

$$y = \left(\frac{x}{\mathrm{e}} \right)^x, \ x > 0 .$$

b) Show that at the point on C where $x = \frac{1}{2}$, the gradient is $-\frac{\ln 2}{\sqrt{2\mathrm{e}}}$.

proof

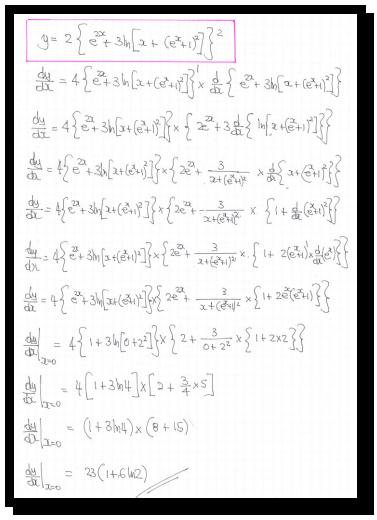


Question 196 **(****+)**

$$y=2\left\{\mathrm{e}^{2x}+3\ln\left[x+\left(\mathrm{e}^{x}+1\right)^{2}\right]\right\}^{2}.$$

Show that the value of $\dfrac{dy}{dx}$ at $x=0$ is $23\left(1+6\ln 2\right)$

[ ], [proof]



Question 197 (**+)**

A curve has equation

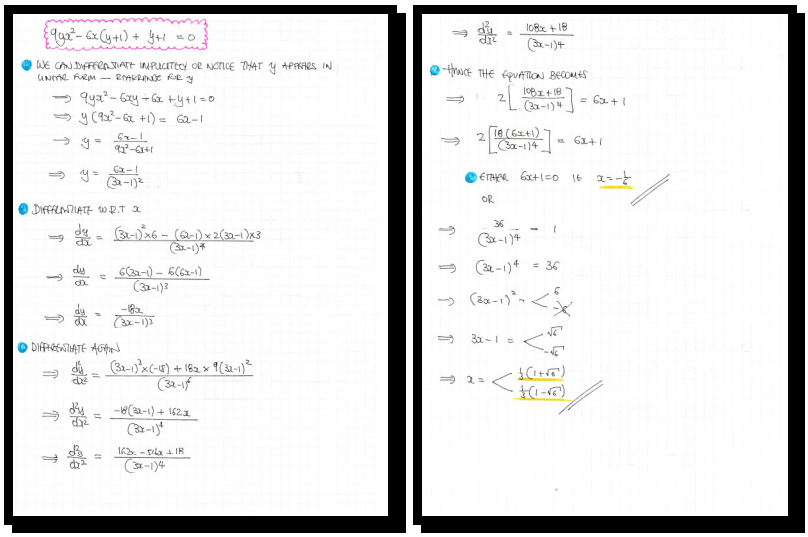
$$9yx^2-6x(y+1)+y+1=0\,,\quad x\in\mathbb{R}\,,\quad x\neq\frac{1}{3}\,.$$

Find, in exact form where appropriate, the three solutions of the equation

$$2\frac{d^2y}{dx^2}=6x+1\,,$$

where $\dfrac{d^2y}{dx^2}$ represents the second derivative of the above equation.

$\square$, $\boxed{x=-\frac{1}{6},\ \frac{1}{3}\left(1\pm\sqrt{6}\right)}$



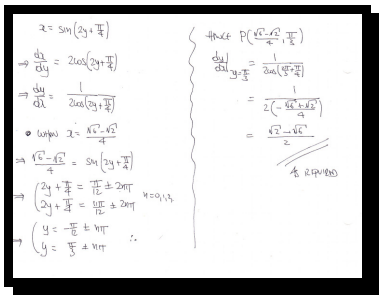
Question 198 (**+)**

The point P, with x coordinate $\dfrac{\sqrt{6}-\sqrt{2}}{4}$, lies on the curve with equation

$$x=\sin\left(2y+\frac{\pi}{4}\right),\ 0\le y\le\frac{\pi}{2}.$$

Show that the value of the gradient at P is $\dfrac{\sqrt{2}-\sqrt{6}}{2}$.

proof



Question 199 **(****+)**

A curve C_1 has equation

$$y=\ln\sqrt{x}+\sqrt{\ln x}\ ,\ \ x>1\ .$$

a) Differentiate y with respect to x, simplifying the answer as far as possible.

A different curve C_2 has equation

$$y=(2x+1)^{\frac{1}{2}}(1-4x)^{-\frac{1}{2}}\ ,\ -\frac{1}{2}\le x<\frac{1}{4}.$$

b) Show that C_2 has no turning points.

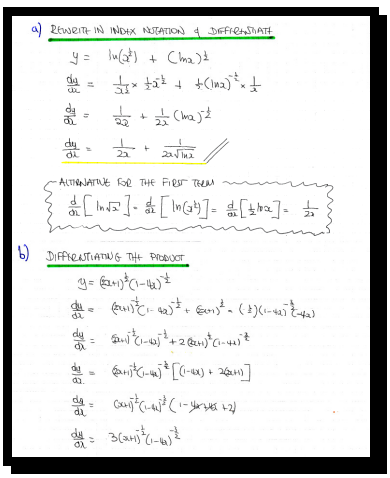
A third curve C_3 has equation

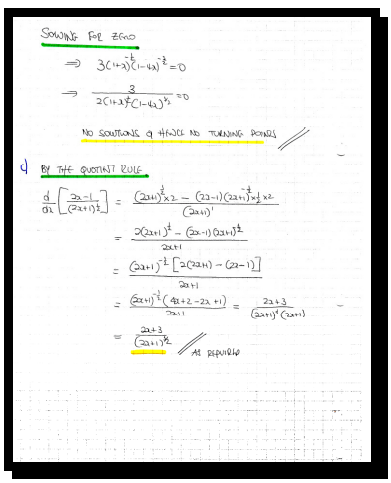
$$y=\frac{2x-1}{\sqrt{2x+1}},\ x\ge-\frac{1}{2}\ .$$

c) Show that

$$\frac{d}{dx}\left(\frac{2x-1}{\sqrt{2x+1}}\right)=\frac{2x+3}{(2x+1)^{\frac{3}{2}}}.$$

$\square$, $\boxed{\dfrac{dy}{dx}=\dfrac{1}{2x}+\dfrac{1}{2x\ln\sqrt{x}}}$



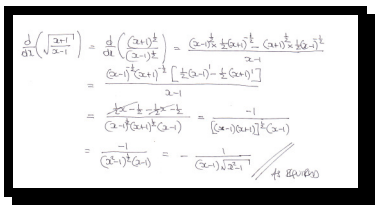


Question 200 (**+)**

Show clearly that

$$\frac{d}{dx}\left(\sqrt{\frac{x+1}{x-1}}\right)=-\frac{1}{(x-1)\sqrt{x^2-1}}.$$

proof

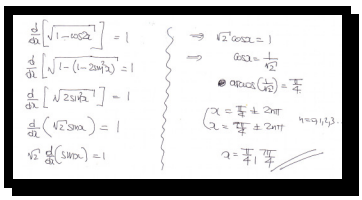


Question 201 (**+)**

Solve the equation

$$\frac{d}{dx}\left(\sqrt{1-\cos 2x}\right)=1,\ 0\le x<2\pi.$$

$$x=\frac{\pi}{4},\frac{5\pi}{4}$$



Question 202 (**+)**

Given that

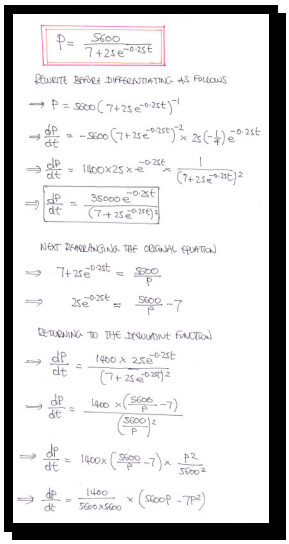
$$P=\frac{5600}{7+25\,\mathrm{e}^{-0.25t}},$$

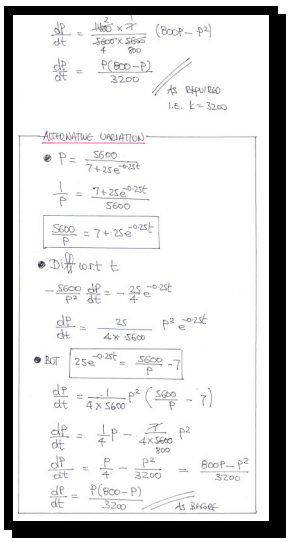
show by a detailed method that

$$\frac{dP}{dt}=\frac{P\left(800-P\right)}{k},$$

where k is an integer to be found.

$\boxed{}$, $\boxed{k=3200}$



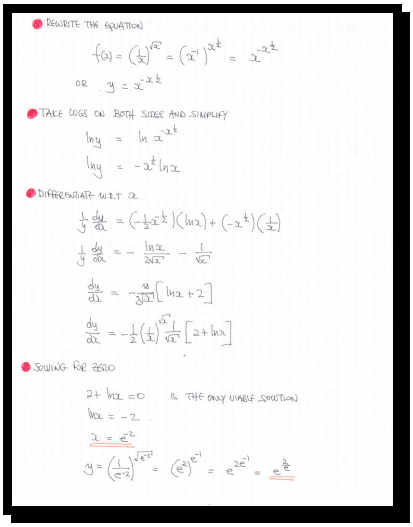


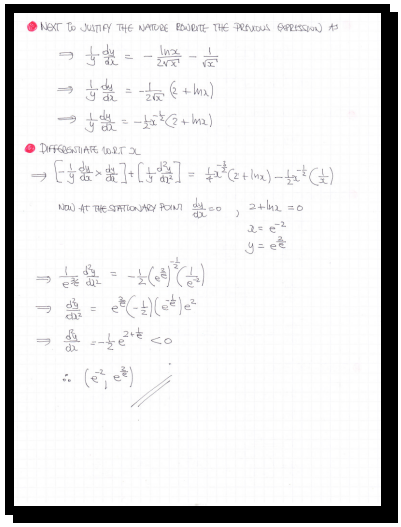
Question 203 **(****+)**

$$f(x)=\left(\frac{1}{x}\right)^{\sqrt{x}},\ x\in\mathbb{R},\ x>0.$$

Determine in exact simplified form the coordinates of the stationary point of $f(x)$, fully justifying its nature.

$\square$, $\boxed{\text{local maximum at}\left(\frac{1}{\mathrm{e}^2},\mathrm{e}^{\frac{2}{\mathrm{e}}}\right)}$



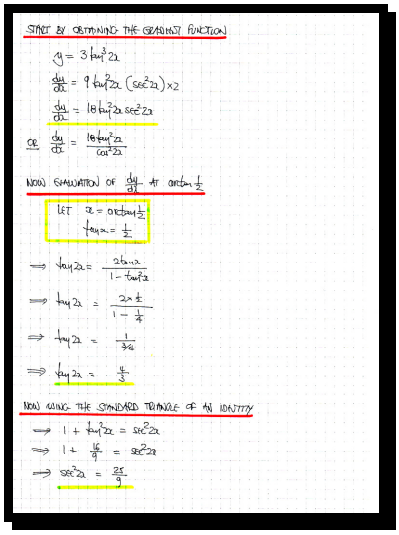


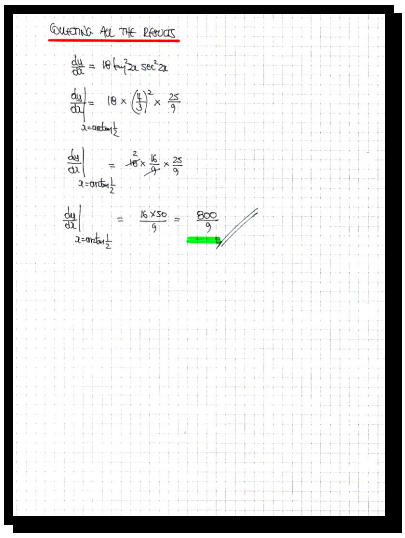
Question 204 **(****+)**

$$y = 3\tan^3 2x\,,\ x\in\mathbb{R}\,,\ x\neq\frac{1}{4}n\pi\,.$$

Determine, by showing detailed workings, the value of $\dfrac{dy}{dx}$, at $x=\arctan\dfrac{1}{2}$.

$\square$, $\left.\dfrac{dy}{dx}\right|_{x=\arctan\frac{1}{2}}=\dfrac{800}{9}$



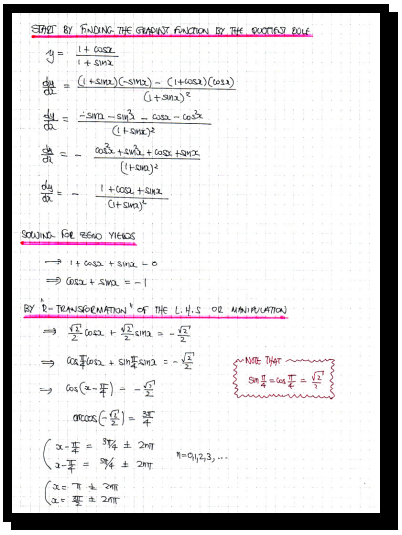


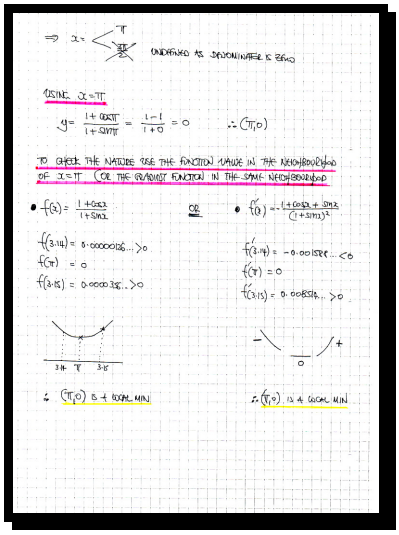
Question 205 (**+)**

$$y=\frac{1+\cos x}{1+\sin x},\ 0\le x<2\pi,\ x\neq\frac{3}{2}\pi .$$

Determine, with full justification, the coordinates of the minimum point of y.

$\boxed{}$, $\boxed{(\pi,0)}$



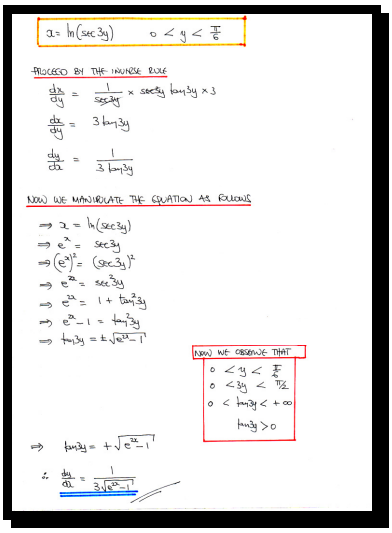


Question 206 (**+)**

$$x=\ln\left(\sec 3y\right),\ 0<y<\frac{1}{6}\pi .$$

Determine, with full justification, an expression for $\dfrac{dy}{dx}$, in terms of x.

$\boxed{}$, $\boxed{\dfrac{dy}{dx}=\dfrac{1}{3\sqrt{\mathrm{e}^{2x}-1}}}$



Question 207 (**+)**

A curve C has equation

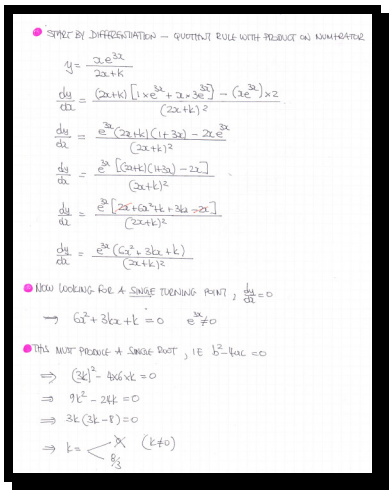
$$y=\frac{x\mathrm{e}^{3x}}{2x+k},\quad x\in\mathbb{R},\quad x\neq k,$$

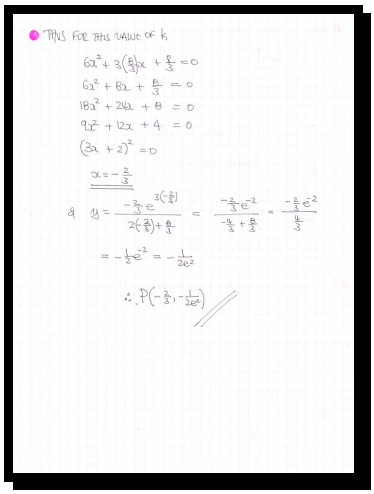
where k is a non zero constant.

It is given that C has a single turning point at P.

Find the exact coordinates of P.

$\boxed{}$, $\boxed{P\left(-\frac{2}{3},-\frac{1}{2}\mathrm{e}^{-2}\right)}$





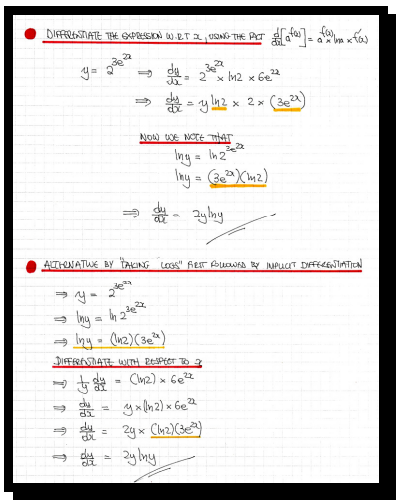
Question 208 (**+)**

A curve has equation

$$2^{3e^{2x}}, \quad x \in \mathbb{R}.$$

Express $\dfrac{dy}{dx}$ in terms of y .

$\square$, $\boxed{\dfrac{dy}{dx} = 2y \ln y}$



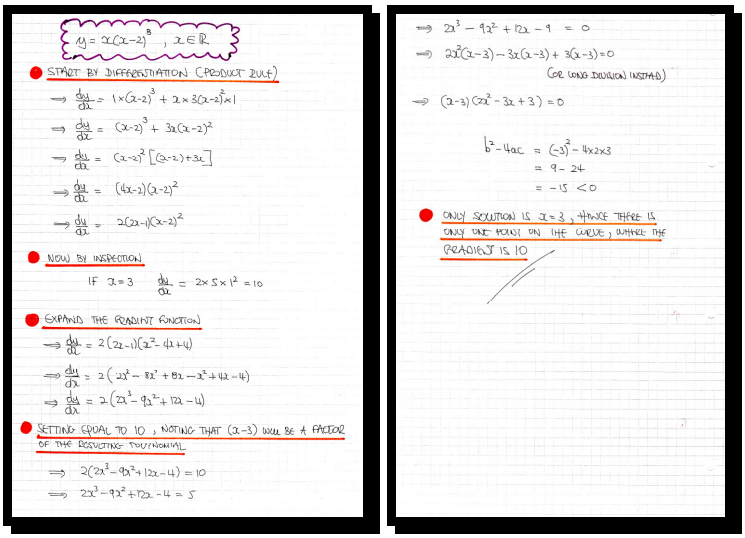
Question 209 (**+)**

A quartic curve C has equation

$$y = x(x-2)^3, \ x \in \mathbb{R}.$$

Show that there is only one point on C where the gradient is 10.

$\boxed{}$, $\boxed{\text{proof}}$



Question 210 (**+)**

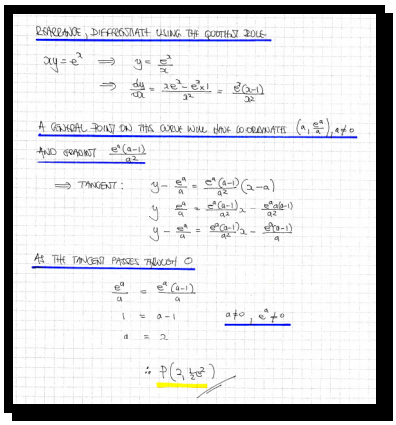
The point P lies on the curve with equation

$$xy = \mathrm{e}^x , \qquad xy > 0 .$$

The tangent to the curve at P passes through the origin O.

Determine the coordinates of P.

$\square$, $\boxed{P\left(2, \frac{1}{2}\mathrm{e}^2\right)}$



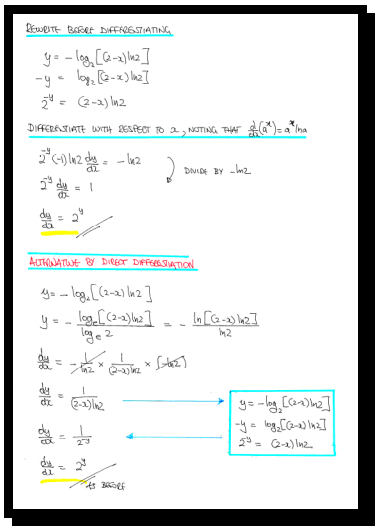
Question 211 (**+)**

A curve has equation

$$y=-\log_2\left[(2-x)\ln 2\right],\ x\in\mathbb{R},\ x<2.$$

Determine a simplified expression for $\dfrac{dy}{dx}$ in terms of y.

$\square$, $\boxed{\dfrac{dy}{dx}=2^y}$



Question 212 **(****+)**

$$y=\arccos x\,,\ x\in\mathbb{R}\,,\ -1\leq x\leq 1\,.$$

a) By writing the above equation in the from $x=f\left(y\right)$, show that

$$\frac{d}{dx}\left(\arccos x\right)=-\frac{1}{\sqrt{1-x^2}}\,.$$

A curve has equation

$$y=\arccos\left(1-x^2\right),\quad x\in\mathbb{R}\,,\ 0<x\leq\sqrt{2}\,.$$

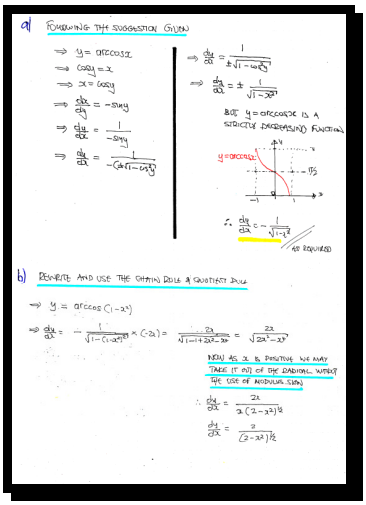
b) Show further that

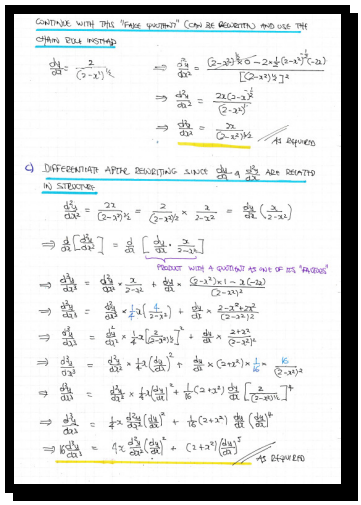
$$\frac{d^2y}{dx^2}=\frac{2x}{\left(2-x^2\right)^{\frac{3}{2}}}\,.$$

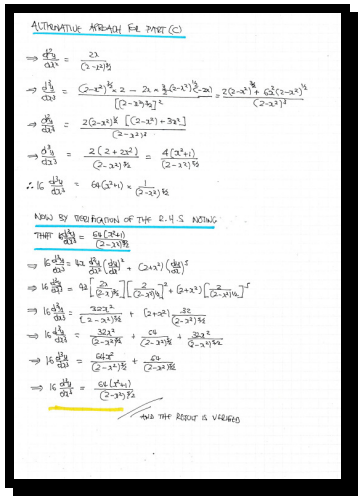
c) Show clearly that

$$16\frac{d^3y}{dx^3}=4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2+\left(2+x^2\right)\left(\frac{dy}{dx}\right)^5\,.$$

[ ], [proof]







Question 213 (*****)

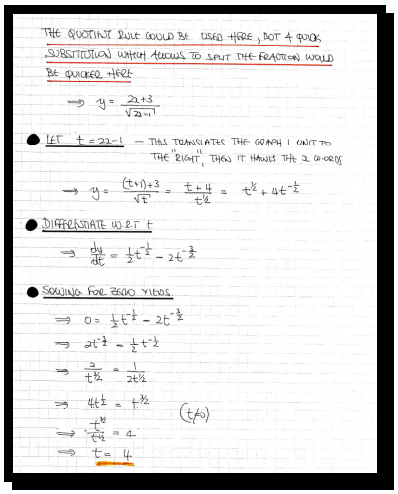
A curve C has equation

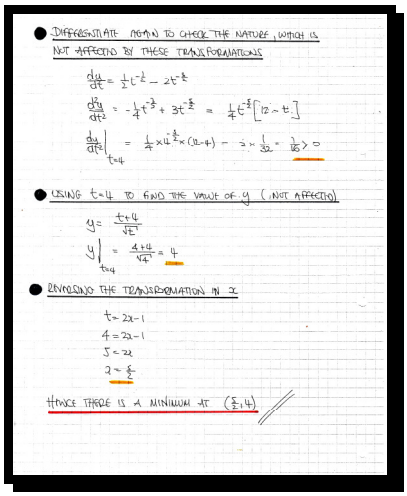
$$y=\frac{2x+3}{\sqrt{2x-1}},\ x\in\mathbb{R},\ x>\frac{1}{2}.$$

Find the coordinates of the stationary point of C, further determining the nature of this point.

You may not use the product rule, the quotient rule or logarithmic differentiation in this question.

$\square$, $\boxed{\min\left(\frac{5}{2},4\right)}$



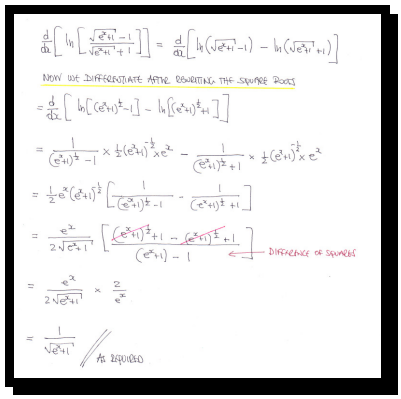


Question 214 (*****)

Show with a detailed method that

$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{\mathrm{e}^x+1}-1}{\sqrt{\mathrm{e}^x+1}+1}\right)\right]=\frac{1}{\sqrt{\mathrm{e}^x+1}}.$$

, proof



Question 215 **(*****)**

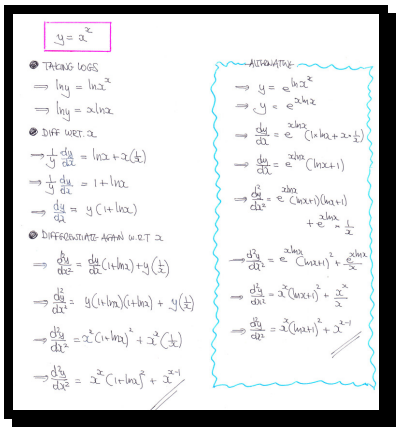
A curve C has equation

$$y = x^x, \quad x \in \mathbb{R}, \quad x > 0.$$

Show that y is a solution of the equation

$$\frac{d^2y}{dx^2} = x^x\left(1+\ln x\right)^2 + x^{x-1}.$$

$\boxed{}$, $\boxed{\text{proof}}$

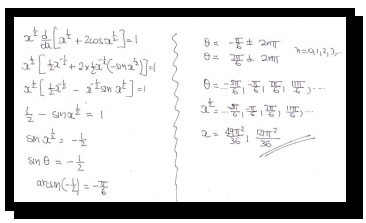


Question 216 **(*****)**

Find, in terms of π, the solutions of the equation

$$\sqrt{x}\,\frac{d}{dx}\left(\sqrt{x}+2\cos\sqrt{x}\right)=1, \quad 0 \le x < 4\pi^2.$$

$$\boxed{x = \frac{49\pi^2}{36}, \frac{121\pi}{36}}$$



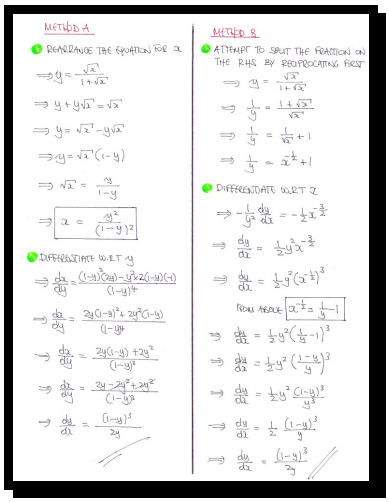
Question 217 **(*****)**

Given that

$$y=\frac{\sqrt{x}}{1+\sqrt{x}},$$

show that $\dfrac{dy}{dx}=f\left(y\right)$, where $f\left(y\right)$ is a function to be determined.

$\square$, $\boxed{f\left(y\right)=\dfrac{\left(1-y\right)^3}{2y}}$



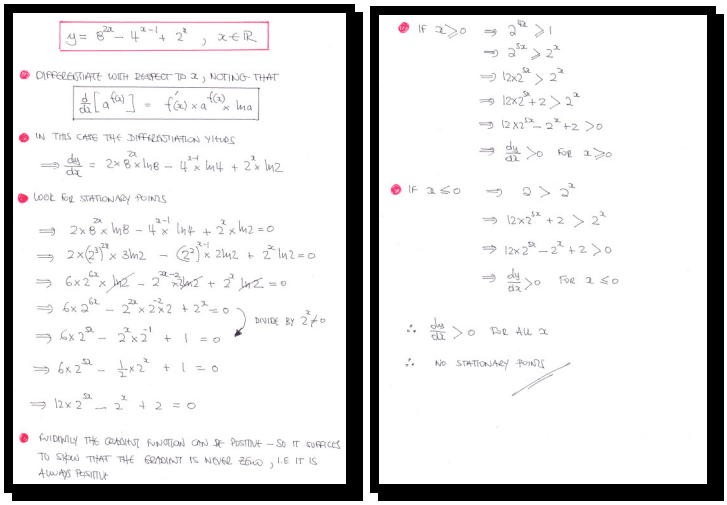
Question 218 (*****)

A curve C has equation

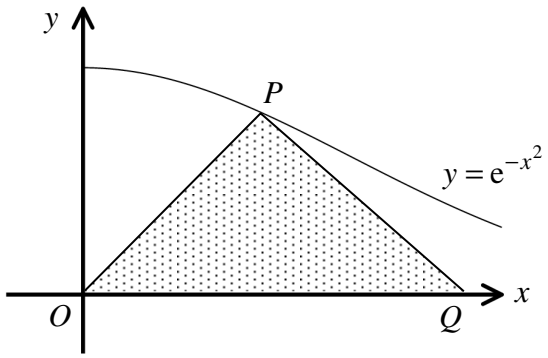
$$y = 8^{2x} - 4^{x-1} + 2^x, \ x \in \mathbb{R}$$

Show that C has no stationary points.

[ ], proof



Question 219 **(*****)**



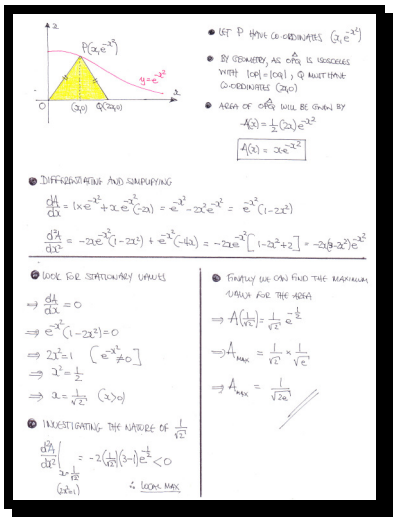
The figure above shows the graph of the curve with equation

$$y=\mathrm{e}^{-x^2}\ ,\ x\geq 0\ .$$

The point P lies on the curve and the point Q lies on the positive x axis so that $|OP|=|PQ|$ where O is the origin.

Show with full justification that the largest area of the triangle OPQ is $\dfrac{1}{\sqrt{2\mathrm{e}}}$.

$\boxed{}$, $\boxed{\text{proof}}$

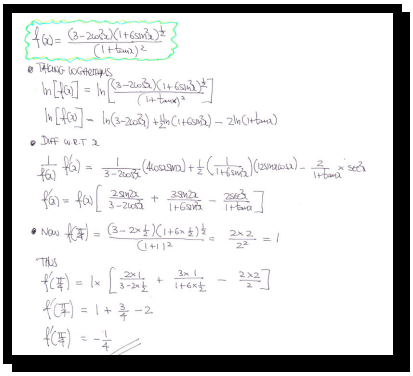


Question 220 (*****)

$$f(x)\equiv\frac{\left(3-2\cos^2 x\right)\left(1+6\sin^2 x\right)^{\frac{1}{2}}}{\left(1+\tan x\right)^2},\ \tan x\neq -1.$$

Determine the value of $f'\left(\frac{\pi}{4}\right)$.

$\square$, $\boxed{f'\left(\frac{\pi}{4}\right)=-\frac{1}{4}}$

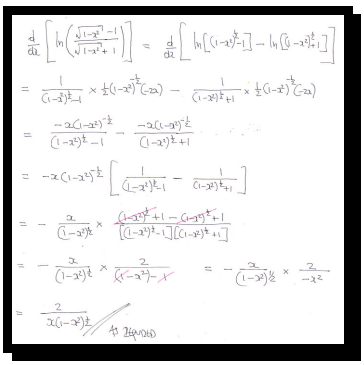


Question 221 (*****)

Show clearly that

$$\frac{d}{dx}\left[\ln\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right)\right]=\frac{2}{x\sqrt{1-x^2}}.$$

proof



Question 222 (*****)

A curve is defined over the largest real domain by the equation

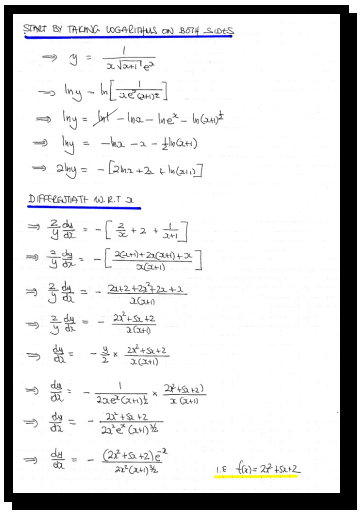
$$y=\frac{1}{x\mathrm{e}^{x}\sqrt{x+1}}$$

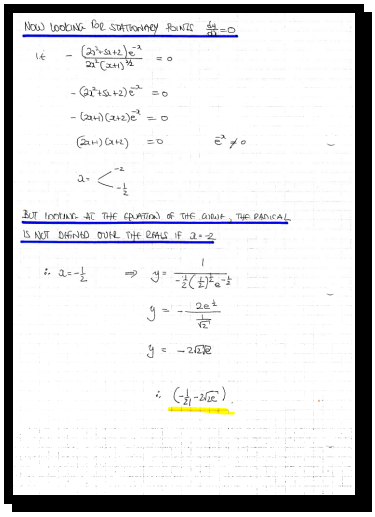
Show that

$$\frac{dy}{dx}=\frac{f\left(x\right)\,\mathrm{e}^{-x}}{2x^{2}\left(x+1\right)^{\frac{3}{2}}},$$

where $f\left(x\right)$ is a quadratic expression and hence find, in exact form, the coordinates of any stationary points of the curve.

$\boxed{}$, $\boxed{f\left(x\right)=2x^{2}+5x+2}$, $\boxed{\left(-\frac{1}{2},-2\sqrt{2\mathrm{e}}\right)}$



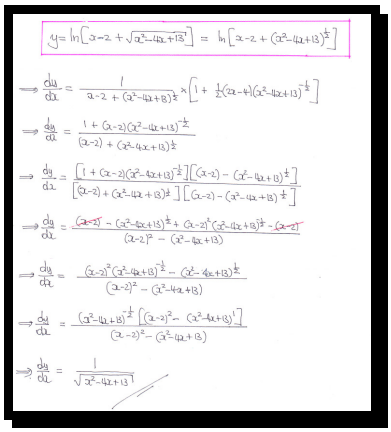


Question 223 (*****)

Show clearly that

$$\frac{d}{dx}\left[\ln\left(x-2+\sqrt{x^2-4x+13}\right)\right]=\frac{1}{\sqrt{x^2-4x+13}}.$$

, proof



Question 224 (*****)

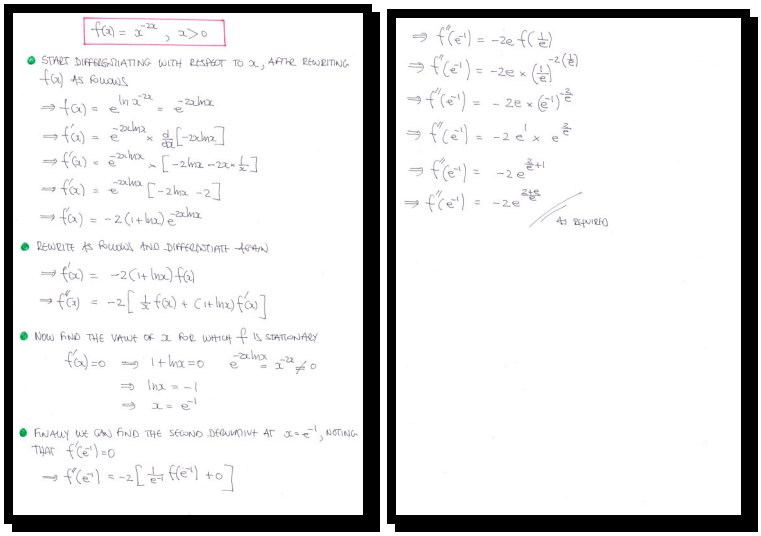
The function f is defined as

$$f(x)=x^{-2x},\quad x\in\mathbb{R},\quad x>0.$$

Show that the value of $f''(x)$ at the stationary point of the function is

$$-2\mathrm{e}^{\frac{\mathrm{e}+2}{\mathrm{e}}}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 225 **(*****)**

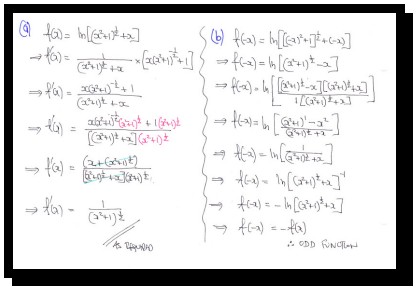
$$f(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right],\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\dfrac{1}{\sqrt{x^2+1}}$.

b) … $f(x)$ is an odd function.

proof



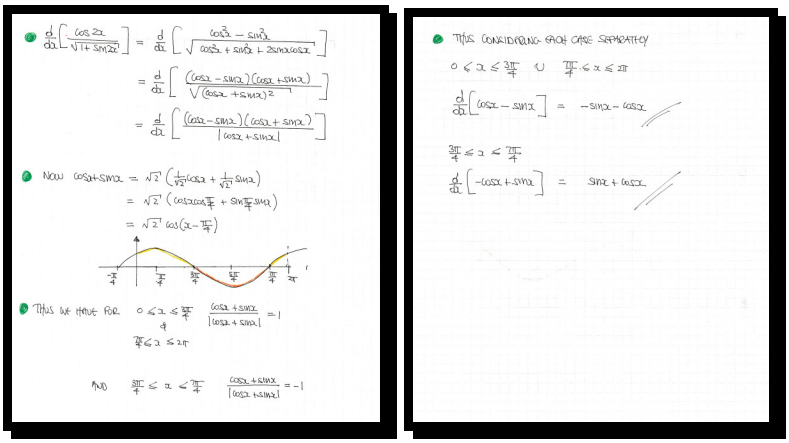
Question 226 **(*****)**

Show clearly that

$$\frac{d}{dx}\left(\frac{\cos 2x}{\sqrt{1+\sin 2x}}\right)=\begin{cases}-\sin x-\cos x & 0\le x\le \alpha\pi \\ \sin x+\cos x & \alpha\pi\le x\le \beta\pi \\ -\sin x-\cos x & \beta\pi\le x\le 2\pi\end{cases}$$

where α and β are constants to be found.

$\boxed{}$, $\boxed{\text{proof}}$



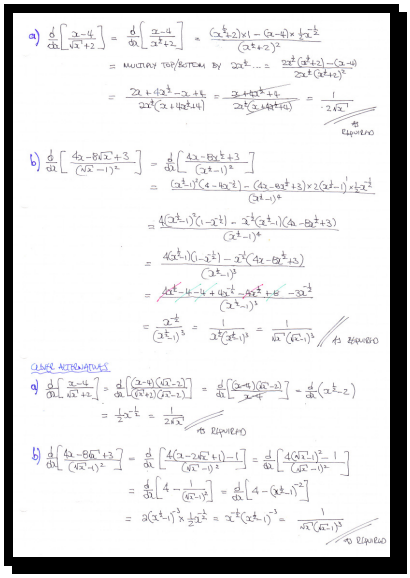
Question 227 (*****)

Show clearly that …

i. … $\dfrac{d}{dx}\left(\dfrac{x-4}{\sqrt{x}+2}\right)=\dfrac{1}{2\sqrt{x}}$

ii. … $\dfrac{d}{dx}\left(\dfrac{4x-8\sqrt{x}+3}{\left(\sqrt{x}-1\right)^2}\right)=\dfrac{1}{\sqrt{x}\left(\sqrt{x}-1\right)^3}$.

$\boxed{}$, $\boxed{\text{proof}}$



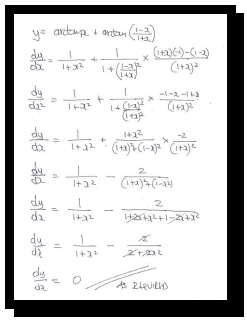
Question 228 (*****)

$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right), \ x \in \mathbb{R}.$$

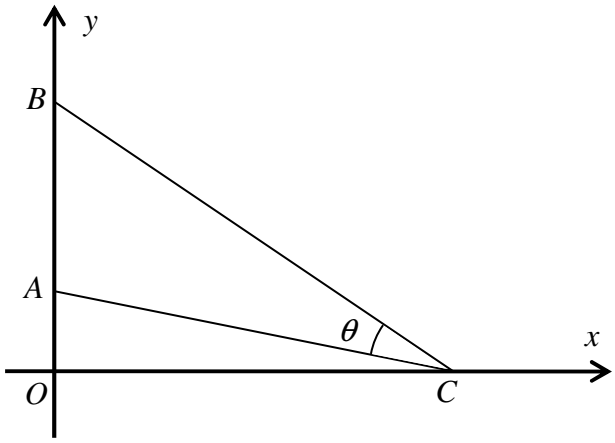
Without simplifying the above expression, use differentiation to show that

$$\frac{dy}{dx} = 0, \text{ for all values of } x.$$

proof



Question 229 **(*****)**

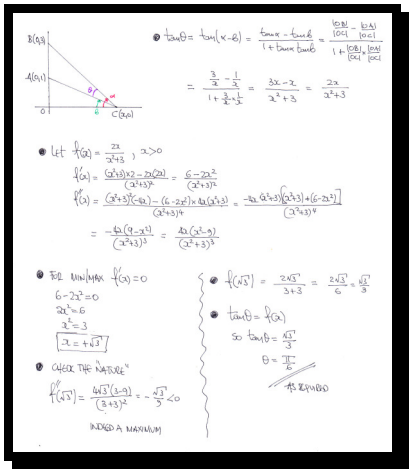


The figure above shows the triangle ABC, where $\angle ACB = \theta$.

The points A and B have respective coordinates $(0,1)$ and $(0,3)$, while the **variable** point $C(x,0)$ lies on the positive x axis.

Show that as C varies, the maximum value of θ is $\dfrac{\pi}{6}$.

[ ], [proof]



Question 230 (*****)

A curve C has equation

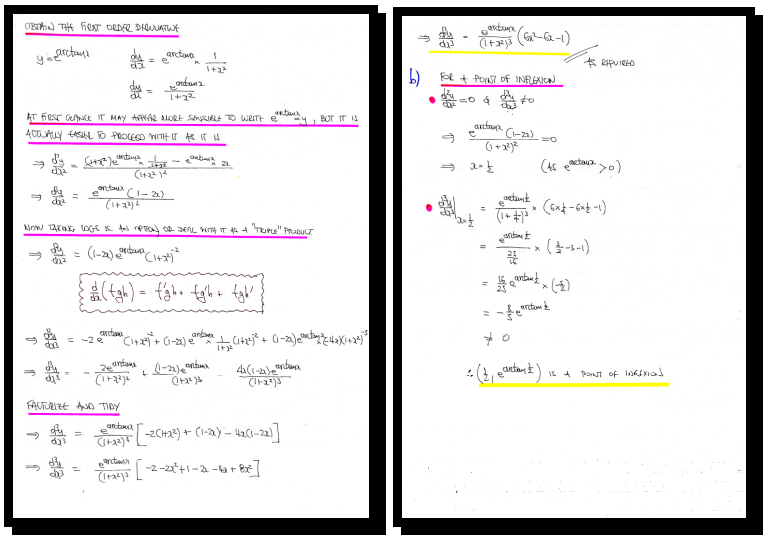
$$y = \mathrm{e}^{\arctan x}, \quad x \in \mathbb{R}.$$

a) Show, with detailed workings, that

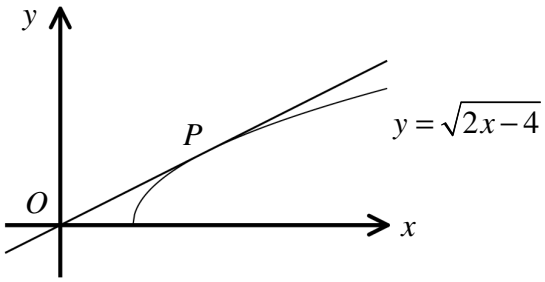
$$\frac{d^3y}{dx^3} = \frac{\left(6x^2 - 6x - 1\right)\mathrm{e}^{\arctan x}}{\left(1+x^2\right)^3}.$$

b) Deduce that C has a point of inflection, stating its coordinates.

$\square$, $\boxed{\left(\frac{1}{2}, \mathrm{e}^{\arctan\frac{1}{2}}\right)}$



Question 231 (*****)



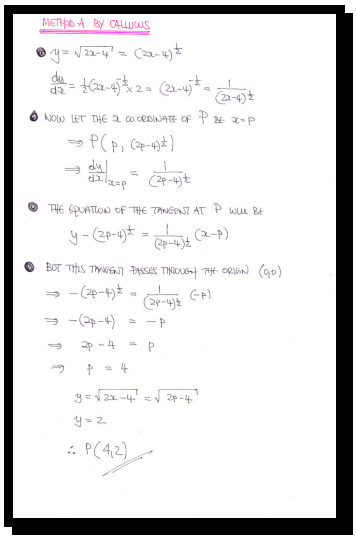
The figure above shows the graph of the curve C with equation

$$y=\sqrt{2x-4}\,,\ x\geq 2\,.$$

The point P lies on C, so that the tangent to the curve at the point P passes through the origin O.

Use a calculus method to find the coordinates of P.

$\square$, $\boxed{P(4,2)}$


METHOD A - BY CALCULUS

$y=\sqrt{2x-4}=(2x-4)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(2x-4)^{-\frac{1}{2}}\times 2=(2x-4)^{-\frac{1}{2}}=\frac{1}{(2x-4)^{\frac{1}{2}}}$

NOW LET THE x CO-ORDINATE OF P BE $x=p$

$\Rightarrow P\left(p,(2p-4)^{\frac{1}{2}}\right)$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=p}=\frac{1}{(2p-4)^{\frac{1}{2}}}$

THE EQUATION OF THE TANGENT AT P WILL BE

$y-(2p-4)^{\frac{1}{2}}=\frac{1}{(2p-4)^{\frac{1}{2}}}(x-p)$

BUT THIS TANGENT PASSES THROUGH THE ORIGIN $(0,0)$

$\Rightarrow -(2p-4)^{\frac{1}{2}}=\frac{1}{(2p-4)^{\frac{1}{2}}}(-p)$

$\Rightarrow -(2p-4)=-p$

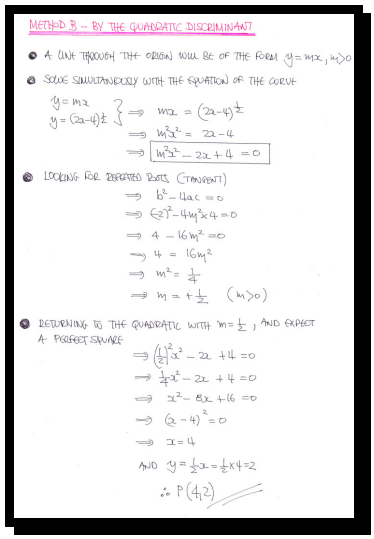
$\Rightarrow 2p-4=p$

$\Rightarrow p=4$

$y=\sqrt{2x-4}=\sqrt{2p-4}$

$y=2$

$\therefore P(4,2)$


METHOD B - BY THE QUADRATIC DISCRIMINANT

A LINE THROUGH THE ORIGIN WILL BE OF THE FORM $y=mx$, $m>0$

SOLVE SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$\left.\begin{matrix}y=mx\\ y=(2x-4)^{\frac{1}{2}}\end{matrix}\right\}\Rightarrow mx=(2x-4)^{\frac{1}{2}}$

$\Rightarrow m^2x^2=2x-4$

$\Rightarrow \boxed{m^2x^2-2x+4=0}$

LOOKING FOR REPEATED ROOTS, (TANGENT)

$\Rightarrow b^2-4ac=0$

$\Rightarrow (-2)^2-4m^2\times 4=0$

$\Rightarrow 4-16m^2=0$

$\Rightarrow 4=16m^2$

$\Rightarrow m^2=\frac{1}{4}$

$\Rightarrow m=+\frac{1}{2}\quad (m>0)$

RETURNING TO THE QUADRATIC WITH $m=\frac{1}{2}$, AND EXPECT A PERFECT SQUARE

$\Rightarrow \left(\frac{1}{2}\right)^2x^2-2x+4=0$

$\Rightarrow \frac{1}{4}x^2-2x+4=0$

$\Rightarrow x^2-8x+16=0$

$\Rightarrow (x-4)^2=0$

$\Rightarrow x=4$

AND $y=\frac{1}{2}x=\frac{1}{2}\times 4=2$

$\therefore P(4,2)$

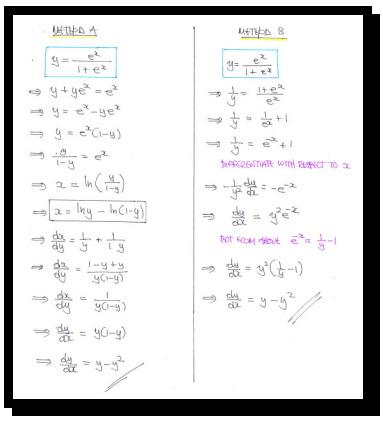
Question 232 (*****)

Given that

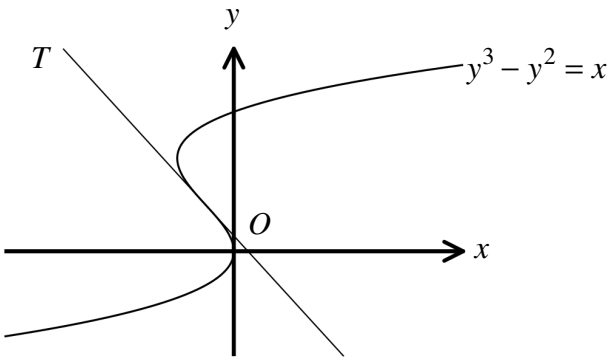
$$y=\frac{\mathrm{e}^{x}}{1+\mathrm{e}^{x}},$$

show that $\dfrac{dy}{dx}=f\left(y\right)$, where $f\left(y\right)$ is a simplified function to be determined.

$\square$, $\boxed{f\left(y\right)=y-y^{2}}$



Question 233 **(*****)**



The figure above shows the graph of a curve with equation

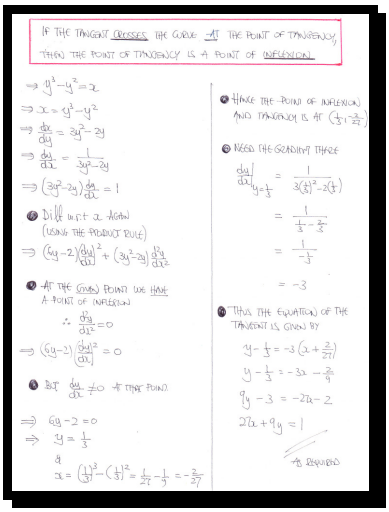
$$y^3 - y^2 = x, \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

There exists a tangent to the curve T, so that this tangent **crosses** the curve at the point of tangency.

Show that an equation of T is

$$27x + 9y = 1.$$

[ ], proof



Question 234 **(*****)**

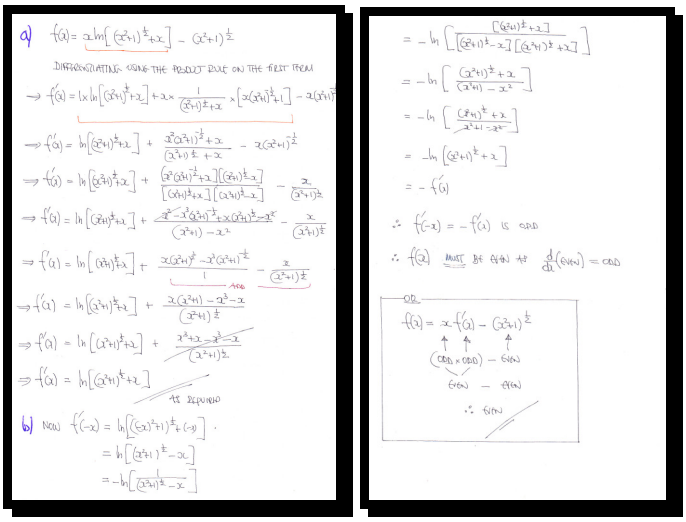
$$f(x)=x\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right]-\left(x^2+1\right)^{\frac{1}{2}},\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right]$.

b) … $f(x)$ is an even function.

proof



Question 235 (*****)

A curve is defined in its largest real domain by the equation

$$y=\arccos\left[\frac{a\cos x+b}{a+b\cos x}\right],$$

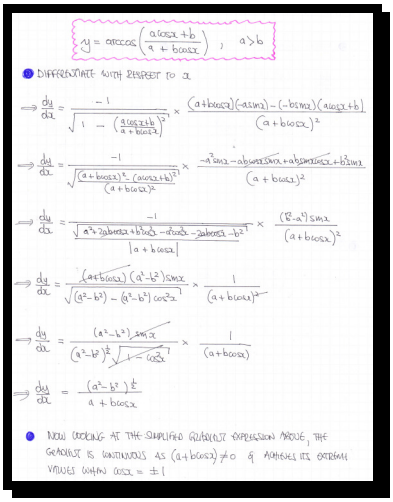
where a and b are constants such $a>b>0$.

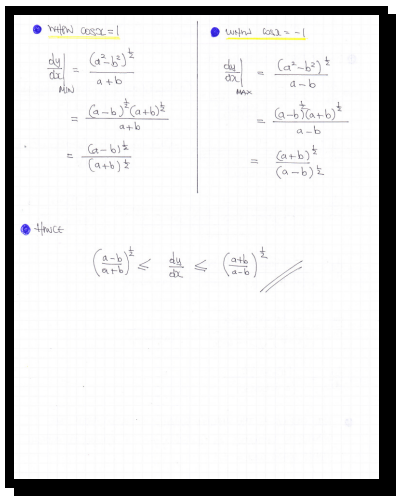
Show that y increases with x at a rate which lies between

$$\sqrt{\frac{a-b}{a+b}} \quad \text{and} \quad \sqrt{\frac{a+b}{a-b}}.$$

You may assume that $\dfrac{d}{dx}[\arccos x]=-\dfrac{1}{\sqrt{1-x^2}}$.

, proof





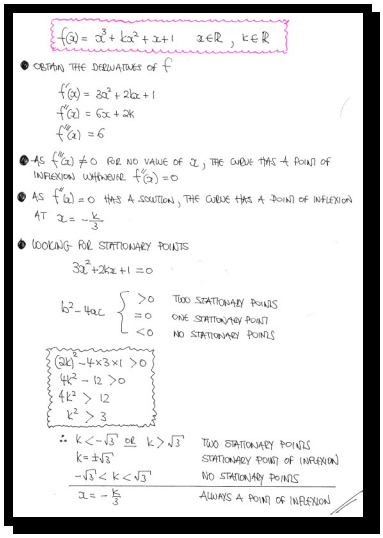
Question 236 (*****)

The function f is defined, in terms of the real constant k, by

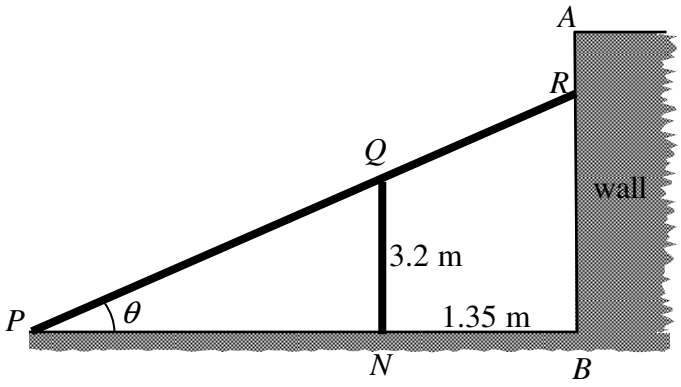
$$f(x) \equiv x^3 + kx^2 + x + 1, \ x \in \mathbb{R}.$$

Investigate the number of turning points of f for different values of k, distinguishing further which ones are stationary

[ ], [investigation]



Question 237 (*****)



The figure above shows the wall AB of a certain structure, which is supported by a straight rigid beam PR, where P is on level ground and R is at some point on the wall.

In order to increase the rigidity of the support, the beam is rested on a steady pole NQ, of height 3.2 metres.

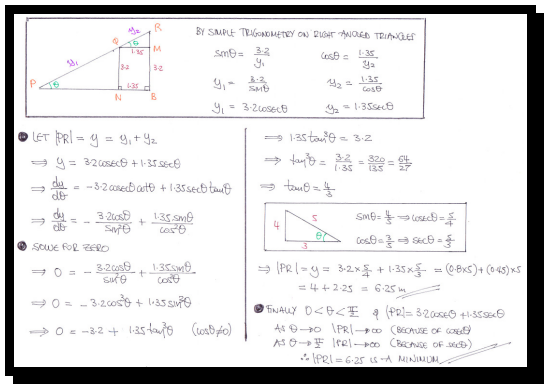
The pole is placed at a distance of 1.35 metres from the bottom of the wall B.

The beam PR is forming an acute angle θ with the horizontal ground PNB.

The angle θ is chosen so that the length of the beam PR, is least.

Determine the least value for the length of the beam PR, assuming that R lies on the wall, fully justifying that this is indeed the minimum value.

$\boxed{}$, $\boxed{6.25}$



Question 238 (*****)

The variables x and y are such so that

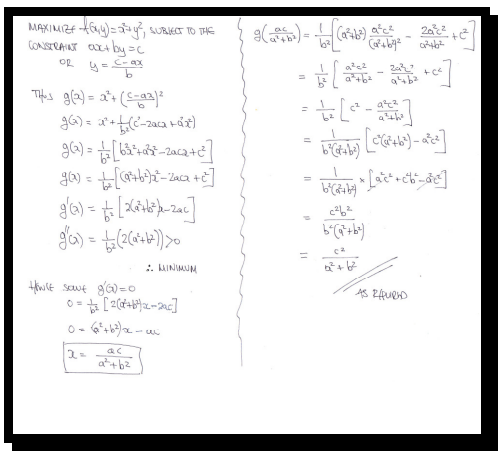
$$ax+by=c\,,$$

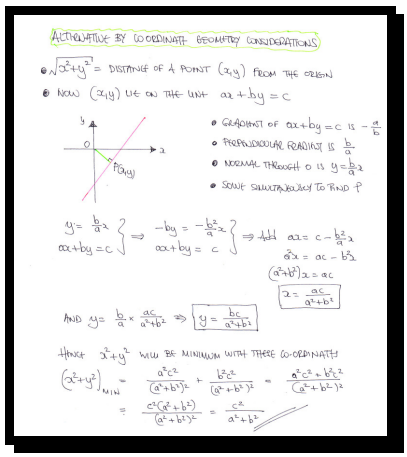
where a, b and c are non zero constants.

Show that the minimum value of x^2+y^2 is

$$\frac{c^2}{a^2+b^2}\,.$$

, proof





Question 239 **(*****)**

The point $A(a,0)$ lies on the circle with Cartesian equation

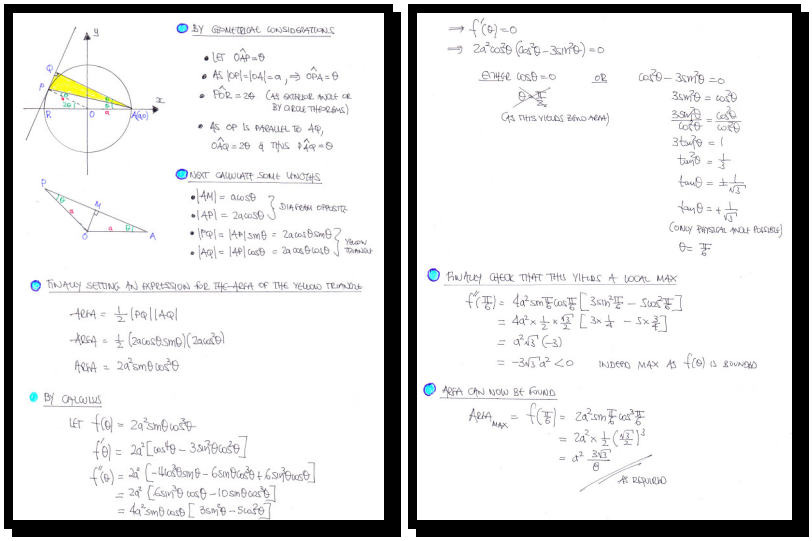
$$x^2+y^2=a^2.$$

The point P is also on the same circle , and the point Q lies on the tangent to the circle through P, so that AQP is a right angle.

Use a calculus method to show that for all possible positions of P, the largest area of the triangle AQP is

$$\frac{3\sqrt{3}}{8}a^2.$$

, proof



Question 240 (*****)

A curve has equation

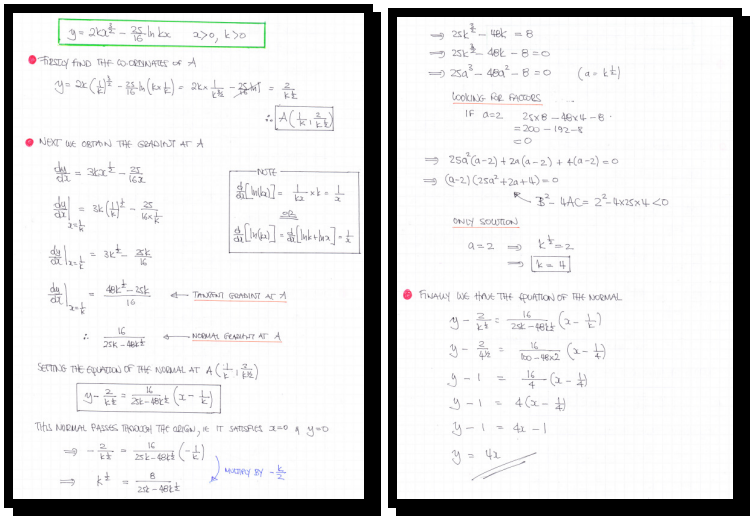
$$y = 2kx^{\frac{3}{2}} - \frac{25}{16}\ln kx\,, \quad x \in \mathbb{R}\,, \quad x > 0\,.$$

where k is a positive constant.

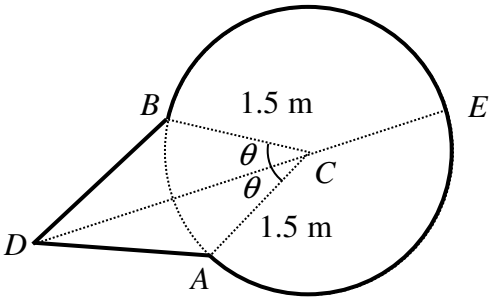
The point A lies on the curve, where $x = \frac{1}{k}$.

Given that the normal to the curve at A passes through the origin O, find an equation of the normal to the curve at A.

$\boxed{}$, $\boxed{y = 4x}$



Question 241 **(*****)**



The figure above shows a circle with centre at C and radius 1.5 metres.

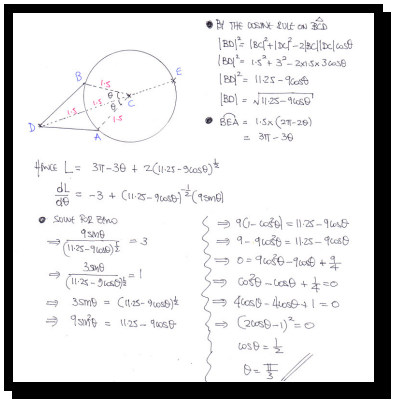
The points A and B lie on the circle so that $\measuredangle BCA = 2\theta$, $0 < \theta < \pi$.

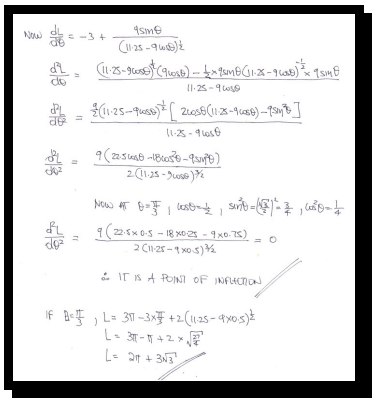
The point D lies outside the circle so that the line segments BD and AD are equal in length and the length of DC is 3 metres. The point E lies on the circle so that DCE is a straight line segment of length 4.5 metres.

The **total length** of the line segment BD, the line segment AD and the circular arc $\overset{\frown}{AEB}$ denoted by L.

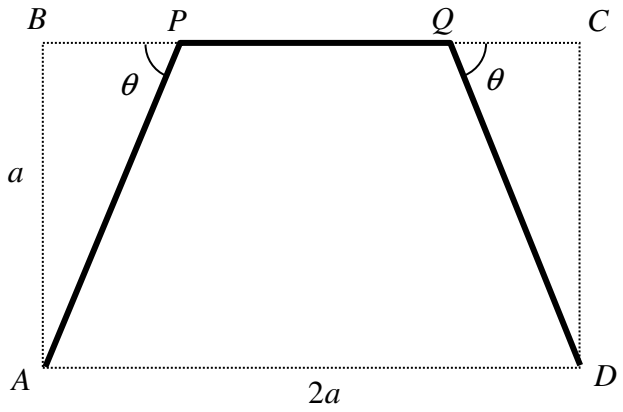
Given that θ varies, show that L has a stationary value when $\theta = \dfrac{\pi}{3}$ and determine further the value L and the nature of this stationary value.

$\square$, $\boxed{\text{inflection at } \left(\dfrac{\pi}{3}, 2\pi + 3\sqrt{3}\right)}$





Question 242 **(*****)**

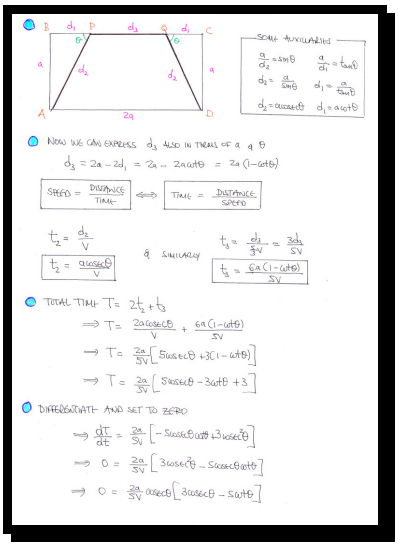


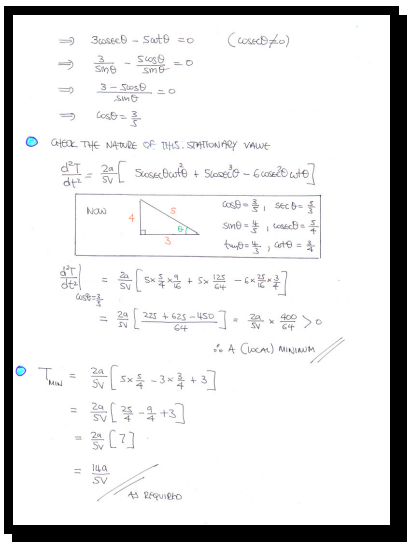
The figure above shows a network $APQD$ inside a rectangle $ABCD$, where $|AB| = a$ and $|BC| = 2a$. The endpoints of the network A and D are fixed. The points P and Q are variable so that they always lie on BC with $|AP| = |QD|$. The angles BPA and CQD are both equal to θ. A particle travels with constant speed v on the sections AP and QD, and with constant speed $\frac{5}{3}v$ on the section PQ.

Let T be the total time for the journey $APQD$.

Given that the positions of the points P and Q can be varied as appropriate, show that the minimum value of T is $\dfrac{14a}{5v}$, fully justifying that this is the minimum value.

[ ], [proof]





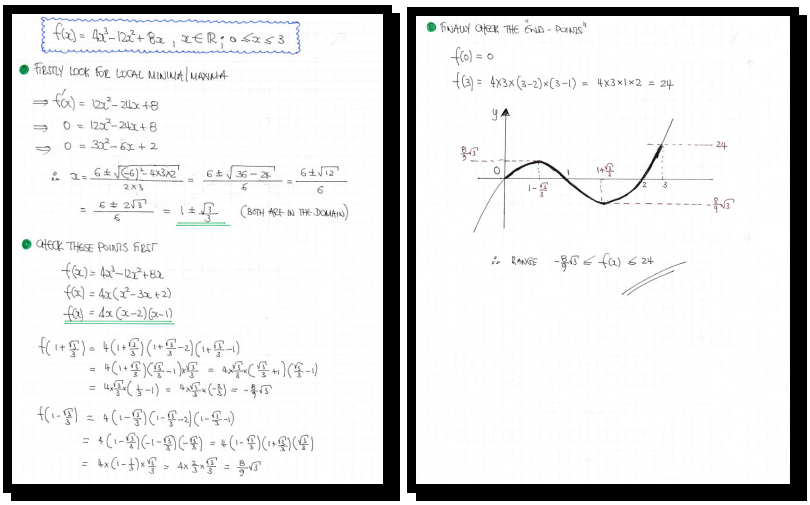
Question 243 **(*****)**

The function f is defined as

$$f(x) \equiv 4x^3 - 12x^2 + 8x, \quad x \in \mathbb{R}, \quad 0 \le x \le 3.$$

Find the range of f, and hence sketch its graph, showing clearly the coordinates of any relevant points.

$\square$, $\boxed{-\frac{8}{9}\sqrt{3} \le f(x) \le 24}$



Question 244 **(*****)**

$$y = \arccos x\,,\ -1 \le x \le 1\,.$$

a) Show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}\,.$$

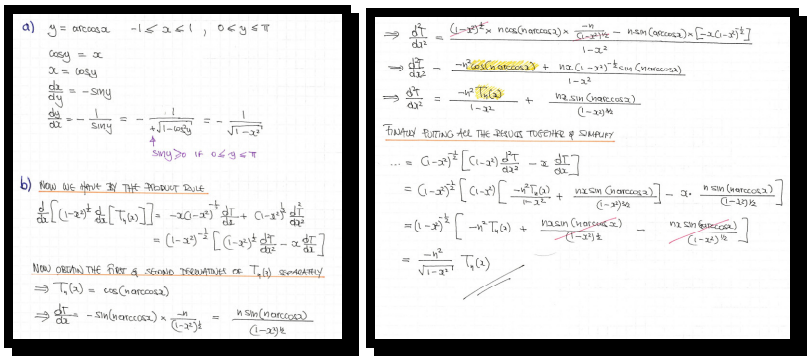
The Chebyshev polynomials of the first kind $T_n(x)$ is a family of functions defined as

$$T_n(x) = \cos\left(n \arccos x\right),\ -1 \le x \le 1,\ n \in \mathbb{N}\,.$$

b) Show further that

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}\left[T_n(x)\right]\right] = \frac{-n^2\,T_n(x)}{\sqrt{1-x^2}}\,.$$

, proof



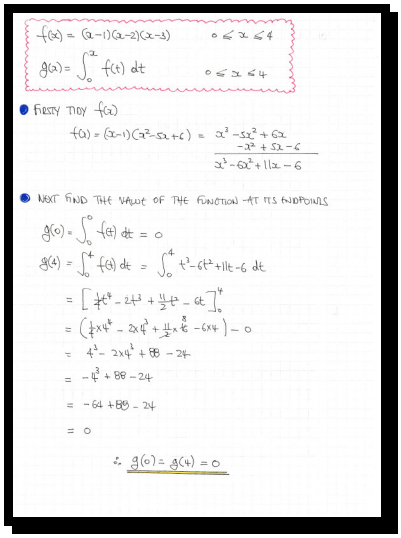
Question 245 (*****)

The real functions f and g have a common domain $0 \le x \le 4$, and defined as

$$f(x) \equiv (x-1)(x-2)(x-3) \qquad \text{and} \qquad g(x) \equiv \int_0^x f(t)\ dt\ .$$

Use a detailed algebraic method to determine the range of g.

$\square$, $\boxed{-\frac{9}{4} \le g(x) \le 0}$



$f(x) = (x-1)(x-2)(x-3) \qquad 0 \le x \le 4$

$g(x) = \int_0^x f(t)\ dt \qquad 0 \le x \le 4$

FIRSTLY TIDY $f(x)$

$f(x) = (x-1)(x^2-5x+6) = x^3-5x^2+6x$

$-x^2+5x-6$

$x^3-6x^2+11x-6$

NEXT FIND THE VALUE OF THE FUNCTION AT ITS ENDPOINTS

$g(0) = \int_0^0 f(t)\ dt = 0$

$g(4) = \int_0^4 f(t)\ dt = \int_0^4 t^3-6t^2+11t-6\ dt$

$= \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^4$

$= \left(\frac{1}{4}\times 4^4 - 2\times 4^3 + \frac{11}{2}\times 4^2 - 6\times 4\right) - 0$

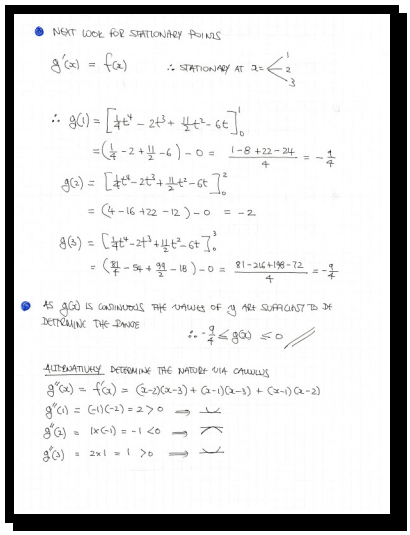
$= 4^3 - 2\times 4^3 + 88 - 24$

$= -4^3 + 88 - 24$

$= -64 + 88 - 24$

$= 0$

$\therefore\ g(0) = g(4) = 0$



NEXT LOOK FOR STATIONARY POINTS

$g'(x) = f(x)$ $\quad \therefore$ STATIONARY AT $x = 1, 2, 3$

$\therefore\ g(1) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^1$

$= \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) - 0 = \frac{1-8+22-24}{4} = -\frac{1}{4}$

$g(2) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^2$

$= (4 - 16 + 22 - 12) - 0 = -2$

$g(3) = \left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^3$

$= \left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right) - 0 = \frac{81-216+198-72}{4} = -\frac{9}{4}$

AS $g(x)$ IS CONTINUOUS THE VALUES OF g ARE SUFFICIENT TO DETERMINE THE RANGE

$\therefore\ -\frac{9}{4} \le g(x) \le 0$

ALTERNATIVELY DETERMINE THE NATURE VIA CALCULUS

$g''(x) = f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$

$g''(1) = (-1)(-2) = 2 > 0 \Rightarrow \cup$

$g''(2) = (1)(-1) = -1 < 0 \Rightarrow \cap$

$g''(3) = 2 \times 1 = 1 > 0 \Rightarrow \cup$

Question 246 (*****)

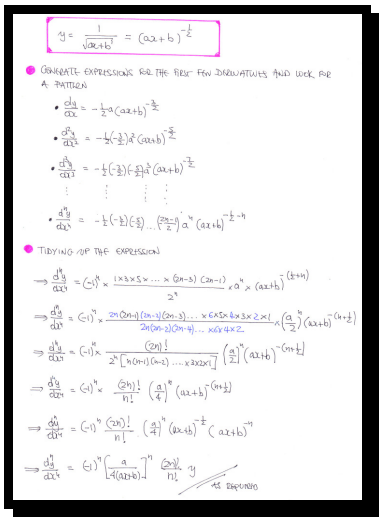
$$y=\frac{1}{\sqrt{ax+b}}\,,\ x\geq 0\,,$$

where a and b are positive constants.

Show, by a detailed method, that

$$\frac{d^n y}{dx^n}=\frac{(-1)^n\, y(2n)!}{n!}\left(\frac{a}{4(ax+b)}\right)^n.$$

, proof



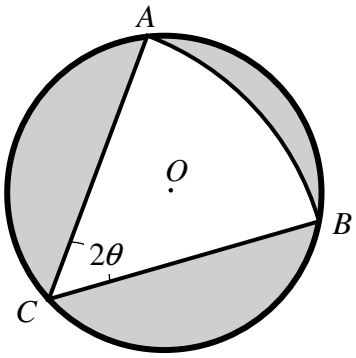
Question 247 **(*****)**

$$2x\tan x=1,\ x\neq\frac{1}{2}n\pi,\ n\in\mathbb{N}$$

a) Show that the above equation has a solution in the interval $(0.6,0.7)$.

b) Use the Newton Raphson method to find the solution of this equation, correct to 5 decimal places.

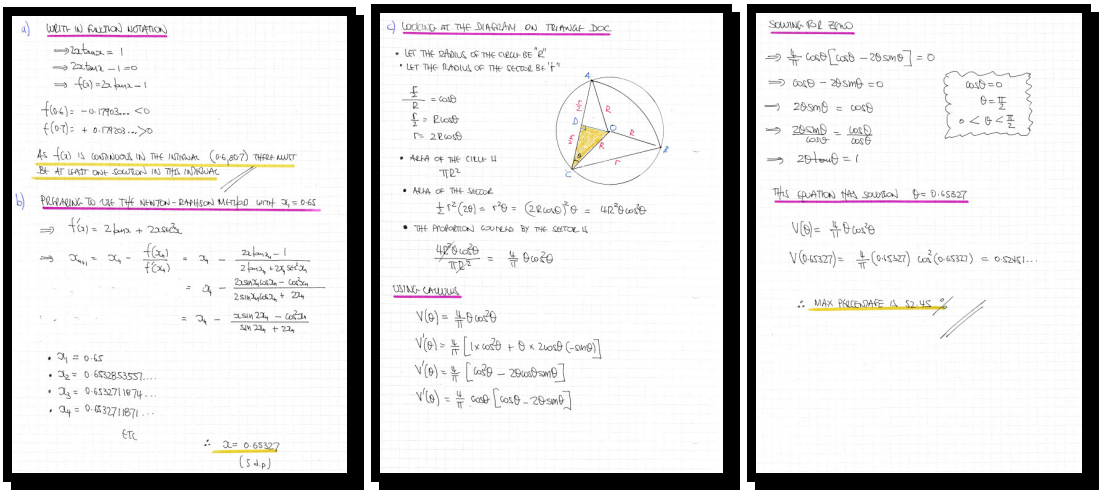
The figure below shows a circle, centre at O. The points A, B and C lie on the circumference of this circle. A circular sector ABC, subtending an angle of 2θ at C, is inscribed in this circle.



c) Determine the greatest proportion of the area of the circle, which can be covered by this sector.

You may give the answer as a percentage, correct to two decimal places

$\boxed{}$, $\boxed{x\approx 0.65327}$, $\boxed{\approx 52.45}$



Question 248 (*****)

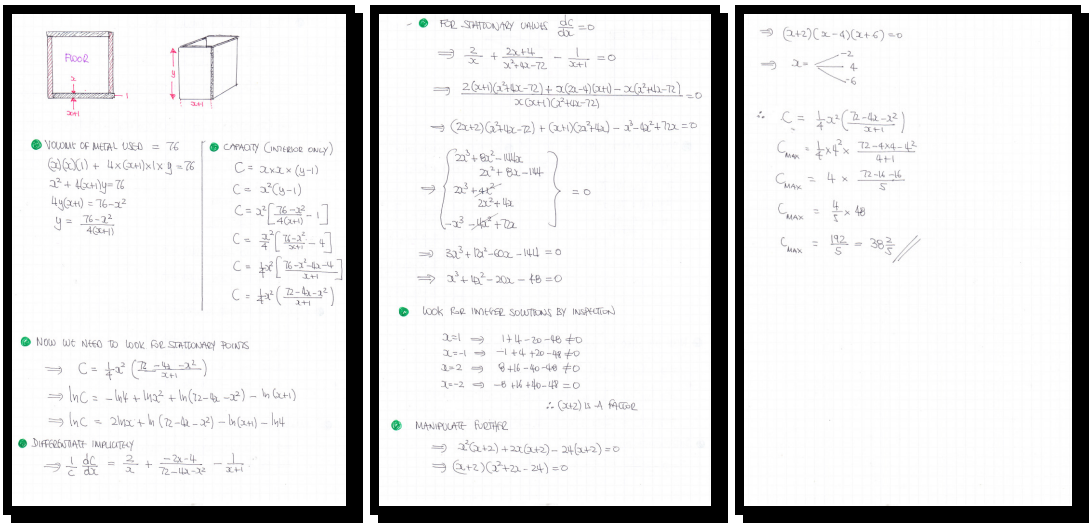
A lump of metal, of volume 76 cubic units is **moulded** into the shape of a cuboidal box, with a square base, rectangular sided and no lid.

All the faces of the box are 1 unit thick.

All the metal is moulded in the construction of this box, and the construction it has maximum capacity.

If the internal width of the box is x, find the value of x which maximises the capacity of he box, and hence determine this maximum capacity.

$\boxed{}$, $\boxed{x=4}$, $\boxed{C_{\max}=38\frac{2}{5}}$



Question 249 **(*****)**

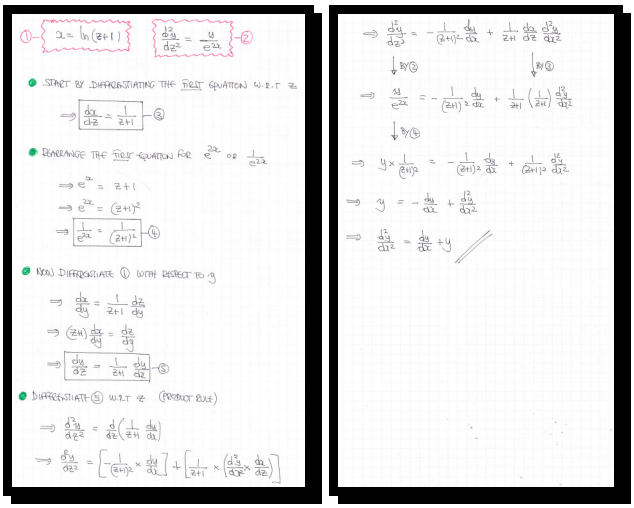
The variables x, y and z satisfy the following relationships.

$$x=\ln\left(z+1\right) \qquad \text{and} \qquad \frac{d^2y}{dz^2}=\frac{y}{\mathrm{e}^{2x}}.$$

Show that

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}+y.$$

, proof



Question 250 (*****)

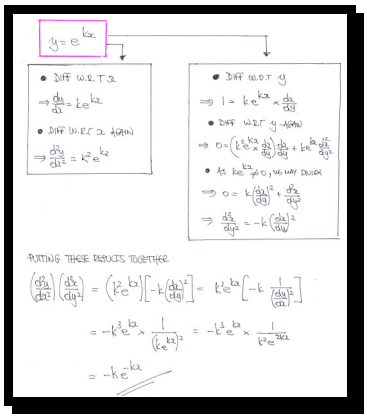
$$y = \mathrm{e}^{kx}, \ k \neq 0 .$$

Find a simplified expression for

$$\left[\frac{d^2y}{dx^2}\right]\left[\frac{d^2x}{dy^2}\right],$$

giving the answer in terms of k and e^{kx}.

$\boxed{}$, proof

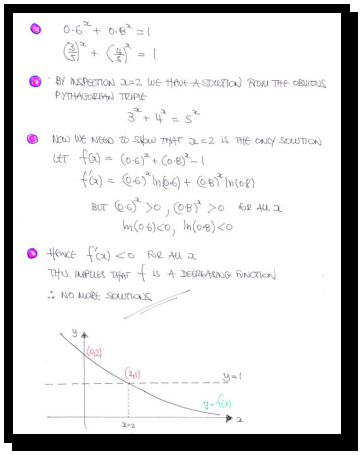


Question 251 (*****)

$$0.6^x + 0.8^x = 1, \ x \in \mathbb{R}.$$

Find the only solution of the above equation, fully justifying the fact that it is the only solution.

$\boxed{}$, $\boxed{x=2}$



Question 252 (*****)

From a thin sheet of metal, a circular sector of area A is removed.

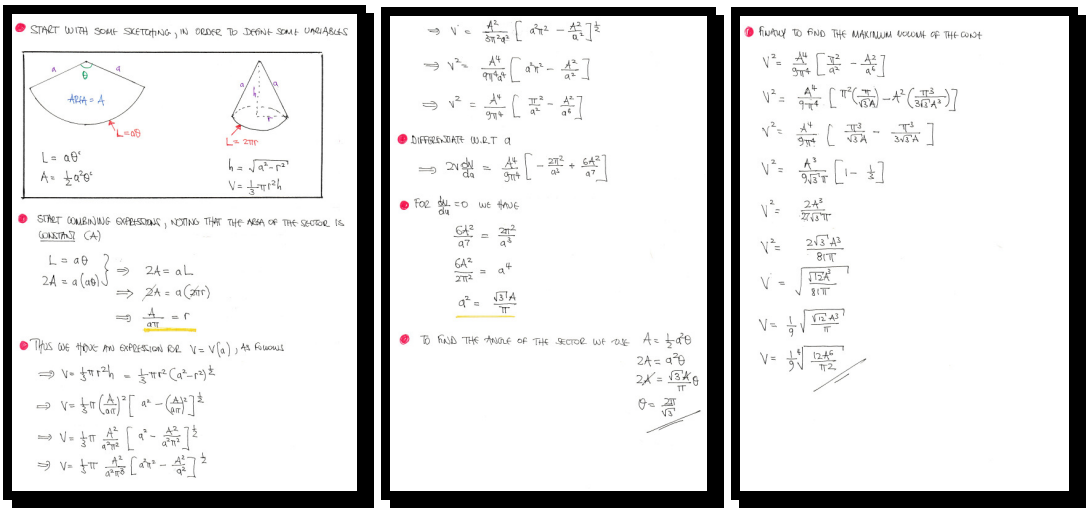
The circular sector is folded without any overlapping into the curved surface of a right circular cone of volume V.

The measurements of the circular sector are such so that V is maximum.

Find the angle subtended by the circular sector at its centre and show further that the maximum value of V is

$$\frac{1}{9}\sqrt[4]{\frac{12A^6}{\pi^2}}\,.$$

$\square$, $\boxed{\theta=\dfrac{2\pi}{\sqrt{3}}}$



Question 253 **(*****)**

The variable point P lies on the positive x axis and the variable point Q lies on the curve with equation

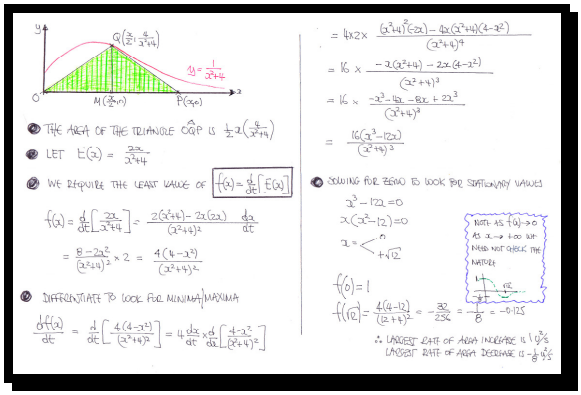
$$y=\frac{1}{x^2+4},\ x\in\mathbb{R},\ x\geq 0.$$

The x coordinate Q is always half the x coordinate of P.

The point P starts at the origin O and begins to move in the positive x direction at constant rate.

Determine the largest rate of area increase and the largest rate of area decrease of the triangle OPQ, as P is moving away from O.

$\boxed{}$, $\boxed{\text{largest rate of area increase } =1}$, $\boxed{\text{largest rate of area decrease } =0.125}$



Question 254 **(*****)**

$$x^3 + px + q = 0\,,\ x \in \mathbb{R}$$

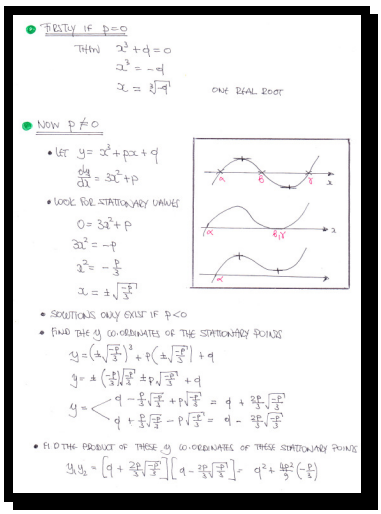
The cubic equation above is given in terms of the real constants, p and q.

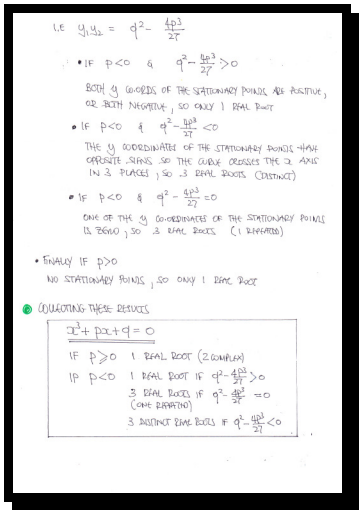
Use differentiation to determine the conditions that p and q must satisfy so that the above equation has …

- … one real root.
- … three real roots, of which one is repeated.
- … three distinct real roots.

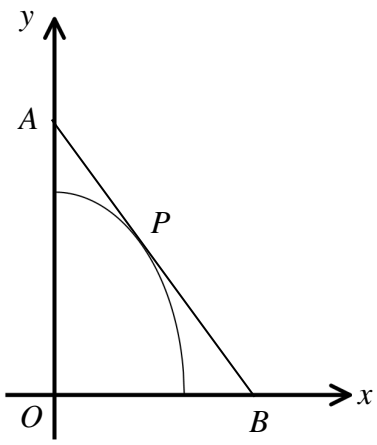
$$\square\,,\quad \boxed{p \geq 0 \ \cup \ \left[p < 0 \ \cap q^2 - \frac{4p^3}{27} > 0 \right]}\,,\quad \boxed{p < 0 \ \cap q^2 - \frac{4p^3}{27} = 0}\,,$$

$$\boxed{p < 0 \ \cap q^2 - \frac{4p^3}{27} < 0}$$





Question 255 **(*****)**



The figure above shows the curve C with equation

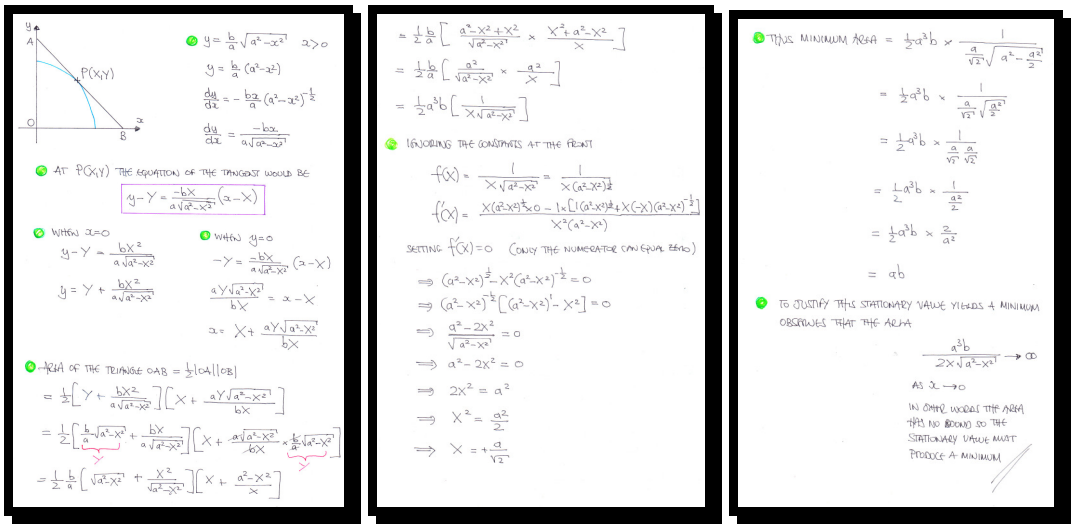
$$y=\frac{b}{a}\sqrt{a^2-x^2}\ ,\ x\geq 0\ ,$$

where a and b are constants such that $b>a>0$.

The point P lies on C and the tangent to C at P meets the coordinate axes at the points A and B, as shown in the figure.

Show with full justification that the minimum area of the triangle AOB, where O is the origin, is ab.

[ ], proof



Question 256 (*****)

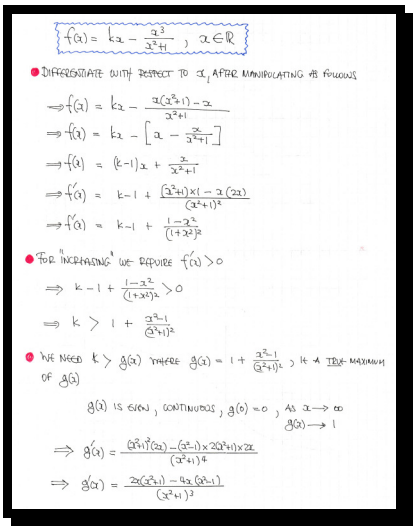
The function f is defined as

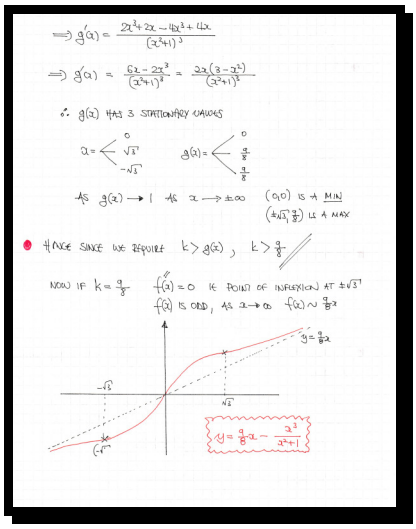
$$f(x)\equiv kx-\frac{x^3}{x^2+1},\ x\in\mathbb{R},$$

where k is a positive constant.

Given that f is increasing for $x\in\mathbb{R}$, show that $k>\frac{9}{8}$ and hence sketch the graph of f, showing clearly the behaviour of f at $\pm\sqrt{2}$.

[ ], [graph]





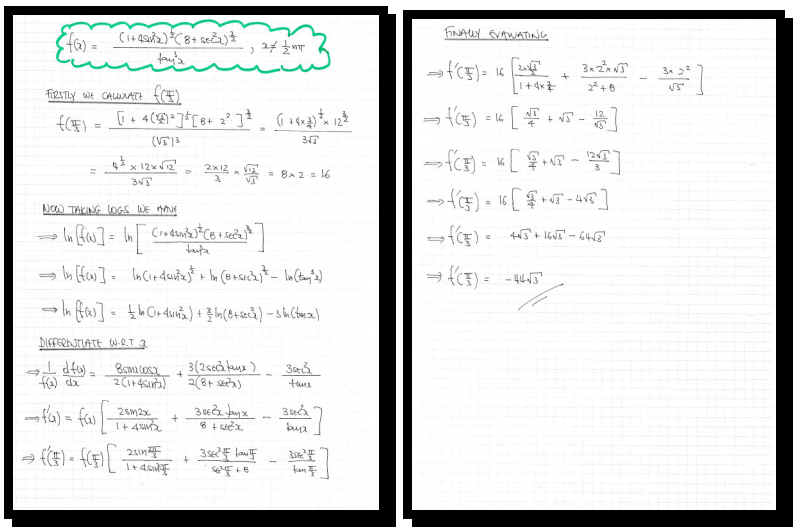
Question 257 (*****)

The function f is defined as

$$f(x)\equiv\frac{\left(1+4\sin^2 x\right)^{\frac{1}{2}}\left(8+\sec^2 x\right)^{\frac{3}{2}}}{\tan^3 x},\quad x\in\mathbb{R},\quad x\neq\frac{1}{2}n\pi.$$

Find in exact simplified form the value of $f'\left(\frac{1}{3}\pi\right)$.

$\square$, $\boxed{f'\left(\frac{1}{3}\pi\right)=-44\sqrt{3}}$



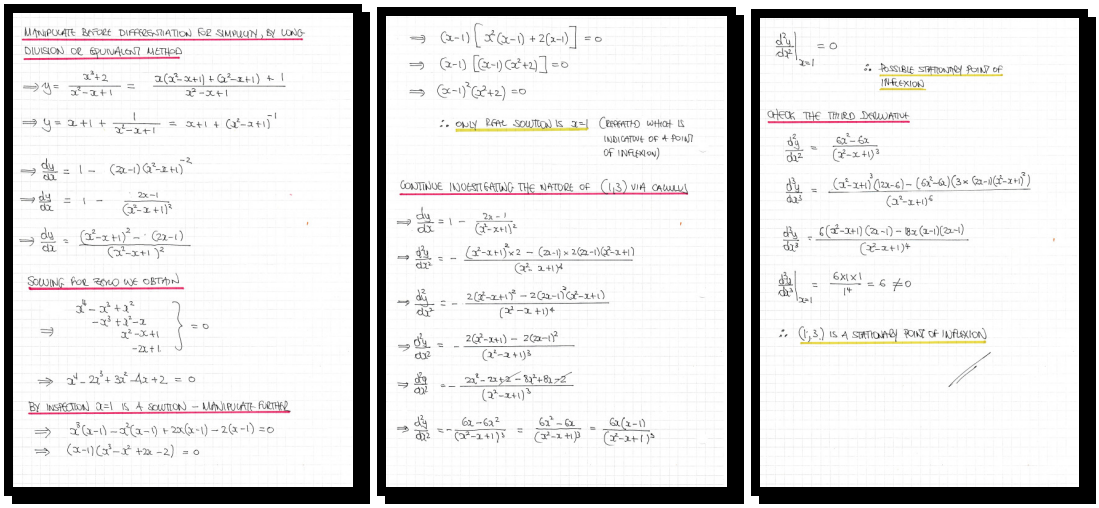
Question 258 (*****)

The curve C has equation

$$y=\frac{x^3+2}{x^2-x+1}.$$

Find the coordinates of the stationary point of C and determine its nature.

$\boxed{}$, $\boxed{\text{point of inflexion at } (1,3)}$



Question 259 **(*****)**

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2 .$$

a) Sketch a detailed graph of C.

The point P, where $x = 2$ lies on C.

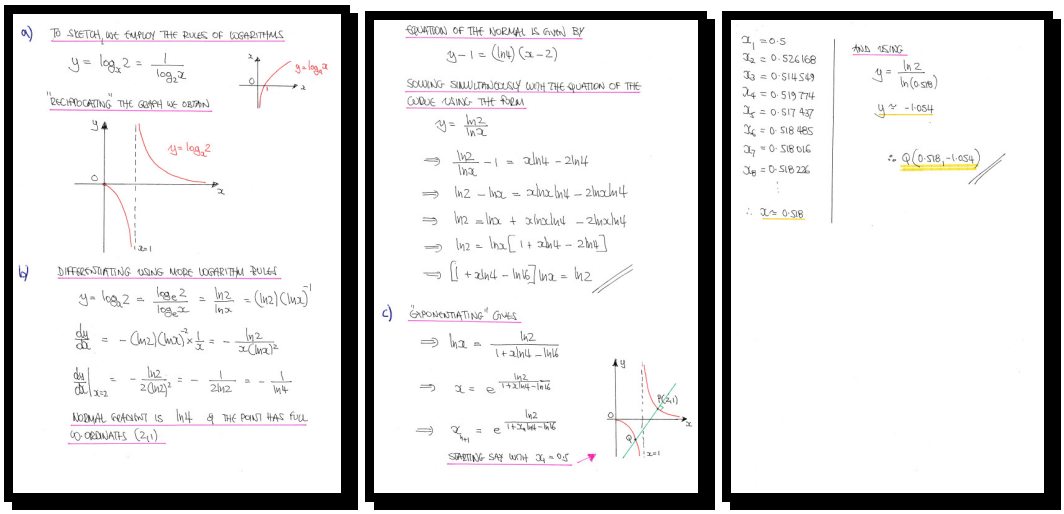
The normal to C at P meets C again at the point Q.

b) Show that the x coordinate of Q is a solution of the equation

$$\left[1 + x\ln 4 - \ln 16\right]\ln x = \ln 2 .$$

c) Use an iterative formula of the form $x_{n+1} = \mathrm{e}^{f(x_n)}$, with a suitable starting value, to find the coordinates of Q, correct to 3 decimal places.

$\square$, $\boxed{Q(0.518, -1.054)}$



Question 260 (*****)

A sphere of radius r, whose centre is at O, is fixed on a horizontal plane.

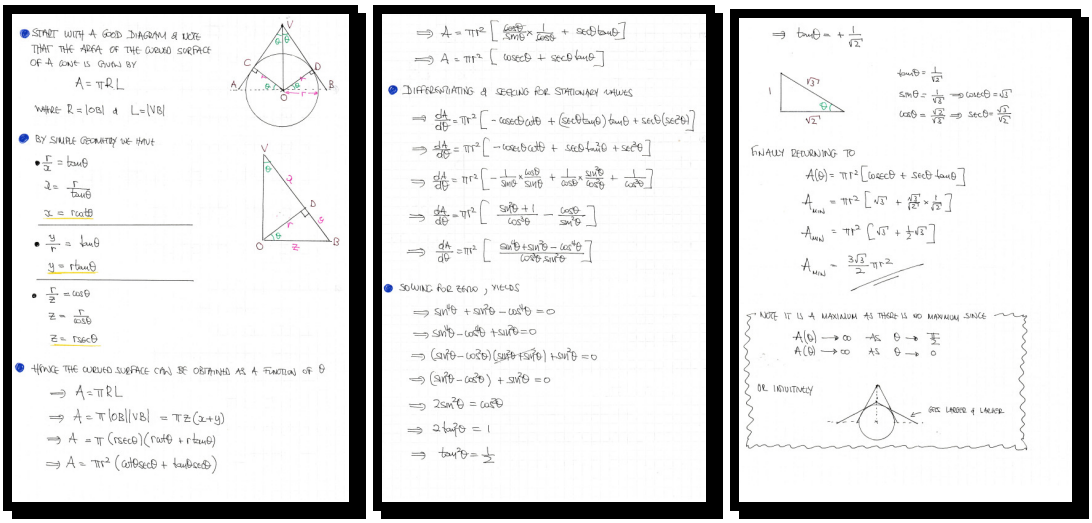
A thin right circular conical shell, **without** a base is placed over the sphere.

The axis of the conical shell is vertical and passes through O. The circumference of the missing base of the conical shell is at the same horizontal level as O.

Show that the minimum value of the outer surface area of the conical shell is

$$\frac{3\sqrt{3}\,\pi r^2}{2}.$$

, proof



Question 261 **(*****)**

The function $y = f(x)$ satisfies the following relationship.

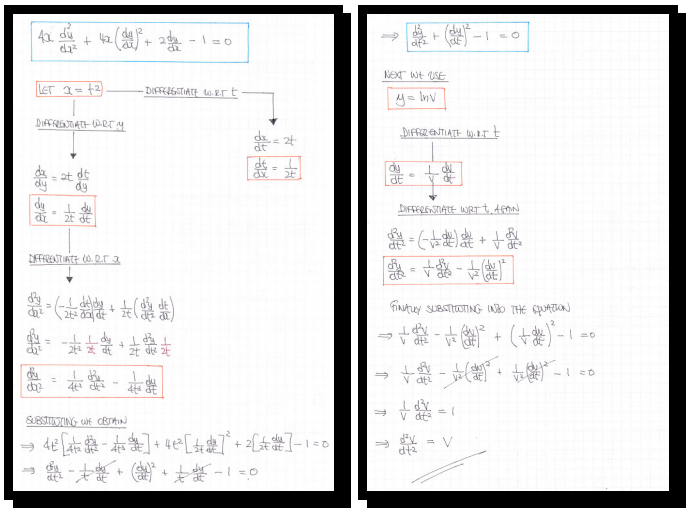
$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0 .$$

It is further given that $x = t^2$ and $y = \ln v$.

Show that

$$\frac{d^2v}{dt^2} = v .$$

, proof

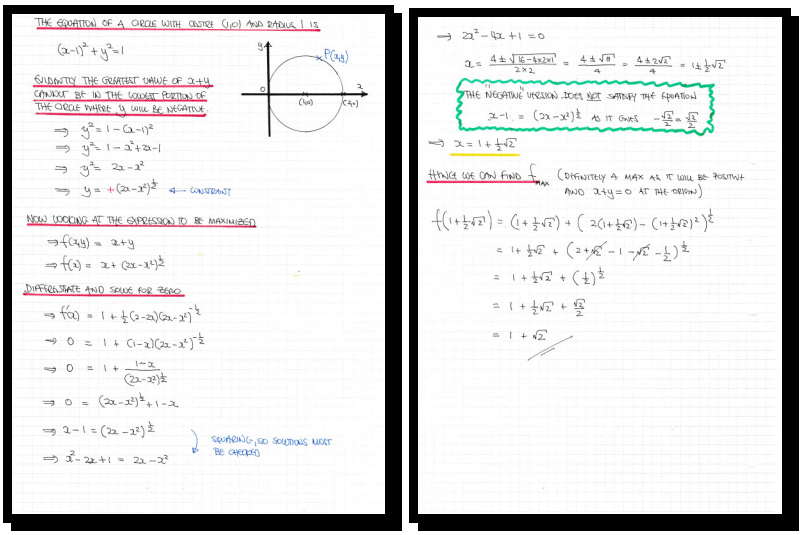


Question 262 **(*****)**

The point $P(x, y)$ lies on a circle with centre at $(1,0)$ and radius 1.

Find, in exact surd form, the greatest value of $x+y$, for all the possible positions of the point P.

$\square$, $\boxed{1+\sqrt{2}}$



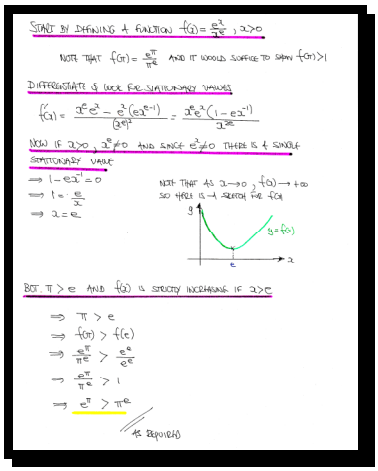
Question 263 (*****)

Assuming that $\pi \approx 3.14$ and $\mathrm{e} \approx 2.7$, show **without** any calculating aid that

$$\mathrm{e}^{\pi} > \pi^{\mathrm{e}} .$$

You must show a detailed method in this question.

□, proof



Question 264 **(*****)**

A general curve C has equation

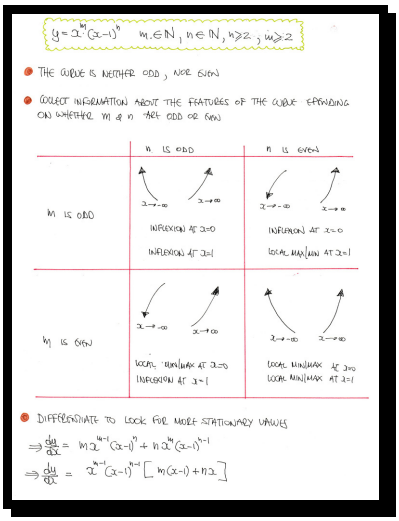
$$y = x^m (x-1)^n,$$

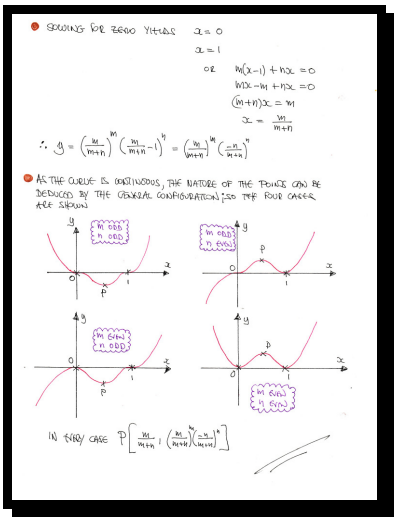
where $x \in \mathbb{R}$, $m \in \mathbb{N}$, $m \geq 2$, $n \in \mathbb{N}$, $n \geq 2$.

Sketch in four separate of axes, the 4 separate shapes which C can take, $m \geq 2$.

The sketches must contain the coordinates of any stationary points.

, graph





Question 265 (*****)

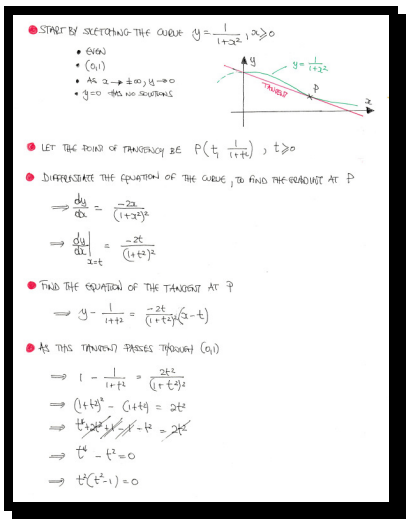
The point P lies on the curve C with equation

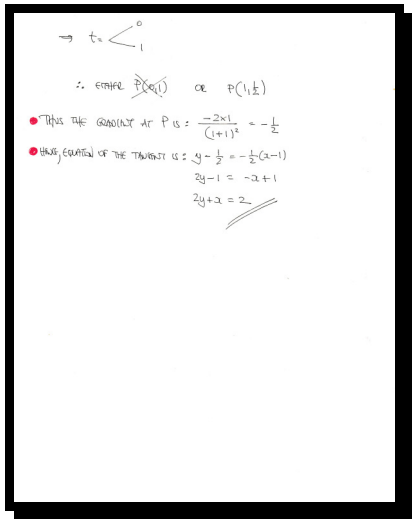
$$y=\frac{1}{1+x^2}, \quad x\in\mathbb{R}, \quad x\geq 0.$$

The straight line L is the tangent to the C at P.

Determine an equation for L, given further that L meets C at the point $(0,1)$.

$\boxed{}$, $\boxed{2y+x=2}$





Question 266 **(*****)**

The point P has rational coordinates and lies on the curve C with equation

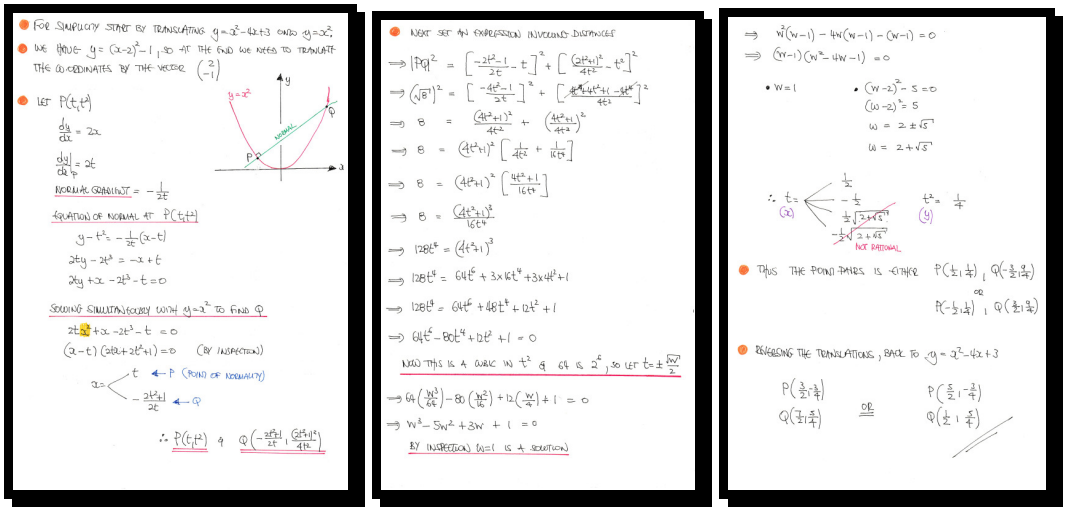
$$y = x^2 - 4x + 3, \quad x \in \mathbb{R}.$$

The straight line L is the normal to the C at P.

L meets the curve again at the point Q.

Given that $|PQ| = \sqrt{8}$, determine the possible coordinates of P and Q.

$\square$, $\boxed{P\left(\frac{3}{2},-\frac{3}{4}\right) \cap Q\left(\frac{7}{2},\frac{5}{4}\right) \quad \cup \quad P\left(\frac{5}{2},-\frac{3}{4}\right) \cap Q\left(\frac{1}{2},\frac{5}{4}\right)}$



Question 267 **(*****)**

Leibniz rule states that the n^{th} derivative of the product of the functions $f(x)$ and $g(x)$ satisfies

$$\left[f(x)\,g(x)\right]^{n}=\sum_{r=0}^{n}\left[\binom{n}{r}\left[\left[f(x)\right]^{(r)}\left[f(x)\right]^{(n-r)}\right]\right],$$

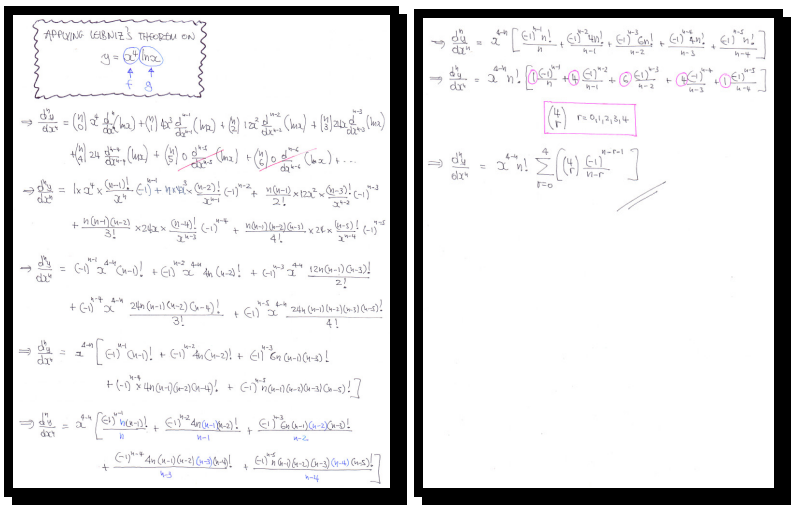
where $f^{0}(x)=f(x)$, $f^{1}(x)=f'(x)$, $f^{2}(x)=f''(x)$, … , $f^{k}(x)=\dfrac{d^{k}}{dx^{k}}\left[f(x)\right]$.

Show, by a detailed method, that

$$\frac{d^{n}}{dx^{n}}\left[x^{4}\ln x\right]=n!\,x^{4-n}\sum_{r=0}^{4}\left[\binom{4}{r}f(n,r)\right],$$

where $f(n,r)$ is a function to be found.

$\square$, $\boxed{\dfrac{d^{n}}{dx^{n}}\left[x^{4}\ln x\right]=n!\,x^{4-n}\displaystyle\sum_{r=0}^{4}\left[\binom{4}{r}\frac{(-1)^{n-r-1}}{n-r}\right]}$



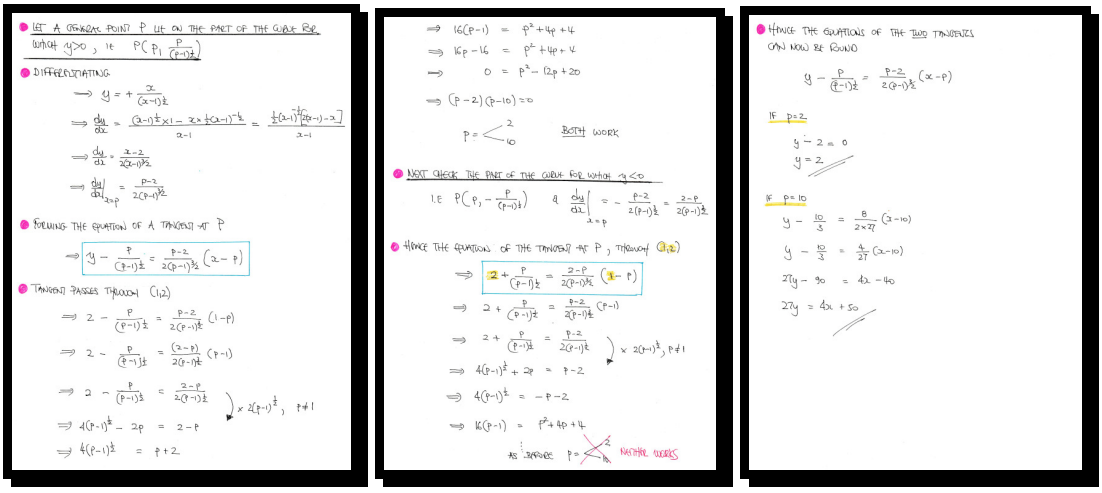
Question 268 **(*****)**

A curve C has equation

$$y^2 = \frac{x^2}{x-1}, \quad x \in \mathbb{R}, \quad x > 1.$$

Show that there exist exactly two tangents to C which pass through the point $(1,2)$, and find their equations.

$\boxed{}$, $\boxed{y=2}$, $\boxed{27y=4x+50}$



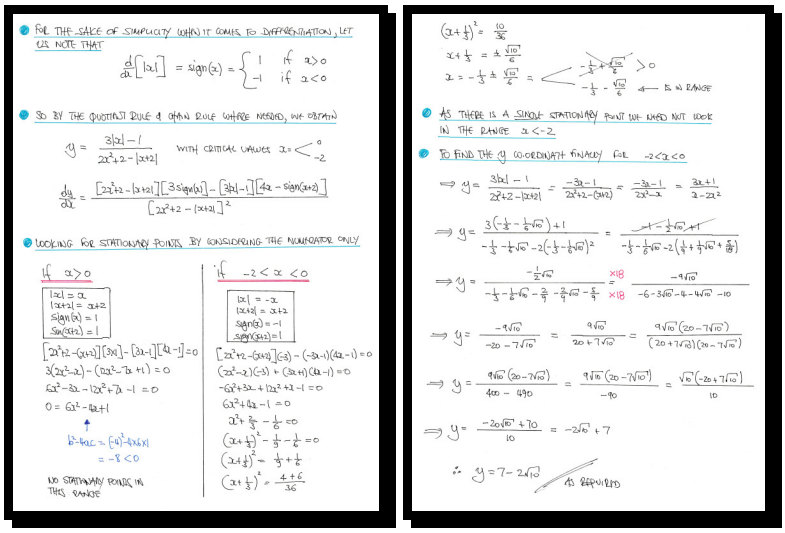
Question 269 **(*****)**

A curve C has equation

$$y=\frac{3|x|-1}{2x^2+2-|x+2|}, \quad x\in\mathbb{R}, \quad x\neq 0, \quad x\neq\frac{1}{2}.$$

Find, in exact simplified surd form, the y coordinate of the stationary point of C.

$\square$, $\boxed{y=7-2\sqrt{10}}$



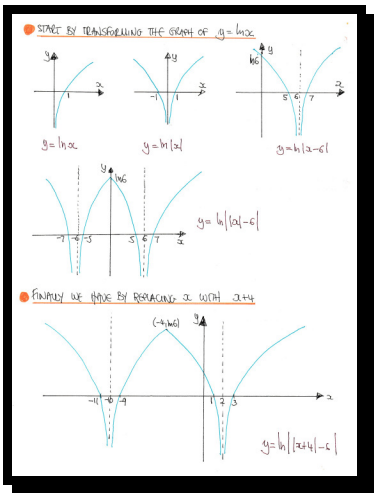
Question 270 (*****)

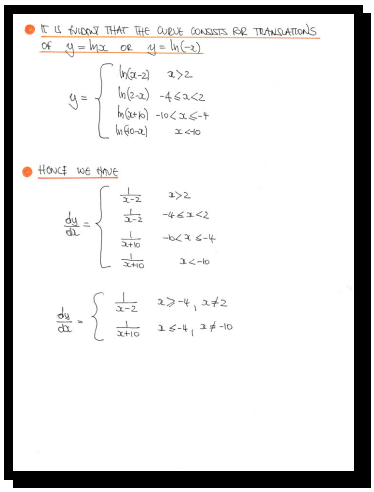
A curve, defined in the largest real domain, has equation

$$y=\ln\left|\left|x+4\right|-6\right|.$$

Determine, in its simplest form, an expression for $\dfrac{dy}{dx}$.

$$\boxed{},\quad \boxed{\frac{dy}{dx}=\begin{cases}\dfrac{1}{x-2}, & x\geq -4,\quad x\neq 2\\ \dfrac{1}{x+10}, & x\leq -4,\quad x\neq -10\end{cases}}$$





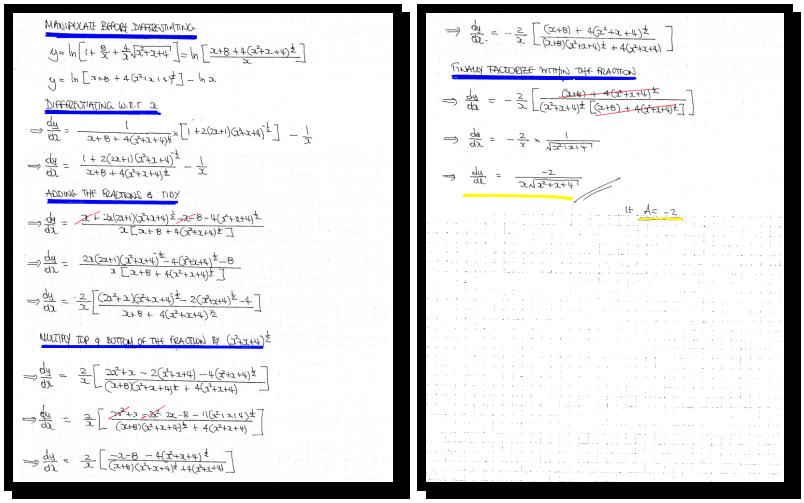
Question 271 **(*****)**

Show that

$$\frac{d}{dx}\left[\ln\left(1+\frac{8}{x}+\frac{4}{x}\sqrt{x^2+x+4}\right)\right] = \frac{A}{x\sqrt{x^2+x+4}},$$

where A is a non zero constant.

$\boxed{}$, $\boxed{A=-2}$



Question 272 **(*****)**

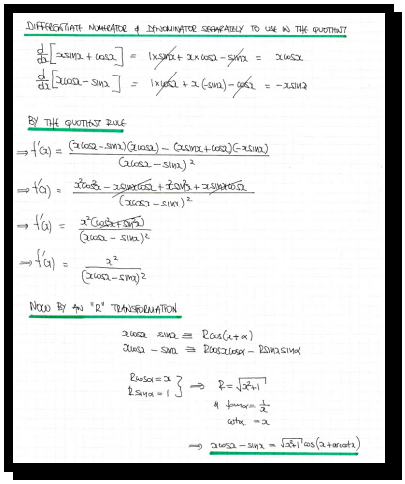
$$f(x) \equiv \frac{x\sin x+\cos x}{x\cos x-\sin x},\ x\in(0,\pi].$$

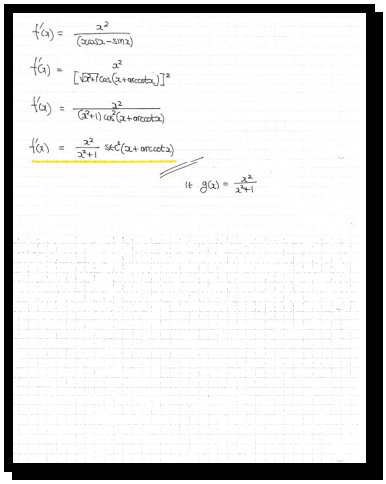
Show that

$$f'(x)=g(x)\sec^2\left[x+\operatorname{arccot}x\right],$$

where $g(x)$ is a rational function to be found.

$\square$, $\boxed{g(x)=\dfrac{x^2}{x^2+1}}$





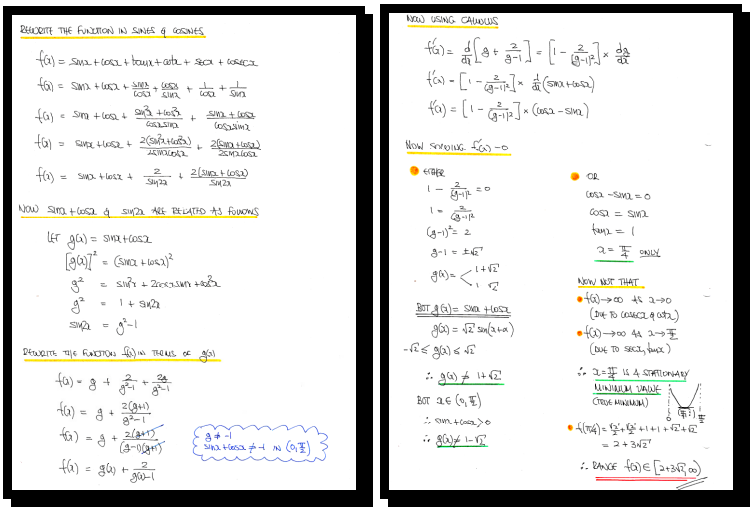
Question 273 (*****)

The function f is defined as

$$f(x)\equiv \sin x+\cos x+\tan x+\cot x+\sec x+\operatorname{cosec} x, \qquad x\in\left(0,\tfrac{1}{2}\pi\right).$$

Determine with full justification the range of f.

$\square$, $\boxed{f(x)\in\left[2+3\sqrt{2},\infty\right)}$



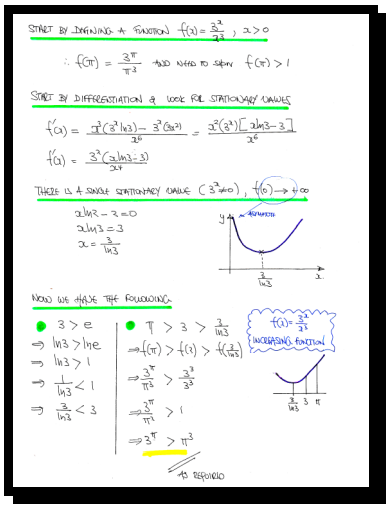
Question 274 **(*****)**

Assuming that $\pi \approx 3.14$, show **without** any calculating aid that

$$3^{\pi} > \pi^3 .$$

You must show a detailed method in this question.

, proof



Question 275 **(*****)**

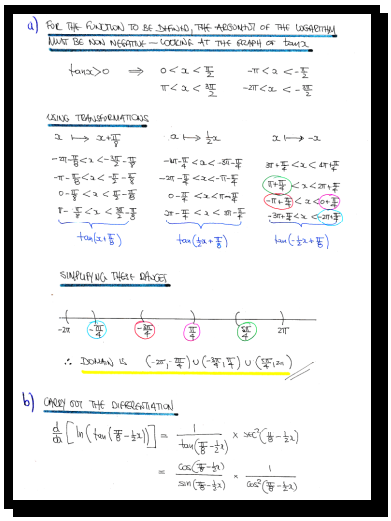
The function f is defined in the largest possible real domain, contained in the interval $(-2\pi, 2\pi)$, and its equation is

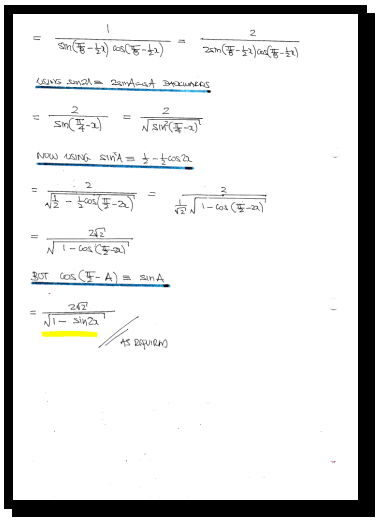
$$f(x) \equiv \ln\left[\tan\left(\frac{1}{8}\pi - \frac{1}{2}x\right)\right].$$

a) Find the domain of f.

b) Show that $f'(x) \equiv \dfrac{k}{\sqrt{1-\sin 2x}}$, for some constant k.

$\square$, $\boxed{\left(-2\pi, -\frac{7}{4}\pi\right) \cup \left(-\frac{3}{4}\pi, \frac{1}{4}\pi\right) \cup \left(\frac{5}{4}\pi, 2\pi\right)}$





Question 276 **(*****)**

A curve has equation

$$y = x^{x^{-1}}, \quad x \in \mathbb{R}.$$

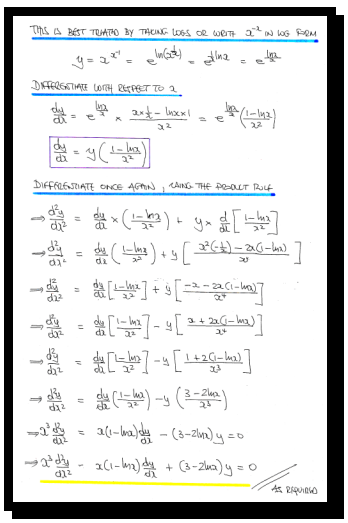
a) Show that y is a solution of the following differential equation.

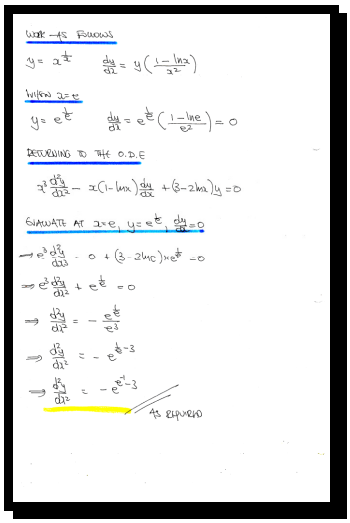
$$x^3 \frac{d^2y}{dx^2} - x(1-\ln x)\frac{dy}{dx} + (3-2\ln x)\, y = 0.$$

b) Show further that

$$\left.\frac{d^2y}{dx^2}\right|_{x=\mathrm{e}} = -\mathrm{e}^{\mathrm{e}^{-1}-3}.$$

[ ], [proof]





Question 277 (*****)

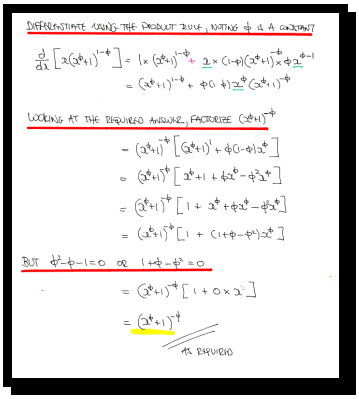
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

Show, with a detailed method, that

$$\frac{d}{dx}\left[x\left(x^{\phi}+1\right)^{1-\phi} \right] = \left(x^{\phi}+1\right)^{-\phi}.$$

□, proof



Question 278 (*****)

The curve C is defined in the greatest real domain by the equation

$$y=\frac{x}{(y-2)(y+1)(y-3)}.$$

a) Show that

$$\frac{dy}{dx}=\frac{1}{2(y-1)\left(ay^2+by+c\right)},$$

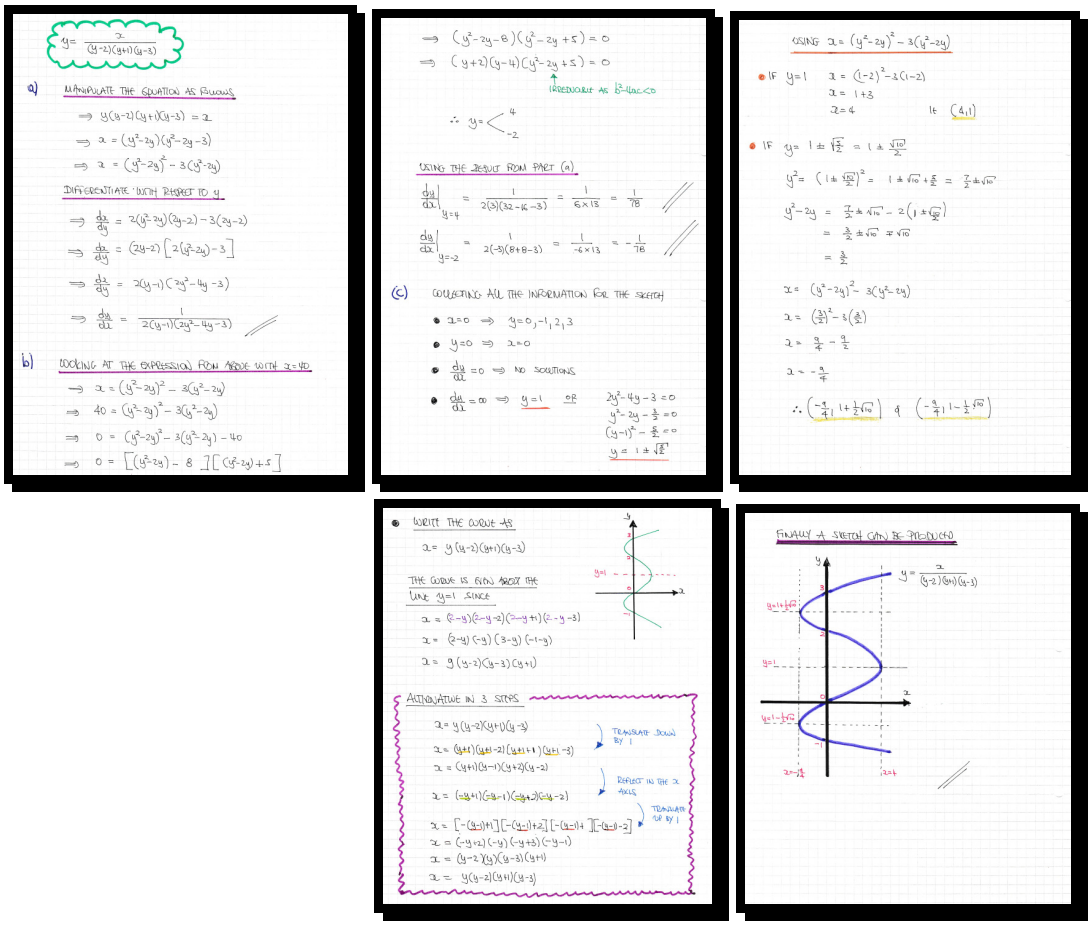
where a, b and c are integers to be found.

b) Determine the exact value of the gradient at the points on C, where $x=40$.

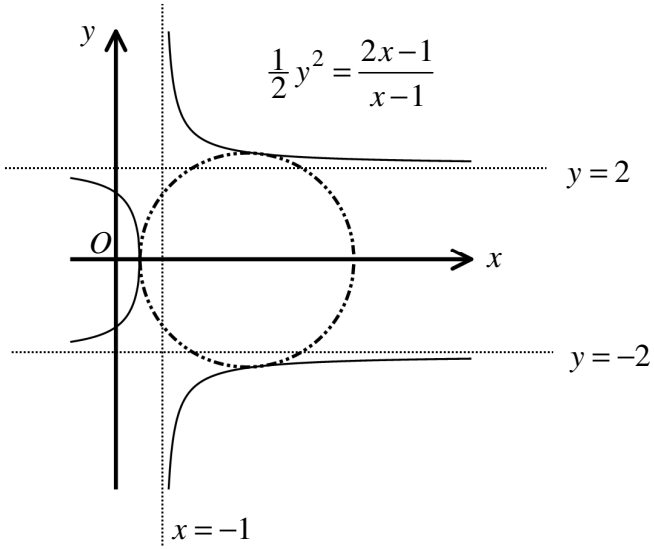
c) Sketch the graph of C.

The sketch must include the coordinates of any points where C meets the coordinate axes, the coordinates of the points of infinite gradient. You must also find, with a full algebraic method, the line of symmetry of C.

$\boxed{}$, $\boxed{a=2,\quad b=-4,\quad c=-3}$, $\boxed{\pm\frac{1}{78}}$



Question 279 (*****)

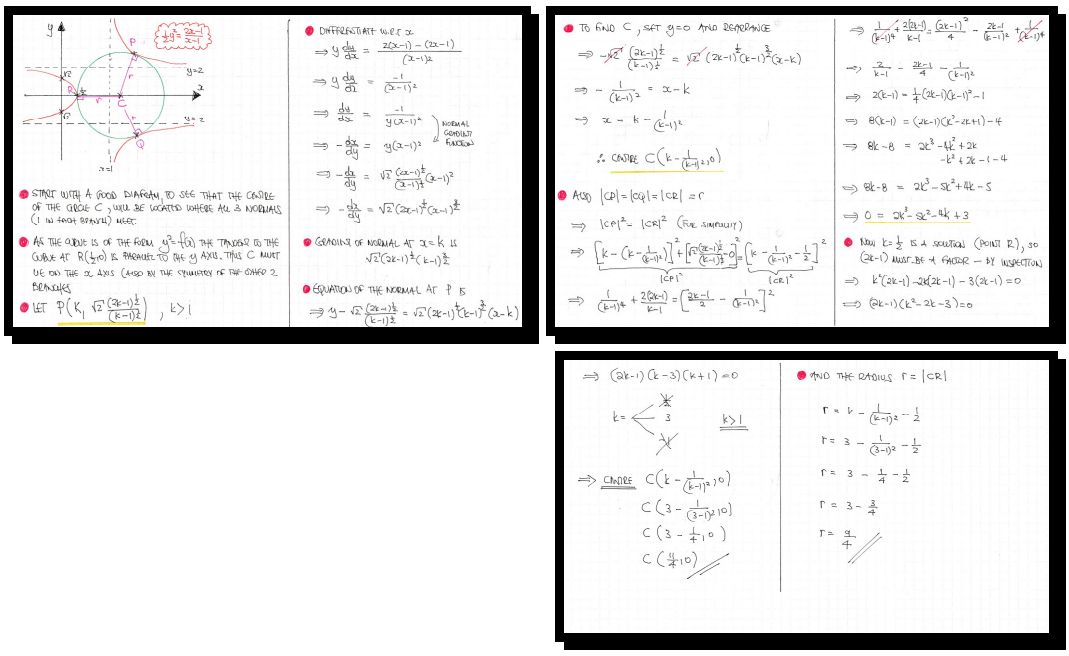


The figure above shows the curve with equation $\frac{1}{2}y^2=\frac{2x-1}{x-1}$, whose three asymptotes are marked with dotted lines.

A circle centred at the point C and of radius r is drawn, so that it touches all three branches of the curve, as shown in the figure.

Determine the coordinates of C and the value of r.

$\square$, $\boxed{C\left(\frac{11}{4},0\right)}$, $\boxed{r=\frac{9}{4}}$



Question 280 (*****)

The function with equation $y = f(x)$ has smooth first and second derivatives.

Show that

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \frac{d^2x}{dy^2} = 0 .$$

, proof

