BINOMIAL SERIES EXPANSIONS

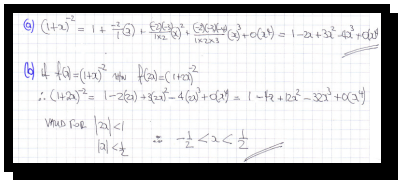
Question 1 (**)

The binomial expression $(1+x)^{-2}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Determine the expansion of $(1+x)^{-2}$, up and including the term in x^3.

b) Use part **(a)** to find the expansion of $(1+2x)^{-2}$, up and including the term in x^3, stating the range of values of x for which this expansion is valid.

$\square$, $\boxed{1-2x+3x^2-4x^3+O\left(x^4\right)}$, $\boxed{1-4x+12x^2-32x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



Question 2 (+)**

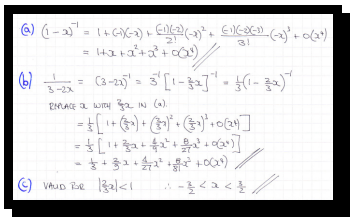
The binomial expression $(1-x)^{-1}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Determine the expansion of $(1-x)^{-1}$, up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $\dfrac{1}{3-2x}$, up and including the term in x^3.

c) State the values of x for which the expansion of $\dfrac{1}{3-2x}$ is valid.

$$\boxed{1+x+x^2+x^3+O\left(x^4\right)}, \quad \boxed{\frac{1}{3}+\frac{2}{9}x+\frac{4}{27}x^2+\frac{8}{81}x^3+O\left(x^4\right)}, \quad \boxed{-\frac{3}{2}<x<\frac{3}{2}}$$



Question 3 (+)**

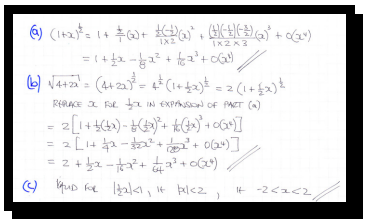
The binomial expression $(1+x)^{\frac{1}{2}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Determine the expansion of $(1+x)^{\frac{1}{2}}$, up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $\sqrt{4+2x}$, up and including the term in x^3.

c) State the range of values of x for which the expansion of $\sqrt{4+2x}$ is valid.

$\square$, $\boxed{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+O\left(x^4\right)}$, $\boxed{2+\frac{1}{2}x-\frac{1}{16}x^2+\frac{1}{64}x^3+O\left(x^4\right)}$, $\boxed{-2<x<2}$



Question 4 (+)**

The binomial expression $(1+x)^{\frac{1}{3}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

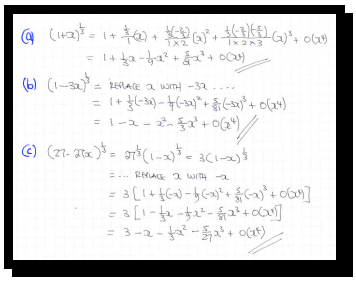
a) Determine the expansion of $(1+x)^{\frac{1}{3}}$, up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $(1-3x)^{\frac{1}{3}}$, up and including the term in x^3.

c) Use the expansion of part **(a)** to find the expansion of $(27-27x)^{\frac{1}{3}}$, up and including the term in x^3.

$$\boxed{1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+O\left(x^4\right)}, \quad \boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)},$$

$$\boxed{3-x-\frac{1}{3}x^2-\frac{5}{27}x^3+O\left(x^4\right)}$$



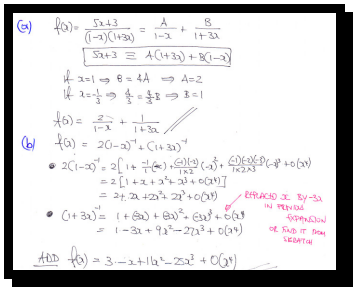
Question 5 (+)**

$$f(x)=\frac{5x+3}{(1-x)(1+3x)},\ |x|<\frac{1}{3}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find the series expansion of $f(x)$, up and including the term in x^3.

$\square$, $\boxed{f(x)=\frac{2}{1-x}+\frac{1}{1+3x}}$, $\boxed{f(x)=3-x+11x^2-25x^3+O\left(x^4\right)}$



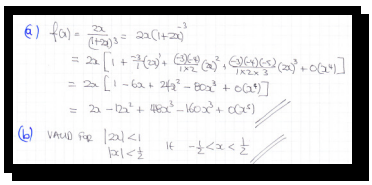
Question 6 (+)**

$$f(x)=\frac{2x}{(1+2x)^3},\ x\neq-\frac{1}{2}.$$

a) Find the first 4 terms in the series expansion of $f(x)$.

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\square$, $\boxed{f(x)=2x-12x^2+48x^3-160x^4+O\left(x^5\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



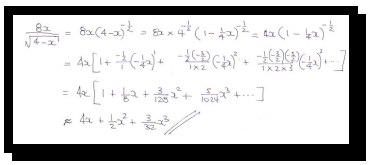
Question 7 (+)**

$$f(x)=\frac{8x}{\sqrt{4-x}}.$$

Show that if x is small, then

$$f(x)\approx 4x+\frac{1}{2}x^2+\frac{3}{32}x^3.$$

$\square$, proof



Question 8 (+)**

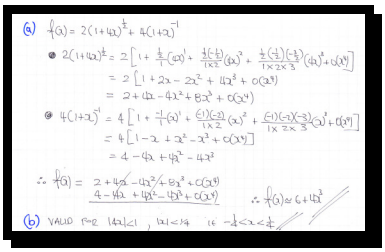
$$f(x)=2\sqrt{1+4x}+\frac{4}{1+x}.$$

a) By combining the first 4 terms in the expansions of $(1+x)^{-1}$ and $(1+4x)^{\frac{1}{2}}$ show that

$$f(x)\approx 6+4x^3.$$

b) State range of values of x for which the expansion of $f(x)$ is valid.

$\square$, $-\frac{1}{4}<x<\frac{1}{4}$



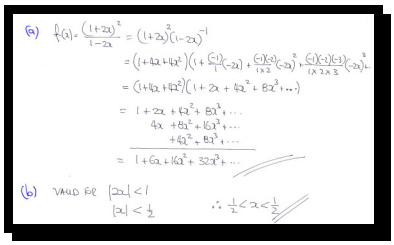
Question 9 (+)**

$$f(x)=\frac{(1+2x)^2}{1-2x},\ x\neq\frac{1}{2}.$$

a) Find the first 4 terms in the series expansion of $f(x)$.

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\square$, $\boxed{f(x)=1+6x+16x^2+32x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



Question 10 (***)

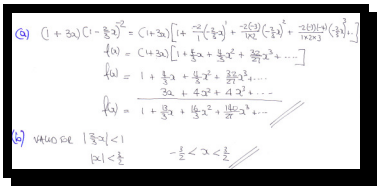
$$f(x)=(1+3x)\left(1-\frac{2}{3}x\right)^{-2}.$$

a) Show that if x is numerically small

$$f(x)\approx 1+\frac{13}{3}x+\frac{16}{3}x^2+\frac{140}{27}x^3.$$

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\square$, $\boxed{-\frac{3}{2}<x<\frac{3}{2}}$



Question 11 (***)

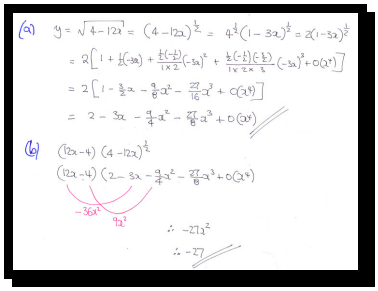
$$y=\sqrt{4-12x}\,,\ -\frac{1}{3}<x<\frac{1}{3}\,.$$

a) Find the binomial expansion of y in ascending powers of x up and including the term in x^3, writing all coefficients in their simplest form.

b) Hence find the coefficient of x^2 in the expansion of

$$(12x-4)(4-12x)^{\frac{1}{2}}\ .$$

$\boxed{}$, $\boxed{y=2-3x-\frac{9}{4}x^2-\frac{27}{8}x^3+O\left(x^4\right)}$, $\boxed{-27}$



Question 12 (***)

The binomial $(1+x)^{-\frac{1}{2}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

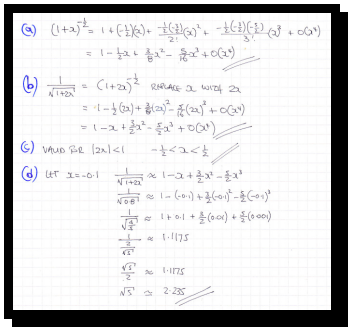
a) Find the series expansion of $(1+x)^{-\frac{1}{2}}$ up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $\dfrac{1}{\sqrt{1+2x}}$, up and including the term in x^3.

c) State the range of values of x for which the expansion of $\dfrac{1}{\sqrt{1+2x}}$ is valid.

d) Use the expansion of $\dfrac{1}{\sqrt{1+2x}}$ with $x=-0.1$ to show that $\sqrt{5}\approx 2.235$.

$$\boxed{1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+O\left(x^4\right)}, \quad \boxed{1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+O\left(x^4\right)}, \quad \boxed{-\frac{1}{2}<x<\frac{1}{2}}$$



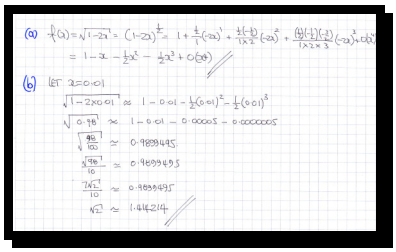
Question 13 (***)

$$f(x)=\sqrt{1-2x}\,, \quad |x|<\frac{1}{2}.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^3.

b) By substituting $x=0.01$ in the expansion, show that $\sqrt{2}\approx 1.414214$.

$$\boxed{f(x)=1-x-\frac{1}{2}x^2-\frac{1}{2}x^3+O\left(x^4\right)}$$



Question 14 **(***)**

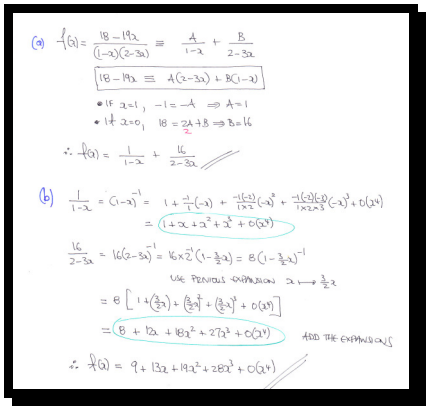
$$f(x)\equiv\frac{18-19x}{(1-x)(2-3x)},\ x\in\mathbb{R},\ |x|<\frac{2}{3}.$$

a) Express $f(x)$ in partial fractions.

b) Hence, or otherwise, show that if x is numerically small

$$f(x)\approx 9+13x+19x^2+28x^3.$$

$\square$, $\boxed{f(x)\equiv\frac{1}{1-x}+\frac{16}{2-3x}}$



Question 15 (***)

$$f(x)\equiv\frac{2-x}{\sqrt{1+x}},\ |x|<1.$$

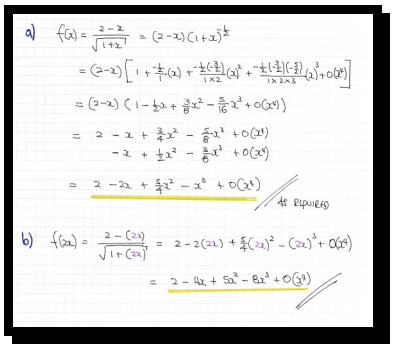
a) Show that the first four terms in the binomial expansion of $f(x)$ are

$$2-2x+\frac{5}{4}x^2-x^3.$$

b) Use the answer of part **(a)** to find the first four terms in the expansion of

$$g(x)=\frac{2-2x}{\sqrt{1+2x}}.$$

$\boxed{}$, $\boxed{g(x)=2-4x+5x^2-8x^3}$



Question 16 (***)

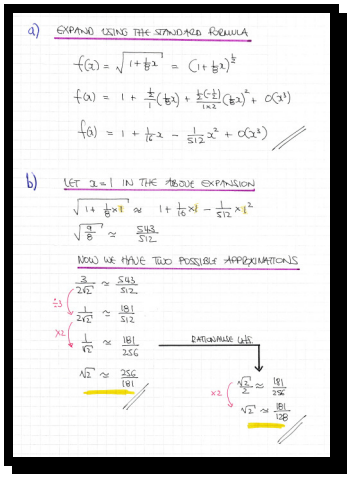
$$f(x)=\sqrt{1+\frac{1}{8}x}\,,\ |x|<8\,.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^2.

b) By substituting $x=1$ in the expansion, show that

$$\sqrt{2}\approx\frac{256}{181}\quad\text{or}\quad\sqrt{2}\approx\frac{181}{128}\,.$$

$\square$, $\boxed{f(x)=1+\frac{1}{16}x-\frac{1}{512}x^2+O\left(x^3\right)}$



Question 17 (***)

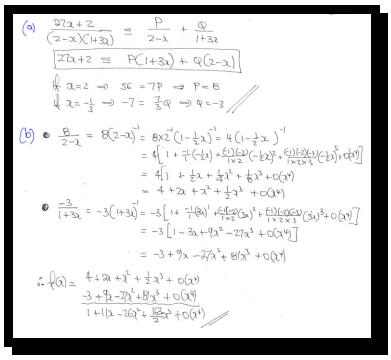
$$\frac{27x+2}{(2-x)(1+3x)} \equiv \frac{P}{2-x}+\frac{Q}{1+3x}.$$

a) Find the value of each of the constants P and Q.

b) Hence show that if x is sufficiently small

$$\frac{27x+2}{(2-x)(1+3x)} \approx 1+11x-26x^2+\frac{163}{2}x^3.$$

$\square$, $\boxed{P=8}$, $\boxed{Q=-3}$



Question 18 (***)

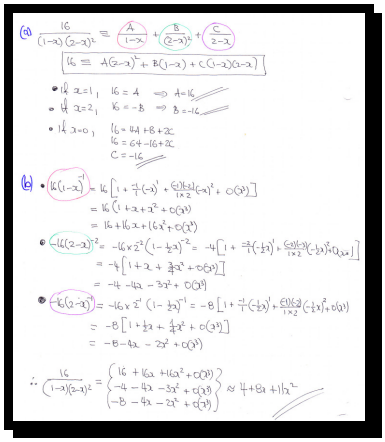
$$\frac{16}{(1-x)(2-x)^2}\equiv\frac{A}{1-x}+\frac{B}{(2-x)^2}+\frac{C}{2-x}.$$

a) Find the value of each of the constants A, B and C.

b) Hence show that if x is sufficiently small

$$\frac{16}{(1-x)(2-x)^2}\approx 4+8x+11x^2\ .$$

$\boxed{}$, $\boxed{A=16}$, $\boxed{B=-16}$, $\boxed{C=-16}$



Question 19 (***)

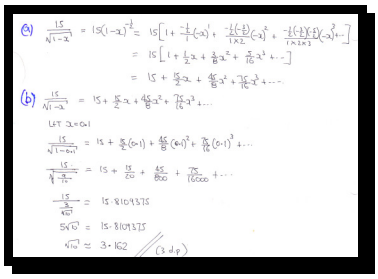
$$f(x)=\frac{15}{\sqrt{1-x}},\ |x|<1.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^3.

b) By substituting $x=0.1$ in the expansion of $f(x)$, show that

$$\sqrt{10}\approx 3.162$$

$$\boxed{f(x)=15+\frac{15}{2}x+\frac{45}{8}x^2+\frac{75}{16}x^3+O\left(x^4\right)}$$



Question 20 (***)

In the convergent binomial expansion of

$$(1+bx)^n, \quad |bx|<1$$

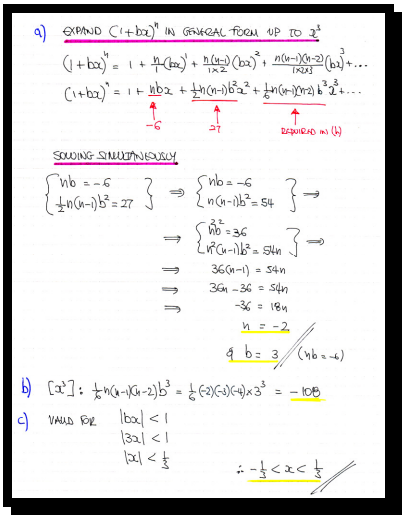
the coefficient of x is -6 and the coefficient of x^2 is 27.

a) Show that $b=3$ and find the value of n.

b) Find the coefficient of x^3.

c) State the range of values of x for which the above expansion is valid.

$\square$, $\boxed{n=-2}$, $\boxed{\left[x^3\right]=-108}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$



Question 21 (***)

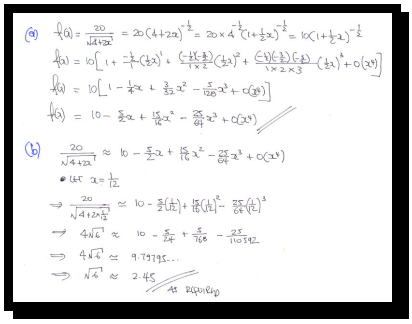
$$f(x)=\frac{20}{\sqrt{4+2x}}\,,\ |x|<2\,.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^3.

b) By substituting $x=\dfrac{1}{12}$ in the above expansion, show that

$$\sqrt{6}\approx 2.45\,.$$

$\square$, $\boxed{f(x)=10-\frac{5}{2}x+\frac{15}{16}x^2-\frac{25}{64}x^3+O\left(x^4\right)}$



Question 22 (***)

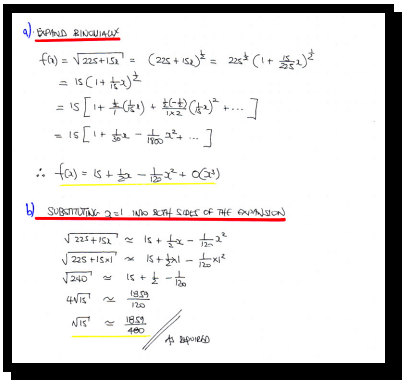
$$f(x)=\sqrt{225+15x}\,,\ |x|<15\,.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^2.

b) By substituting $x=1$ in the expansion of $f(x)$, show that

$$\sqrt{15}\approx\frac{1859}{480}\,.$$

$\square$, $\boxed{f(x)=15+\frac{1}{2}x-\frac{1}{120}x^2+O\left(x^3\right)}$



Question 23 (***)

$$f(x)=\frac{4x+1}{(1-2x)(1+x)},\ |x|<\frac{1}{2}.$$

a) Find the first four terms in the series expansion of $(1+x)^{-1}$.

b) Hence, find the first four terms in the series expansion of $(1-2x)^{-1}$.

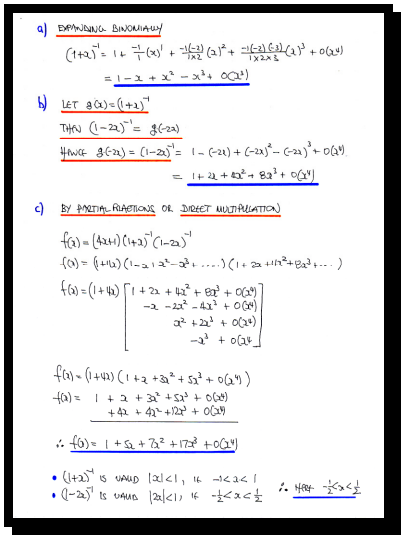
c) Hence show that

$$f(x)\approx 1+5x+7x^2+17x^3,$$

stating the range of values of x for which the above approximation is valid.

$\square$, $\boxed{(1+x)^{-1}=1-x+x^2-x^3+O\left(x^4\right)}$, $\boxed{(1-2x)^{-1}=1+2x+4x^2+8x^3+O\left(x^4\right)}$,

$\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



a) EXPANDING BINOMIALLY

$(1+x)^{-1} = 1 + \frac{-1}{1}(x)^1 + \frac{-1(-2)}{1\times 2}(x)^2 + \frac{-1(-2)(-3)}{1\times 2\times 3}(x)^3 + O(x^4)$

$= 1 - x + x^2 - x^3 + O(x^4)$

b) LET $g(x)=(1+x)^{-1}$

THEN $(1-2x)^{-1} = g(-2x)$

HENCE $g(-2x) = (1-2x)^{-1} = 1-(-2x)+(-2x)^2-(-2x)^3+O(x^4)$

$= 1+2x+4x^2+8x^3+O(x^4)$

c) BY PARTIAL FRACTIONS OR DIRECT MULTIPLICATION

$f(x) = (4x+1)(1+x)^{-1}(1-2x)^{-1}$

$f(x) = (4x+1)(1-x+x^2-x^3+\ldots)(1+2x+4x^2+8x^3+\ldots)$

$f(x) = (1+4x)\left[\begin{array}{l} 1+2x+4x^2+8x^3+O(x^4) \\ -x-2x^2-4x^3+O(x^4) \\ x^2+2x^3+O(x^4) \\ -x^3+O(x^4)\end{array}\right]$

$f(x) = (1+4x)(1+x+3x^2+5x^3+O(x^4))$

$f(x) = 1+x+3x^2+5x^3+O(x^4)$
$\quad +4x+4x^2+12x^3+O(x^4)$

$\therefore f(x) = 1+5x+7x^2+17x^3+O(x^4)$

- $(1+x)^{-1}$ IS VALID $|x|<1$, IE $-1<x<1$
- $(1-2x)^{-1}$ IS VALID $|2x|<1$, IE $-\frac{1}{2}<x<\frac{1}{2}$

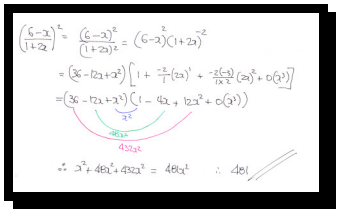
$\therefore$ VALID $-\frac{1}{2}<x<\frac{1}{2}$

Question 24 (***)

$$f(x)=\left(\frac{6-x}{1+2x}\right)^2,\ |x|<\frac{1}{2}.$$

Determine the value of the coefficient of x^2 in the binomial expansion of $f(x)$.

$\square$, $\boxed{481}$



Question 25 (***)

$$f(x)=(1-x)^{\frac{1}{3}},\ -1<x<1.$$

a) Find the binomial expansion of $f(x)$ in ascending powers of x up and including the term in x^2.

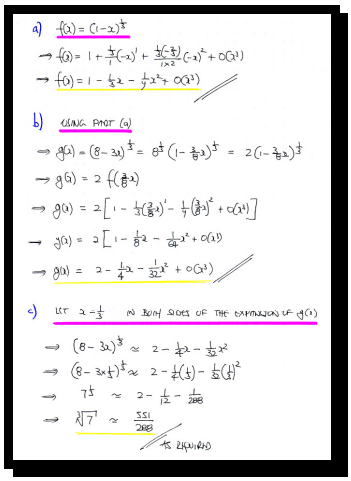
$$g(x)=(8-3x)^{\frac{1}{3}},\ -\frac{8}{3}<x<\frac{8}{3}.$$

b) Use the result of part **(a)** to find the binomial expansion of $g(x)$ in ascending powers of x up and including the term in x^2.

c) Hence, show that

$$\sqrt[3]{7}\approx\frac{551}{288}.$$

$\square$, $\boxed{f(x)=1-\frac{1}{3}x-\frac{1}{9}x^2+O\left(x^3\right)}$, $\boxed{g(x)=2-\frac{1}{4}x-\frac{1}{32}x^2+O\left(x^3\right)}$



Question 26 (***)

In the series expansion of

$$(1+ax)^n,\ |ax|<1,$$

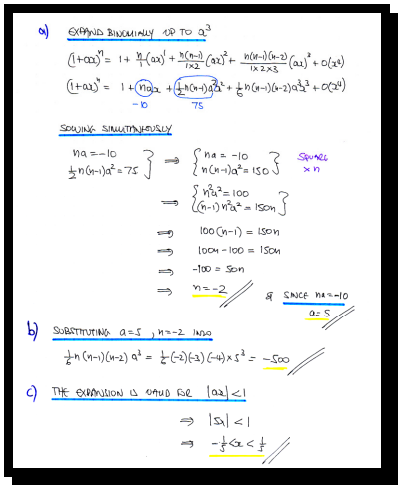
the coefficient of x is -10 and the coefficient of x^2 is 75.

a) Show that $n=-2$ and find the value of a.

b) Find the coefficient of x^3.

c) State the range of values of x for which the above expansion is valid.

$\boxed{\ }$, $\boxed{a=5}$, $\boxed{\left[x^3\right]:-500}$, $\boxed{-\frac{1}{5}<x<\frac{1}{5}}$



Question 27 (***)

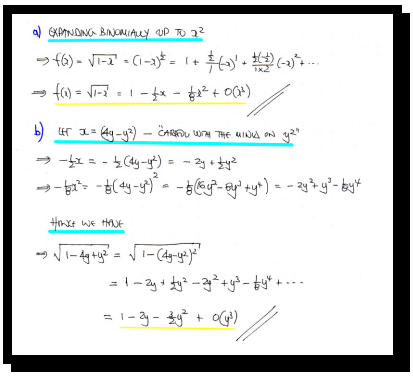
$$f(x)=\sqrt{1-x},\ -1<x<1,$$

a) Expand $f(x)$ in ascending powers of x, up and including the term in x^2.

b) Use the expansion of part **(a)** to show that if y is numerically small

$$\sqrt{1-4y+y^2}\approx 1-2y-\frac{3}{2}y^2.$$

$\square$, $\boxed{f(x)=1-\frac{1}{2}x-\frac{1}{8}x^2+O\left(x^3\right)}$



Question 28 (***)

$$f(x)=\frac{4+x}{(1+3x)^2},\ |x|<\frac{1}{3}.$$

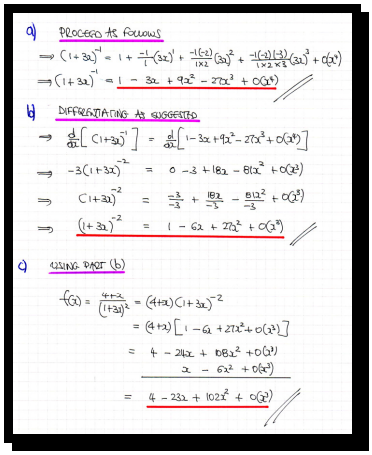
a) Find the series expansion of $(1+3x)^{-1}$, up and including the term in x^3.

b) By differentiating both sides of the expansion found in part **(a)**, show that

$$(1+3x)^{-2}=1-6x+27x^2+...$$

c) Hence find the first three terms in the series expansion of $f(x)$.

$\boxed{}$, $\boxed{(1+3x)^{-1}=1-3x+9x^2-27x^3+O\left(x^4\right)}$, $\boxed{f(x)=4-23x+102x^2+O\left(x^3\right)}$



Question 29 (***)

$$f(x)=(1-2x)^{-\frac{1}{2}}.$$

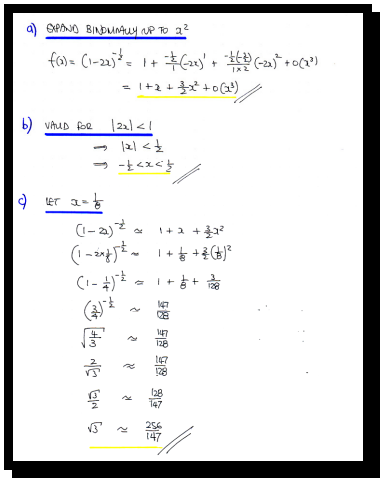
a) Expand $f(x)$ up and including the term in x^2.

b) State the values of x for which the expansion is valid.

c) By substituting $x=\dfrac{1}{8}$ in the expansion of part **(a)** show that

$$\sqrt{3}\approx\frac{256}{147}.$$

$\square$, $\boxed{1+x+\frac{3}{2}x^2+O\left(x^3\right)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$



Question 30 (*+)**

$$f(x)=(1+ax)^n,\ a\in\mathbb{R},\ n\in\mathbb{R}.$$

It is given that the series expansion of $f(x)$ is

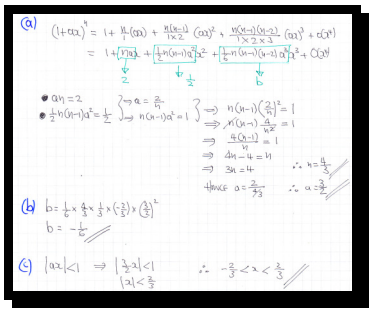
$$1+2x+\frac{1}{2}x^2+bx^3+O\left(x^4\right).$$

a) Show that $a=\frac{3}{2}$ and find the value of n.

b) Find the value of b.

c) State the range of values of x for which the above expansion is valid.

$\square$, $\boxed{n=\frac{4}{3}}$, $\boxed{b=-\frac{1}{6}}$, $\boxed{-\frac{2}{3}<x<\frac{2}{3}}$



Question 31 (*+)**

In the series expansion of

$$(1+ax)^n,\ |ax|<1,\ \ a,n\in\mathbb{R},$$

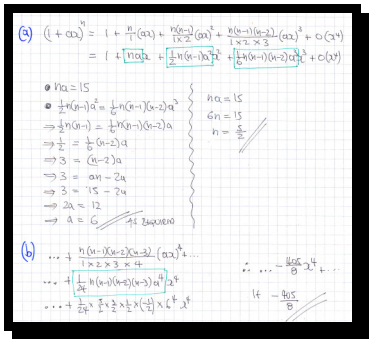
the coefficient of x is 15 and the coefficients of x^2 and x^3 are equal.

a) Given that n is not a positive integer, show that $a=6$.

b) Find the value of n.

c) Find the coefficient of x^4.

$\square$, $\boxed{n=\frac{5}{2}}$, $\boxed{\left[x^4\right]=-\frac{405}{8}}$



Question 32 (*+)**

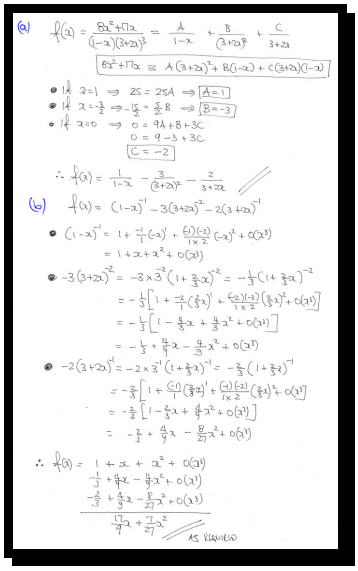
$$f(x)=\frac{8x^2+17x}{(1-x)(3+2x)^2},\ |x|<1.$$

a) Express $f(x)$ into partial fractions.

b) Hence show that

$$f(x)\approx\frac{1}{27}x(7x+51).$$

$\square$, $\boxed{f(x)=\dfrac{1}{(1-x)}-\dfrac{3}{(3+2x)^2}-\dfrac{2}{(3+2x)}}$



(a) $f(x)=\frac{8x^2+17x}{(1-x)(3+2x)^2}\equiv\frac{A}{1-x}+\frac{B}{(3+2x)^2}+\frac{C}{3+2x}$

$8x^2+17x\equiv A(3+2x)^2+B(1-x)+C(3+2x)(1-x)$

- $x=1\Rightarrow 25=25A\Rightarrow \boxed{A=1}$
- $x=-\frac{3}{2}\Rightarrow -15=\frac{5}{2}B\Rightarrow \boxed{B=-3}$
- $x=0\Rightarrow 0=9A+B+3C$
  $0=9-3+3C$
  $\boxed{C=-2}$

$\therefore f(x)=\frac{1}{1-x}-\frac{3}{(3+2x)^2}-\frac{2}{3+2x}$

(b) $f(x)=(1-x)^{-1}-3(3+2x)^{-2}-2(3+2x)^{-1}$

- $(1-x)^{-1}=1+\frac{-1}{1}(-x)+\frac{(-1)(-2)}{1\times 2}(-x)^2+O(x^3)$
  $=1+x+x^2+O(x^3)$
- $-3(3+2x)^{-2}=-3\times 3^{-2}\left(1+\frac{2}{3}x\right)^{-2}=-\frac{1}{3}\left(1+\frac{2}{3}x\right)^{-2}$
  $=-\frac{1}{3}\left[1+\frac{-2}{1}\left(\frac{2}{3}x\right)+\frac{(-2)(-3)}{1\times 2}\left(\frac{2}{3}x\right)^2+O(x^3)\right]$
  $=-\frac{1}{3}\left[1-\frac{4}{3}x+\frac{4}{3}x^2+O(x^3)\right]$
  $=-\frac{1}{3}+\frac{4}{9}x-\frac{4}{9}x^2+O(x^3)$
- $-2(3+2x)^{-1}=-2\times 3^{-1}\left(1+\frac{2}{3}x\right)^{-1}=-\frac{2}{3}\left(1+\frac{2}{3}x\right)^{-1}$
  $=-\frac{2}{3}\left[1+\frac{(-1)}{1}\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{1\times 2}\left(\frac{2}{3}x\right)^2+O(x^3)\right]$
  $=-\frac{2}{3}\left[1-\frac{2}{3}x+\frac{4}{9}x^2+O(x^3)\right]$
  $=-\frac{2}{3}+\frac{4}{9}x-\frac{8}{27}x^2+O(x^3)$

$\therefore f(x)=1+x+x^2+O(x^3)$
$-\frac{1}{3}+\frac{4}{9}x-\frac{4}{9}x^2+O(x^3)$
$-\frac{2}{3}+\frac{4}{9}x-\frac{8}{27}x^2+O(x^3)$
$\frac{17}{9}x+\frac{7}{27}x^2$ AS REQUIRED

Question 33 (*+)**

The algebraic expression $\sqrt[3]{1-3x}$ is to be expanded as an infinite convergent series, in ascending powers of x.

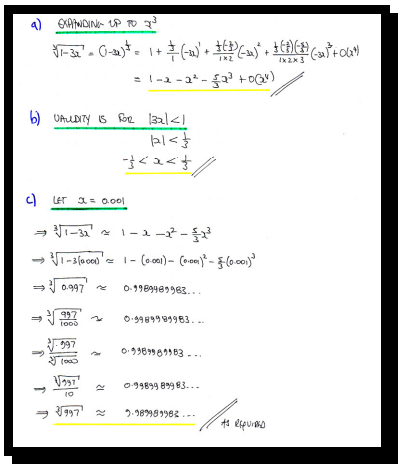
a) Find the first 4 terms in the series expansion of $\sqrt[3]{1-3x}$.

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the expansion, show that

$$\sqrt[3]{997} \approx 9.989989983 .$$

$\boxed{}$, $\boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$



Question 34 (*+)**

The binomial expression $(1+12x)^{\frac{3}{4}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

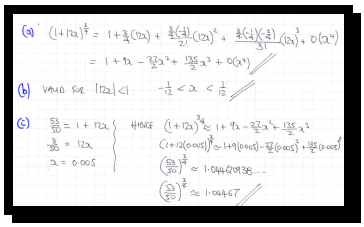
a) Find the first 4 terms in the expansion of $(1+12x)^{\frac{3}{4}}$.

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the expansion show that

$$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467 .$$

$\square$, $\boxed{1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{12}<x<\frac{1}{12}}$



Question 35 (*+)**

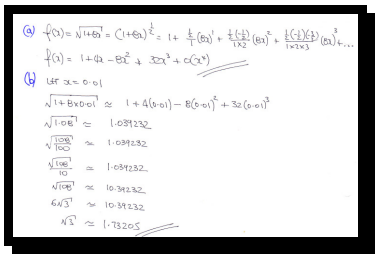
$$f(x)=\sqrt{1+8x}\ ,\ |x|<\frac{1}{8}$$

a) Expand $f(x)$ up and including the term in x^3.

b) By considering $\sqrt{1.08}$ and the series obtained in part (a), show that

$$\sqrt{3}\approx 1.73205\ .$$

$$\boxed{f(x)=1+4x-8x^2+32x^3+O\left(x^4\right)}$$



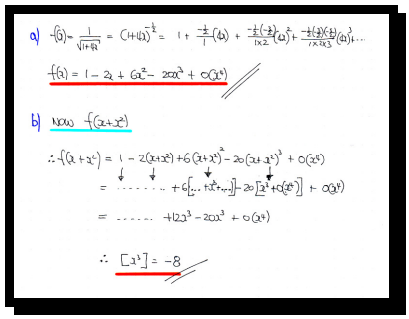
Question 36 (*+)**

$$f(x)=\frac{1}{\sqrt{1+4x}}\,,\ -\frac{1}{4}<x<\frac{1}{4}\,.$$

a) Find the binomial series expansion of $f(x)$ up and including the term in x^3.

b) Hence determine the coefficient of x^3 in the binomial expansion of $f\left(x+x^2\right)$.

$\square$, $\boxed{f(x)=1-2x+6x^2-20x^3+O\left(x^4\right)}$, $\boxed{\left[x^3\right]=-8}$



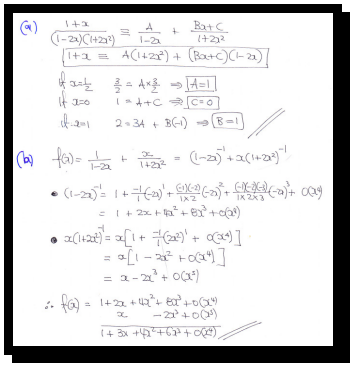
Question 37 (*+)**

$$f(x)\equiv\frac{1+x}{(1-2x)(1+2x^2)}\equiv\frac{A}{1-2x}+\frac{Bx+C}{1+2x^2},\ |x|<\frac{1}{2}$$

a) Find the value of each of the constants A, B and C.

b) Find the binomial expansion of $f(x)$, up and including the term in x^3.

$$\boxed{A=1,\ B=1,\ C=0},\quad \boxed{f(x)=1+3x+4x^2+6x^3+O\left(x^4\right)}$$



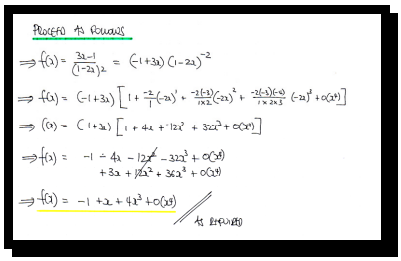
Question 38 (*+)**

$$f(x)=\frac{3x-1}{(1-2x)^2},\ |x|<\frac{1}{2}$$

Show that if x is small, then

$$f(x)\approx -1-x+4x^3.$$

$$\boxed{},\quad \boxed{\text{proof}}$$



Question 39 (*+)**

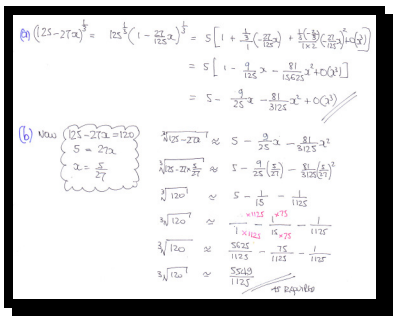
$$(125-27x)^{\frac{1}{3}},\ |x|<\frac{125}{27}$$

a) Find the first three terms in the series expansion of $f(x)$.

b) Use first three terms in the series expansion of $f(x)$ to show that

$$\sqrt[3]{120}\approx\frac{5549}{1125}.$$

$\square$, $\boxed{f(x)=5-\frac{9}{25}x-\frac{81}{3125}x^2+O\left(x^3\right)}$



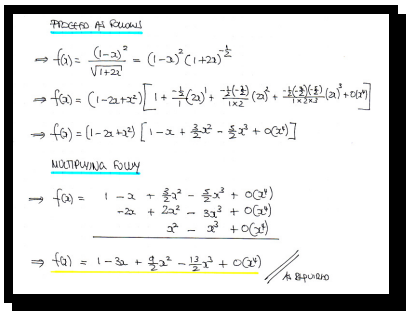
Question 40 (*+)**

$$f(x)=\frac{(1-x)^2}{\sqrt{1+2x}},\quad |x|<\frac{1}{2}$$

Show that if x is small, then

$$f(x)\approx 1-3x+\frac{9}{2}x^2-\frac{13}{2}x^3.$$

[ ], [proof]



Question 41 (*+)**

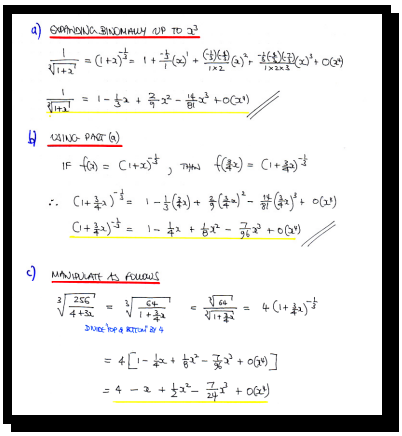
The algebraic expression $\dfrac{1}{\sqrt[3]{1+x}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Expand $\dfrac{1}{\sqrt[3]{1+x}}$ up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $\left(1+\dfrac{3}{4}x\right)^{-\frac{1}{3}}$ up and including the term in x^3.

c) Hence find the expansion of $\sqrt[3]{\dfrac{256}{4+3x}}$ up and including the term in x^3.

$\square$, $\boxed{1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+O\left(x^4\right)}$, $\boxed{1-\frac{1}{4}x+\frac{1}{8}x^2-\frac{7}{96}x^3+O\left(x^4\right)}$, $\boxed{4-x+\frac{1}{2}x^2-\frac{7}{24}x^3+O\left(x^4\right)}$



Question 42 (*+)**

$$f(x)\equiv\frac{1}{(2-3x)^3},\ |x|<\frac{2}{3}$$

a) Find the series expansion of $f(x)$, up and including the term in x^2.

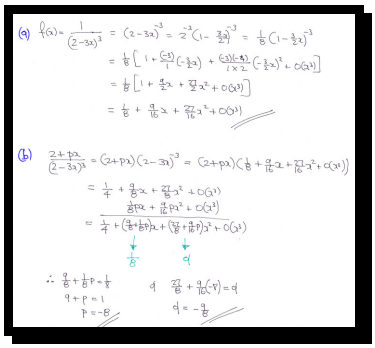
It is given that

$$\frac{2+px}{(2-3x)^3}\equiv\frac{1}{4}+\frac{1}{8}x+qx^2+\ldots$$

where p and q are non zero constants.

b) Determine the value of p and the value of q.

$\square$, $\boxed{f(x)=\frac{1}{8}+\frac{9}{16}x+\frac{27}{16}x^2+O\left(x^3\right)}$, $\boxed{p=-8,\ q=-\frac{9}{8}}$



Question 43 (*+)**

$$f(x)\equiv\left(\frac{1}{4}-x\right)^{-\frac{3}{2}},\ |x|<\frac{1}{4}$$

a) Find the series expansion of $f(x)$, up and including the term in x^3.

b) Use the result of part **(a)** to obtain the series expansion of

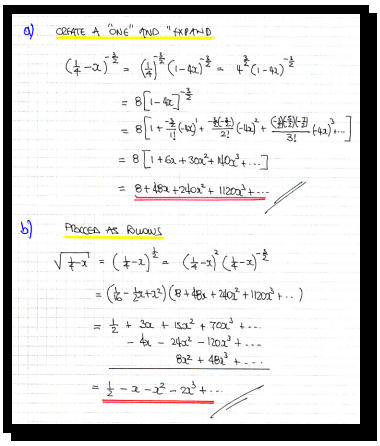
$$\sqrt{\frac{1}{4}-x},\ |x|<\frac{1}{4},$$

up and including the term in x^3.

No credit will be given for obtaining a direct expansion in this part.

$\square$, $\boxed{f(x)=8+48x+240x^2+1120x^3+O\left(x^4\right)}$,

$\boxed{\sqrt{\frac{1}{4}-x}=\frac{1}{2}-x-x^2-2x^3+O\left(x^4\right)}$



Question 44 (*+)**

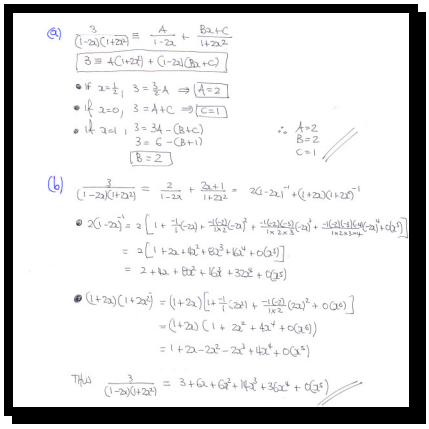
$$\frac{3}{(1-2x)(1+2x^2)}\equiv\frac{A}{1-2x}+\frac{Bx+C}{1+2x^2},\ |x|<\frac{1}{2}.$$

a) Find the value of each of the constants A, B and C.

b) Hence or otherwise find the first five terms in the binomial expansion of

$$\frac{3}{(1-2x)(1+2x^2)}.$$

$\boxed{A=2}$, $\boxed{B=2}$, $\boxed{C=1}$, $\boxed{3+6x+6x^2+14x^3+36x^4+O\left(x^5\right)}$



Question 45 (*+)**

The function $f(x)$ is defined in terms of the non zero constant n, by

$$f(x)=(3+2x)^n,\ -\frac{3}{2}<x<\frac{3}{2}.$$

a) Given that n is not a positive integer, find in terms of n the ratio of the coefficient of x^3 to the coefficient of x^2 in binomial expansion of $f(x)$.

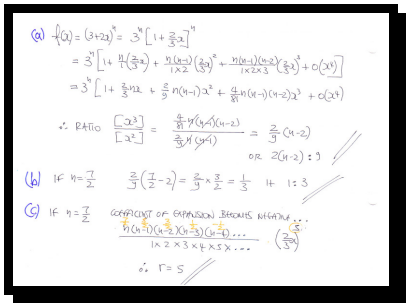
It is now given that $n=\frac{7}{2}$.

b) Evaluate the ratio found in part **(a)**.

The coefficient of x^r in the binomial expansion of $f(x)$ is negative.

c) Find the smallest value of r.

$\square$, $\boxed{[x^3]:[x^2]=2(n-2):9}$, $\boxed{[x^3]:[x^2]=1:3}$, $\boxed{r=5}$



Question 46 (****)

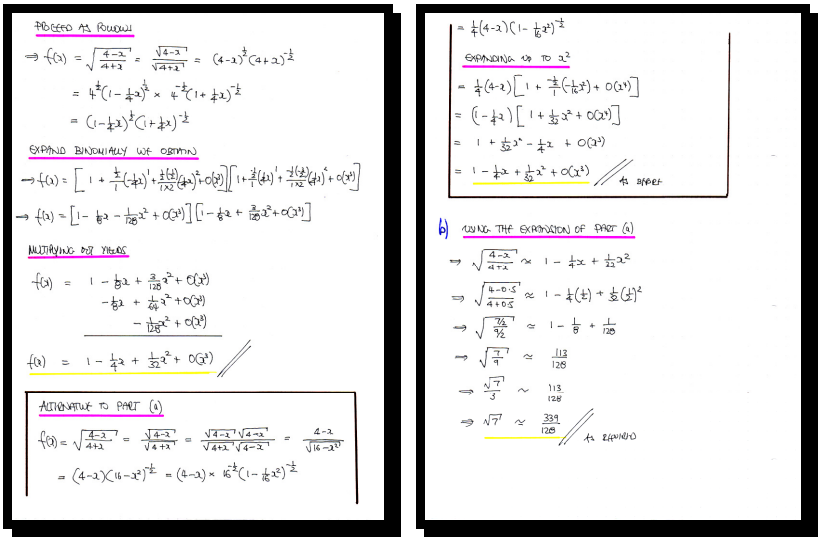
$$f(x)=\sqrt{\frac{4-x}{4+x}}\ ,\ |x|<4\ .$$

a) Expand $f(x)$ as an infinite convergent series, up and including the term in x^2.

b) By substituting $x=0.5$ in the expansion of part **(a)**, show that

$$\sqrt{7}\approx\frac{339}{128}\ .$$

$\square$, $\boxed{f(x)=1-\frac{1}{4}x+\frac{1}{32}x^2+O\left(x^3\right)}$



Question 47 (****)

$$f(x)\equiv(1-8x)^{\frac{1}{4}},\ |x|<\frac{1}{8}.$$

a) Find the first four terms in the binomial series expansion of $f(x)$.

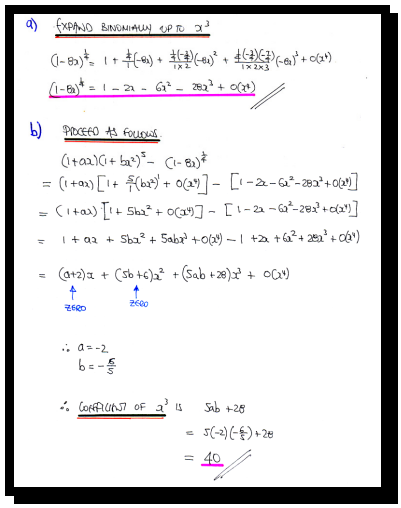
The term of lowest degree in the series expansion of

$$(1+ax)\left(1+bx^2\right)^5-f(x),$$

is the term in x^3.

b) Determine the value of each of the constants a and b, and hence state the coefficient of x^3.

$\boxed{}$, $\boxed{1-2x-6x^2-28x^3+O\left(x^4\right)}$, $\boxed{a=-2}$, $\boxed{b=-\frac{6}{5}}$, $\boxed{\left[x^3\right]=40}$



Question 48 (****)

$$\frac{2-3x^2}{(2x+1)\left(x^2+1\right)}\equiv\frac{A}{2x+1}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1}.$$

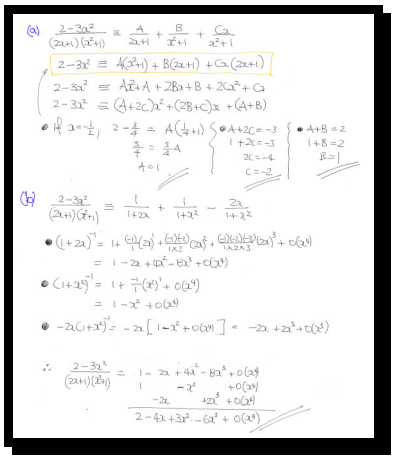
a) Find the value of each of the constants A, B and C in the above identity.

b) Hence, or otherwise, determine the series expansion of

$$\frac{2-3x^2}{(2x+1)\left(x^2+1\right)},$$

up and including the term in x^3.

$$\boxed{A=1},\ \boxed{B=1},\ \boxed{C=-2},\ \boxed{2-4x+3x^2-6x^3+O\left(x^4\right)}$$



Question 49 (****)

$$f(x)=\sqrt{1-x}\,,\ -1<x<1\,.$$

a) Expand $f(x)$ up and including the term in x^3.

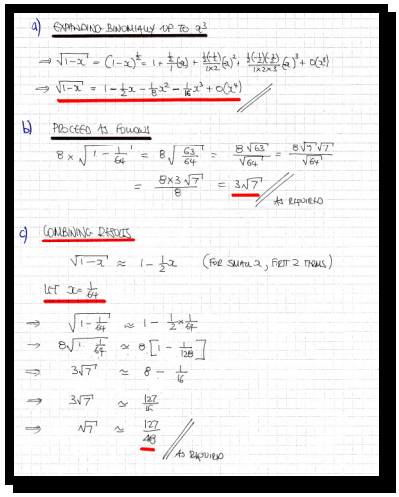
b) Show clearly that

$$8\times\sqrt{1-\frac{1}{64}}=3\sqrt{7}\,.$$

c) By using the **first two** terms of the expansion obtained in part **(a)** and the result shown in part **(b)**, show further that

$$\sqrt{7}\approx\frac{127}{48}\,.$$

$\square$, $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)$



Question 50 (****)

The algebraic expression $\dfrac{1+x}{1+3x}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Find the first 4 terms in the binomial expansion of

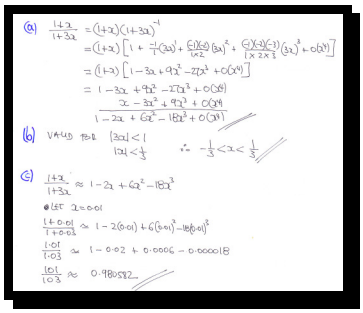
$$\frac{1+x}{1+3x}.$$

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the above expansion show that

$$\frac{101}{103}\approx 0.980582\,.$$

$$\boxed{1-2x+6x^2-18x^3+O\left(x^4\right)}\,,\qquad \boxed{-\frac{1}{3}<x<\frac{1}{3}}$$



Question 51 (****)

The algebraic expression $\sqrt{9-x}$ is to be expanded as an infinite convergent series, in ascending powers of x.

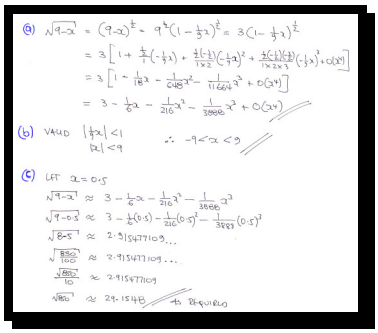
a) Find the first 4 terms in the series expansion of $\sqrt{9-x}$.

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the expansion show that

$$\sqrt{850} \approx 29.1548 .$$

$$\boxed{3-\frac{1}{6}x-\frac{1}{216}x^2-\frac{1}{3888}x^3+O\left(x^4\right)}, \quad \boxed{-9<x<9}$$



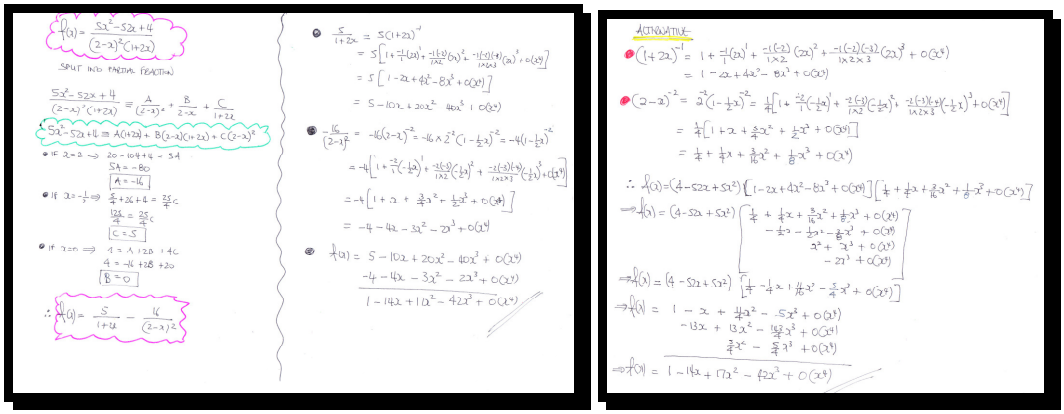
Question 52 (****)

$$f(x)=\frac{5x^2-52x+4}{(1+2x)(2-x)^2},\ |x|<\frac{1}{2}.$$

Show that if x is numerically small

$$f(x)\approx 1-14x+17x^2-42x^3.$$

proof



Question 53 (****)

$$\frac{2x^2-3}{(3-2x)(1-x)^2} \equiv \frac{A}{3-2x}+\frac{B}{1-x}+\frac{C}{(1-x)^2}.$$

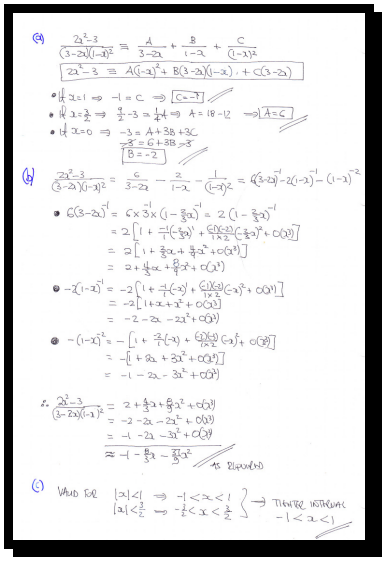
a) Find the value of each of the constants A, B and C.

b) Hence show that for small x

$$\frac{2x^2-3}{(3-2x)(1-x)^2} \approx -1-\frac{8}{3}x-\frac{37}{9}x^2 .$$

c) State the range of values of x for which the approximation in part **(b)** is valid.

$\boxed{A=6,\ B=-2,\ C=-1}$, $\boxed{-1<x<1}$



Question 54 (****)

$$f(x)=\frac{18-20x}{8x^2-18x+9},\ -\frac{3}{4}<x<\frac{3}{4}\ .$$

a) Express $f(x)$ into partial fractions.

b) Hence show that

$$f(x)\approx 2+\frac{16}{9}x+\frac{16}{9}x^2+\frac{160}{81}x^3\ .$$

$$\boxed{f(x)=\frac{2}{3-4x}+\frac{4}{3-2x}}$$



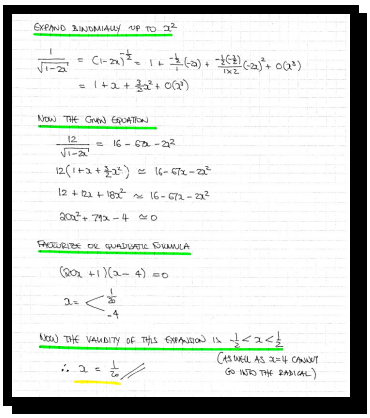
Question (****)

$$f(x)=\frac{12}{\sqrt{1-2x}},\quad x\in\mathbb{R},\quad x\leq\frac{1}{2}.$$

Use a quadratic approximation for $f(x)$ to solve the equation

$$f(x)=16-67x-2x^2.$$

$\square$, $\boxed{x\approx\frac{1}{20}}$



Question 55 (****)

$$f(x)=\sqrt{1-x}\,,\ -1<x<1\,.$$

a) Expand $f(x)$ up and including the term in x^3.

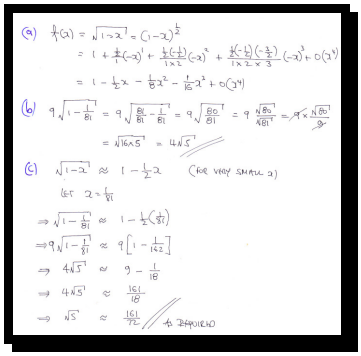
b) Show clearly that

$$9\sqrt{1-\frac{1}{81}}=4\sqrt{5}\,.$$

c) By using the **first two** terms of the expansion obtained in part **(a)** and the result shown in part **(b)**, show further that

$$\sqrt{5}\approx\frac{161}{72}\,.$$

$$\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$$



Question 56 (****)

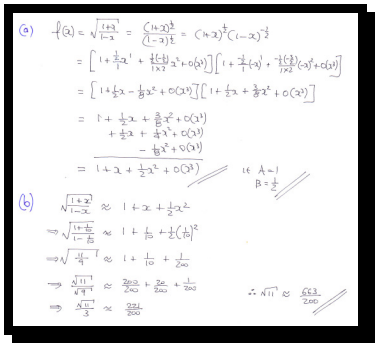
$$f(x)=\sqrt{\frac{1+x}{1-x}}\approx 1+Ax+Bx^2\text{, for small } x\text{.}$$

a) Show that $B=\frac{1}{2}$ and find the value of A.

b) By using $x=\frac{1}{10}$ in the above expansion, show clearly that

$$\sqrt{11}\approx\frac{663}{200}.$$

$$\boxed{A=1}$$



Question 57 (****)

$$f(x)=\sqrt{1-x},\ -1<x<1.$$

a) Expand $f(x)$ up and including the term in x^3.

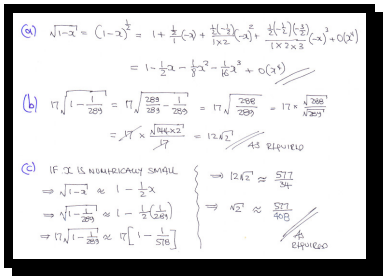
b) Show carefully that

$$17\sqrt{1-\frac{1}{289}}=12\sqrt{2}.$$

c) By using the **first two** terms of the expansion obtained in part **(a)** and the result shown in part **(b)**, show further that

$$\sqrt{2}\approx\frac{577}{408}.$$

$$\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$$



Question 58 (****)

$$f(x)\equiv\frac{4x(9x-10)}{(2-x)(2-3x)^2},\ x\in\mathbb{R},\ |x|<\frac{2}{3},\ x\neq 0.$$

a) Find the values of the constants A, B and C given that

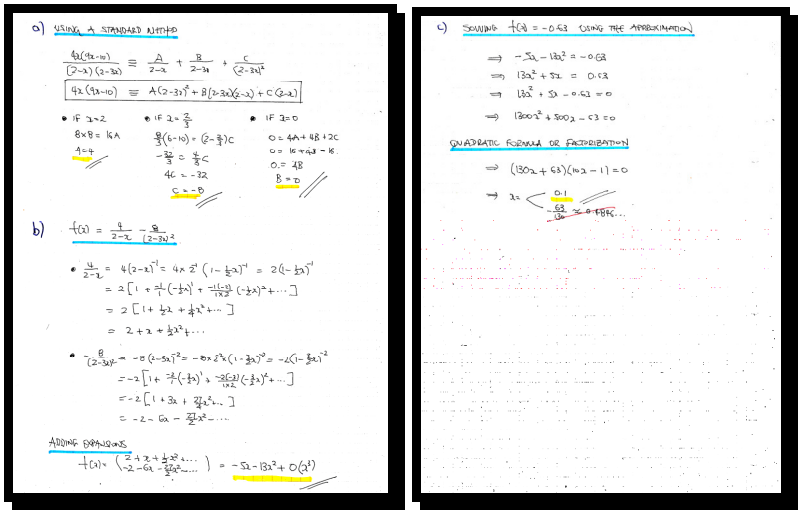
$$f(x)\equiv\frac{A}{2-x}+\frac{B}{2-3x}+\frac{C}{(2-3x)^2}.$$

b) Hence, or otherwise, find the binomial series expansion of $f(x)$, up and including the term in x^2.

The equation $f(x)=-0.63$ is known to have a positive solution which is further known to be numerically small.

c) Use part **(b)** to find this solution.

$\boxed{}$, $\boxed{A=4}$, $\boxed{B=0}$, $\boxed{C=-8}$, $\boxed{f(x)=-5x-13x^2+O\left(x^3\right)}$, $\boxed{x=0.1}$



Question 59 (****)

$$f(x)=(1+kx)^{-3},\ |kx|<1,$$

where k is a non zero constant.

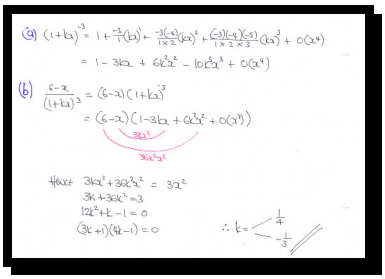
a) Expand $f(x)$, in terms of k, as an infinite convergent series up and including the term in x^3.

$$g(x)=\frac{6-x}{(1+kx)^3},\ |kx|<1.$$

The coefficient of x^2 in the expansion of $g(x)$ is 3.

b) Find the possible values of k.

$\square$, $\boxed{1-3kx+6k^2x^2-10k^3x^3+O\left(x^4\right)}$, $\boxed{k=-\frac{1}{3},\ \frac{1}{4}}$



Question 60 (****)

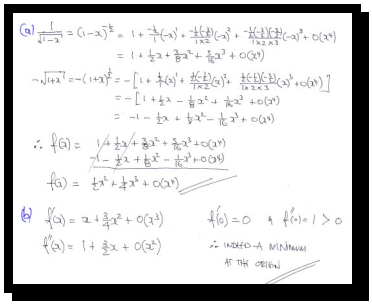
$$f(x)=\frac{1}{\sqrt{1-x}}-\sqrt{1+x}\,,\ |x|<1.$$

a) Show clearly that

$$f(x)=\frac{1}{2}x^2+\frac{1}{4}x^3+O\left(x^4\right).$$

b) Hence show that $f(x)$ has a minimum at the origin.

proof



Question 61 (****)

$$f(x)\equiv\frac{16x^2+3x-2}{x^2(3x-2)},\ x\in\mathbb{R},\ |x|<\frac{2}{3},\ x\neq 0.$$

a) Determine the value of each of the constants A, B and C given that

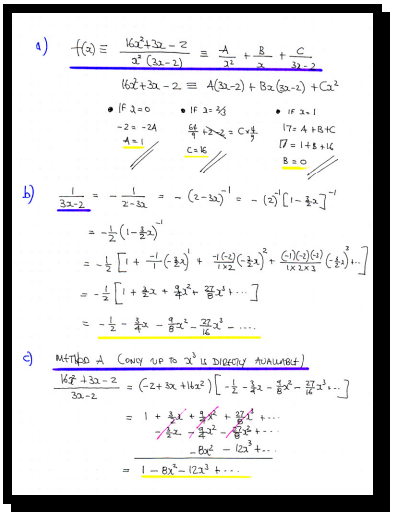
$$f(x)\equiv\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(3x-2)}.$$

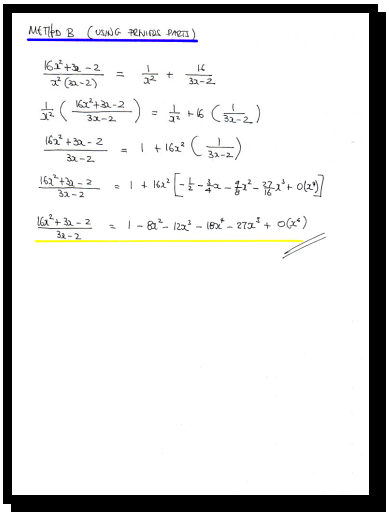
b) Find the binomial series expansion of $\dfrac{1}{3x-2}$, up and including the term in x^3.

c) Hence, or otherwise, show that if x is numerically small

$$\frac{16x^2+3x-2}{(3x-2)}\approx 1-8x^2-12x^3-18x^4-27x^5.$$

$\boxed{}$, $\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=16}$, $\boxed{-\frac{1}{2}-\frac{3}{4}x-\frac{9}{8}x^2-\frac{27}{16}x^3+O\left(x^4\right)}$





Question 62 (****)

$$f(x)=\sqrt{1-x}\,,\ -1<x<1\,.$$

a) Expand $f(x)$ up and including the term in x^3.

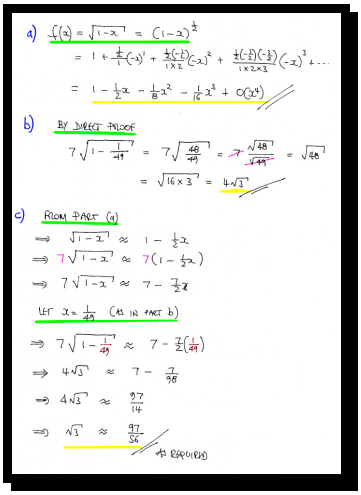
b) Show clearly that

$$7\sqrt{1-\frac{1}{49}}=4\sqrt{3}\,.$$

c) By using the **first two** terms of the expansion obtained in part **(a)** and the result obtained in part **(b)**, show further that

$$\sqrt{3}\approx\frac{97}{56}\,.$$

$\square$, $\boxed{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+O\left(x^4\right)}$



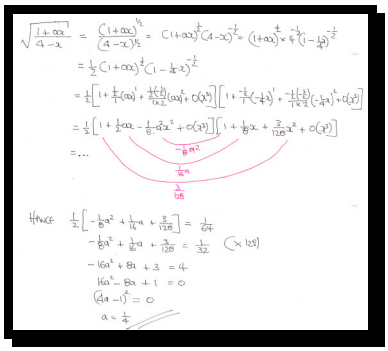
Question 63 (**+)**

$$f(x)=\sqrt{\frac{1+ax}{4-x}}\ ,\ -1<x<1\ .$$

The value of the constant a is such so that the coefficient of x^2 in the convergent binomial expansion of $f(x)$ is $\dfrac{1}{64}$.

Find the value of a.

$\square$, $\boxed{a=\frac{1}{4}}$



Question 64 (**+)**

$$f(x) \equiv \frac{1}{\sqrt{1-ax}} - \sqrt{1+bx}\,,$$

where a and b are constants so that $a > b > 0$.

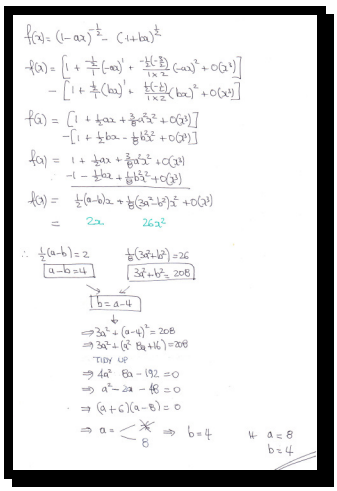
The function f is defined in a suitable domain of x, and furthermore the values of x are small enough so that $f(x)$ has a binomial series expansion.

Given that

$$f(x) \approx 2x + 26x^2\,,$$

determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=8}$, $\boxed{b=4}$



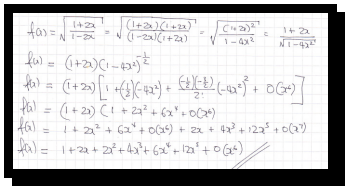
Question 65 (**+)**

$$f(x)=\sqrt{\frac{1+2x}{1-2x}}\,,\quad |x|<\frac{1}{2}.$$

By writing $f(x)$ in the form $f(x)=\dfrac{1+ax}{\sqrt{1+bx^2}}$, show that

$$f(x)=1+2x+2x^2+4x^3+6x^4+12x^5+O\left(x^6\right).$$

proof



Question 66 (**+)**

$$f(x)=\left(\frac{1}{2}-x\right)^{-3},\quad |x|<\frac{1}{2}.$$

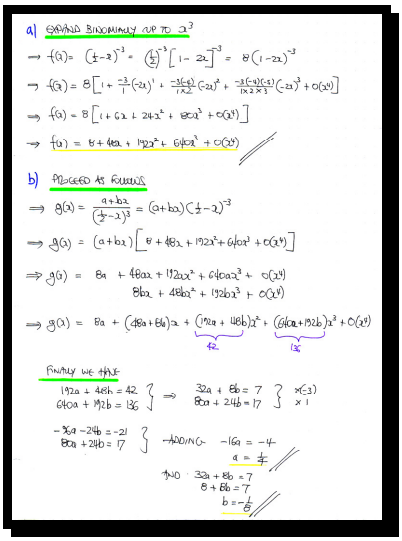
a) Expand $f(x)$, up and including the term in x^3.

$$g(x)=\frac{a+bx}{\left(\frac{1}{2}-x\right)^3}.$$

The coefficients of x^2 and x^3 in the expansion of $g(x)$ are 42 and 136 respectively.

b) Show that $a=\frac{1}{4}$ and find the value of b.

$\boxed{}$, $\boxed{f(x)=8+48x+192x^2+640x^3+O\left(x^4\right)}$, $\boxed{b=-\frac{1}{8}}$



Question 67 (**+)**

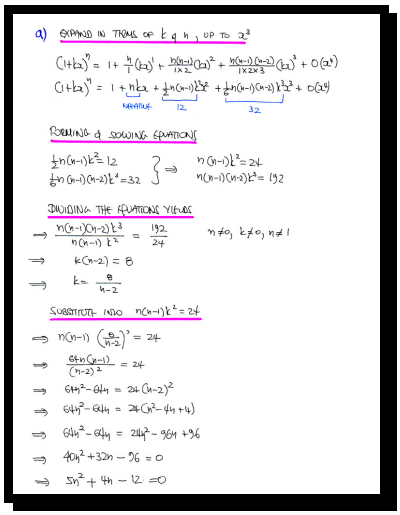
In the convergent expansion of

$$(1+kx)^n, \quad |kx|<1,$$

where k and n are non zero constants, the coefficient of x^2 is 12 and the coefficient of x^3 is 32.

Given the coefficient of x is negative determine the values of k and n.

$\square$, $\boxed{n=\frac{6}{5}}$, $\boxed{k=-10}$


a) EXPANSION IN TERMS OF k & n, UP TO x^3

$(1+kx)^n = 1 + \frac{n}{1}(kx)^1 + \frac{n(n-1)}{1\times 2}(kx)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(kx)^3 + O(x^4)$

$(1+kx)^n = 1 + nkx + \frac{1}{2}n(n-1)k^2x^2 + \frac{1}{6}n(n-1)(n-2)k^3x^3 + O(x^4)$

NEGATIVE 12 32

FORMING & SOLVING EQUATIONS

$\frac{1}{2}n(n-1)k^2 = 12$ $\Rightarrow$ $n(n-1)k^2 = 24$

$\frac{1}{6}n(n-1)(n-2)k^3 = 32$ $\Rightarrow$ $n(n-1)(n-2)k^3 = 192$

DIVIDING THE EQUATIONS YIELDS

$\Rightarrow \frac{n(n-1)(n-2)k^3}{n(n-1)k^2} = \frac{192}{24}$ $\quad n\neq 0,\ k\neq 0,\ n\neq 1$

$\Rightarrow k(n-2) = 8$

$\Rightarrow k = \frac{8}{n-2}$

SUBSTITUTE INTO $n(n-1)k^2 = 24$

$\Rightarrow n(n-1)\left(\frac{8}{n-2}\right)^2 = 24$

$\Rightarrow \frac{64n(n-1)}{(n-2)^2} = 24$

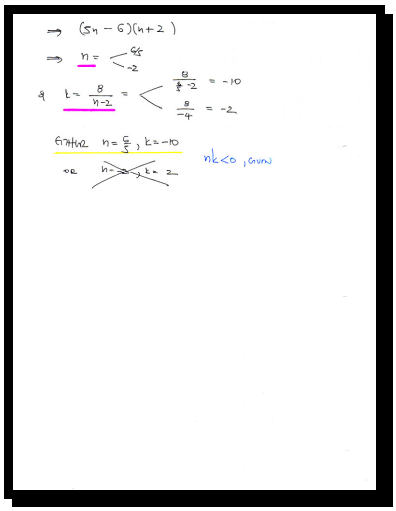
$\Rightarrow 64n^2 - 64n = 24(n-2)^2$

$\Rightarrow 64n^2 - 64n = 24(n^2 - 4n + 4)$

$\Rightarrow 64n^2 - 64n = 24n^2 - 96n + 96$

$\Rightarrow 40n^2 + 32n - 96 = 0$

$\Rightarrow 5n^2 + 4n - 12 = 0$


$\Rightarrow (5n-6)(n+2)$

$\Rightarrow n = \frac{6}{5}, -2$

$k = \frac{8}{n-2} = \frac{8}{\frac{6}{5}-2} = -10$ or $\frac{8}{-4} = -2$

EITHER $n=\frac{6}{5}$, $k=-10$ OR ~~$n=-2$, $k=-2$~~

$nk<0$, (GIVEN)

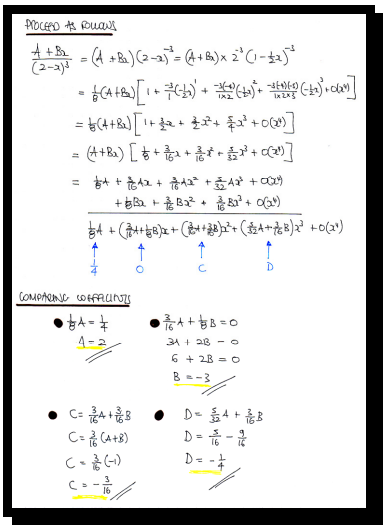
Question 68 (**+)**

$$\frac{A+Bx}{(2-x)^3}\equiv\frac{1}{4}+Cx^2+Dx^3+\ldots,$$

where A, B, C and D are constants, and $|x|<2$

Determine the value of A, B, C and D.

$\boxed{}$, $\boxed{A=2}$, $\boxed{B=-3}$, $\boxed{C=-\frac{3}{16}}$, $\boxed{D=-\frac{1}{4}}$



Question 69 (**+)**

$$f(x)\equiv\sqrt[3]{1+12x}\,.$$

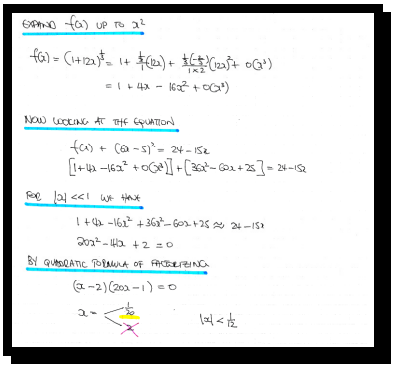
It is given that the equation

$$f(x)+(6x-5)^2=24-15x$$

has a solution α, which is numerically small.

Use a quadratic approximation for $f(x)$ to find an approximate value for α.

$\square$, $\boxed{\alpha\approx\frac{1}{20}}$



Question 70 (**+)**

The function f is defined as

$$f(x)\equiv\frac{ax+b}{(1-x)(1+2x)},\ x\in\mathbb{R},\ |x|<\frac{1}{2},$$

where a and b are constants.

a) Find the values of the constants P and Q in terms of a and b, given that

$$f(x)\equiv\frac{P}{(1-x)}+\frac{Q}{(1+2x)}.$$

The binomial series expansion of $f(x)$, up and including the term in x^3 is

$$f(x)=1+13x+Ax^2+Bx^3+\ldots,$$

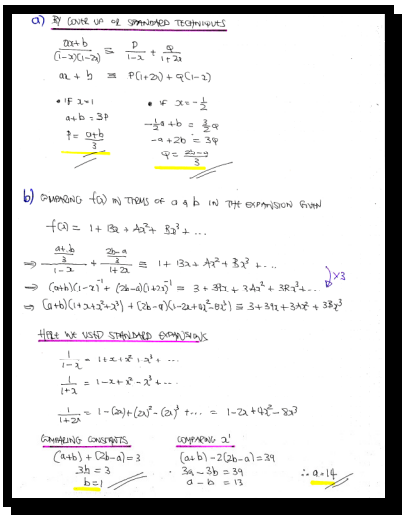
where A and B are constants.

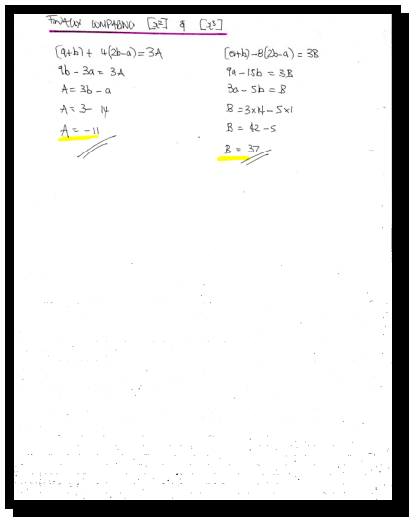
b) Determine the value of the constants …

i. … a and b.

ii. … A and B.

$\boxed{}$, $\boxed{P=\frac{a+b}{3}}$, $\boxed{P=\frac{2b-a}{3}}$, $\boxed{a=14}$, $\boxed{b=1}$, $\boxed{A=-11}$, $\boxed{B=37}$





Question 71 (**+)**

The function f is defined as

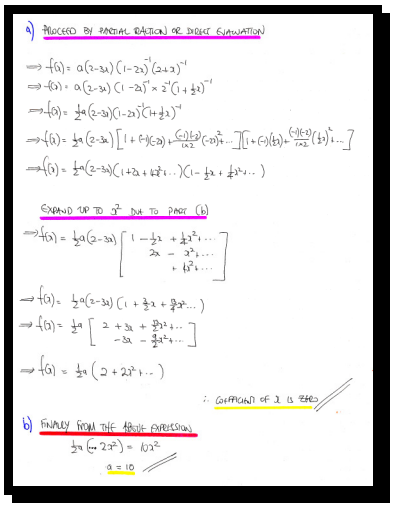
$$f(x)\equiv\frac{a(2-3x)}{(1-2x)(2+x)},\ x\in\mathbb{R},\ |x|<\frac{1}{2},\ x\neq 0.$$

where a is a non zero constant.

a) Show that for all values of the constant a, the coefficient of x in the binomial series expansion of $f(x)$, is zero.

b) Find the value of a, given that the coefficient of x^2 in the binomial series expansion of $f(x)$, is 10.

$\boxed{}$, $\boxed{a=10}$



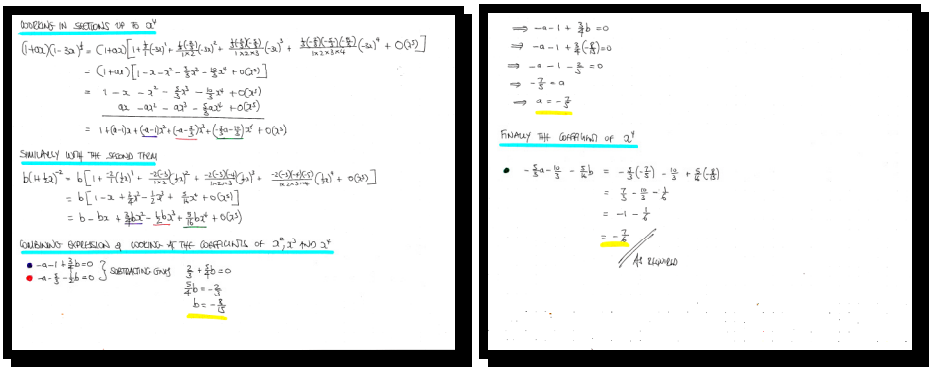
Question 72 (**+)**

$$f(x)=(1+ax)(1-3x)^{\frac{1}{3}}+\frac{b}{\left(1+\frac{1}{2}x\right)^2},\ |3x|<1,\ |ax|<1.$$

In the binomial expansion of $f(x)$ the coefficients of x^2 and x^3 are both zero.

Show clearly that the coefficient of x^4 is $-\frac{7}{6}$.

$\boxed{}$, $\boxed{\text{proof}}$



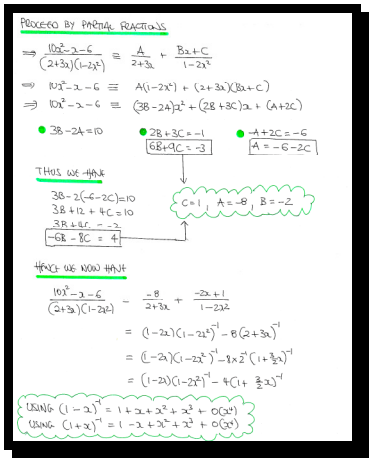
Question 73 **(****+)**

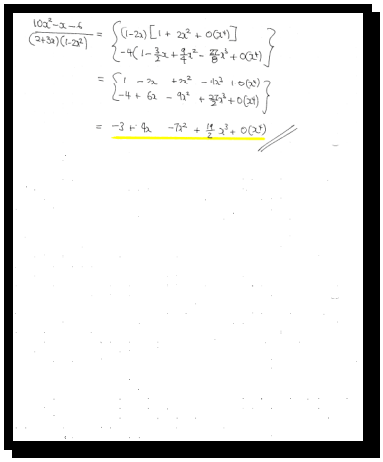
If x is sufficiently small find the series expansion of

$$\frac{10x^2-x-6}{(2+3x)\left(1-2x^2\right)},$$

up and including the term in x^3.

$\square$, $\dfrac{10x^2-x-6}{(2+3x)\left(1-2x^2\right)}=-3+4x-7x^2+\dfrac{19}{2}x^3+O\left(x^4\right)$





Question 74 (**+)**

In the convergent expansion of

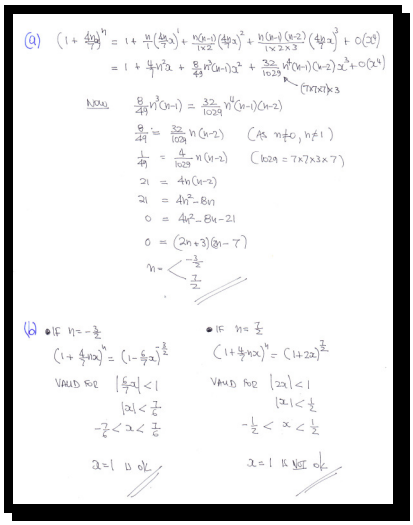
$$\left(1+\frac{4}{7}nx\right)^n,\ n\in\mathbb{R},\ n\notin\mathbb{N},\ n\neq 0,$$

the coefficients of x^2 and x^3 are non zero and equal.

a) Determine the possible values of n.

b) State with justification which value, values or indeed if any of the values of n produces a valid expansion for $x=1$.

$\square$, $\boxed{n=-\frac{3}{2},\ \frac{7}{2}}$, $\boxed{\text{only } n=-\frac{3}{2} \text{ produces a valid expansion for } x=1}$



Question 75 (*****)

$$f(x) \equiv \frac{1}{(1-5x)^2}, \quad |x| < \frac{1}{5}.$$

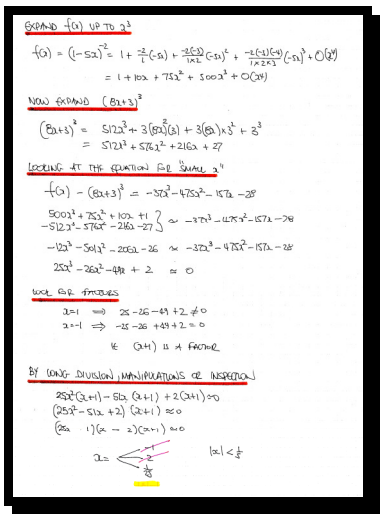
It is given that the equation

$$f(x) - (8x+3)^3 = -37x^3 - 475x^2 - 157x + 27$$

has a solution α, which is numerically small.

Find an approximate value for α.

$\square$, $\boxed{\alpha \approx \frac{1}{25}}$



Question 76 (*****)

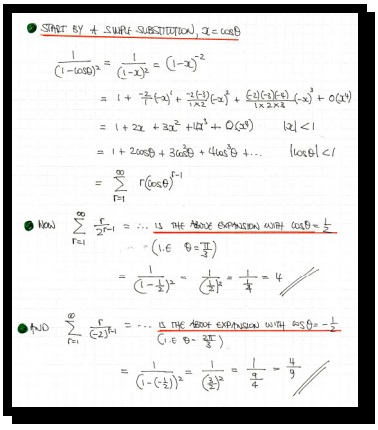
By considering the binomial expansion of

$$\frac{1}{(1-\cos\theta)^2},$$

sum each of the following series.

- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]$.
- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]$.

$\square$, $\boxed{\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]=4}$, $\boxed{\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]=\frac{4}{9}}$



Question 77 (*****)

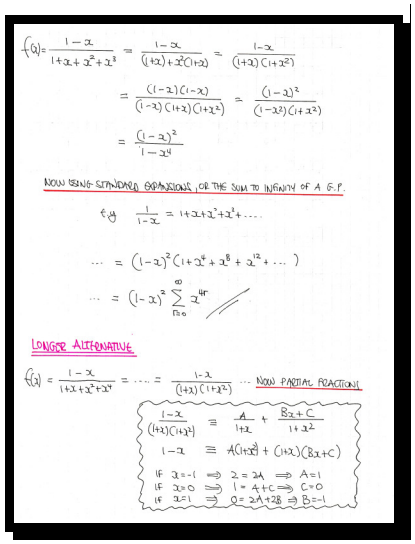
$$f(x)\equiv\frac{1-x}{1+x+x^2+x^3},\ -1<x<1.$$

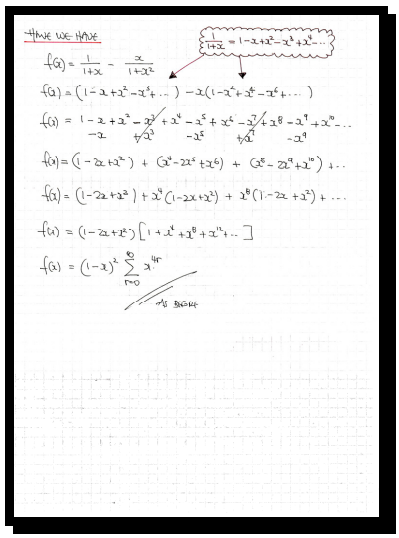
Show that $f(x)$ can be written in the form

$$f(x)=g(x)\sum_{r=0}^{\infty}\left(x^{4r}\right),$$

where $g(x)$ is a simplified function to be found.

$\square$, $\boxed{g(x)=(1-x)^2}$



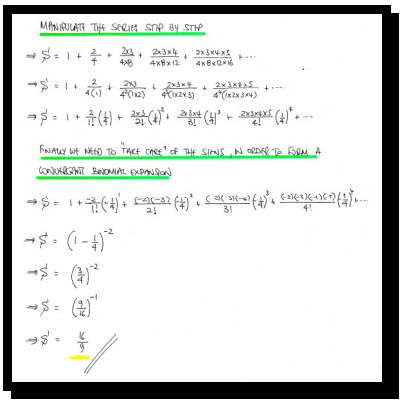


Question 78 (*****)

$$S=1+\frac{2}{4}+\frac{2\cdot 3}{4\cdot 8}+\frac{2\cdot 3\cdot 4}{4\cdot 8\cdot 12}+\frac{2\cdot 3\cdot 4\cdot 5}{4\cdot 8\cdot 12\cdot 16}+...$$

By considering a suitable binomial series, or other wise, find the sum to infinity of S.

V, , $\boxed{S_{\infty}=\frac{16}{9}}$

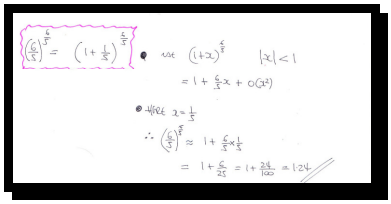


Question 79 (*****)

Without the use of any calculating aid and by showing full workings, show that

$$\left(\frac{6}{5}\right)^{\frac{6}{5}}\approx 1.24\,.$$

proof

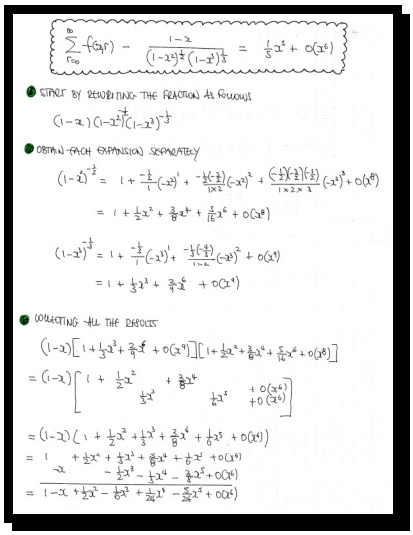


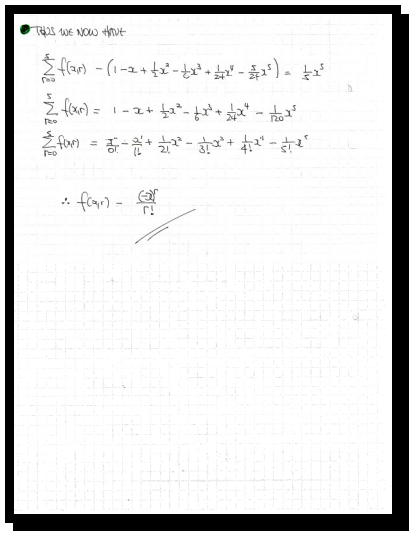
Question 80 (*****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x,r) \; - \; \frac{1-x}{\sqrt{1-x^2}\sqrt[3]{1-x^3}}, \; -1<x<1.$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x,r)$.

$\square$, $\boxed{f(x,r)=\dfrac{(-x)^r}{r!}}$



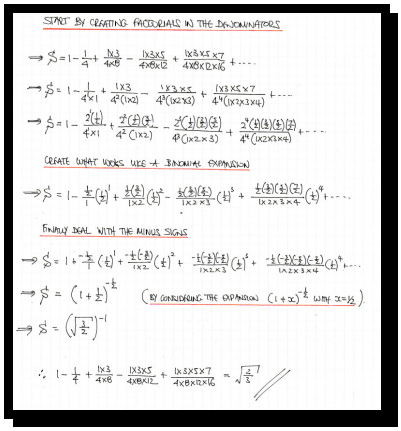


Question 81 **(*****)**

$$S=1-\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}-\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16}-\ldots$$

Find the sum to infinity of S, by considering the binomial series expansion of $(1+x)^n$ for suitable values of x and n.

$\square$, $\boxed{S_\infty=\sqrt{\frac{2}{3}}}$

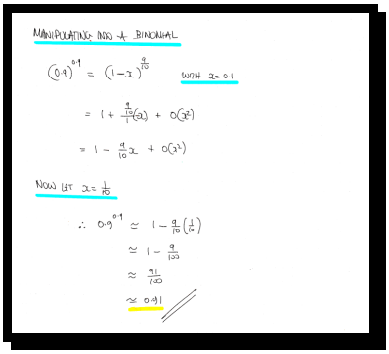


Question 82 **(*****)**

Without the use of any calculating aid and by showing full workings, show that

$$(0.9)^{0.9}\approx 0.91.$$

$\square$, proof



Question 83 **(*****)**

$$f(x)=\frac{1}{\sqrt{1-x}},\ -1<x<1.$$

a) By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x)=\sum_{r=0}^{\infty}\binom{2r}{r}\left(\frac{1}{4}x\right)^r.$$

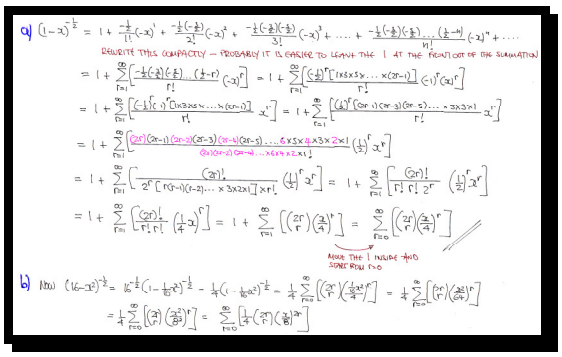
b) Find a similar expression for $\dfrac{1}{\sqrt{16-x^2}}$ and show further that

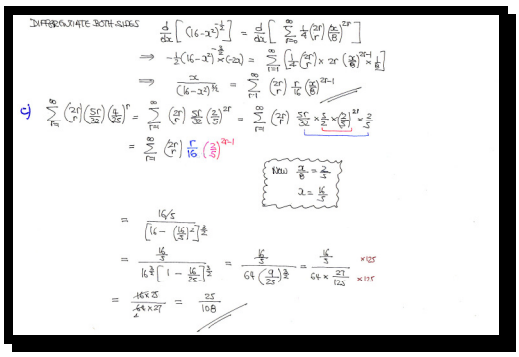
$$\frac{x}{\left(16-x^2\right)^{\frac{3}{2}}}=\sum_{r=1}^{\infty}\binom{2r}{r}\left(\frac{1}{16}r\right)\left(\frac{1}{8}x\right)^{2r-1}.$$

c) Determine the exact value of

$$\sum_{r=1}^{\infty}\binom{2r}{r}\left(\frac{5}{32}r\right)\left(\frac{4}{25}\right)^r.$$

$\square$, $\boxed{\dfrac{25}{108}}$





Question 84 **(*****)**

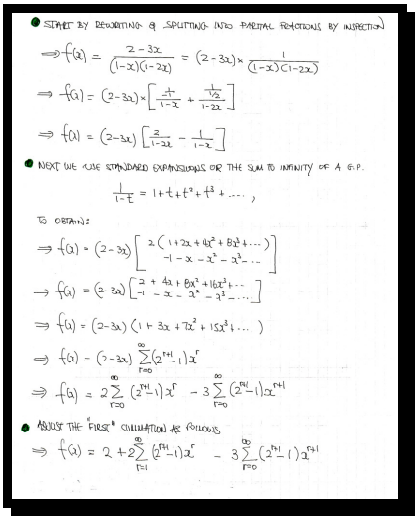
$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, \ -\frac{1}{2} < x < \frac{1}{2}.$$

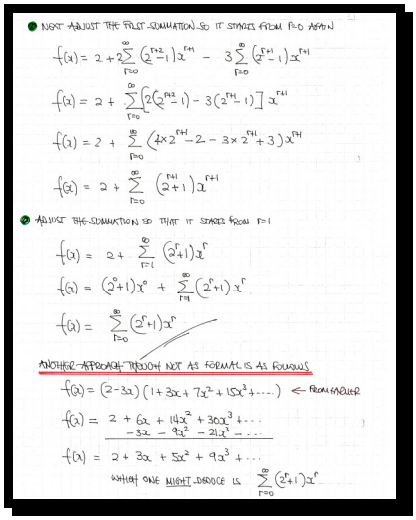
Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty} \left[x^r \, g(r) \right],$$

where $g(r)$ is a simplified function to be found.

$\square$, $\boxed{g(r) = 2^r + 1}$



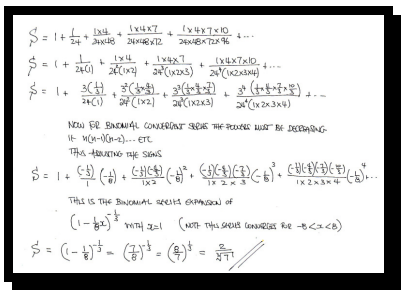


Question 85 (*****)

Show by considering a suitable binomial expansion that

$$1+\frac{1}{24}+\frac{1\cdot 4}{24\cdot 48}+\frac{1\cdot 4\cdot 7}{24\cdot 48\cdot 72}+\frac{1\cdot 4\cdot 7\cdot 10}{24\cdot 48\cdot 72\cdot 96}-\ldots=\frac{2}{\sqrt[3]{7}}.$$

V, [ ], proof

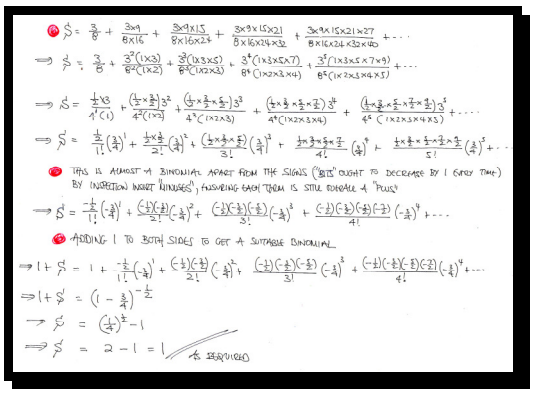


Question 86 (*****)

$$S=\frac{3}{8}+\frac{3\times 9}{8\times 16}+\frac{3\times 9\times 15}{8\times 16\times 24}+\frac{3\times 9\times 15\times 21}{8\times 16\times 24\times 32}+\frac{3\times 9\times 15\times 21\times 27}{8\times 16\times 24\times 32\times 40}\ldots$$

By considering a suitable binomial expansion, show that $S=1$.

[ ], proof



Question 87 **(*****)**

The function f is defined in terms of the real constants, a, b and c, by

$$f(x)=\left(a+bx+cx^2\right)(1-x)^{-3}, \quad x\in\mathbb{R}, \quad |x|<1.$$

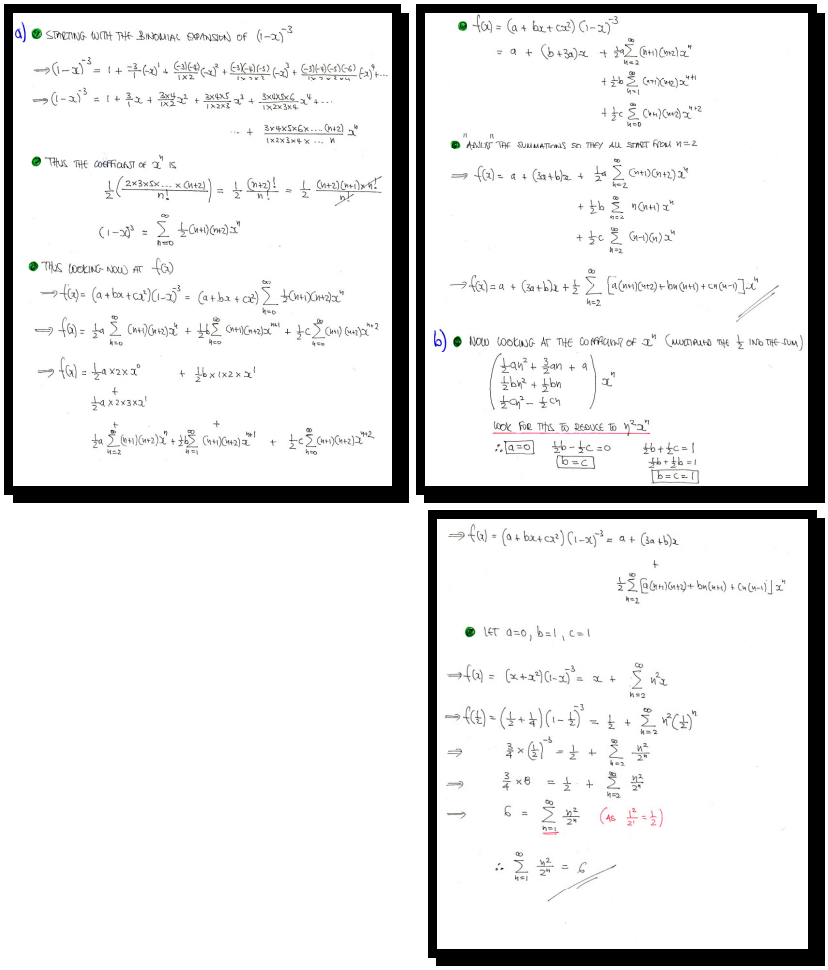
a) Show that

$$f(x)=a+(3a+b)x+\frac{1}{2}\sum_{n=2}^{\infty}\left[\left[a(n+1)(n+2)+bn(n+1)+cn(n-1)\right]x^n\right].$$

b) Use the expression of part **(a)** to deduce the value of

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}.$$

$\boxed{}$, 6



Question 88 (*****)

The first three terms of a series S are

$$S=7+9x+8x^2+...$$

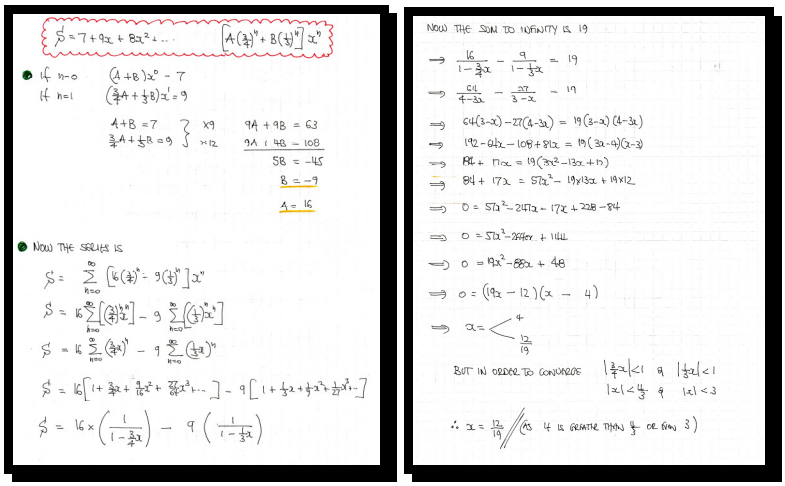
The n^{th} term of S is given by

$$A\left(\frac{3}{4}x\right)^n+B\left(\frac{1}{3}x\right)^n,$$

where A and B are non zero constants.

Given that the sum to infinity of S is 19, determine the value of x.

$\square$, $\boxed{x=\frac{12}{19}}$



Question 89 **(*****)**

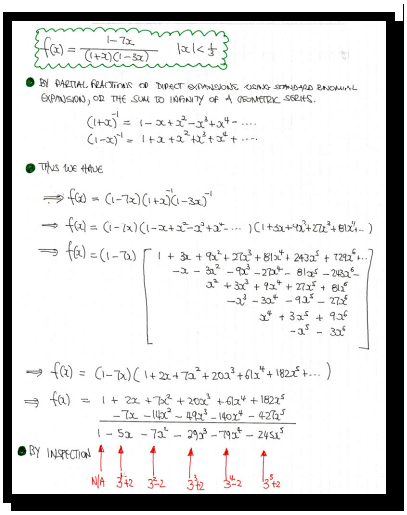
$$f(x)\equiv\frac{1-7x}{(1+x)(1-3x)},\ -\frac{1}{3}<x<\frac{1}{3}.$$

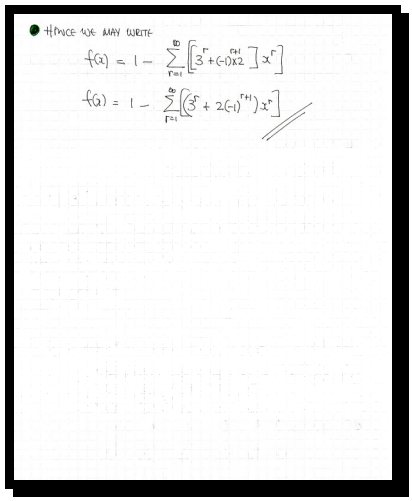
Show that $f(x)$ can be written in the form

$$f(x)=1-\sum_{r=1}^{\infty}\left[x^r\,g(r)\right],$$

where $g(r)$ is a simplified function to be found.

$\square$, $\boxed{g(r)=3^r+2\times(-1)^{r+1}}$





Question 90 (*****)

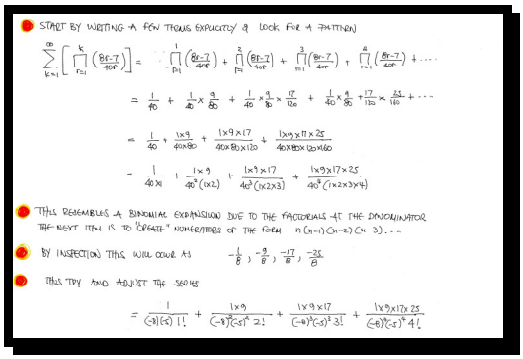
The product operator $\prod$, is defined as

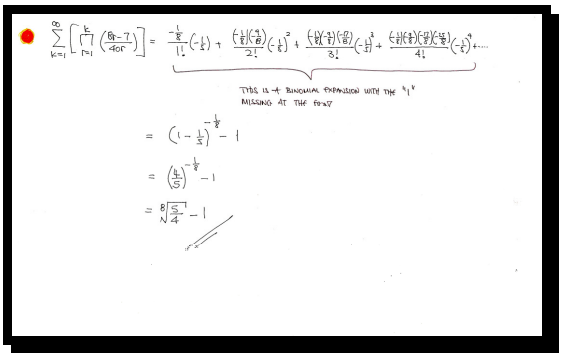
$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k \, .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

$$\square \, , \quad \sqrt[8]{\frac{5}{4}}-1$$





Question 91 (*****)

It is given that for $x \in \mathbb{R}$, $-\dfrac{1}{k} < x < \dfrac{1}{k}$, $k > 0$,

$$f(x,k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

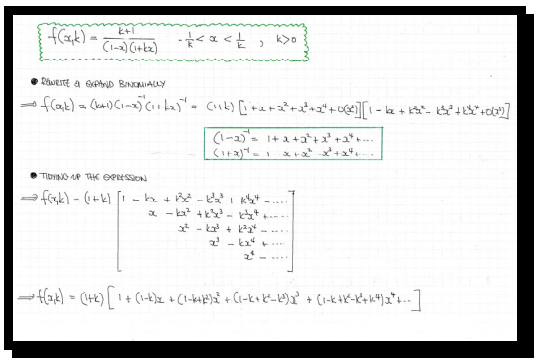
$$f(x,k) \equiv \sum_{r=0}^{\infty}\left[a_r x^r\right],$$

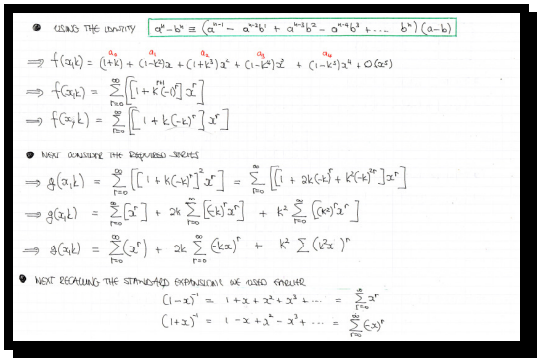
where a_r are functions of k, show that

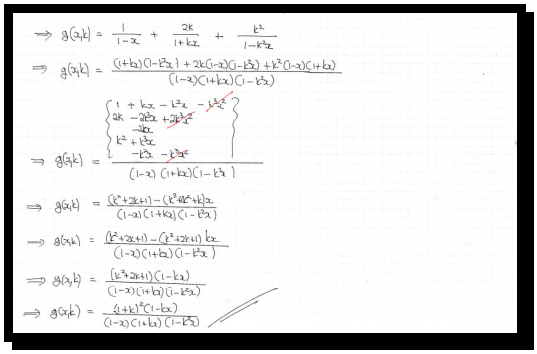
$$\sum_{r=0}^{\infty}\left[a_r{}^2 x^r\right] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)\left(1-k^2x\right)}.$$

You may assume that $\displaystyle\sum_{r=0}^{\infty}\left[a_r{}^2 x^r\right]$ converges.

, proof







Question 92 (*****)

Consider the following infinite series, S.

$$S=\frac{5}{18}-\frac{5\times 8}{18\times 24}+\frac{5\times 8\times 11}{18\times 24\times 30}-\frac{5\times 8\times 11\times 14}{18\times 24\times 30\times 36}+\frac{5\times 8\times 11\times 14\times 17}{18\times 24\times 30\times 36\times 42}-\ldots$$

Given that S converges, show that

$$S=9A-41,$$

where A is an exact simplified surd.

$\square$, $\boxed{A=\sqrt[3]{96}}$

