POLAR CORDINATES
and
CENTRAL FORCES

Question 1 (**)

A particle P is moving on a cardioid with polar equation

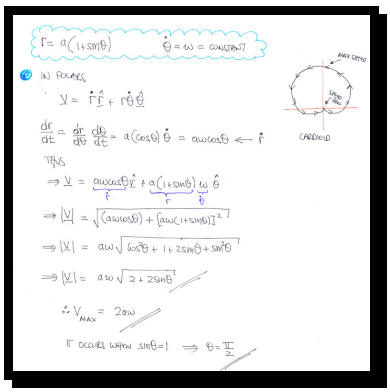
$$r = a(1+\sin\theta),\ 0 \le \theta < 2\pi,$$

where a is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Find an expression for the speed of P in terms of a, ω and θ, and hence determine the maximum speed of the speed of P and the value of θ when this maximum speed occurs.

$$\boxed{|\mathbf{v}| = a\omega\sqrt{2+2\sin\theta}},\ \boxed{|\mathbf{v}|_{\max} = 2a\omega},\ \boxed{\theta = \frac{\pi}{2}}$$



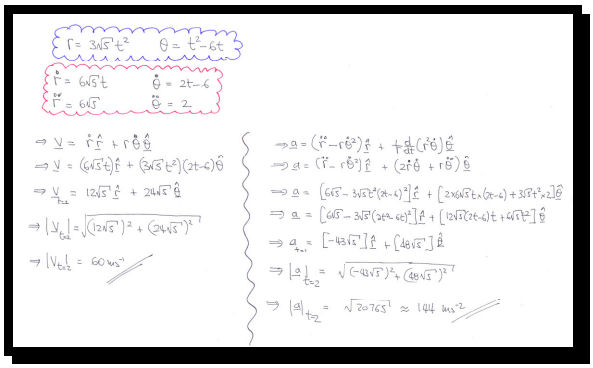
Question 2 (**)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), by the parametric equations

$$r=3\sqrt{5}t^2\,,\quad \theta=t^2-6t\,,\quad t\geq 0\,.$$

Determine the speed of P and the magnitude of its acceleration when $t=2$.

$$\boxed{\left|\mathbf{v}\right|_{t=2}=60\text{ ms}^{-1}}\,,\quad \boxed{\left|\mathbf{a}\right|_{t=2}=\sqrt{20765}\approx 144\text{ ms}^{-2}}$$



Question 3 (**)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

The path of P traces the spiral with polar equation

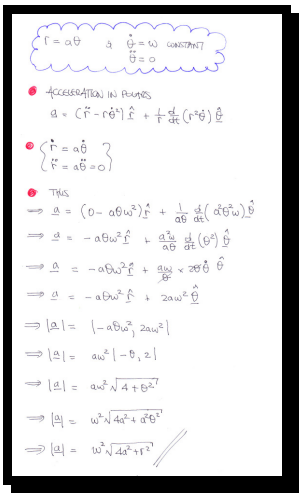
$$r = a\theta,$$

where a is a positive constant.

The radius vector OP rotates with constant angular speed ω.

Determine a simplified expression for the magnitude of the acceleration of P in terms of a, ω and r.

$$\boxed{|\mathbf{a}| = \omega^2\sqrt{4a^2 + r^2}}$$



Question 4 (**)

A particle P is moving on a cardioid with polar equation

$$r = a(1-\sin\theta),\ 0 \le \theta < 2\pi,$$

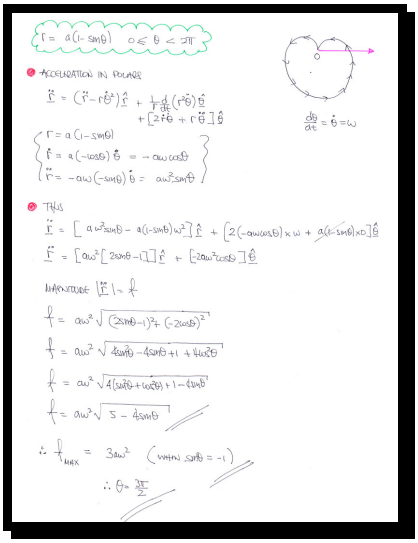
where a is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

The magnitude of the acceleration of P is denoted by f.

Find an expression for f in terms of a, ω and θ, and hence state the greatest value of f and the value of θ when this greatest value of f occurs.

$$\boxed{f = a\omega^2\sqrt{5-4\sin\theta}},\ \boxed{f_{\max} = 3a\omega^2},\ \boxed{\theta = \frac{3\pi}{2}}$$



Question 5 (**)

A particle P is moving on the curve with polar equation

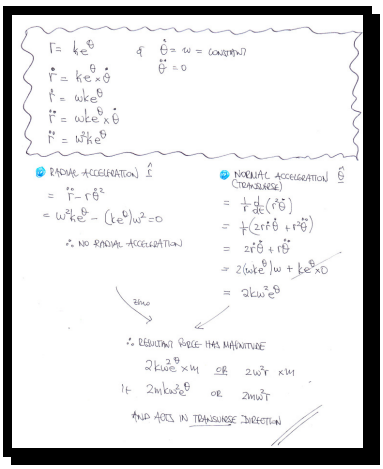
$$r = k\,\mathrm{e}^{\theta}\,,\ 0 \le \theta < 2\pi\,,$$

where k is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Find the magnitude and direction of the acceleration acting on P.

$$\boxed{|\mathbf{a}| = 2mk\omega^2\,\mathrm{e}^{\theta} = 2mr\omega^2, \quad \text{transversly}}$$



Question 6 **(**+)**

In a plane polar coordinate system (r,θ), the base unit vectors are defined as $\hat{\mathbf{r}}$ in the direction of r increasing, and $\hat{\boldsymbol{\theta}}$ perpendicular to $\hat{\mathbf{r}}$, in the direction of θ increasing.

a) Given that the position vector $\mathbf{r}$ of a particle P is given by $\mathbf{r}=r\hat{\mathbf{r}}$, derive expressions for the velocity and acceleration of P in plane polar coordinates.

You may assume standard differentiation results for $\hat{\mathbf{r}}$ *and* $\hat{\boldsymbol{\theta}}$.

b) If $r^2\dfrac{d\theta}{dt}$ is constant state what can be deduced about the force acting on P.

P is moving on the curve with polar equation

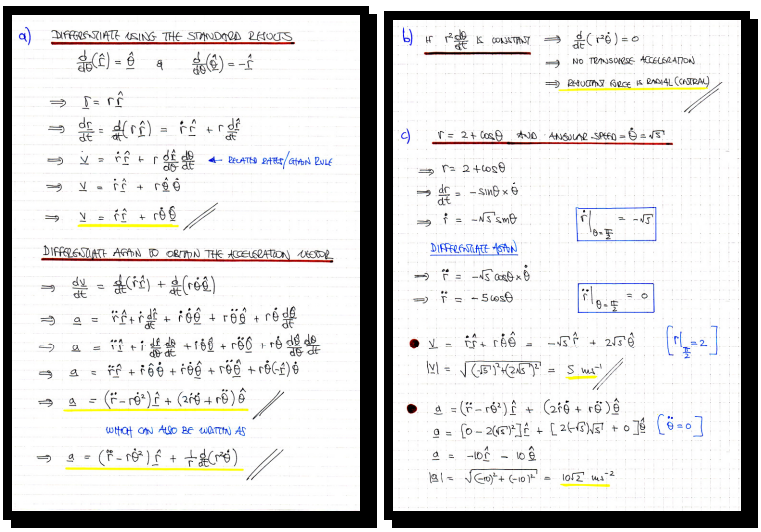
$$r=2+\cos\theta\,,\ 0\le\theta<2\pi\,,$$

with **constant** angular speed $\sqrt{5}$ rad s^{-1}.

c) Find the speed and the magnitude of the acceleration of P, when $\theta=\dfrac{\pi}{2}$.

$\boxed{}$, $\boxed{\mathbf{v}=\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}}$, $\boxed{\mathbf{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{\mathbf{r}}+\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\boldsymbol{\theta}}}$, $\boxed{\text{no transverse force}}$,

$\boxed{|\mathbf{v}|=5\text{ ms}^{-1}}$, $\boxed{|\mathbf{a}|=10\sqrt{2}\text{ ms}^{-2}}$



Question 7 (***)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

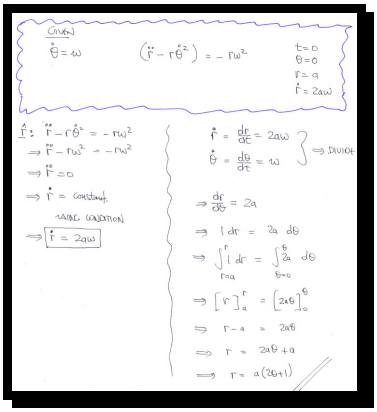
The radius vector OP rotates with constant angular speed ω.

The radial component of the acceleration of P has magnitude $r\omega^2$, and is directed towards O.

Initially, P is at the point with coordinates $(a,0)$, where a is a positive constant, and has radial velocity $2a\omega$.

Determine a polar equation for the path of P, in terms of a.

$$\boxed{r=a(2\theta+1)}$$



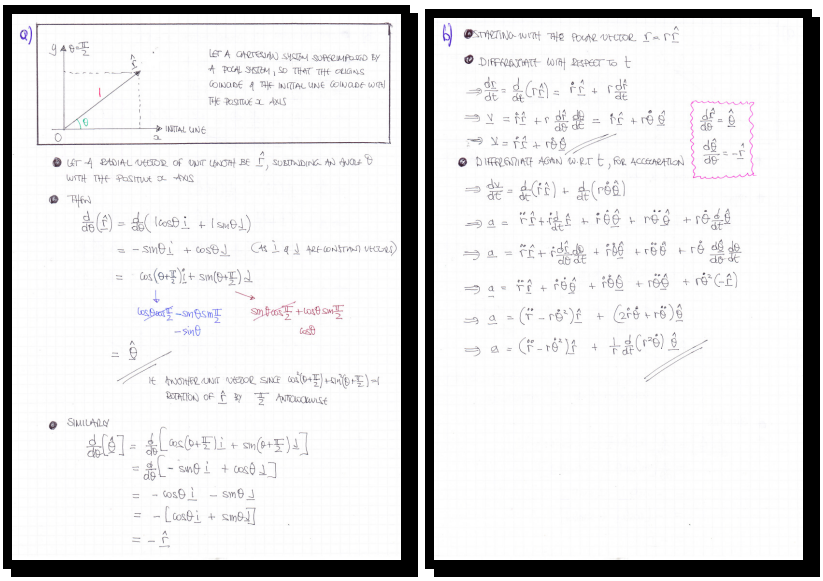
Question 8 (***)

In a plane polar coordinate system (r,θ), the base unit vectors are defined as $\hat{\mathbf{r}}$ in the direction of r increasing, and $\hat{\boldsymbol{\theta}}$ perpendicular to $\hat{\mathbf{r}}$, in the direction of θ increasing.

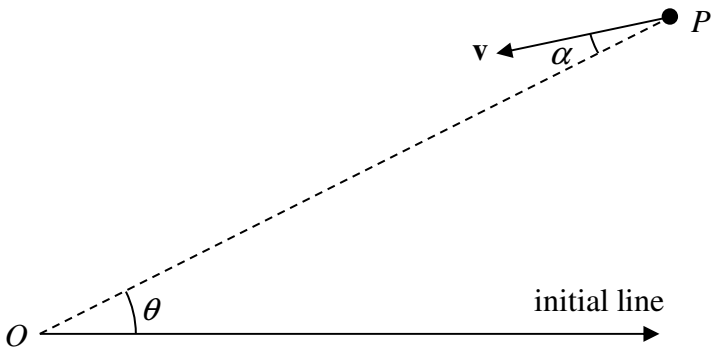
a) Find expressions for $\dfrac{d}{d\theta}(\hat{\mathbf{r}})$ and $\dfrac{d}{d\theta}\left(\hat{\boldsymbol{\theta}}\right)$

b) Given that the position vector $\mathbf{r}$ of a particle P is given by $\mathbf{r}=r\hat{\mathbf{r}}$, derive expressions for the velocity and acceleration of P in plane polar coordinates.

$\boxed{\hat{\boldsymbol{\theta}}}$, $\boxed{-\hat{\mathbf{r}}}$, $\boxed{\mathbf{v}=\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}}$, $\boxed{\mathbf{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{\mathbf{r}}+\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\boldsymbol{\theta}}}$



Question 9 (***)



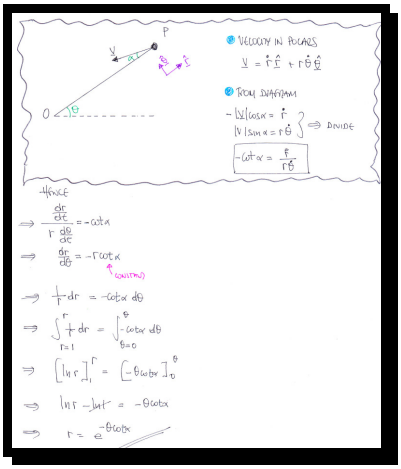
A particle P is moving on a polar plane (r,θ) so that its velocity vector $\mathbf{v}$ forms a constant angle α with OP, where O is the pole, as shown in the figure above.

Given further that P crosses the initial line at $r=1$, show that the polar equation of the path of P is

$$r=\mathrm{e}^{-\theta\cot\alpha}.$$

You may not use verification in this question.

proof



Question 10 (***)

A particle P, of mass m, is moving on a path with polar equation

$$r = a\mathrm{e}^{k\theta},\ 0 \le \theta < 2\pi,$$

where a and k are positive constants.

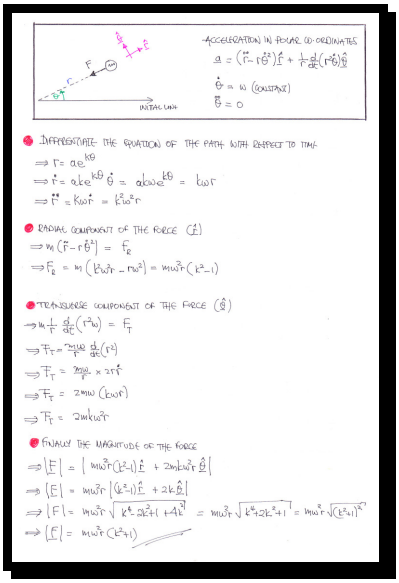
The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Show that the magnitude of the resultant force acting on the plane of its polar path is

$$m\omega^2 r\left(k^2+1\right),$$

where r is the distance OP.

proof



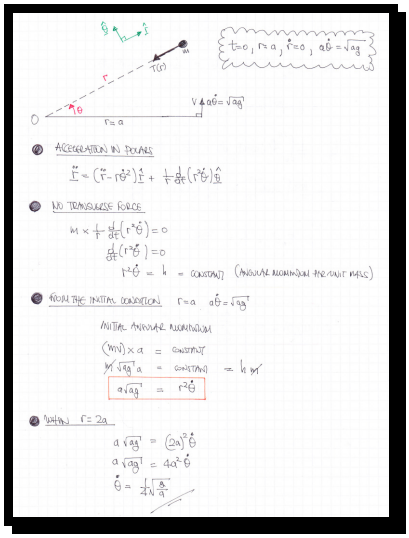
Question 11 (***)

A particle P rests on a smooth horizontal surface attached to a fixed point O on the surface by a light elastic string of natural length a.

When $|OP| = a$ the particle is projected with speed $\sqrt{ag}$ along the surface, in a direction perpendicular to OP.

Find the angular speed of P at the instant when $|OP| = 2a$.

$$\sqrt{\frac{g}{16a}}$$



Question 12 (***)

A particle P is moving on the curve with equation

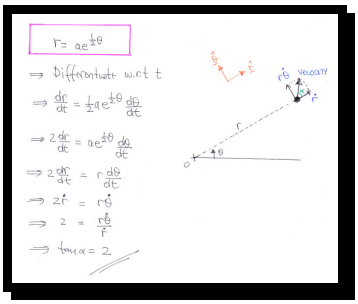
$$r = a\mathrm{e}^{\frac{1}{2}\theta},$$

where (r,θ) are plane polar coordinates, and a is a positive constant.

The angle the velocity of P makes with OP, where O is the pole, is denoted by α.

Determine the value of $\tan\alpha$.

$\boxed{\tan\alpha = 2}$



Question 13 (*+)**

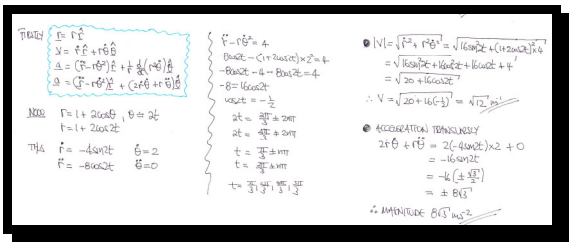
A particle is moving on path whose polar equation is

$$r = 1 + 2\cos\theta\,,\ 0 \le \theta < 2\pi\,.$$

The particle is moving in such a way so that $\theta = 2t$, where t represents the time in s, measured after a given instant. All distances are measure in m.

Determine the speed of the particle and the magnitude of its transverse acceleration when its radial acceleration is $4\ \text{ms}^{-1}$.

$$\boxed{\sqrt{12}\ \text{ms}^{-1}}\,,\ \boxed{8\sqrt{3}\ \text{ms}^{-2}}$$



Question 14 (*+)**

At time $t=0$, a particle is on the initial line of a standard polar coordinate system (r,θ), and moving on a path with polar equation

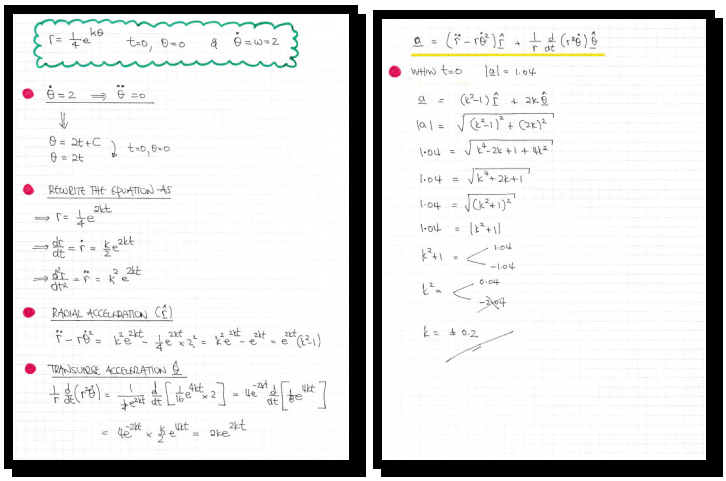
$$r=\frac{1}{4}\mathrm{e}^{k\theta},\theta\geq 0,$$

where k is a constant.

Relative to the pole O, the particle has a constant angular velocity of 2 rads^{-1}, throughout the motion.

Given that the initial magnitude of the acceleration of the particle is 1.04 ms^{-2}, determine the possible values of k.

$\square$, $\boxed{\pm\frac{1}{5}}$



Question 15 (*+)**

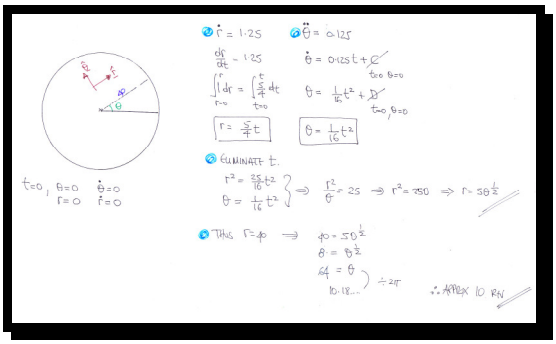
A man is standing at the centre at O of a circular platform, whose radius is 40 m, which is initially at rest.

At time $t=0$ the platform begins to rotate about O with constant angular acceleration of 0.125 rad s^{-1}, and at the same time the man begins to walk with constant speed 1.25 ms^{-1}, radially outwards relative to the platform.

Let r be the radial distance of the man from O and θ the angle by which the platform has turned.

Determine a polar equation for the path of the man, relative to the ground, in the form $r=f(\theta)$ and hence show that the platform has completed 10 revolutions by the time the man reaches the edge of the platform.

$$\boxed{r=5\theta^{\frac{1}{2}}}$$



Question 16 (****)

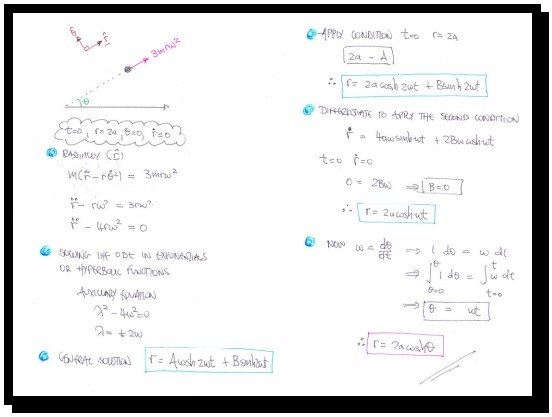
A particle of mass m is moving with constant angular velocity ω on a polar plane (r,θ), with pole at O. The only force acting on the particle has magnitude $3mr\omega^2$, which acts radially outwards.

When $t=0$, the particle is at the point $(2a,0)$, where a is a positive constant, and has no radial speed.

By forming and solving a suitable differential equation, show that the equation of the path of the particle is

$$r=2a\cosh\theta\,.$$

proof



Question 17 (****)

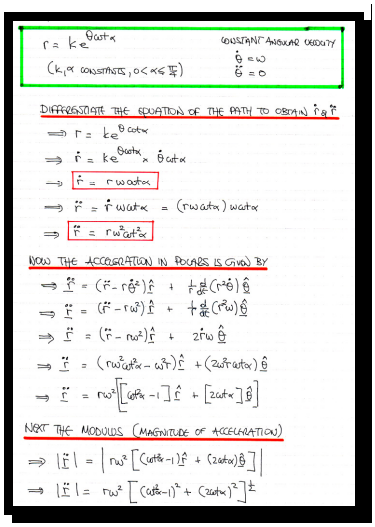
Relative to a fixed origin O, a particle P is moving with constant angular velocity ω on the curve with polar equation

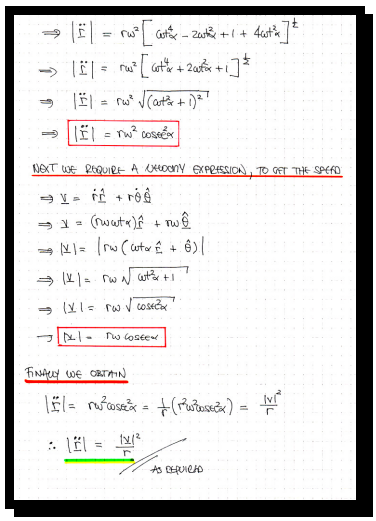
$$r = k\,\mathrm{e}^{\theta\cot\alpha},$$

where k and α are positive constants with $0<\alpha<\frac{1}{4}\pi$.

Show that the magnitude of the acceleration of the particle is $\dfrac{v^2}{r}$, where v is the speed of the particle and r is the distance OP.

, proof





Question 18 (****)

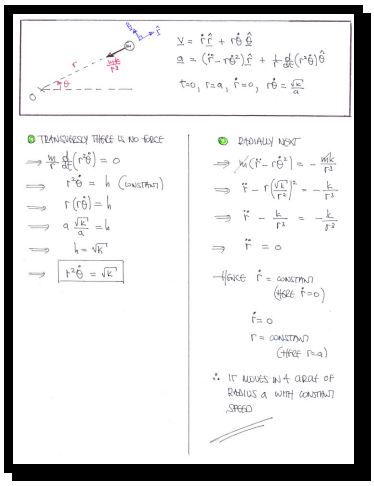
A particle P, of mass m, moves in a plane under the action of a force F which is directed towards a fixed origin O.

The magnitude of F is $\dfrac{mk}{r^3}$, where $r=|OP|$ and k is a positive constant.

Initially $r=a$ and the particle has speed $\dfrac{\sqrt{k}}{a}$ in a direction perpendicular to OP.

Use polar coordinates to describe the motion and path of P

moving on a circle of radius a with constant speed



Question 19 **(****)**

A particle P of mass m is moving on a polar plane (r,θ), with pole at O.

The path of P traces the spiral with polar equation

$$r = a\mathrm{e}^{k\theta},$$

where a and k are positive constants.

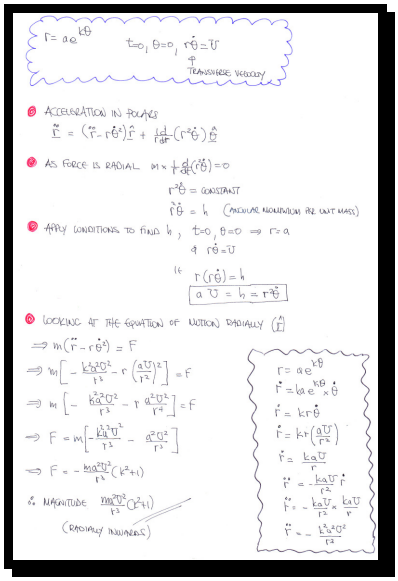
A variable force acts on P, acting in the radial direction with magnitude F.

Initially $\theta = 0$, and at that instant the transverse speed of P is U.

Show that

$$F = \frac{ma^2U^2}{r^3}\left(k^2+1\right).$$

proof



Question 20 (****)

A particle of mass 0.1 kg is attached to one end of a light elastic string and the other end is attached to a fixed point O on a smooth horizontal surface. The string has natural length 0.8 m and modulus of elasticity 61.74 N.

The string is then extended to 3.2 m and the particle is projected with speed u ms^{-1} at right angles to the string. In the subsequent motion, the polar coordinates of the particle relative to O are (r,θ).

a) Express $r^2\dot{\theta}$ in terms of u.

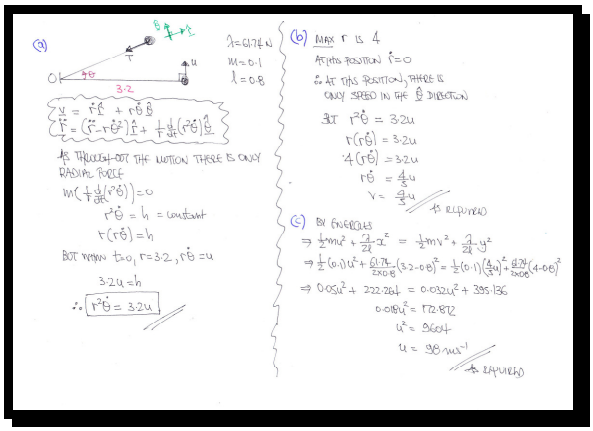
During the motion the maximum value of r is 4 m and at that position the particle has speed v ms^{-1}.

b) Show clearly that

$$v=\frac{4}{5}u\,.$$

c) By considering energies in two suitable positions, show that $v=98$

$$\boxed{r^2\dot{\theta}=\frac{16}{5}u=3.2u}$$



Question 21 (****)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

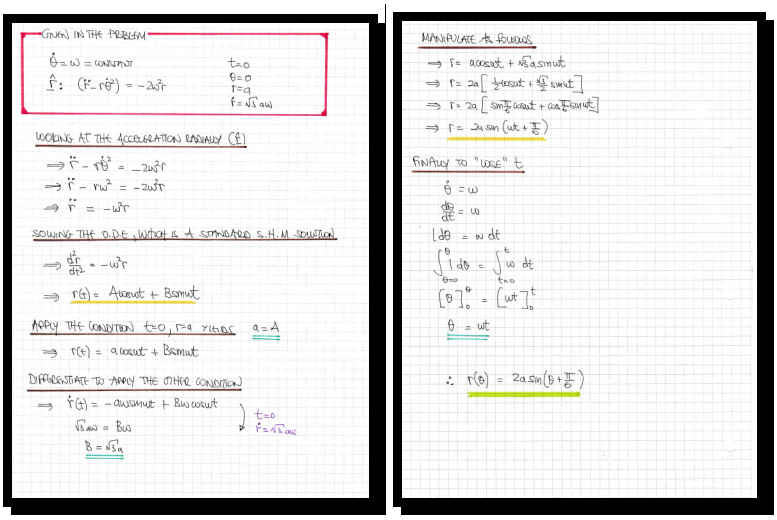
The radius vector OP rotates with constant angular speed ω.

The radial component of the acceleration of P has magnitude $2r\omega^2$, and is directed towards O.

Initially, P is at the point with coordinates $(a,0)$, where a is a positive constant, and has radial velocity $\sqrt{3}\,a\omega$.

Determine, in terms of a, a polar equation for the path of P.

$\boxed{}$, $\boxed{r=2a\sin\left(\theta+\frac{\pi}{6}\right)}$

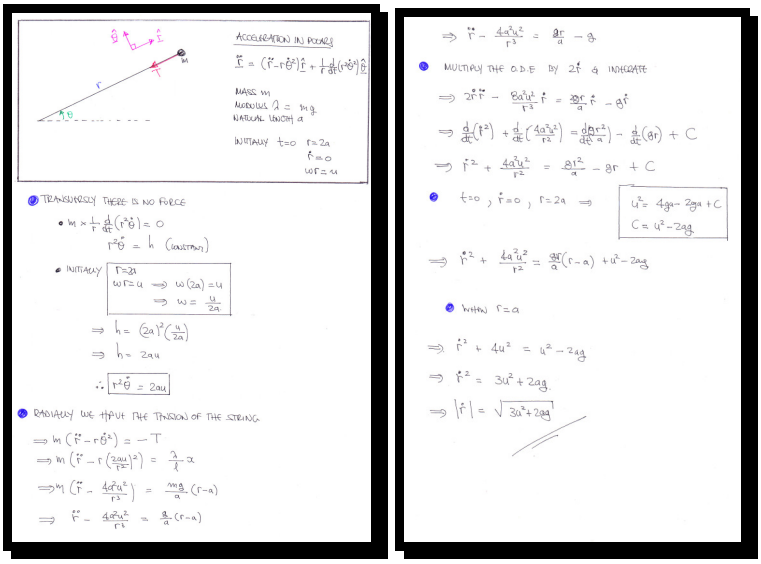


Question 22 (****)

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity mg. The other end of the string is attached to a fixed point O on a smooth horizontal surface. The particle is held in contact with the horizontal table so that $|OP| = 2a$ and projected with horizontal speed u in a direction perpendicular to OP.

Show that when $r = a$ and the radial speed of P is $\sqrt{3u^2 + 2ag}$.

proof



Question 23 (****)

A particle P of mass 0.45 kg is attached to another particle Q of mass 2 kg by a light inextensible string of length 1.2 m.

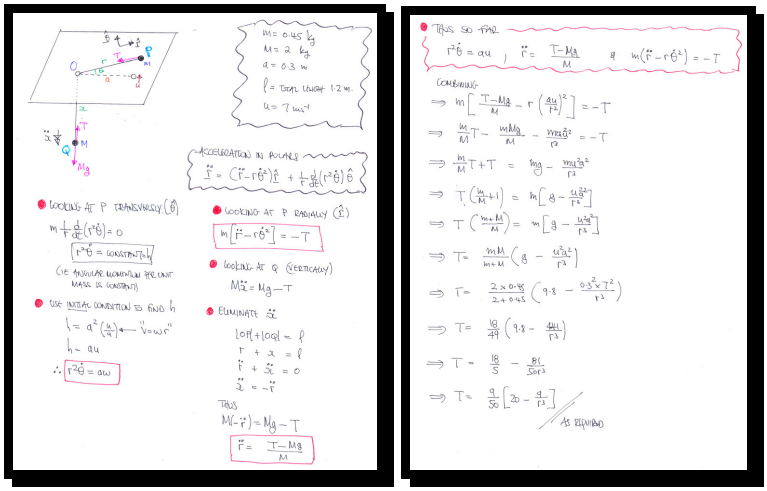
The string passes through a small smooth hole O on a smooth large table, so and P lies on the table and Q is hanging vertically below O.

When $|OP| = 0.3$ m, P is projected with horizontal speed 7 ms^{-1} at right angles to the taut string.

Show that when $|OP| = r$ m, the tension in the string T satisfies

$$T = \frac{9}{50}\left[20 - \frac{9}{r^3}\right].$$

proof



Question 24 (**+)**

A particle P of mass 0.5 kg is moving on the circle with equation

$$(x-1)^2+y^2=1.$$

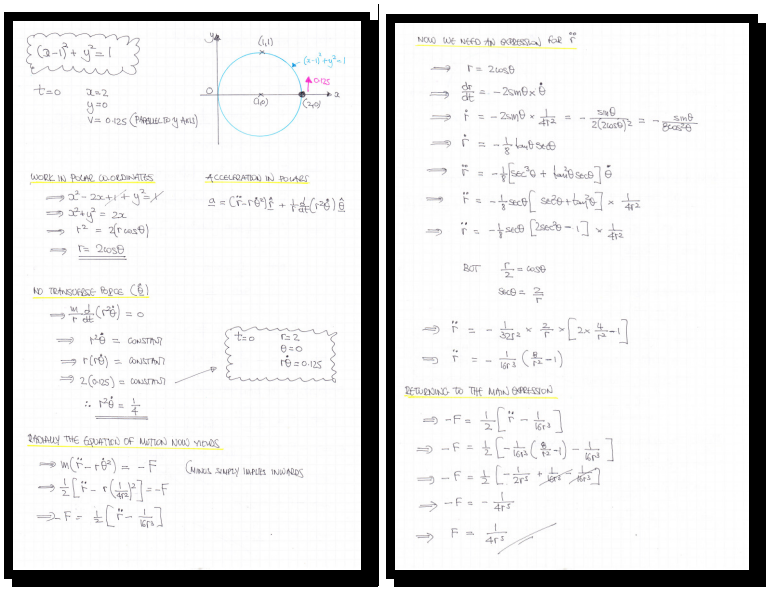
The particle is subject to a force of magnitude F, which always acts in the direction PO, where O is the origin.

The particle is observed passing through the point $(2,0)$ with speed 0.125 ms^{-1}, tangential to the circle and parallel to the y axis.

Show that if $|OP|=r$ m, then

$$F=\frac{1}{4r^5}.$$

proof



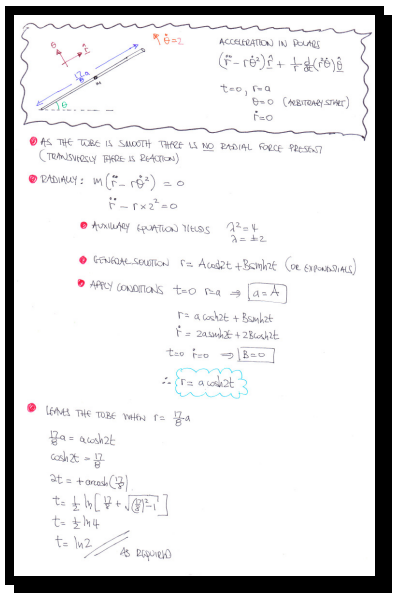
Question 25 (**+)**

A particle of mass m is placed inside a smooth tube OA of length $\frac{17}{8}a$. Initially the particle is at rest at a distance a from O

The tube is made to rotate with constant angular velocity ω, in a horizontal plane through a vertical axis passing through O. The particle reaches A in time T.

Show that $T = \ln 2$.

proof



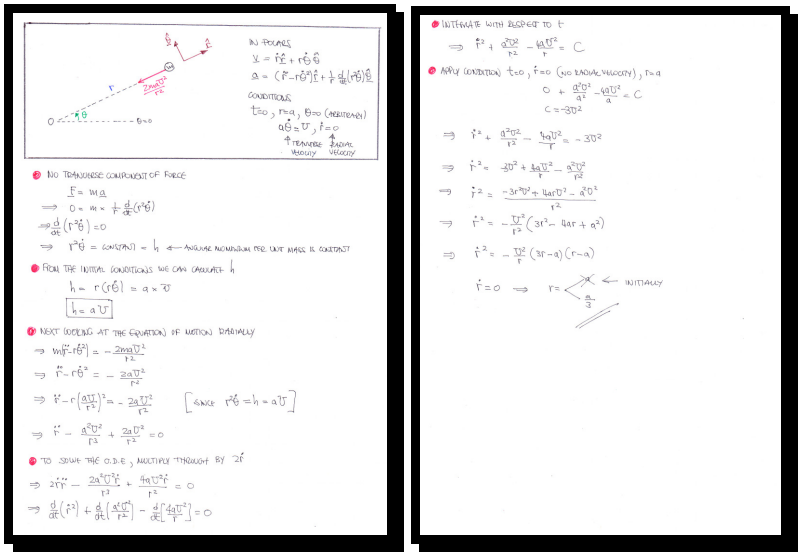
Question 26 (**+)**

A particle P, of mass m, is moving on a plane passing though a fixed origin O under the action of a force F, which acts radially in the direction PO.

The distance PO at time t s is denoted by r. At time $t=0$, $r=a$ and the speed of P is U, pointing in a direction perpendicular to PO.

Given that $F=\dfrac{2maU^2}{r^2}$ determine the least value of r in the subsequent motion.

$$\boxed{r_{\min}=\frac{1}{3}a}$$



Question 27 (**+)**

A particle P, of mass m, is moving on a plane passing though a fixed origin O under the action of a force F, which acts radially in the direction PO. The distance PO at time t s is denoted by r. The path of P has polar equation

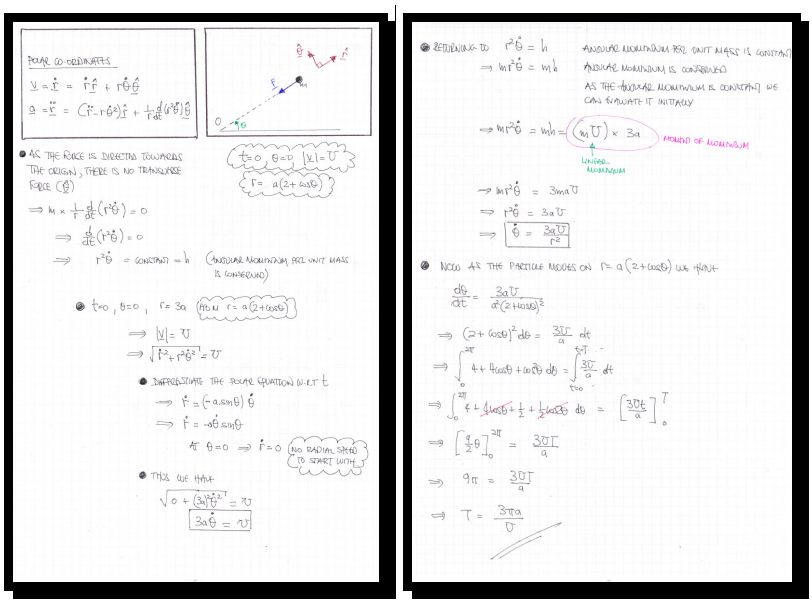
$$r = a\left(2+\cos\theta\right),$$

where a is a positive constant.

At time $t = 0$, $\theta = 0$ and the speed of P is U.

Find, in terms of π, a and U, the time it takes P to return to its starting position.

$$\boxed{t = \frac{3\pi a}{U}}$$



Question 28 (**+)**

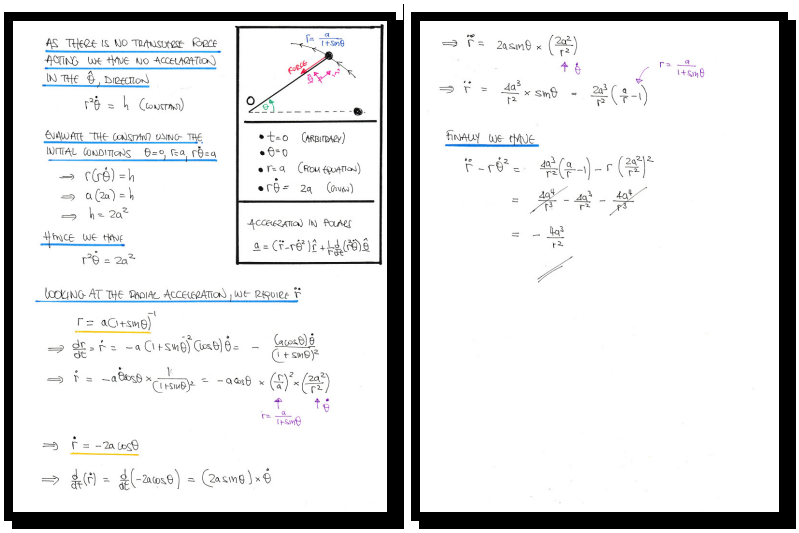
When a particle is on the initial line of a standard polar coordinate system (r,θ), it has transverse velocity a, where a is a positive constant.

The particle is moving on a path with polar equation

$$r=\frac{a}{1+\sin\theta},\ -\pi<\theta<\pi.$$

If the particle experiences a force, which directed towards the pole at all times, show that the radial acceleration of the particle is $-\dfrac{4a^3}{r^2}$.

, proof



Question 29 (**+)**

A circular rough wire of radius a and centre O is fixed with the plane of the wire in a horizontal position. A particle of mass m is threaded on the wire. The system lies in a field which exerts a vertical force on the particle in such a way so that the particle is weightless whilst inside the field. The coefficient of friction between the particle and the wire is μ.

When $t=0$, the particle is at the point with polar coordinates $(r,\theta)=(a,0)$, and is given an initial angular speed ω.

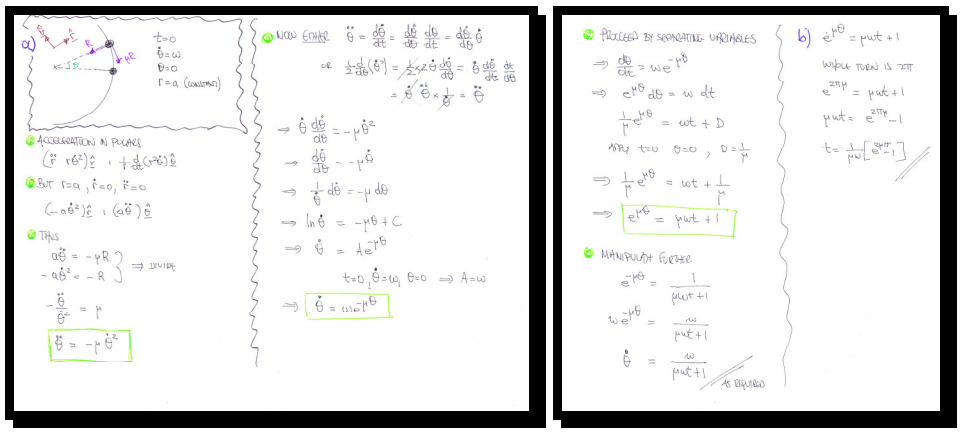
a) By forming and solving a suitable differential equation, show that the angular speed of the particle $\dfrac{d\theta}{dt}$, in time t satisfies

$$\frac{d\theta}{dt}=\frac{\omega}{\mu\omega t+1}.$$

b) Show further that the time T it takes the particle to complete its first revolution is given by

$$T=\frac{e^{2\mu\pi}-1}{\mu\omega}.$$

proof



Question 30 (**+)**

A particle P, of mass m, is moving around a fixed origin O under the action of a single force of magnitude $\dfrac{mk}{r^2}$, where k is a positive constant.

This force is always directed along PO towards O.

At time t the length OP is r and the angular velocity of P around O is $\dfrac{d\theta}{dt}$.

a) Show that if h is a positive constant

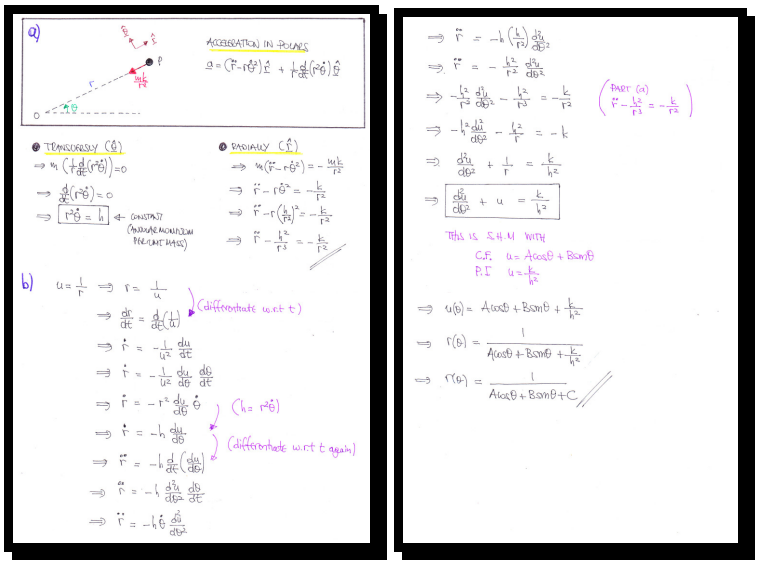
$$\frac{d^2r}{dt^2}-\frac{h^2}{r^3}=-\frac{k}{r^2}.$$

b) By using the substitution $u=\dfrac{1}{r}$ show further that

$$r=\frac{1}{A\cos\theta+B\sin\theta+C},$$

where A, B and C are constants.

proof



Question 31 (*****)

A particle of mass m is free to move on a smooth horizontal surface. The particle is attached to one end of a light elastic spring of natural length l and modulus of elasticity λ. The other end of the spring is attached to a fixed point O, on the surface.

The particle is held on the surface with the spring at its natural length and then is projected with speed U at right angles to the spring.

Ignoring air resistance and assuming that in standard S.I. units $m=1$, $l=1$, $9\lambda=8$ and $U=2$, determine the range of values of the length of the spring in the subsequent motion.

$$\boxed{1 \le \text{length} \le 3}$$

