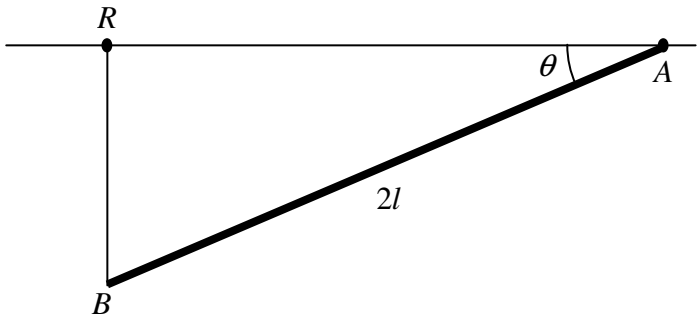
EQUILIBRIUM
and
POTENTIAL ENERGY

Question 1 (**)

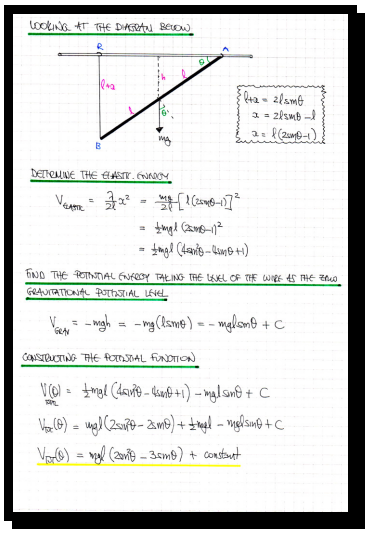


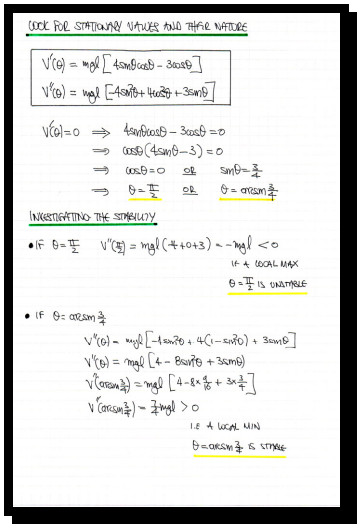
A small light smooth ring is attached to the end A of a uniform rod AB, of mass m and length $2l$. The ring is threaded on a smooth horizontal wire. Another small light smooth ring R is threaded on the same wire and a light elastic spring has one end attached to R and the other end attached to B. The system rest is equilibrium with B vertically below R, as shown in the figure above.

The angle BAR is denoted by θ, $0 \leq \theta \leq \frac{\pi}{2}$.

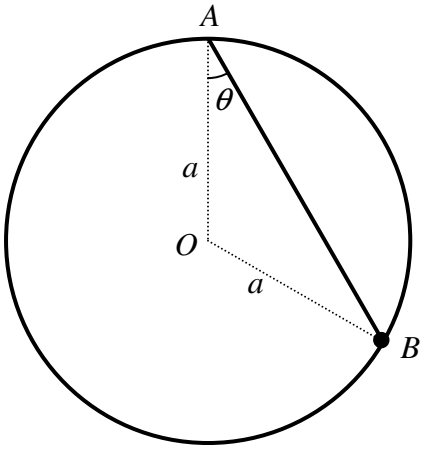
Find the two positions of equilibrium of the system and determine their stability.

$\boxed{}$, $\boxed{\theta = \frac{\pi}{2}, \text{ unstable}}$, $\boxed{\theta = \arcsin\left(\frac{3}{4}\right), \text{ stable}}$





Question 2 (+)**



The figure above shows a small bead B of mass m, threaded on a smooth circular wire of radius a, which is fixed in a vertical plane. The centre of the circle is at O, and the highest point of the circle is at A.

A light elastic string of natural length a and modulus of elasticity $3mg$ has one end attached to A and the other end attached to B.

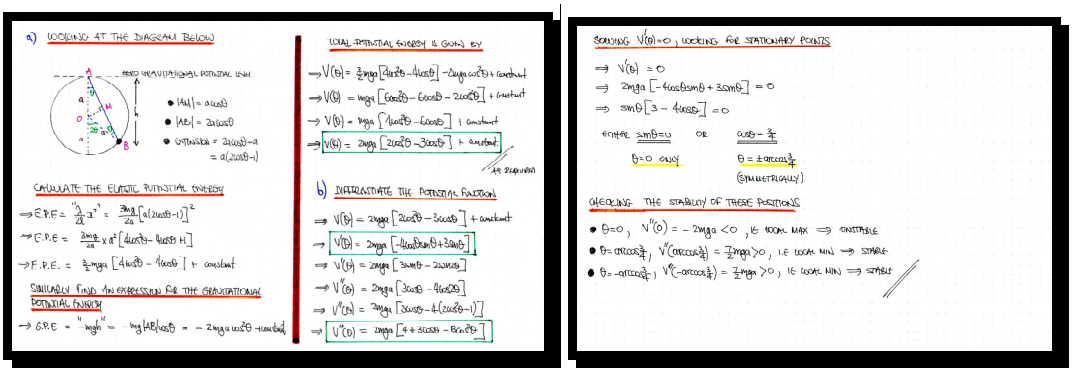
The acute angle OAB is denoted by θ.

a) Given that the string is taut, show that the potential energy of the system is

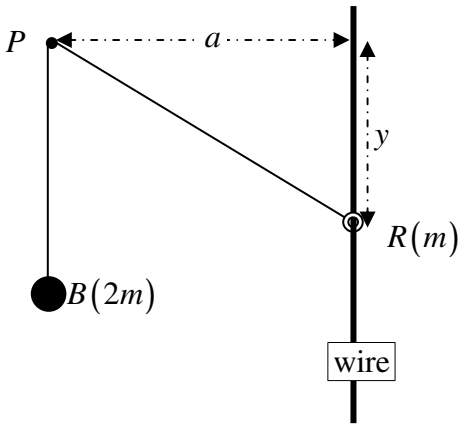
$$2mga\left(2\cos^2\theta-3\cos\theta\right)+\text{constant}\,.$$

b) Hence find the three positions of equilibrium of the system and determine their stability.

$\boxed{}$, $\boxed{\theta=0,\text{ unstable}}$, $\boxed{\theta=\arccos\left(\frac{3}{4}\right),\text{ stable}}$, $\boxed{\theta=-\arccos\left(\frac{3}{4}\right),\text{ stable}}$



Question 3 (***)



A ring R, of mass m, is threaded on a smooth wire which is securely taut in a vertical direction. The ring is connected to a particle B, of mass $2m$, by a light inextensible string which passes over a smooth peg P. The peg is at a distance a away from the wire.

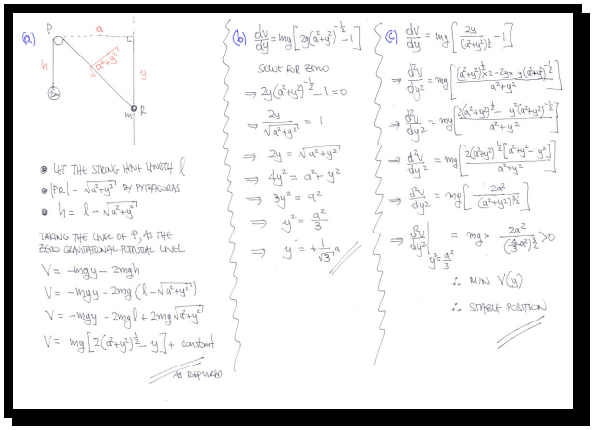
a) Given that y is the vertical distance of the R below the level of P, show that the total potential energy of the system V, is given by

$$V = mg\left[2\sqrt{a^2+y^2} - y\right] + \text{constant}\,.$$

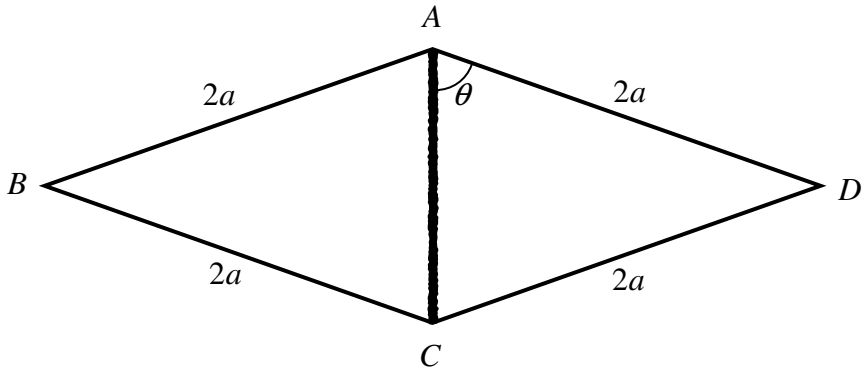
b) Find the value y, in terms of a for which the system is in equilibrium.

c) Determine the stability of this position of equilibrium.

$$\boxed{y = \frac{1}{\sqrt{3}}a,\ \text{stable}}$$



Question 4 (***)



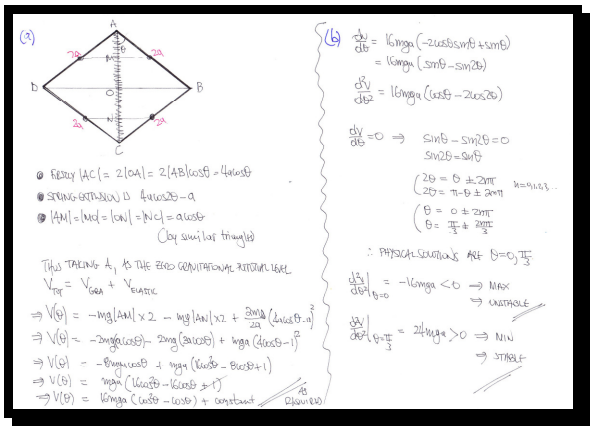
Four identical uniform rods each of mass m and length $2a$ are smoothly joined together to form a rhombus $ABCD$. The vertex of the rhombus is smoothly pinned at a fixed point A and a light elastic string connects A and C of the rhombus. The string has natural length a and modulus of elasticity $2mg$. The rhombus hangs in equilibrium with C vertically below A

a) Given that $\angle CAD = \theta$, show that the total potential energy of the system V, is given by

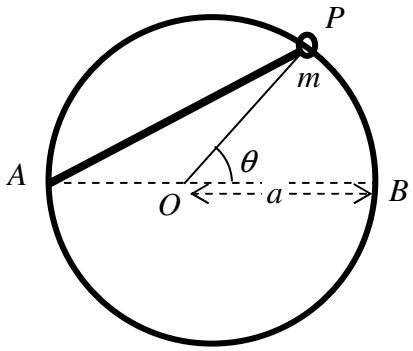
$$V = 16mga\left(\cos^2\theta - \cos\theta\right) + \text{constant}.$$

b) Use calculus to find the value or values of θ for which the system is in equilibrium, determining further the stability of these positions.

$$\boxed{\theta = 0,\ \text{unstable}}, \quad \boxed{\theta = \frac{\pi}{3},\ \text{stable}}$$



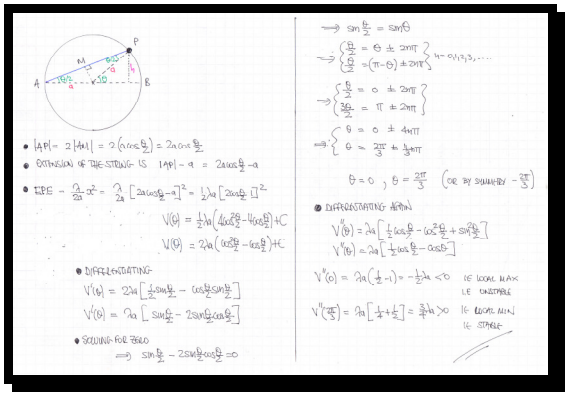
Question 5 (***)



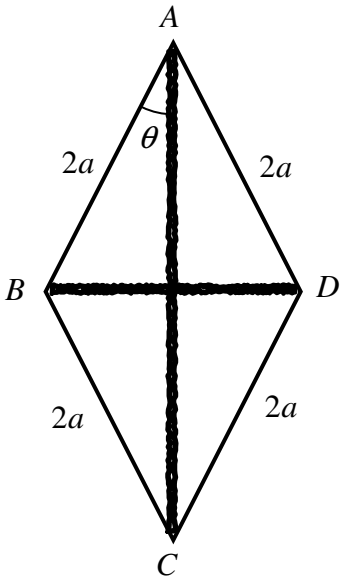
A small light ring P is free to slide on a smooth wire, bent into the shape of a circular hoop of radius a, centred at O. The wire is fixed in a vertical plane. The point B lies on the wire so that AOB is a horizontal diameter. A light elastic string has one end attached to A and the other end attached to P. The string has natural length a and modulus of elasticity λ.

Given that the angle POB is denoted by θ, find each of the values of θ, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 0°, \text{ unstable}}$, $\boxed{\theta = \pm 120°, \text{ stable}}$



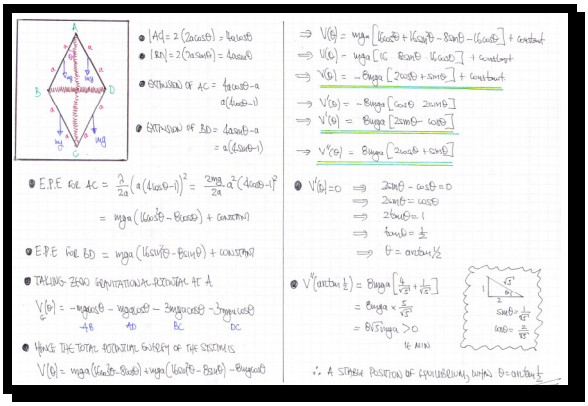
Question 6 (***)



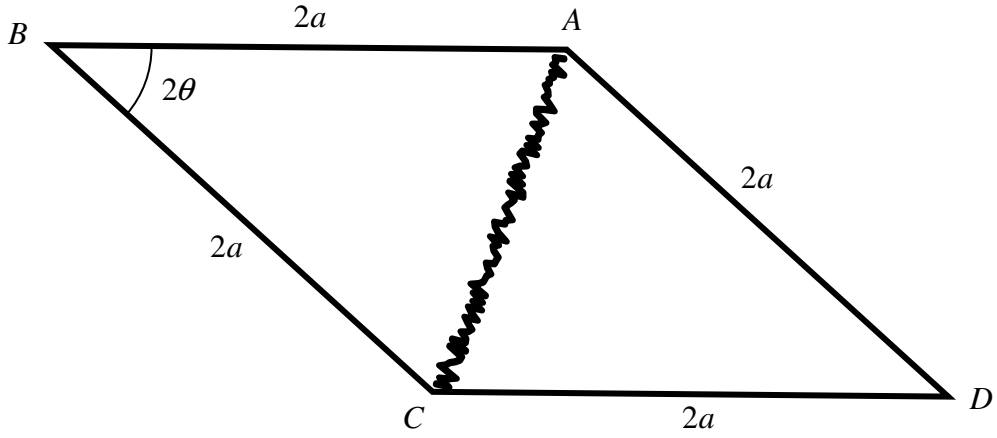
Four identical uniform rods each of mass m and length $2a$ are smoothly joined together to form a rhombus $ABCD$. The vertex of the rhombus is smoothly pinned at a fixed point A. A light elastic string connects A and C with an identical string connecting B to D. Each of the two strings has natural length a and modulus of elasticity $2mg$. The rhombus hangs in equilibrium with C vertically below A, as shown in the figure above.

Given that $\angle BAC = \theta$, use calculus to find the value or values of θ for which the system is in equilibrium, determining further the stability of these positions.

$$\boxed{\theta = \arctan\frac{1}{2},\ \text{stable}}$$



Question 7 (***)



Four identical uniform rods each of mass m and length $2a$ are smoothly joined together to form a rhombus $ABCD$.

The rod AB is fixed in a horizontal position and a light elastic string connects the joints A and C.

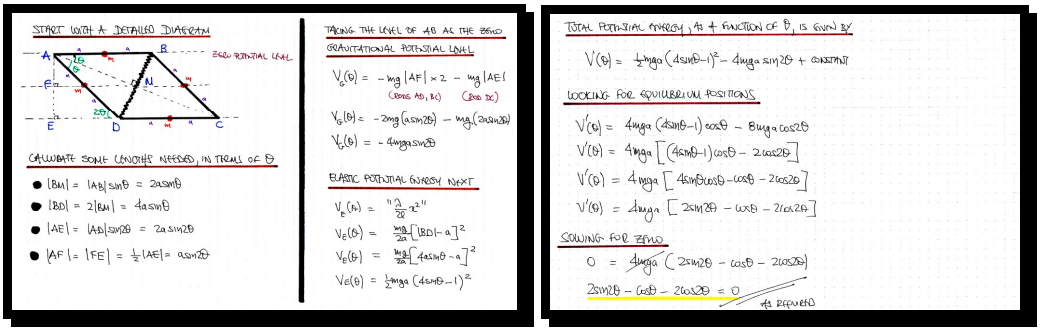
The string has natural length a and modulus of elasticity mg.

The rhombus hangs in equilibrium with C and D at the same horizontal level, vertically below AB.

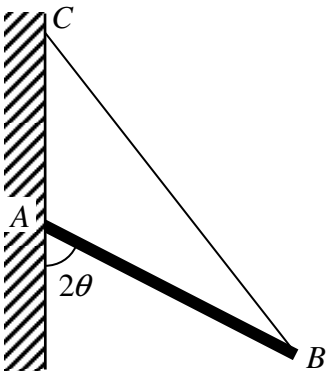
Given that $\angle ABC = 2\theta$, $0° < \theta < 45°$, show that the positions of equilibrium of the system are solutions of the equation

$$2\sin 2\theta - 2\cos 2\theta - \cos\theta = 0 .$$

[ ], proof



Question 8 (***)



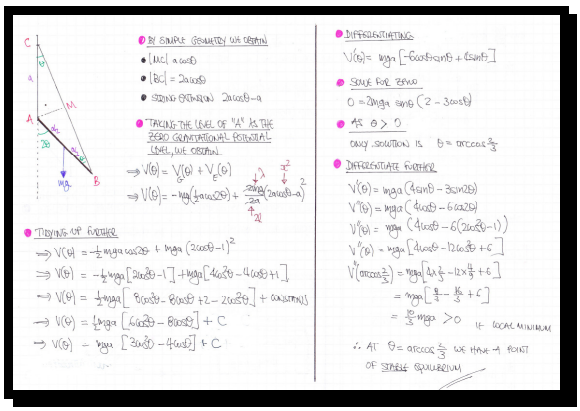
The figure above shows a uniform rod AB of length a metres and mass m, having its end A smoothly hinged against a vertical wall.

A light elastic string BC is attached to a point C on the wall which lies vertically above A, such that $|AC| = |AB|$.

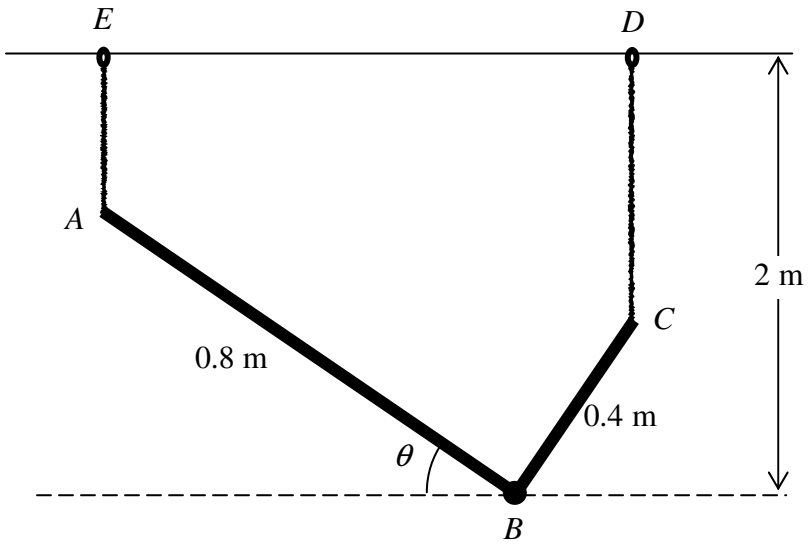
The plane ABC is perpendicular to the wall and the angle AB makes with the downward vertical is 2θ, where $\theta > 0$.

Given that the natural length of the string is a and its modulus of elasticity is $2mg$, find the value of θ when the rod is in equilibrium and determine its stability.

$$\boxed{\text{stable at } \theta = \arccos\frac{2}{3}}$$



Question 9 (***)



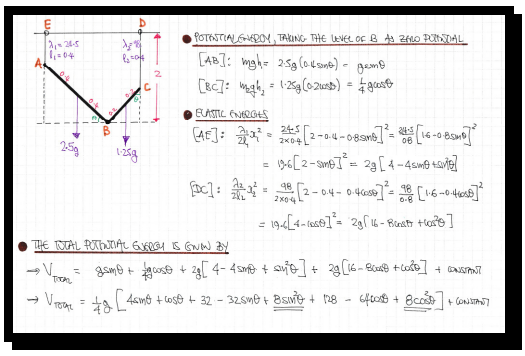
A uniform rod AB, of mass 2.5 kg and length 0.8 m, is rigidly joined to another uniform rod BC, of mass 1.25 kg and length 0.4 m, so that $\measuredangle ABC = 90°$. The rigid structure ABC is freely pivoted at B, so it can rotate in a vertical plane.

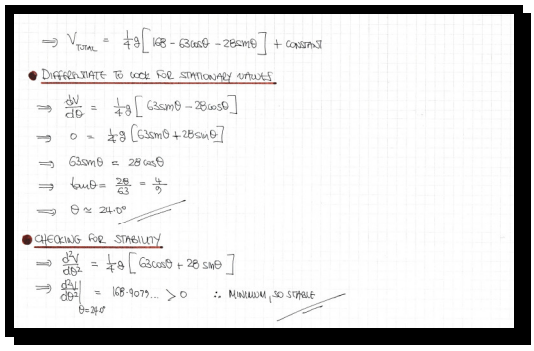
A light elastic string AE, of natural length 0.4 m and modulus of elasticity 24.5 N, and another light elastic string CD, of natural length 0.4 m and modulus of elasticity 98 N, are attached to the rigid structure at A and C, respectively. A smooth horizontal wire is threaded through at D and E, so that AE and CD remain vertical at all times, as shown in the figure above.

The vertical distance of B below the wire is 2 m and the acute angle AB makes with the horizontal is denoted by θ, $0 \le \theta \le 90°$.

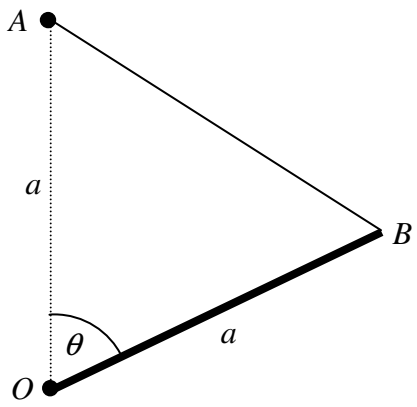
By considering potential energy, find the value of θ for which ABC is in equilibrium, and determine its stability.

$\boxed{}$, $\boxed{\theta \approx 24.0°, \text{ stable}}$





Question 10 (*+)**



The figure above shows a uniform rod OB of mass m and length a, freely hinged at O, so that the rod can rotate about O in a vertical plane.

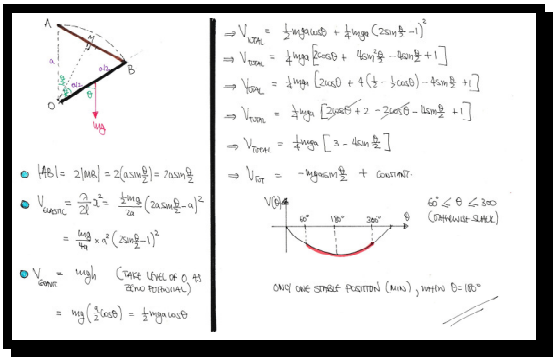
The point A is fixed at a distance a, vertically above O.

A light elastic string of natural length a and modulus of elasticity $\frac{1}{2}mg$ has one end attached to A and the other end attached to B.

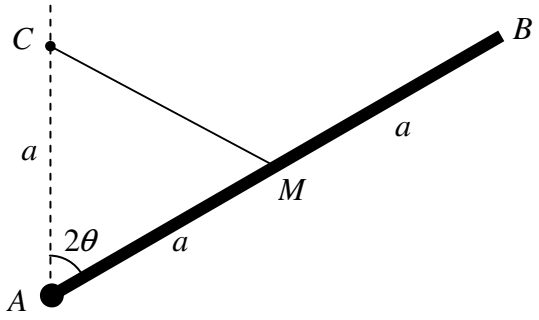
The acute angle OAB is denoted by θ.

By considering potential energy find the value of θ at any positions of equilibrium of the system and determine their stability.

$\boxed{}$, $\boxed{\theta = 180°, \text{ stable}}$



Question 11 (*+)**



A uniform rod AB, of mass m and length $2a$, is smoothly hinged to a fixed point A so it can rotate in a vertical plane. A light elastic string of natural length a and modulus of elasticity λ has one of its ends attached to the midpoint of the rod, M. The other end of the string is attached to a fixed point C which lies at a vertical distance a above A, as shown in the figure. The potential energy of the system is V.

a) Given that $\angle CAM = 2\theta$, with $0 \le \theta < \pi$, show that

i. … $V'(\theta) = 2a\cos\theta\left[2(\lambda - mg)\sin\theta - \lambda\right]$

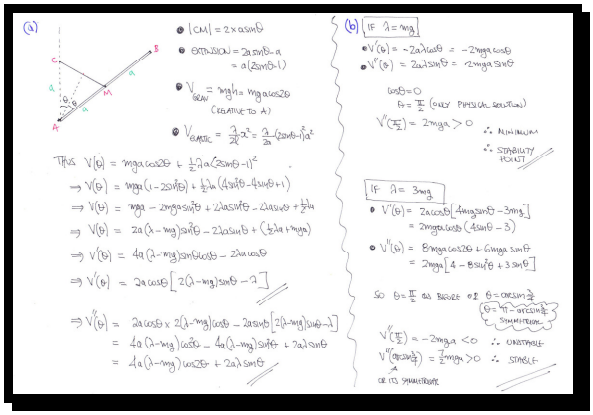
ii. … $V''(\theta) = 4a(\lambda - mg)\cos 2\theta + 2a\lambda\sin\theta$.

b) Find the values of θ for which the system is in equilibrium and investigate their stability in each of the cases

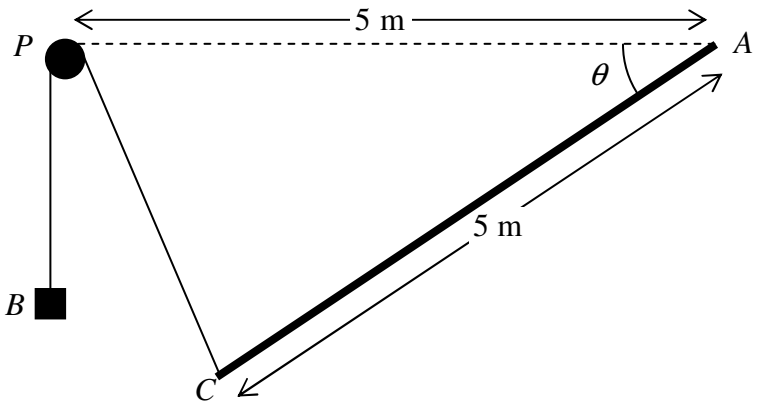
i. $\lambda = mg$.

ii. $\lambda = 3mg$.

$\boxed{\lambda = mg,\ \text{stable at } \theta = \frac{\pi}{2}}$, $\boxed{\lambda = 3mg,\ \text{unstable at } \theta = \frac{\pi}{2},\ \text{stable at } \theta = \arcsin\frac{3}{4}}$



Question 12 **(***+)**



The figure above shows a uniform rod AC, of length 5 m and mass 18 kg, with its end A smoothly hinged to a fixed point.

One end of a light inextensible string is attached to the other end of the rod B. The string passes over a small smooth pulley which is fixed at the point P, where AP is horizontal where $|AP| = 5$ m.

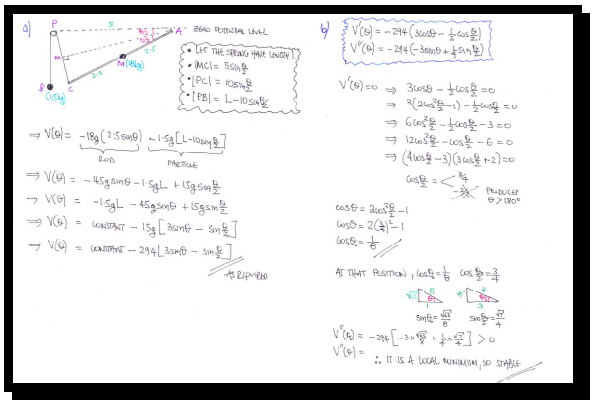
A small box B, of mass 1.5 kg is attached to the other end of the string and the particle hangs vertically below P. The acute angle PAC is denoted by θ.

a) Show that the potential energy of the system is

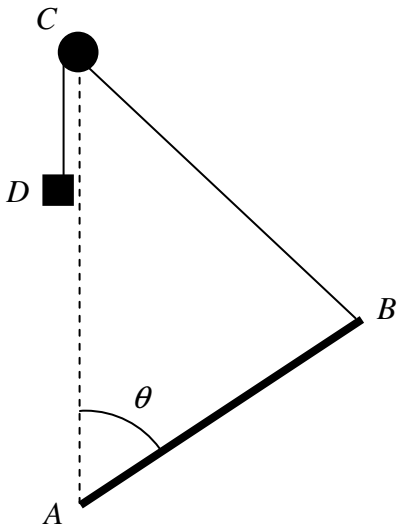
$$\text{constant} - 294\left[3\sin\theta - \sin\left(\tfrac{1}{2}\theta\right)\right].$$

b) Find the exact value of $\cos\theta$ when the system is in equilibrium and determine the stability of this equilibrium position.

$$\boxed{\cos\theta = \frac{1}{8}, \text{ stable}}$$



Question 13 **(***+)**



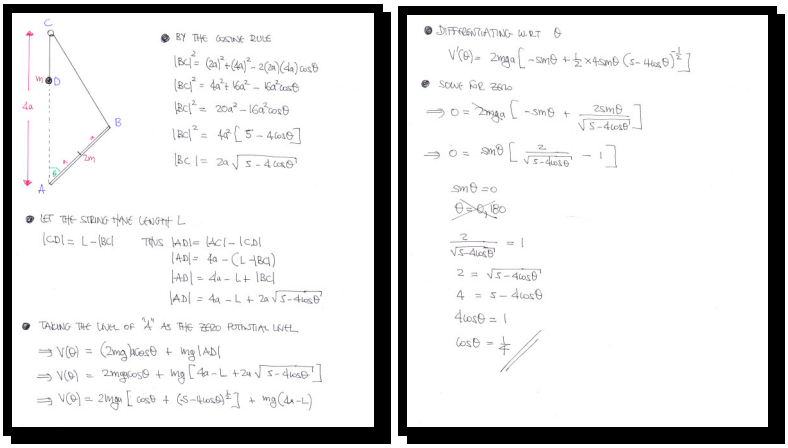
The figure above shows a uniform rod AB, of length $2a$ and mass m, with its end A smoothly hinged to a fixed point.

One end of a light inextensible string is attached to the other end of the rod B. The string passes over a small smooth pulley which is fixed at the point C, where AC is vertical with $|AC| = 4a$.

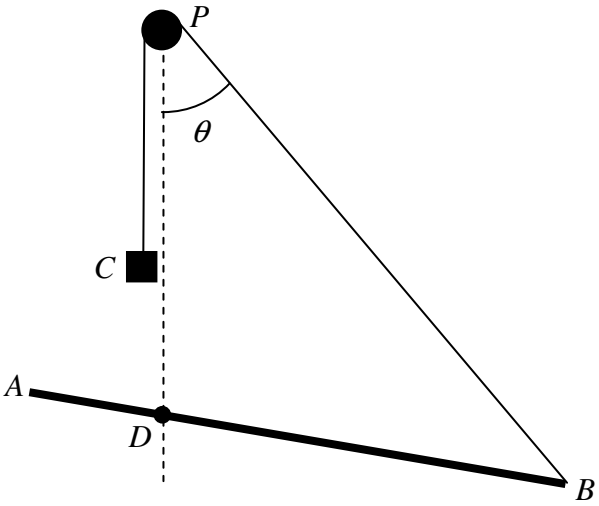
A particle D, of mass $2m$ is attached to the other end of the string and the particle hangs vertically below C. The acute angle CAB is denoted by θ, where $0 < \theta° < 180$.

Find the exact value of $\cos\theta$ when the system is in equilibrium.

$$\boxed{\cos\theta = \frac{1}{4}}$$



Question 14 **(***+)**

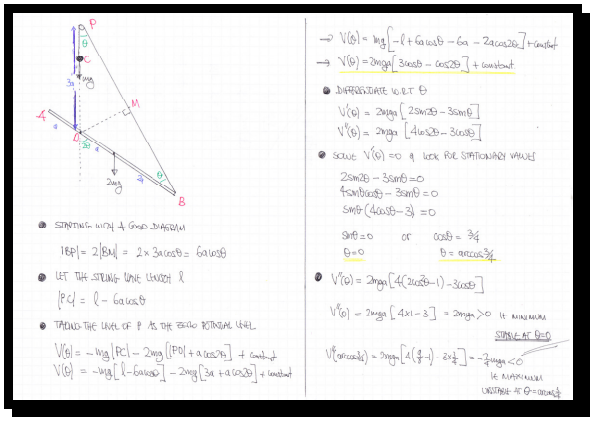


The figure above shows a uniform rod AB, of length $4a$ and mass $2m$, smoothly hinged to a fixed point D, so that $|AD| = a$. One end of a light inextensible string is attached to the end of the rod B. The string passes over a small smooth pulley which is fixed at the point P, where DP is vertical with $|DP| = 3a$. A particle C, of mass m is attached to the other end of the string and the particle hangs vertically below P.

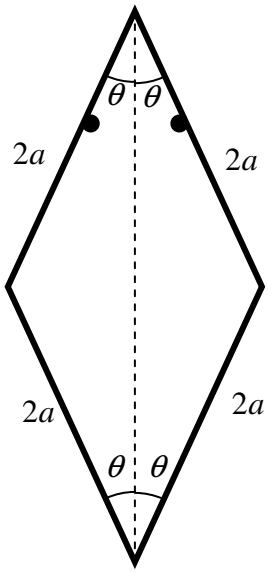
The acute angle CPB is denoted by θ, where $0 < \theta < \frac{1}{2}\pi$.

Find the two values of θ for which the system is in equilibrium and determine the stability of these equilibrium positions.

$$\boxed{\text{unstable at } \theta = \arccos\frac{3}{4}}, \quad \boxed{\text{stable at } \theta = 0}$$



Question 15 (*+)**

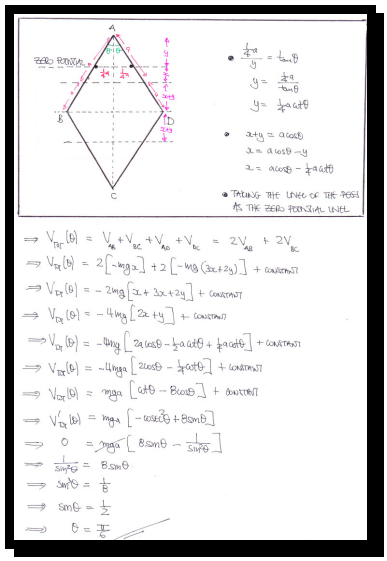


Four identical uniform rods, each of mass m and length $2a$, are smoothly joined to form a rhombus. The system of the four rods is placed over two smooth pegs and rests in equilibrium as shown in the figure above.

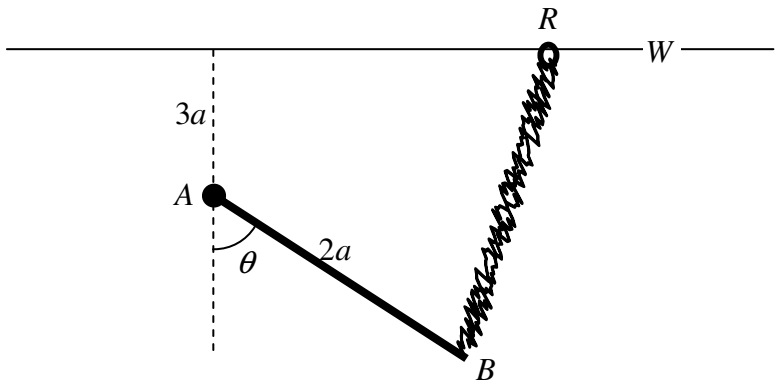
The pegs are in the same horizontal level at a distance $\frac{1}{2}a$ apart. Each of the rods makes an angle θ with the upward vertical.

Find the value of θ for which the system is in equilibrium.

$$\boxed{\theta=\frac{\pi}{6}}$$



Question 16 (*+)**



A uniform rod AB has mass m and length $2a$.

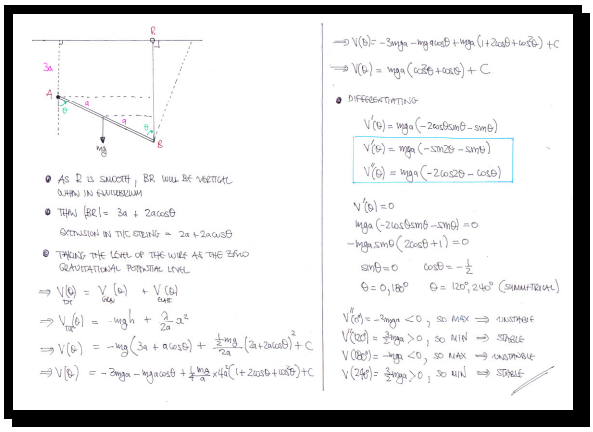
The end A is freely hinged at a fixed point, which is at a distance $3a$ below a fixed smooth rigid horizontal wire W.

The end B is attached to one end of an elastic string of natural length a and modulus of elasticity $\frac{1}{2}mg$. The other end of the string is connected to a smooth ring R which is threaded to W as shown in the figure above.

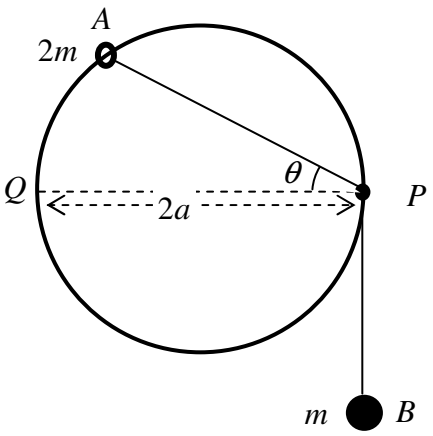
The angle the rod makes with the downward vertical through A is denoted by $\theta°$.

Find each of the values of θ, $0 \le \theta° < 360$, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 0°,\ \text{unstable}}$, $\boxed{\theta = 120°,\ \text{stable}}$, $\boxed{\theta = 180°,\ \text{unstable}}$, $\boxed{\theta = 240°,\ \text{stable}}$



Question 17 (*+)**

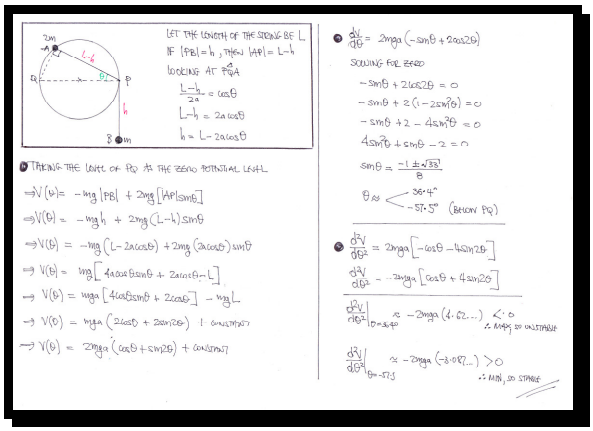


A small ring A, of mass m, is free to slide on a smooth wire, bent into the shape of a circular hoop of radius a. The wire is fixed in a vertical plane. A light inextensible string has one end attached to A and passes over a smooth pulley P, located at the end of a horizontal diameter of the wire, QP. The other end of the string is attached to a particle of mass $2m$, which hangs freely as shown in the figure above.

The angle QPA is denoted by $\theta°$.

Find each of the values of θ, $0 \le \theta° < 360$, for which the above described system is in equilibrium and determine their stability.

$\boxed{\theta = 36.4°, \text{ unstable}}$, $\boxed{\theta = -57.5°, \text{ stable}}$



Question 18 (****)

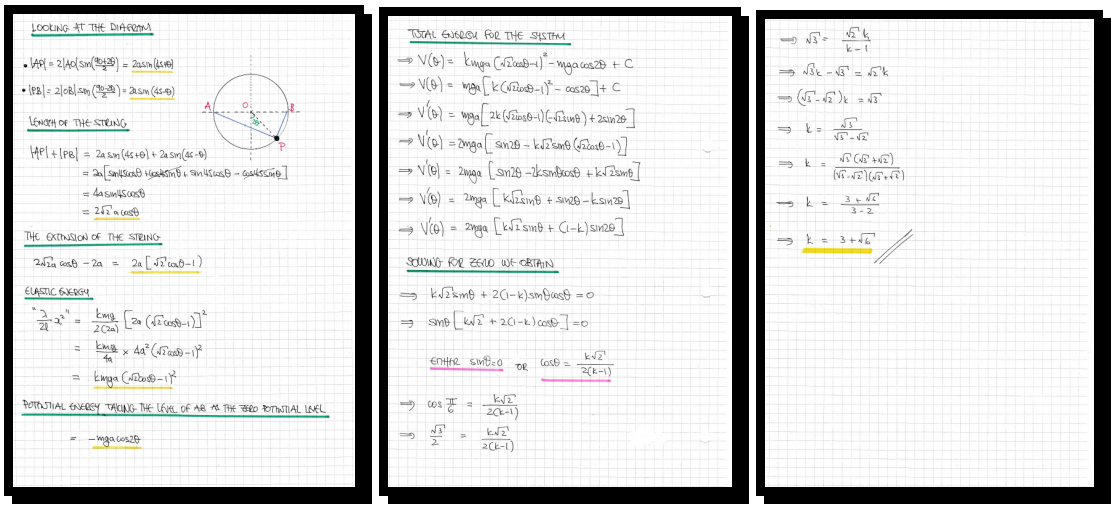
A smooth wire is bent into the shape of a circle of radius a, centre at O. The wire is fixed in a vertical plane and the straight line AOB is a horizontal diameter of the circle. A small ring P, of mass m, is threaded on the wire and a light elastic string is also threaded through the ring. The two ends of the string are attached at A and B.

The natural length of the string is $2a$ and its modulus of elasticity is kmg, $k>0$.

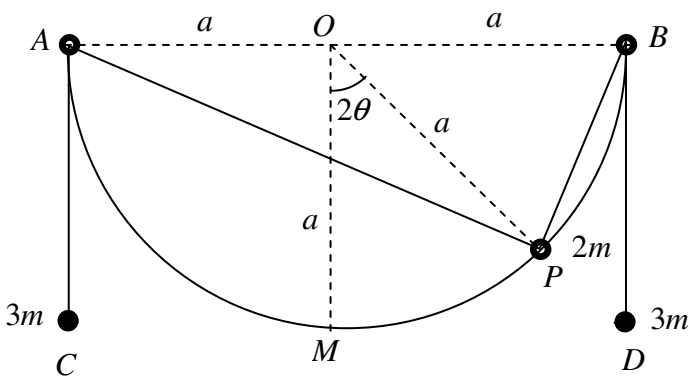
The angle between the radius OP and the downward vertical through O is denoted by 2θ, where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

If $\theta = \frac{1}{6}\pi$ is a position of equilibrium for the above described system determine the exact value of k.

$\boxed{}$, $\boxed{k=3+\sqrt{6}}$



Question 19 **(****)**



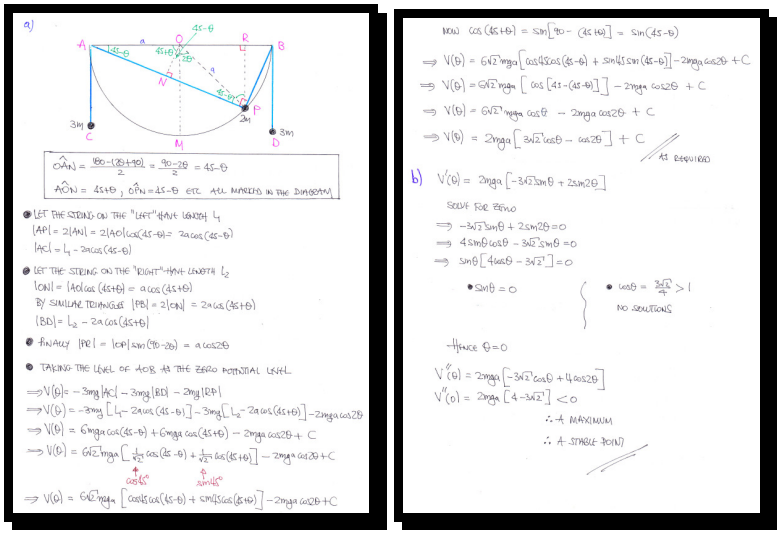
A smooth wire, with ends A and B, is in the shape of a semicircle of radius a. The line AB is horizontal and the midpoint of AB is O. The wire is fixed in a vertical plane. A small ring P of mass $2m$ is threaded on the wire and is attached to two light inextensible strings. One string passes through a small smooth ring fixed at A and is attached to a particle of mass $3m$. The other string passes through a small smooth ring fixed at B and is attached to another particle of mass $3m$. The particles hang freely under gravity, as shown in the figure above. The angle between the radius OP and the downward vertical OM is 2θ, where $-45^\circ \le \theta \le 45^\circ$.

a) Show that the potential energy of the system is

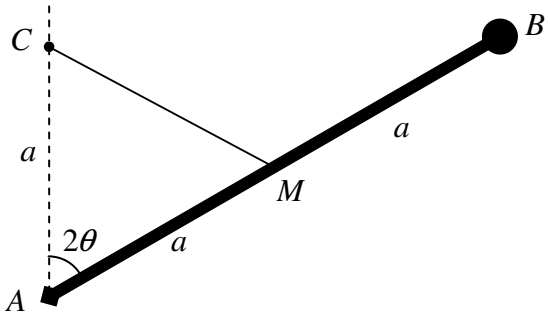
$$2mga\left[3\sqrt{2}\cos\theta-\cos 2\theta\right]+\text{constant}.$$

b) Find the value of θ for which the above described system is in equilibrium and determine its stability.

$\theta = 0$, stable



Question 20 (**+)**



A uniform rod AB, of mass m and length $2a$, is smoothly hinged to a fixed point A so it can rotate in a vertical plane. A particle of mass km, where k is a positive constant, is attached at B. A light elastic string of natural length a and modulus of elasticity $2mg$ has one of its ends attached to the midpoint of the rod, M. The other end of the string is attached to a fixed point C which lies at a vertical distance a above A, as shown in the figure. The potential energy of the system is V.

Given that $\measuredangle CAM = 2\theta$ show clearly that …

a) … $V = 2mga\left[\left(1-2k\right)\sin^2\theta - 2\sin\theta\right] + \text{constant}$.

b) … the system is in equilibrium when $\cos\theta = 0$ or $\sin\theta = \dfrac{1}{1-2k}$.

c) … there no physical positions of equilibrium if $\sin\theta = \dfrac{1}{1-2k}$.

d) … there is a stable position of equilibrium for all positive values of k.

proof

