MOMENT OF INERTIA CALCULATIONS

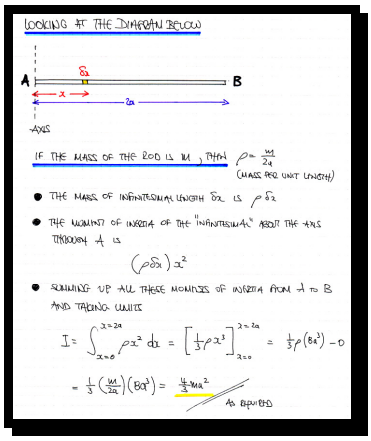
MOMENT OF INERTIA BY CALCULUS

Question 1 (**)

Use integration to show that the moment of inertia I of a thin uniform rod AB, of length $2a$ and mass m, about an axis through A and perpendicular to the length of the rod is given by

$$I=\frac{4}{3}ma^2 .$$

[ ], proof



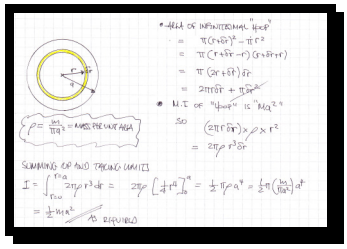
Question 2 (**)

A uniform circular disc, of mass m and radius a, is free to rotate about an axis L, through the centre of the disc and perpendicular to the plane of the disc

Prove, using integration, that the moment of inertia of the disc about L is $\frac{1}{2}ma^2$.

[*You may assume without proof that the standard result of the moment of inertia of a uniform hoop about an axis through its centre and perpendicular to its plane.*]

proof



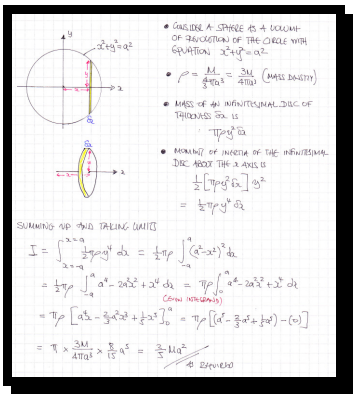
Question 3 (+)**

Show that the moment of inertia of a uniform solid sphere of mass m and radius a, about one of its diameters is

$$\frac{2}{5}ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

proof



Question 4 (***)

A uniform circular lamina L has mass m and radius a.

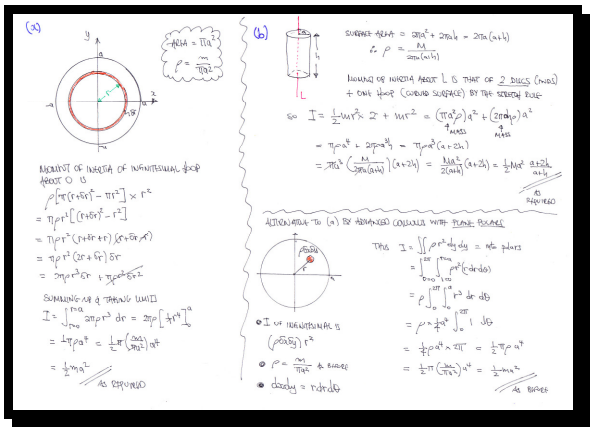
a) Show by integration that the moment of inertia of L about a perpendicular axis through the plane of the lamina and though its centre is $\frac{1}{2}ma^2$.

A closed hollow cylinder C has mass M, radius a and height h. The entire cylinder is made of the same material with uniform density.

b) Show that the moment of inertia of C about its axis of symmetry is

$$\frac{1}{2}Ma^2\left(\frac{a+2h}{a+h}\right).$$

proof



Question 5 (*+)**

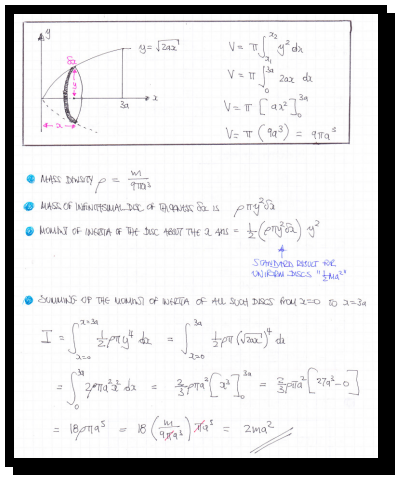
The finite region R is bounded by the x axis, the straight line with equation $x=3a$ and the curve with equation $y=\sqrt{2ax}$, where a is a positive constant.

A uniform solid S is generated by fully rotating R in the x axis.

If the mass of S is m, determine the moment of inertia of S about the x axis.

[*In this question, you may assume standard results for the moment of inertia of uniform circular discs.*]

$$\boxed{I=2ma^2}$$



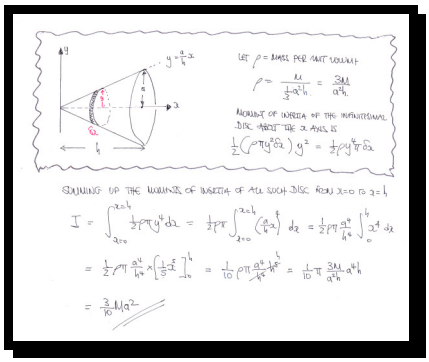
Question 6 (*+)**

Show by integration that the moment of inertia of a uniform solid circular cone of mass M, height h and base radius a, about its axis of symmetry, is given by

$$\frac{3}{10}Ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

proof

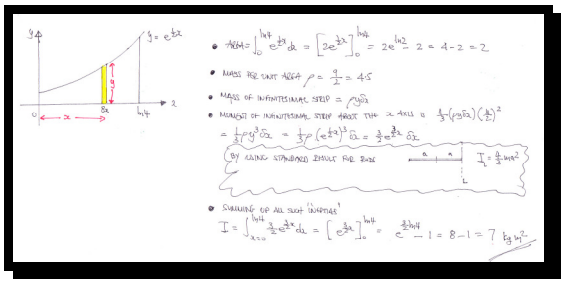


Question 7 (*+)**

A uniform lamina of mass 9 kg occupies the finite region bounded by the coordinate axes, the straight line with equation $x=\ln 4$ and the curve with equation $y=\mathrm{e}^{\frac{1}{2}x}$.

Given that distances are measured in metres, calculate the moment of inertia of the lamina about the x axis.

$I=7\text{ kg m}^2$



Question 8 (*+)**

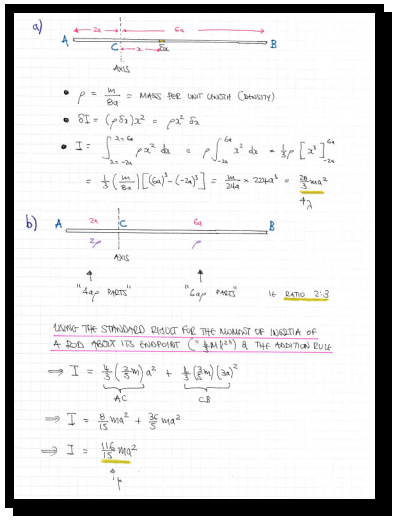
A uniform rod AB, of mass m and length $8a$, is free to rotate about an axis L which passes through the point C, where $|AC| = 2a$.

a) Given that the moment of inertia of the rod about L is λma^2, use integration to find the value of λ.

A different rod AB, also of mass m and length $8a$ is free to rotate about a smooth fixed axis L', which passes through the point C, where $|AC| = 2a$. The mass density of the section AC is twice as large as the mass density of the section CB.

b) Given that the moment of inertia of this rod about L' is μma^2, determine the value of μ.

$\square$, $\boxed{\lambda = \frac{28}{3}}$, $\boxed{\mu = \frac{115}{15}}$



Question 9 (****)

A triangular lamina OAB has $|OA|=|OB|$ and $|AB|=2a$. The height of the lamina drawn from O to AB has length h.

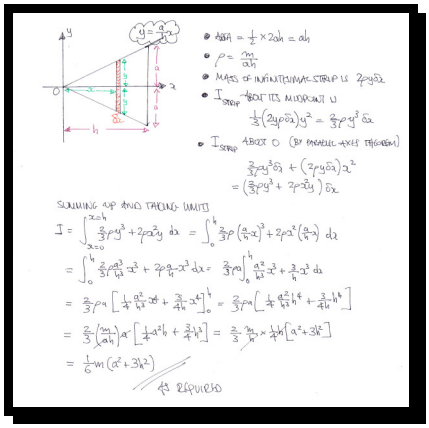
Show by integration that the moment of inertia of the lamina about an axis through its vertex through O and perpendicular to the plane of the lamina, is given by

$$\frac{1}{6}m\left(a^2+3h^2\right),$$

where m is the mass of the lamina.

[*In this proof, you may assume standard results for the moment of inertia of uniform rods.*]

proof



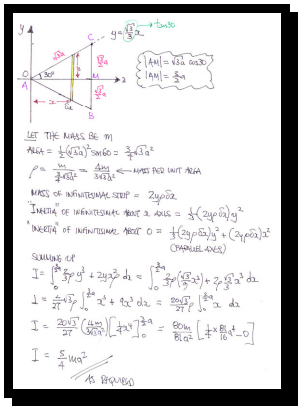
Question 10 (****)

A uniform equilateral triangular lamina ABC has mass m and side length of $\sqrt{3}a$.

Show, by integration, that the moment of inertia of the lamina about an axis through one of its vertices and perpendicular to the plane of the lamina is $\frac{5}{4}ma^2$.

[*In this proof, you may assume standard results for the moment of inertia of uniform rods.*]

proof

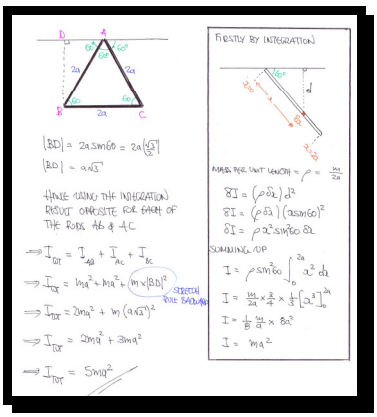


Question 11 (****)

A framework, in the shape of an equilateral triangle ABC, is formed by rigidly joining three uniform rods, each of mass m and length $2a$.

Find the moment of inertia of the framework about an axis passing through A, and parallel to BC

$$\boxed{5ma^2}$$



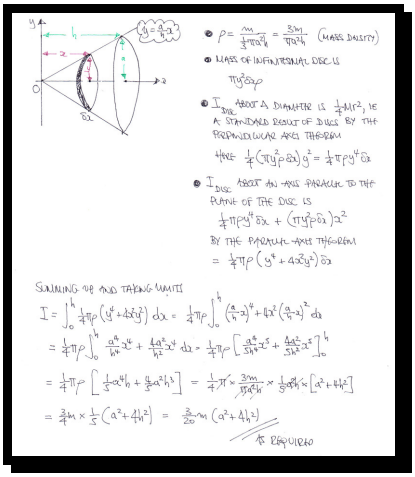
Question 12 (****)

Show by integration that the moment of inertia of a uniform solid circular cone of mass m, height h and base radius a, about an axis through its vertex and parallel to its base, is given by

$$\frac{3}{20}m\left(a^2+4h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular discs.*]

proof



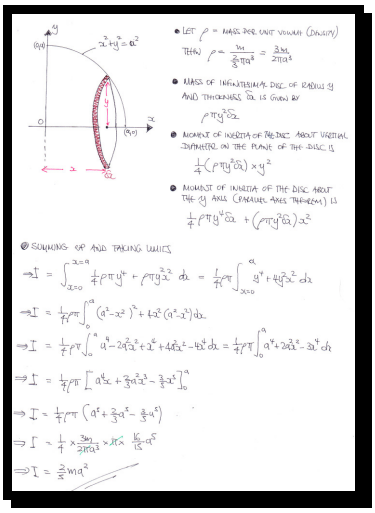
Question 13 (****)

Show by integration that the moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its plane face, is

$$\frac{2}{5}ma^2 .$$

[*In this proof, you may assume standard results for the moment of inertia of a uniform circular disc about one of its diameters.*]

proof



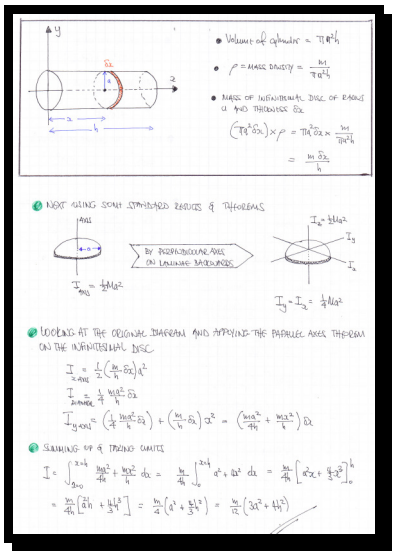
Question 14 (****)

Show by integration that the moment of inertia of a uniform solid right circular cylinder of mass m and radius a about a diameter of its plane face, is

$$\frac{m}{12}\left(3a^2+4h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of a uniform circular disc about one of its diameters.*]

proof



Question 15 (****)

A thin uniform shell in the shape of a right circular cylinder of radius a and height h has both its circular ends removed.

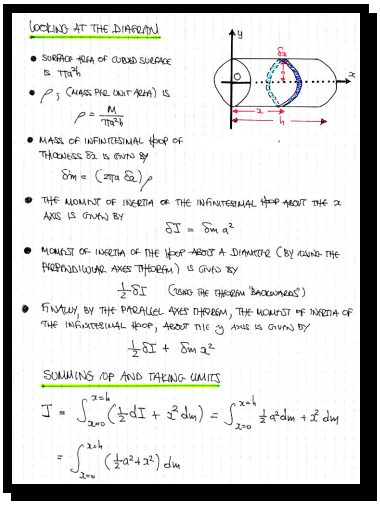
The resulting open cylindrical shell has mass M .

Show by integration that the moment of inertia of this shell about a diameter coplanar with one of its removed circular ends, is given by

$$\frac{1}{6}M\left(3a^2+2h^2\right).$$

[*In this proof, you may assume standard results for the moment of inertia of uniform circular hoops.*]

, proof



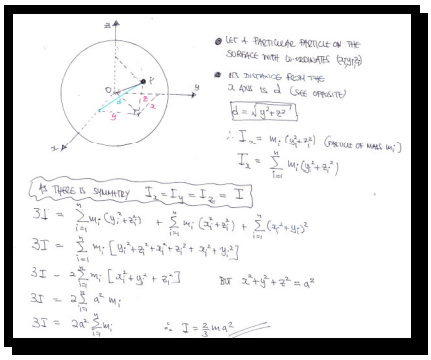
Question 16 (*****)

Show that the moment of inertia of a thin uniform spherical shell of mass m and radius a, about one of its diameters is

$$\frac{2}{3}ma^2 .$$

proof

[*In this proof, you may use valid symmetry arguments instead of calculus.*]



MOMENT OF INERTIA
BY
STANDARD RESULTS

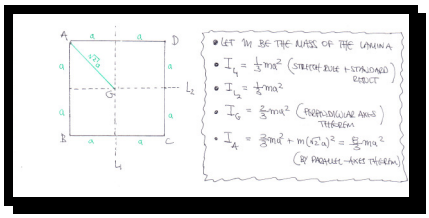
Question 1 (**)

A uniform square lamina has mass m and side length $2a$.

The lamina is free to rotate in about an axis L, which is perpendicular to the lamina and passes through one of the vertices of the lamina.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I=\frac{8}{3}ma^2}$$



Question 2 (**)

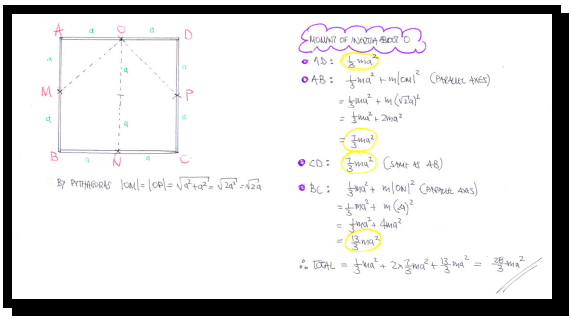
A square frame $ABCD$ consists of four uniform rods AB, BC, CD and DA.

Each of the rods has mass m and length $2a$.

The frame is free to rotate about an axis L, which is perpendicular to the plane of the frame and passes through the midpoint of AB.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I=\frac{28}{3}ma^2}$$



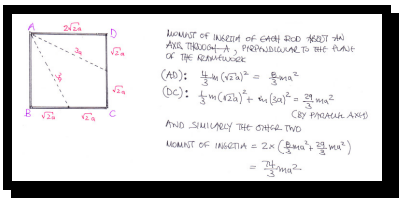
Question 3 (**)

Four uniform rods, each of mass m and length $2\sqrt{2}\,a$, are rigidly joined together to form a square framework $ABCD$.

The framework is free to rotate about an axis L, which is perpendicular to plane of the framework and passes through A.

Calculate the moment of inertia of $ABCD$ about L, in terms of m and a.

$$\boxed{I=\frac{74}{3}ma^2}$$

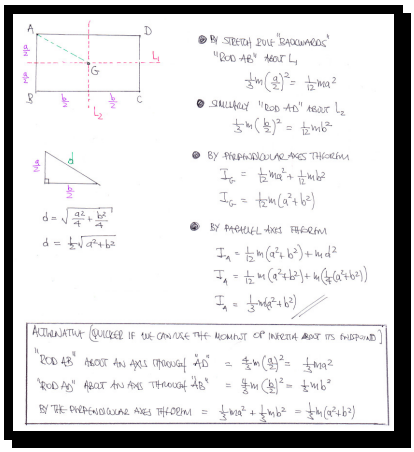


Question 4 (**)

A rectangular lamina $ABCD$, has mass m, length a and width b are rigidly joined together to form a square framework.

Calculate the moment of inertia of $ABCD$ about an axis through A perpendicular to the plane of $ABCD$.

$$\boxed{I=\frac{1}{3}m\left(a^2+b^2\right)}$$



Question 5 (**)

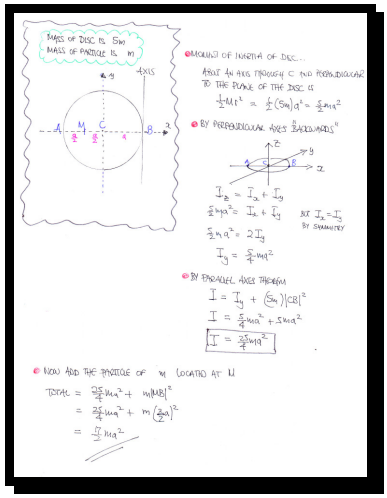
A uniform circular disc, with centre C, has mass $5m$ and radius a.

The straight line AB is a diameter of the disc.

A particle of mass m is attached to the disc at the point M, where M is the midpoint of AC. The disc is free to rotate about an axis L, which lies in the plane of the disc and is a tangent to the disc at B.

Find the moment of inertia of the loaded disc about L.

$$\boxed{I=\frac{17}{2}ma^2}$$



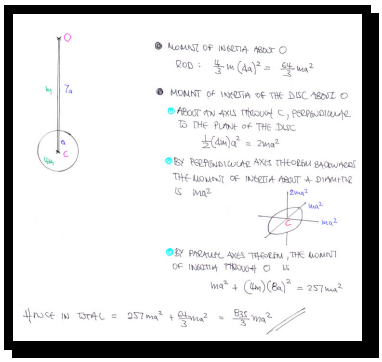
Question 6 (+)**

A compound pendulum consists of a thin uniform rod OC of length $8a$ and mass m is rigidly attached at C to the centre of a thin uniform circular disc of radius a and mass $4m$. The rod is in the same vertical plane as the disc. The pendulum is free to rotate in this vertical plane, through a smooth vertical axis through O, perpendicular to the plane of the disc.

Show that the moment of inertia of the pendulum about the above described axis is

$$\frac{835}{3}ma^2.$$

proof

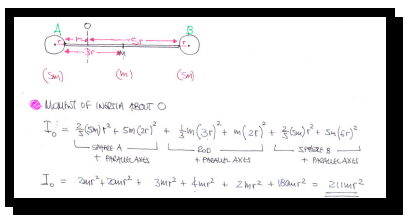


Question 7 (+)**

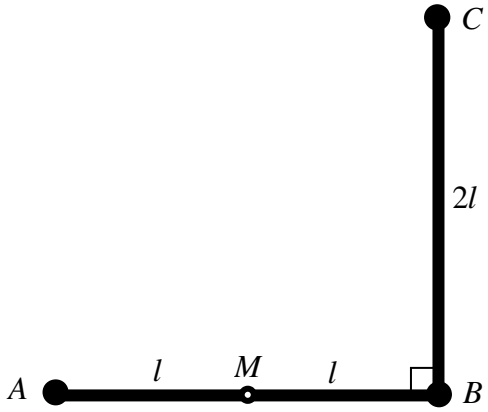
Two uniform spheres, each of mass $5m$ and radius r, are attached to each of the ends of a thin uniform rod AB, of mass m and length $6r$. The centres of the spheres are collinear with AB, and are located $8r$ apart. The system is free to rotate about an axis on a point on the rod O, where $|AO| = r$. This axis is perpendicular to AB.

Determine the moment of inertia of the system about O.

$211mr^2$



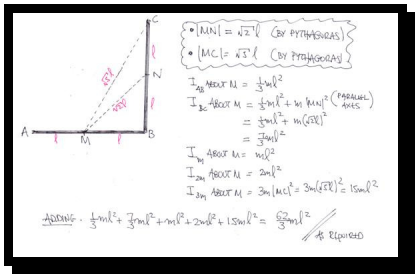
Question 8 (+)**



Two identical uniform rods AB and BC, each of mass m and length l are rigidly joined at B, so that $\angle ABC = 90°$. Three particles of masses m, $2m$ and $3m$ are fixed at A, B and C, respectively. The system of the two rods and the three particles can rotate freely in a vertical plane about a horizontal axis through M, where M is the midpoint of AB.

Show clearly that the moment of inertia of the system about an axis through M and perpendicular to the plane ABC is $\frac{62}{3}ml^2$.

proof



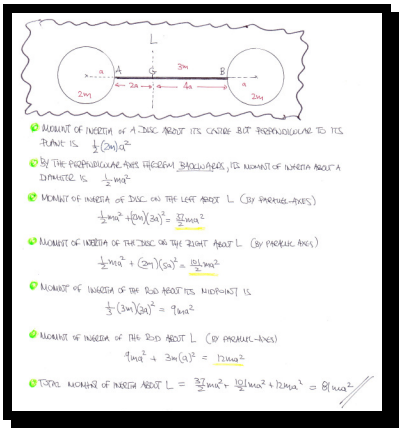
Question 9 (+)**

Two uniform discs, each of mass $2m$ and radius a, are attached to each of the ends of a thin uniform rod AB, of mass $3m$ and length $6a$. The system lies in the same plane the centres of the discs being collinear with AB, and located $8a$ apart.

The system is free to rotate about an axis on a point on the rod C, where $|AC| = 2a$. This axis is perpendicular to AB but lies in the same plane as the system.

Determine the moment of inertia of the system about C.

$$81ma^2$$



Question 10 (+)**

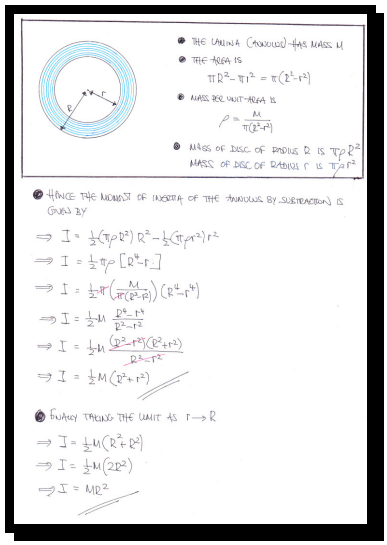
A disc of radius r and centre O is removed from a larger uniform disc of radius R and centre O, forming an annulus of mass M.

Use standard results to show that the moment of inertia of the annulus about an axis through O and perpendicular to its plane, is

$$\frac{1}{2}M\left(R^2-r^2\right),$$

and use this result to deduce the moment of inertia of a circular hoop of mass M, about an axis through its centre and perpendicular to the plane of the hoop.

proof



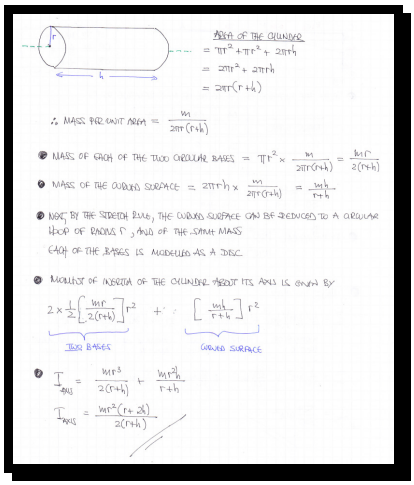
Question 11 (+)**

A thin uniform shell in the shape of a right circular cylinder of radius r and height h, with both its circular ends made of the same material and having the same thickness.

The resulting closed cylindrical shell has mass m.

Find the moment of inertia of the shell about its axis of symmetry.

$$\boxed{\frac{mr^2(r+2h)}{2(r+h)}}$$

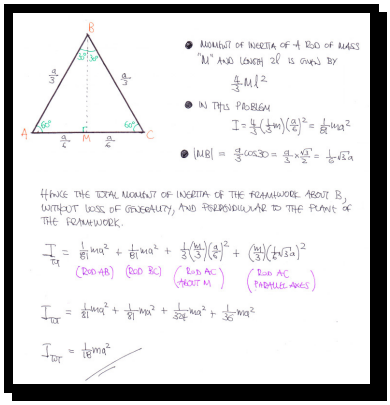


Question 12 (***)

A thin uniform wire AB, of mass m and length $3a$, is bent into the shape of an equilateral triangle.

Find the moment of inertia of the triangle about an axis through one of its vertices and perpendicular to the plane of the triangle.

$$\boxed{I=16ma^2}$$

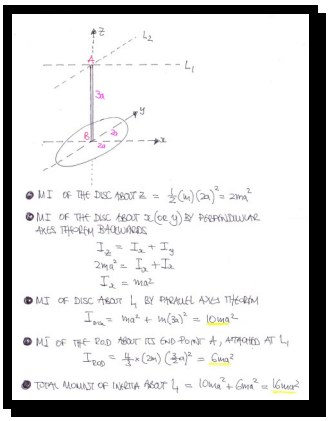


Question 13 (***)

A composite body consists of a thin uniform rod AB, of mass m and length $3a$, with the end B rigidly attached to the centre O of a uniform circular lamina, of radius $2a$ and mass m. The rod is perpendicular to the plane of the lamina. The body is free to rotate in a vertical plane about a horizontal axis through A, and perpendicular to AB.

Find the moment of inertia of the body about the above described axis.

$$\boxed{I=16ma^2}$$



Question 14 (***)

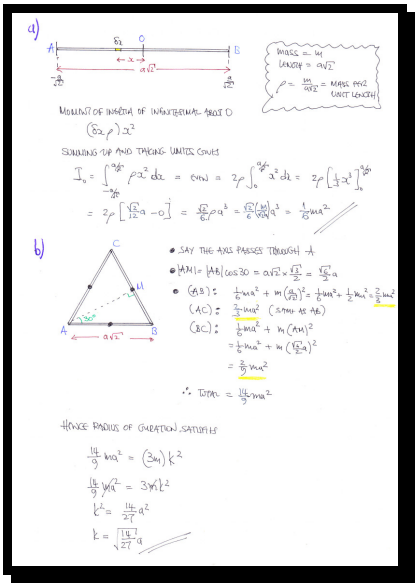
A uniform rod AB, has mass m and length $\sqrt{2}a$.

a) Use integration to find the moment of inertia of the rod about an axis through its midpoint O.

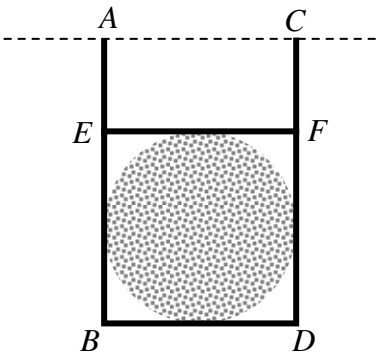
Three rods, identical to AB, are joined together to form an equilateral triangle ABC. The triangle is free to rotate about a fixed smooth axis L, which is perpendicular to the plane of ABC and passes through one of the vertices of ABC.

b) Determine the radius of gyration of ABC about L.

$$\boxed{I_O=\frac{1}{6}ma^2}\,,\qquad \boxed{k=\sqrt{\frac{14}{27}}\,a}$$



Question 15 (***)

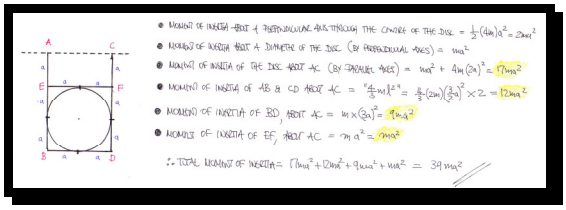


A shop sign is in the shape of a uniform circular disc of mass $4m$ and radius a.

It is suspended vertically by two uniform rods AB and CD, each of length $3a$ and mass $2m$. Two more rods EF and BD, each of length $2a$ and mass m are placed around the sign. All the rods are tangents to the disc so that $BFED$ is a square as shown in the figure.

Use standard results to determine the moment of inertia of the shop sign and the 4 rods, about a horizontal axis through A and C.

$$\boxed{39ma^2}$$



Question 16 (***)

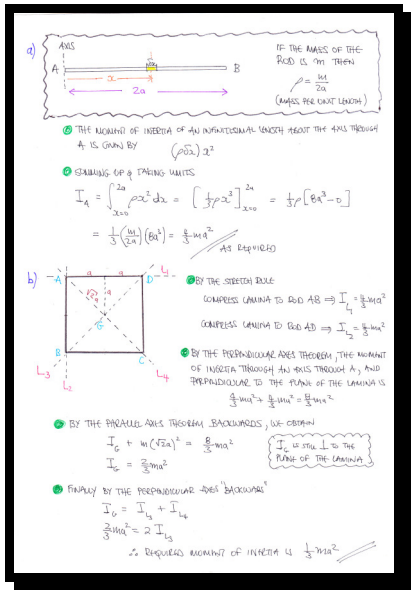
A thin uniform rod AB, of length $2a$ and mass m, is free to rotate about an axis L, which passes through A and is perpendicular to the length of the rod.

a) Use integration to show that the moment of inertia I of this rod about L is

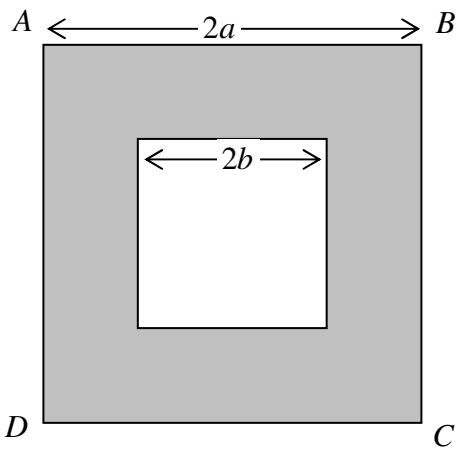
$$I = \frac{4}{3}ma^2 .$$

b) Use this result and moment of inertia theorems, to determine the moment of inertia of a uniform square lamina, of side length $2a$ and mass m, about one of its diagonals.

$$\boxed{\frac{1}{3}ma^2}$$



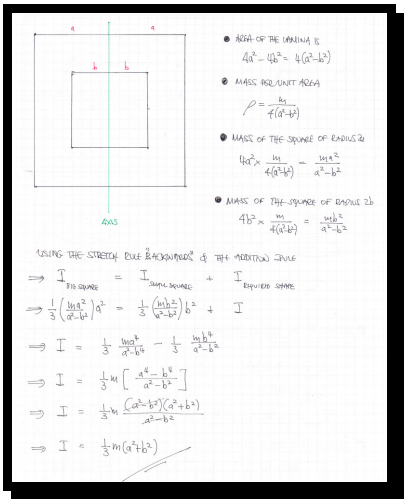
Question 17 (***)



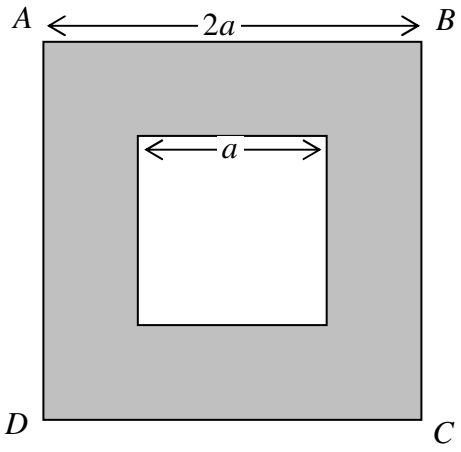
A uniform lamina, of mass m, is formed from a square lamina $ABCD$ of side $2a$, by removing a square of side $2b$ so both squares have parallel sides and share the same centre, as shown in the figure above.

Find the moment of inertia of this lamina about an axis passing through the midpoint of AB and the midpoint of DC.

$$\boxed{\frac{1}{3}m\left(a^2+b^2\right)}$$



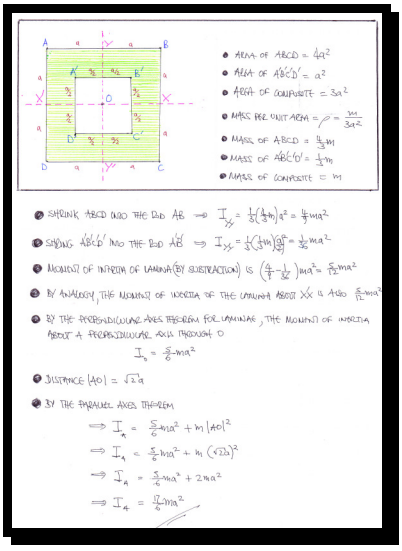
Question 18 (***)



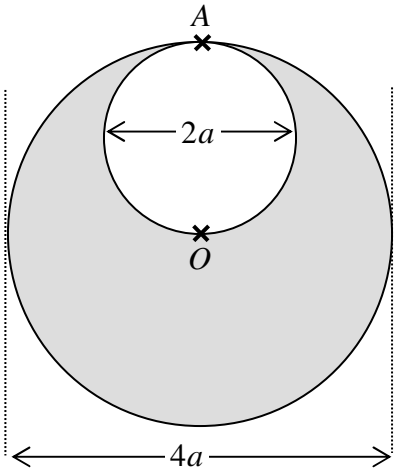
A uniform lamina, of mass m, is formed from a square lamina $ABCD$ of side $2a$, by removing a square of side a so both squares have parallel sides and share the same centre, as shown in the figure above.

Find the moment of inertia of this lamina about an axis through A and perpendicular to the plane of the lamina.

$$\boxed{\frac{17}{6}ma^2}$$



Question 19 (***)



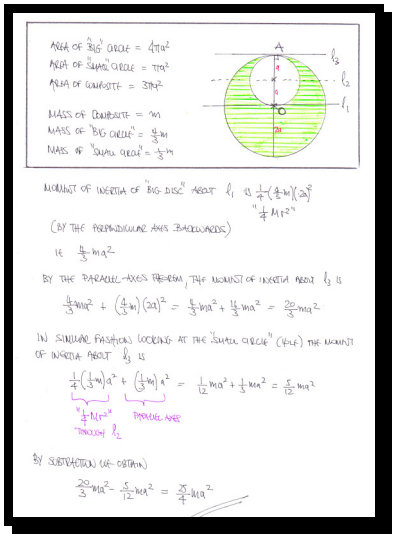
The point A lies on the circumference of a uniform circular disc of diameter $4a$.

A smaller circular disc with diameter OA is removed from the larger disc, where O is the centre of the larger disc, as shown in the figure above.

The remaining composite lamina L has mass m.

Determine its moment of inertia of L about an axis lying on the plane on L, the axis passing through A and being perpendicular to AO.

$$\frac{25}{4}ma^2$$



Question 20 (*+)**

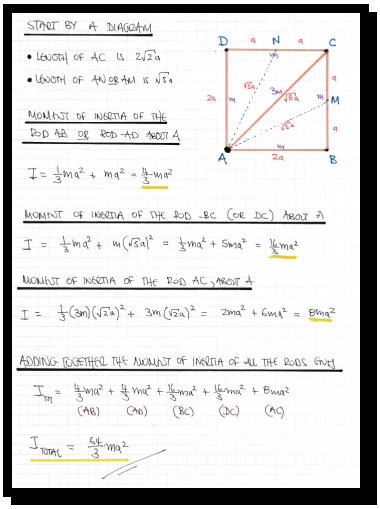
Four identical rods, each of mass m and length $2a$ are joined together to form a square rigid framework $ABCD$.

A fifth rod AC, of mass $3m$, is added to the framework for extra support.

The 5 rod framework is free to rotate about an axis L, which passes through A, and is perpendicular to the plane of $ABCD$.

Determine the moment of inertia of the framework about L.

$\square$, $\boxed{I=\frac{64}{3}ma^2}$

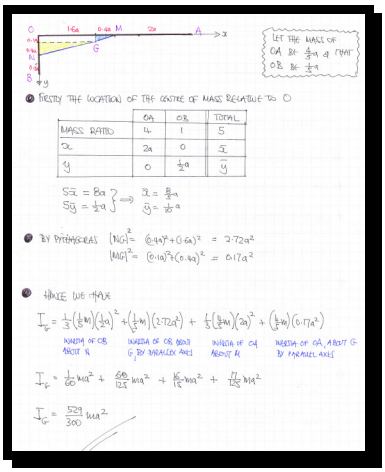


Question 21 (*+)**

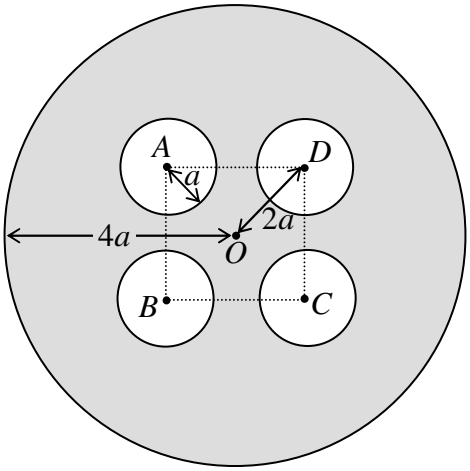
A uniform rod AB is bent at the point O, so that in the resulting $L-$shaped rigid object $\angle AOB = \frac{1}{2}\pi$, $|AO| = 4a$ and $|OB| = a$.

Find the moment of inertia of the resulting object, about an axis through its centre of mass and perpendicular to the plane AOB.

$$\boxed{\frac{529}{300}ma^2}$$



Question 22 (*+)**



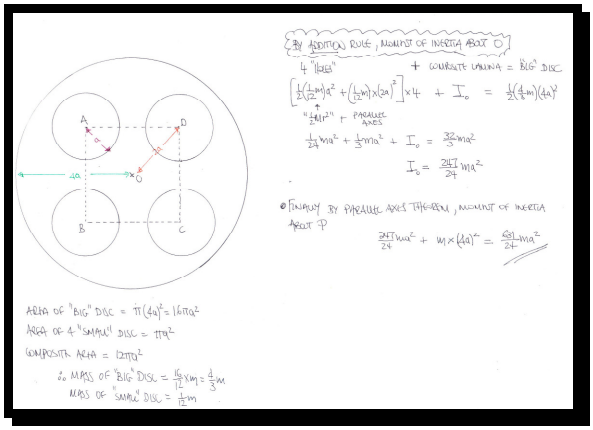
A uniform circular lamina has radius $4a$ and centre O. The points A, B, C and D lie on the lamina and are vertices of a square whose centre is at O so that $|OD| = 2a$.

Four circular discs, each of radius a, with centres A, B, C and D are removed from the lamina. The remaining lamina forms a new composite lamina of mass m.

The new lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the lamina and passes through a point P at the circumference of the lamina.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I = \frac{631}{24}ma^2}$$



Question 23 (*+)**

A uniform lamina has mass m and is in the shape of a semicircle of radius a, centred at the point O. The centre of mass of the lamina is at the point G.

The lamina is free to rotate about a fixed smooth horizontal axis L, which is perpendicular the plane of the lamina and passes through G.

Calculate the moment of inertia of the lamina about L, in terms of m and a.

$$\boxed{I=\frac{ma^2}{18\pi^2}\left(9\pi^2-32\right)}$$

