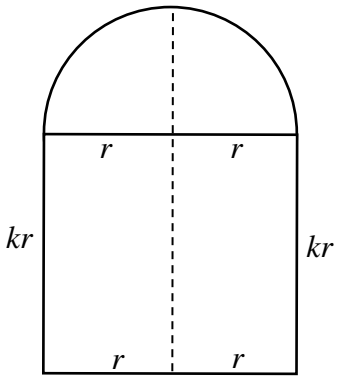
CENTRE OF MASS OF SOLIDS

Question 1 (**)



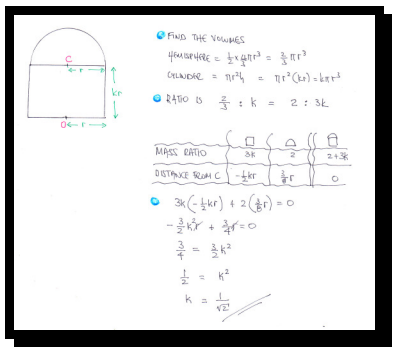
A uniform composite solid S consists of a solid hemisphere of radius r and a solid circular cylinder of radius r and height kr, where k is a positive constant

The circular face of the hemisphere is joined to one of the circular faces of the cylinder, so that the centres of the two faces coincide. The other circular face of the cylinder has centre O.

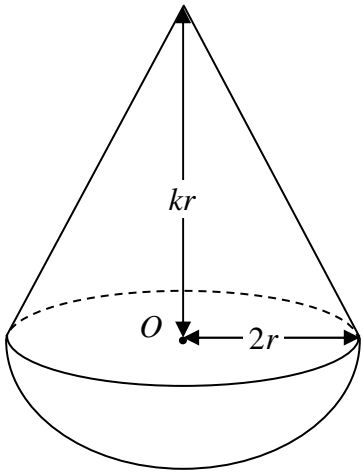
The centre of mass of S lies on the common plane of the cylinder and the hemisphere.

Determine the exact value of k.

$$\boxed{k=\frac{1}{\sqrt{2}}}$$



Question 2 **(**)**



A uniform solid S, consists of a hemisphere of radius $2r$ and a right circular cone of radius $2r$ and height kr, where k is a constant such that $k>2\sqrt{3}$. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone, as shown in the figure above.

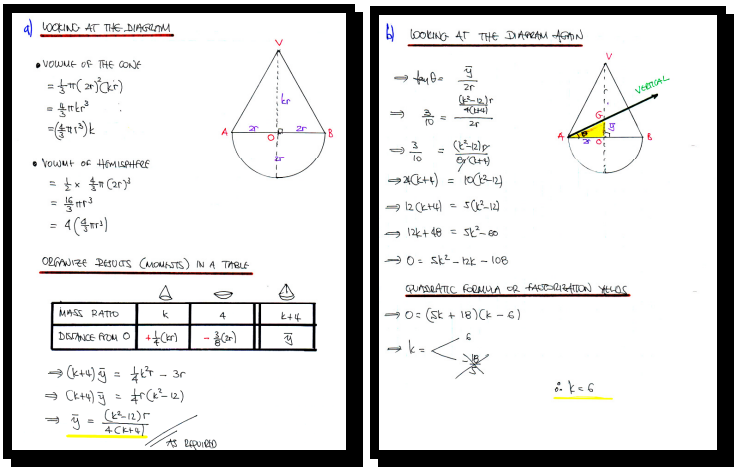
a) Show that the distance of the centre of mass of S from O is

$$\frac{k^2-12}{4(k+4)}r\,.$$

The point P lies on the circumference of the base of the cone. S is suspended by a string and hangs freely in equilibrium. The angle between OP and the vertical when S is in equilibrium is θ.

b) Given that $\tan\theta=0.3$ determine the value of k.

$\boxed{}$, $\boxed{k=6}$



Question 3 **(***)**

A uniform solid spindle is made up by joining together the circular faces of two right circular cones.

The common circular face of the two cones has radius r and centre at the point O.

The smaller cone has height h and the larger cone has height kh, $k>1$.

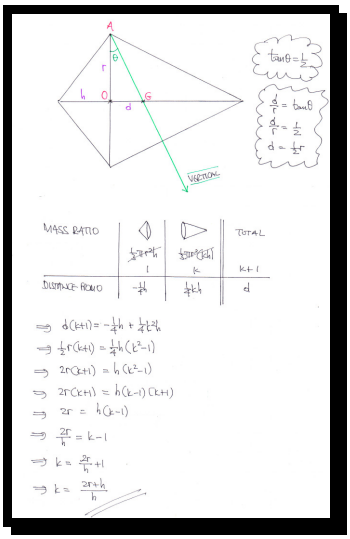
The point A lies on the circumference of the common circular face of the two cones.

The spindle is suspended from A and hangs freely in equilibrium with AO at an angle of $\arctan\frac{1}{2}$ to the vertical.

Show that

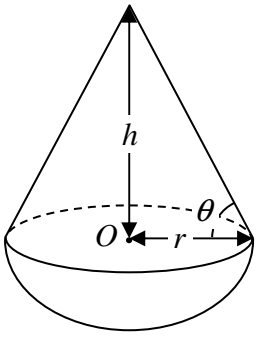
$$k=\frac{2r+h}{h}.$$

proof



Question 4 **(***)**

figure 1



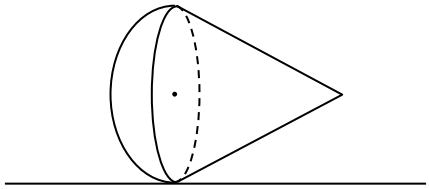


figure 2

Figure 1 above, shows a uniform solid S, consisting of a hemisphere of radius r and a right circular cone of radius r and height h. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone. The curved surface of the cone makes an angle θ with its base. The distance of the centre of mass of S above the level of O is denoted by $\overline{y}$.

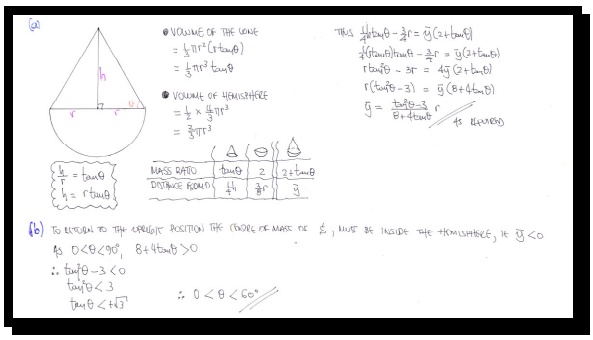
a) Show clearly that

$$\overline{y}=\frac{\tan^2\theta-3}{8+4\tan\theta}r\,.$$

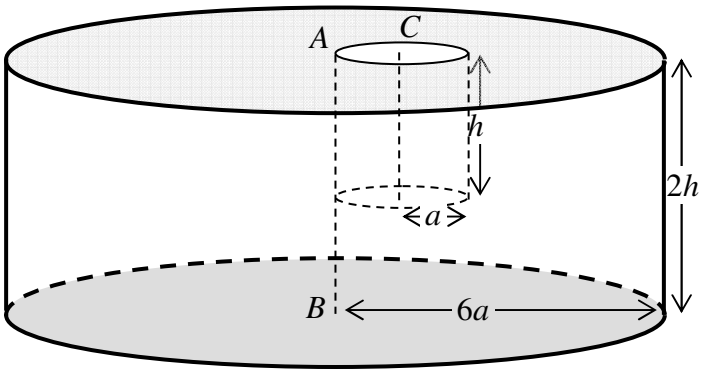
Figure 2 shows S held still on a horizontal surface so that the common plane of the hemisphere and the cone is perpendicular to the surface. When S is released it eventually returns to an upright position.

b) Determine the range of values of θ.

$$0<\theta<60^\circ$$



Question 5 **(***)**



A uniform solid right circular cylinder has height $2h$ and radius $6a$. The centre of one plane face is A and the centre of the other plane face is B.

A hole is made by removing a solid right circular cylinder of radius a and height h from the end with centre A. The axis of the cylindrical hole made is parallel to AB and meets the end with centre A at the point C, where $AC = a$.

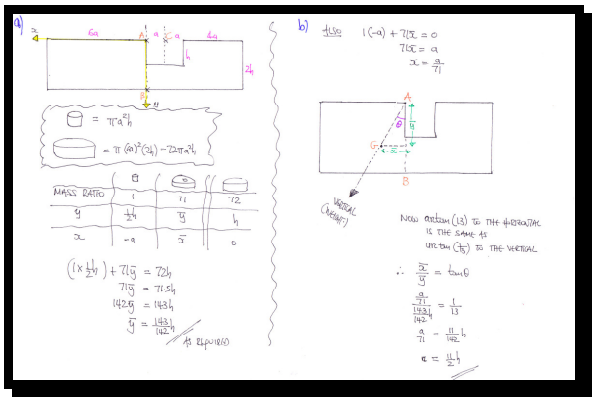
One plane face of the cylindrical hole made, coincides with the plane face through A of the cylinder. The resulting composite solid is shown in the figure above.

a) Show that the centre of mass of the composite is at a vertical distance $\frac{143}{142}h$ from the plane face containing A.

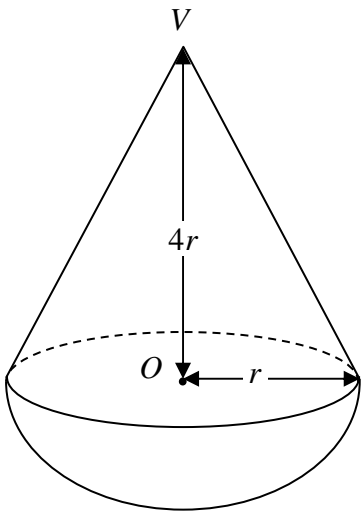
The composite is freely suspended from A and hangs in equilibrium with the axis AB inclined at an angle $\arctan(13)$ to the horizontal.

b) Express a in terms of h.

$$\boxed{a = \frac{11}{2}h}$$



Question 6 **(***+)**

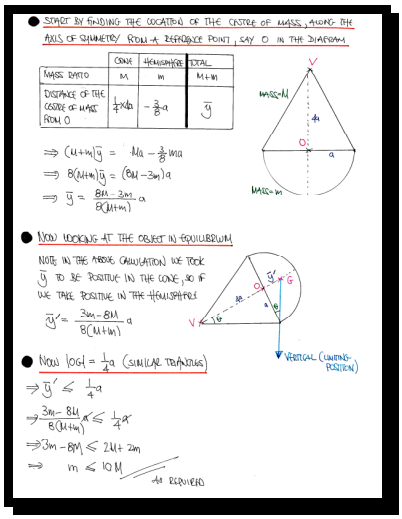


A uniform solid S, consists of a hemisphere of radius r and mass M, and a right circular cone of radius r, height $4r$ and mass m. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone, as shown in the figure above. The point P lies on the circumference of the base of the cone. S is placed on a horizontal surface, so that VP is in contact with the surface, where VP is the vertex of the cone.

Given that S remains in equilibrium in that position, show that

$$m \leq 10M \ .$$

[ ], [proof]



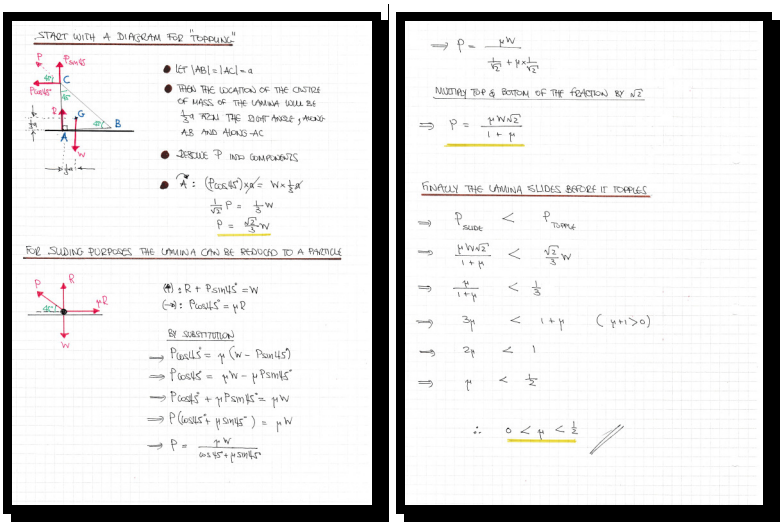
Question 7 (****)

A uniform lamina is in the shape of an isosceles right angled triangle ABC, where $\measuredangle BAC = 90°$.

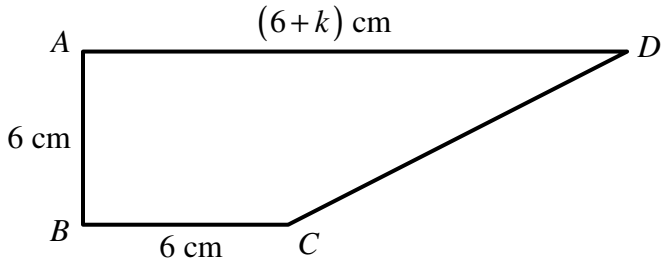
The lamina is placed with AB in contact with a rough horizontal plane and C vertically above A. A gradually increasing force is applied at C, in the direction BC, until equilibrium is broken. The line of action of this force lies in the vertical plane containing the lamina.

Given that the lamina slides before it topples determine the range of possible values of the coefficient of friction between the lamina and the plane.

$\boxed{\phantom{0<\mu<\frac{1}{2}}}$, $\boxed{0<\mu<\frac{1}{2}}$



Question 8 **(****)**



The figure above shows the cross section of a solid uniform right prism which in the shape of a right angled trapezium $ABCD$. It is further given that $|AB|=|BC|=6$ cm, $|AD|=(6+k)$ cm and $\measuredangle DAB=\measuredangle ABC=90°$.

a) By treating $ABCD$ as a uniform lamina, find in terms of the constant k the position of the centre of mass of $ABCD$, relative to the vertex A.

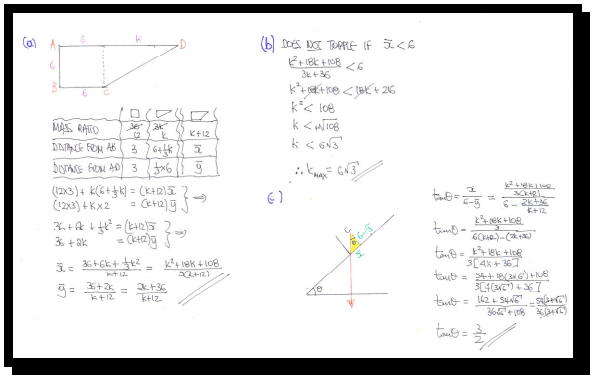
The prism is resting with $ABCD$ perpendicular to a horizontal surface and the face which contains BC, in contact with this horizontal surface.

b) Calculate the greatest value of k which allows the prism **not** to topple.

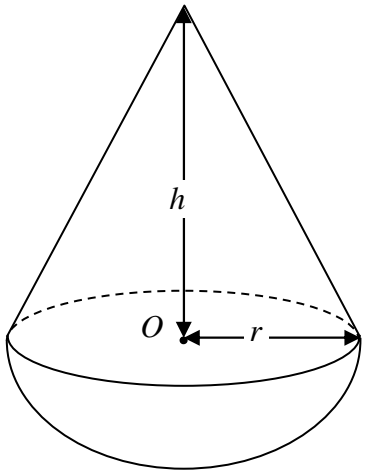
The prism is the placed on a rough plane inclined at θ to the horizontal, with BC lying on the line of greatest slope of the plane. The value of k is taken to be $3\sqrt{6}$.

c) Given the prism is about to topple, determine the exact value of $\tan\theta$.

$$\boxed{\overline{x}_{AB}=\frac{k^2+18k+108}{3k+36}},\ \boxed{\overline{y}_{AB}=\frac{2k+36}{k+12}},\ \boxed{k_{\max}=6\sqrt{3}},\ \boxed{\tan\theta=\frac{3}{2}}$$



Question 9 **(****)**



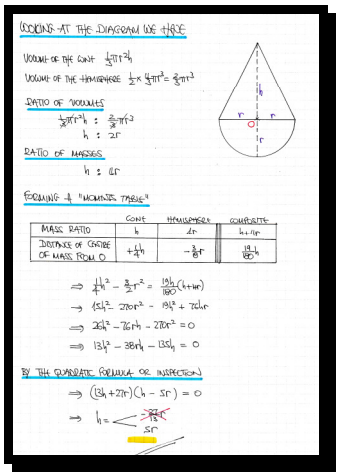
A solid S, consists of a hemisphere of radius r and a right circular cone of radius r and height h. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone.

Both the cone and the hemisphere are of uniform density, but the density of the hemisphere is twice as large as that of the cone.

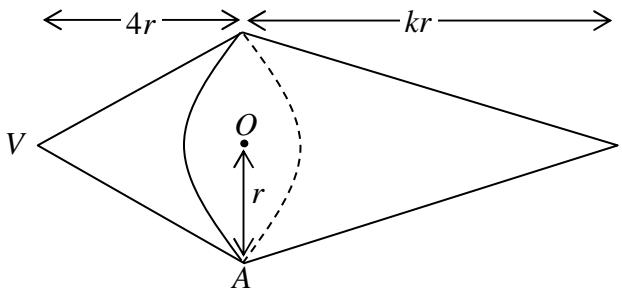
The centre of mass of S lies inside the cone, at a distance of $\dfrac{19h}{180}$ from O.

Express h in terms of r.

$\boxed{}$, $\boxed{h=5r}$



Question 10 **(****)**



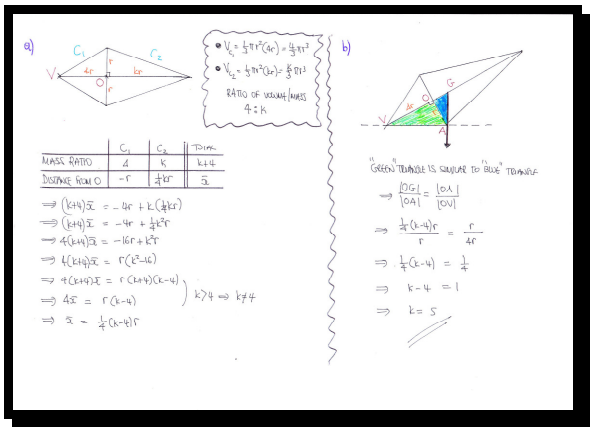
A uniform solid S, is formed by joining the plane faces of two solid right circular cones, both of radius r, so that the centres of their bases coincide at the point O, as shown in the figure above. The cone with vertex at the point V has height $4r$ and the other cone has radius kr, $k>4$.

a) Determine the distance of the centre of mass of S from O.

The point A lies on the circumference of the common base of the two cones. When S is placed with AV in contact with a horizontal surface, it rests in equilibrium.

b) Find the greatest possible value of k.

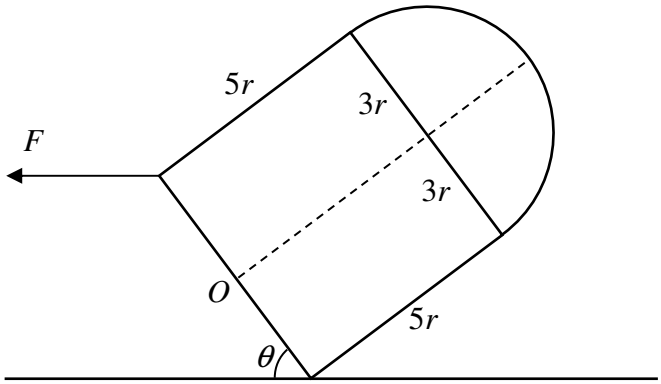
$$\boxed{\bar{x}=\frac{1}{4}(k-4)r}\,,\quad \boxed{k=5}$$



Question 11 (**+)**

A composite solid C consists of a uniform solid hemisphere of radius $3r$ and a uniform solid circular cylinder of radius $3r$ and height $5r$. The circular face of the hemisphere is joined to one of the circular faces of the cylinder, so that the centres of the two faces coincide. The other circular face of the cylinder has centre O.

a) Determine, in terms of r, the distance of the centre of mass of C from O.

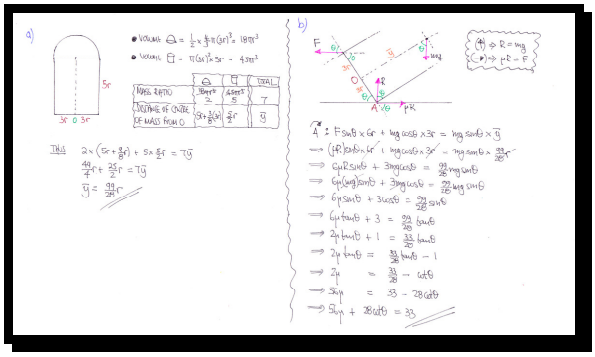


The composite is held in equilibrium by a horizontal force of magnitude F. The circular face of C has one point in contact with a fixed rough horizontal plane and is inclined at an angle θ to the horizontal. The force acts through the highest point of the circular face of C and in the vertical plane through the axis of the cylinder, as shown in the figure above. The coefficient of friction between C and the plane is μ.

b) Given that C is on the point of slipping along the plane in the same direction as F, show clearly that

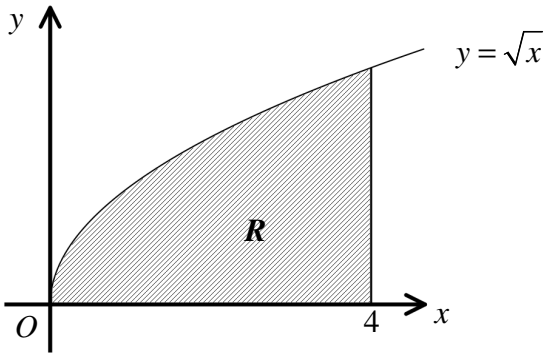
$$56\mu + 28\cot\theta = 33 .$$

$$\boxed{\overline{y} = \frac{99}{28} r}$$



CENTRE OF MASS BY CALCULUS

Question 1 (**)

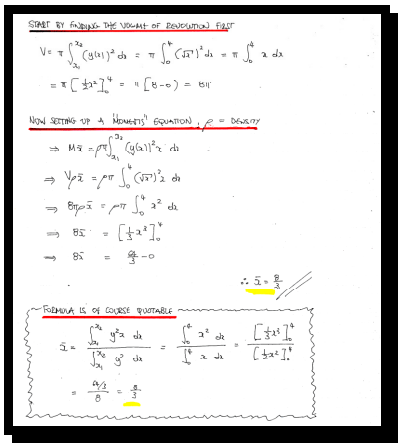


The figure above shows the finite region R bounded by the x axis, the curve with equation $y=\sqrt{x}$ and the straight line with equation $x=4$.

R is rotated about the x axis forming a solid of revolution S.

Use integration to determine the x coordinate of the centre of mass of S.

$\square$, $\boxed{x=\frac{8}{3}}$


START BY FINDING THE VOLUME OF REVOLUTION FIRST

$V=\pi\int_{y_1}^{y_2}(y(x))^2\,dx=\pi\int_0^4(\sqrt{x})^2\,dx=\pi\int_0^4 x\,dx$

$=\pi\left[\frac{1}{2}x^2\right]_0^4=\pi(8-0)=8\pi$

NOW SETTING UP A "MOMENTS" EQUATION, ρ = DENSITY

$\Rightarrow M\bar{x}=\rho\pi\int_{y_1}^{y_2}(y(x))^2 x\,dx$

$\Rightarrow V\rho\bar{x}=\rho\pi\int_0^4(\sqrt{x})^2 x\,dx$

$\Rightarrow 8\rho\pi\bar{x}=\rho\pi\int_0^4 x^2\,dx$

$\Rightarrow 8\bar{x}=\left[\frac{1}{3}x^3\right]_0^4$

$\Rightarrow 8\bar{x}=\frac{64}{3}-0$

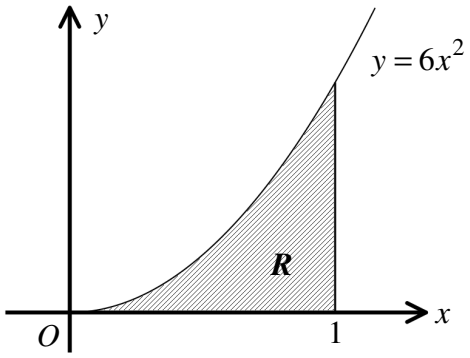
$\therefore \bar{x}=\frac{8}{3}$

FORMULA IS OF COURSE TABULATED

$\bar{x}=\frac{\int_{y_1}^{y_2}y^2 x\,dx}{\int_{y_1}^{y_2}y^2\,dx}=\frac{\int_0^4 x^2\,dx}{\int_0^4 x\,dx}=\frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4}$

$=\frac{64/3}{8}=\frac{8}{3}$

Question 2 (**)

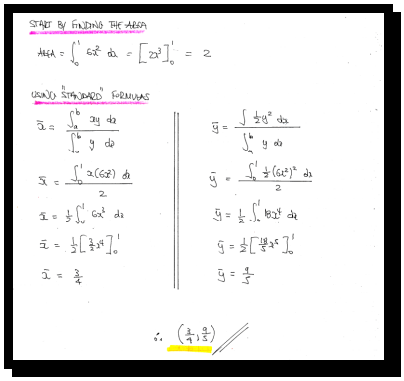


The figure above shows the finite region R bounded by the x axis, the curve with equation $y = 6x^2$ and the straight line with equation $x = 1$.

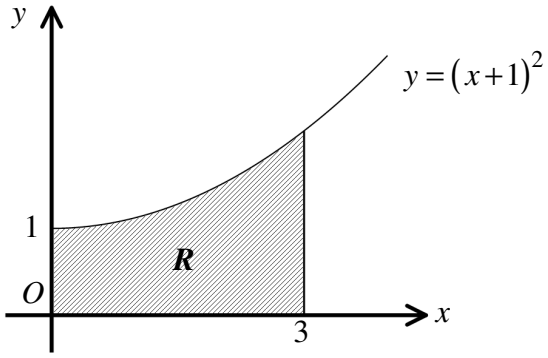
The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the coordinates of G.

$\square$, $\boxed{G\left(\frac{3}{4}, \frac{9}{5}\right)}$



Question 3 (+)**

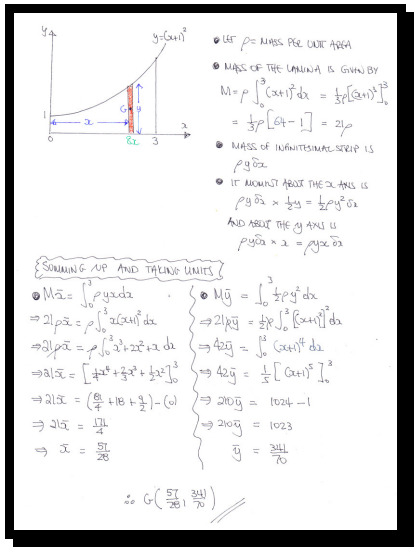


The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y=(x+1)^2$ and the straight line with equation $x=3$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use a detailed calculus method to determine the coordinates of G.

$$\boxed{G\left(\frac{57}{28},\frac{341}{70}\right)}$$



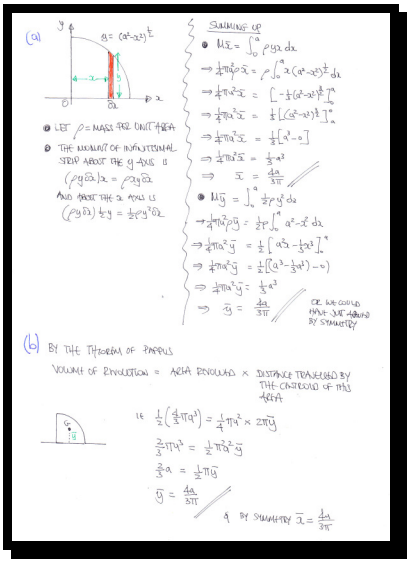
Question 4 (***)

A uniform lamina is in the shape of a quarter circle of radius a.

a) Use a calculus method to show that the centre of mass of the lamina is at a distance of $\frac{4a}{3\pi}$ from both the straight edges of the lamina.

b) Use one of the theorems of Pappus to verify the result of part **(a)**.

proof

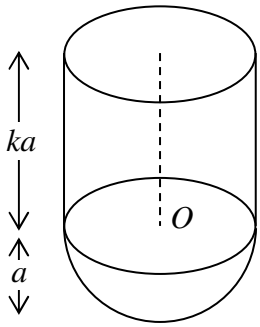


Question 5 (***)

The volume of a hemisphere of radius r is $\frac{1}{2}\pi r^3$.

a) Show, by calculus, that the centre of mass of a hemisphere of radius r is at a distance of $\frac{3}{8}r$ from the centre of its plane face.

A composite C is formed by joining a uniform solid right circular cone, of base radius a and height ka, where k is a positive constant, to a uniform solid hemisphere of radius a. The plane face of the hemisphere and one of the plane faces of the cylinder coincide, both having O as a centre, as shown in the figure below.

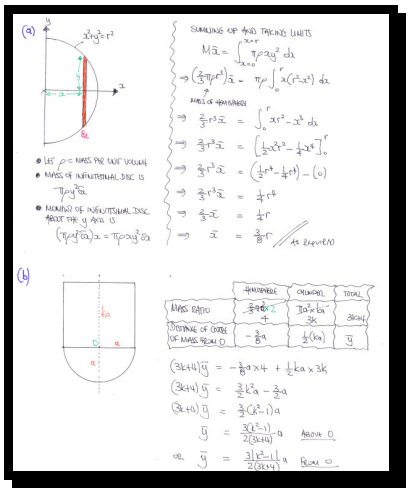


The mass density of the hemisphere is twice the mass density of the cylinder.

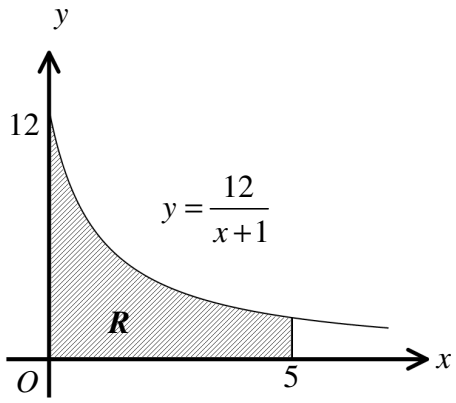
b) Show that that the distance of the centre of mass of C from O is

$$\frac{3|k^2-1|a}{2(3k+4)}.$$

proof



Question 6 **(***+)**

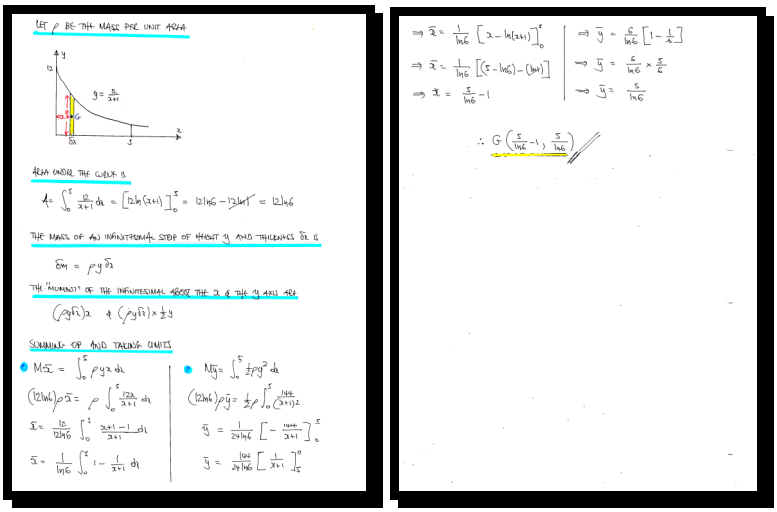


The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y=\frac{12}{x+1}$ and the straight line with equation $x=5$.

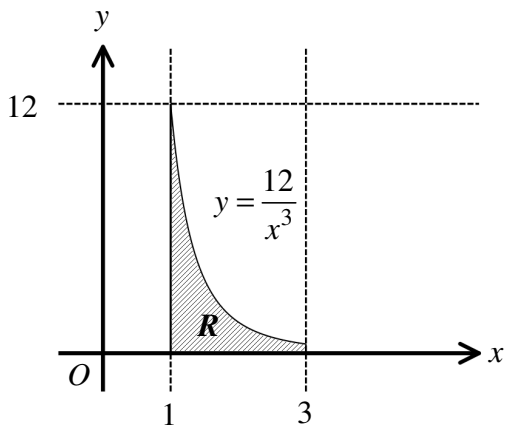
The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the exact coordinates of G.

$\square$, $\boxed{G\left(\frac{5}{\ln 6}-1,\ \frac{5}{\ln 6}\right)}$



Question 7 **(***+)**

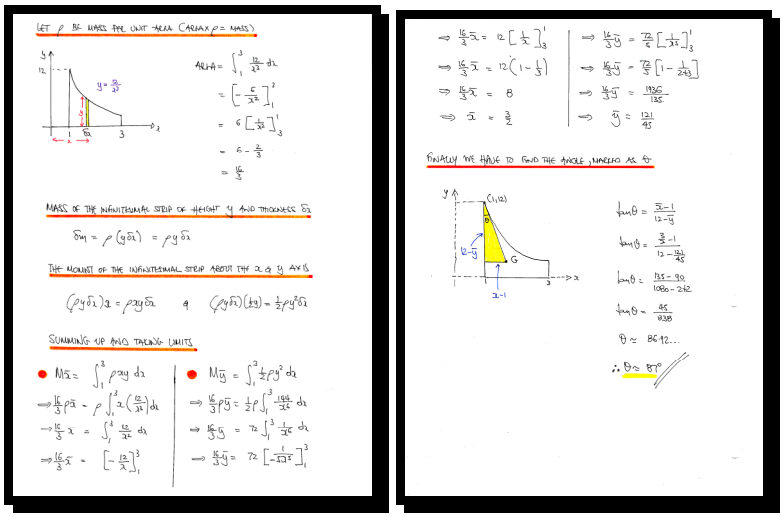


The figure above shows the finite region R bounded by the x axis, the curve with equation $y=\dfrac{60}{x^3}$ and the straight lines with equations $x=1$ and $x=3$.

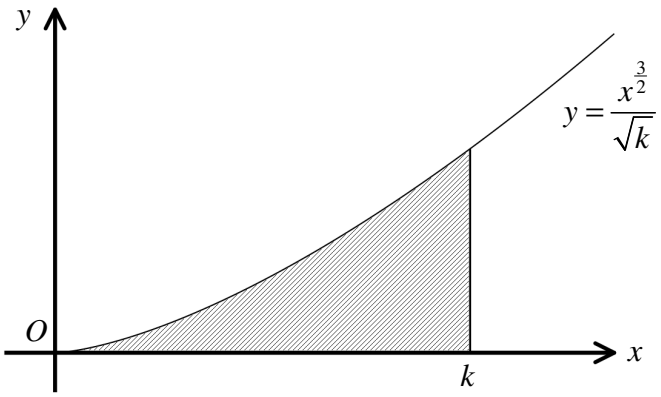
A uniform lamina whose shape is that of R, is suspended from the point $(1,12)$ and hangs freely under gravity.

Determine the angle the longer straight edge of the lamina makes with the vertical.

$\boxed{}$, $\boxed{\theta \approx 87^\circ}$



Question 8 **(***+)**



The figure above shows the curve with equation

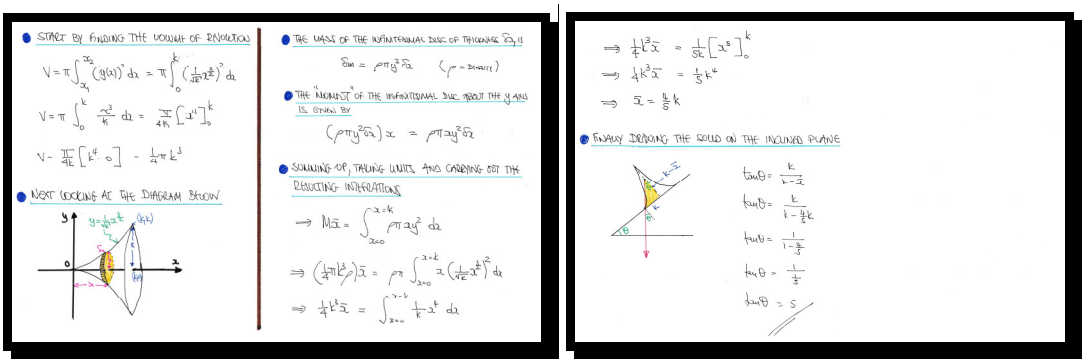
$$y=\frac{x^{\frac{3}{2}}}{\sqrt{k}} \ ,$$

where k is a positive constant

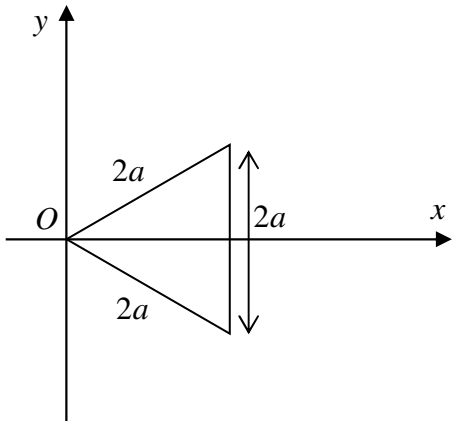
The finite region bounded by the curve, the coordinate axes and the straight line with equation $x=k$ region is revolved by 360° about the x axis, forming a solid of revolution. This solid is carefully placed with its plane face on a rough plane inclined at an angle θ to the horizontal and is at the point of toppling without any slipping.

Determine the value of $\tan\theta$.

$\boxed{}$, $\boxed{\tan\theta=5}$



Question 9 **(****)**

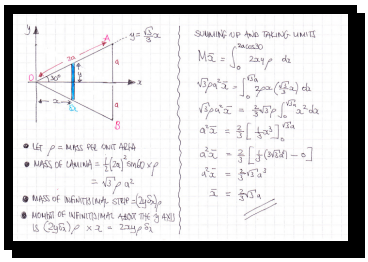


The figure above shows a uniform lamina in the shape of an equilateral triangle of side length $2a$.

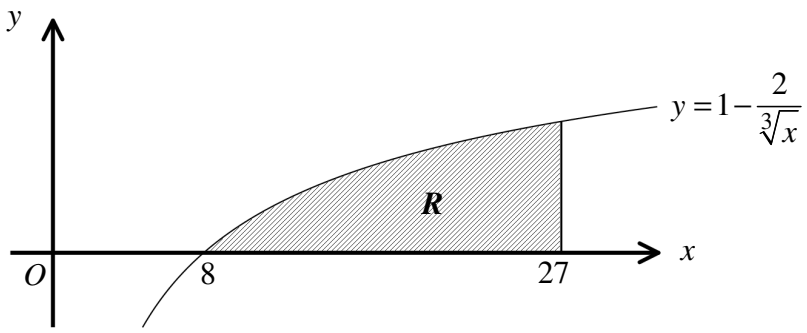
The lamina is referred in the Cartesian plane with one vertex at the origin O and an axis of symmetry along the x axis.

Use integration, with a detailed method, to find, in terms of a, the x coordinate of the centre of mass of the lamina.

$$\boxed{\bar{x}=\frac{2}{3}\sqrt{3}a}$$



Question 10 **(***+)**



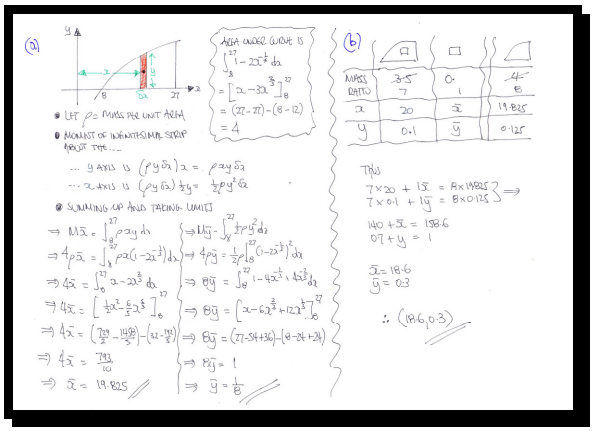
The figure above shows the finite region R, bounded by the x axis, the curve with equation $y=1-\frac{2}{\sqrt[3]{x}}$ and the line with equation $x=27$.

a) Use integration to determine, in exact form the coordinates, of the centre of mass of a lamina whose shape is that of R.

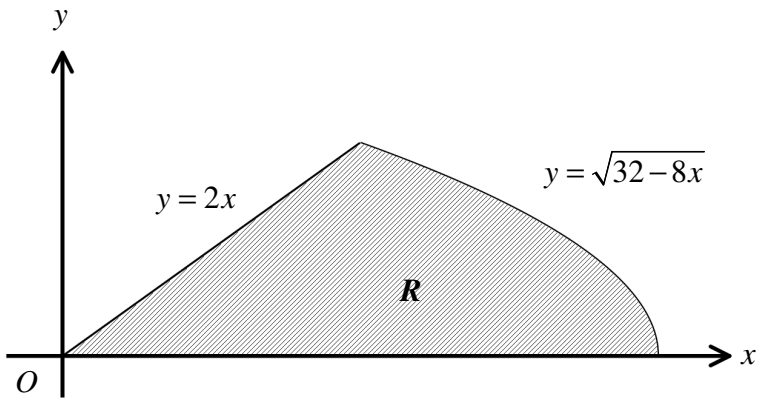
A shape whose area is 0.5 square units is removed from R so that the coordinates of the **resulting** shape are now located at $(20,0.1)$.

b) Determine the coordinates of the centre of mass of the shape that was removed.

$$\left(\frac{793}{40},\frac{1}{8}\right)=(19.825,0.125), \quad (18.6,0.3)$$



Question 11 (****)

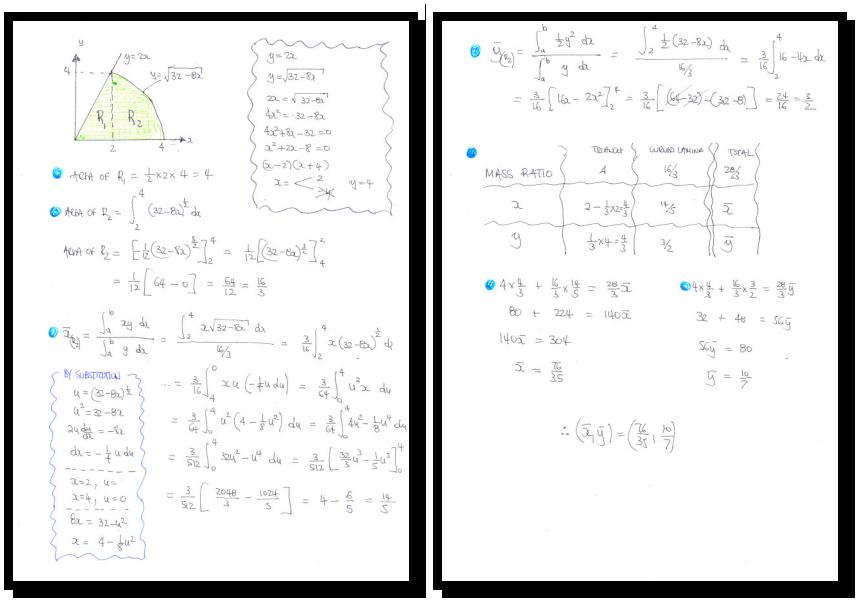


The figure above shows the finite region R bounded by the coordinate axes, the straight line with equation $y = 2x$ and the curve with equation $y = \sqrt{32 - 8x}$.

The centre of mass of a uniform lamina whose shape is that of R, is denoted by G.

Use integration to determine the exact coordinates of G.

$$\boxed{G\left(\frac{75}{35}, \frac{10}{7}\right)}$$



Question 12 (****)

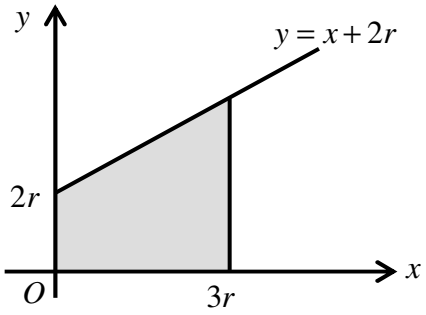


figure 1

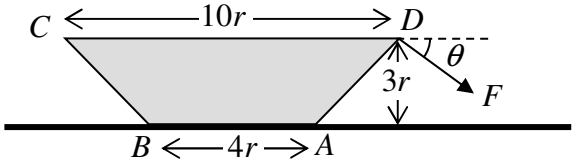


figure 2

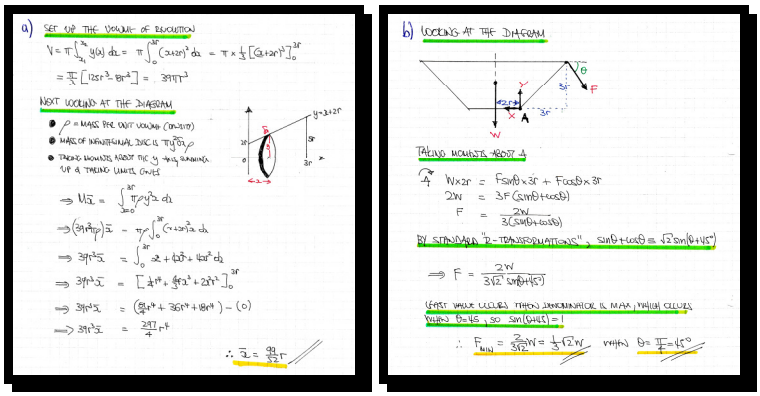
The finite region bounded by the coordinate axes and the lines with equations $y = x + 2r$ and $x = 3r$ is shown shaded in figure 1. This region is rotated about the x axis to form a frustum of a uniform right circular cone.

a) Use integration to find the distance of the centre of mass of the frustum of the cone from O.

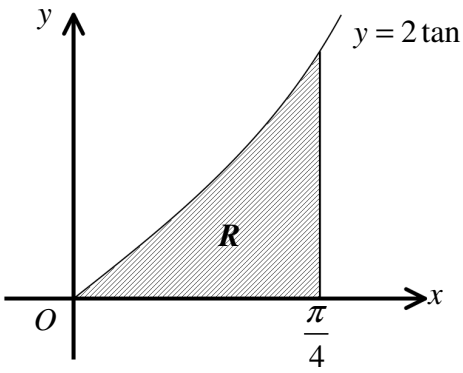
The resulting frustum has weight W. The frustum is then placed on a rough horizontal surface with the plane surface of the frustum of radius $3r$ in contact with the surface. A force of magnitude , inclined at angle θ below the horizontal, is acting at a point on the circumference of the plane surface of the frustum of radius $5r$, as shown in figure 2. The frustum is at the point of toppling without sliding.

b) Given that θ can vary, determine the least value for F and the value of θ for which F takes this least value.

$\boxed{}$, $\boxed{\overline{x} = \frac{99r}{52}}$, $\boxed{F_{\min} = \frac{\sqrt{2}}{3}W}$, $\boxed{\theta = 45°}$



Question 13 **(****)**



The figure above shows the finite region R bounded by the coordinate axes, the curve with equation $y = 2\tan x$ and the line with equation $x = \frac{\pi}{4}$.

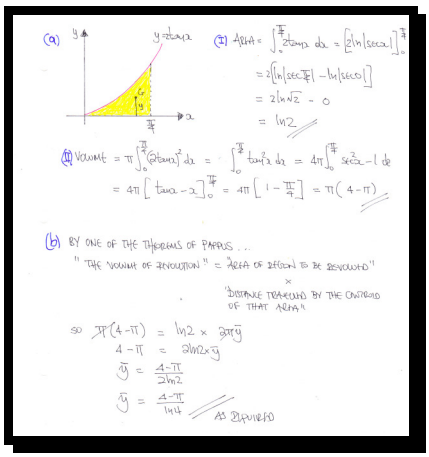
a) Use integration to determine in exact form…

i. … the area of R.

ii. … the volume of the solid generated when R is revolved by a full turn about the x axis.

b) Hence, or otherwise, show that the y coordinate of the centre of mass of a lamina whose shape is that of R is $\frac{4-\pi}{\ln 4}$.

$\boxed{\text{area} = \ln 2}$, $\boxed{\text{volume} = \pi(4-\pi)}$



Question 14 **(****)**

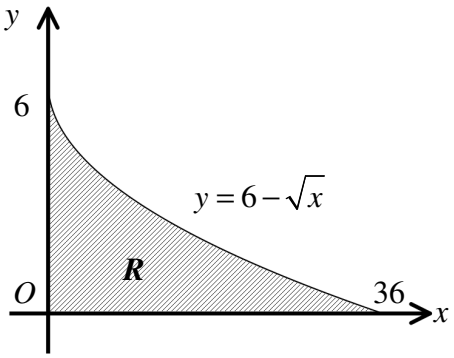


Figure 1

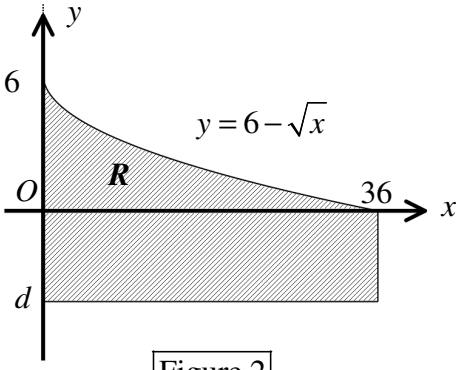


Figure 2

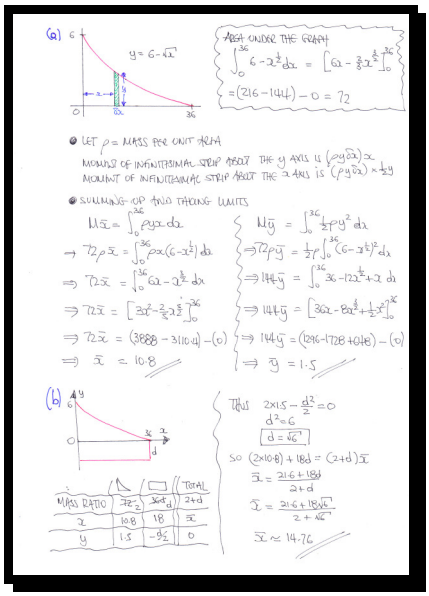
Figure 1 above shows the finite region R bounded by the coordinate axes and the curve with equation $y = 6 - \sqrt{x}$.

a) Use integration to determine the coordinates of the centre of mass of a uniform lamina of identical shape and measurements as that of R.

A rectangular lamina of identical thickness and density as R, measuring 36 units by d units is attached to R, as shown in figure 2. The centre of mass of the resulting composite now lies on the x axis.

b) Determine the x coordinate of the centre of mass of the composite

$$\boxed{(10.8, 1.5)}, \quad \boxed{\bar{x} \approx 14.76}$$



Question 15 **(****)**

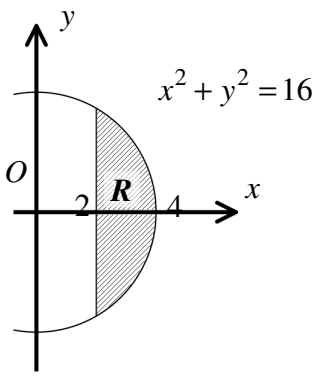


Figure 1

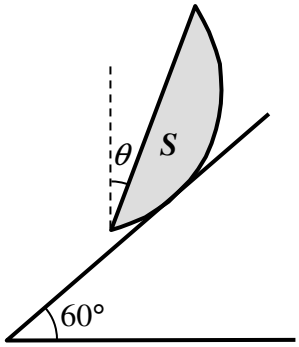


Figure 2

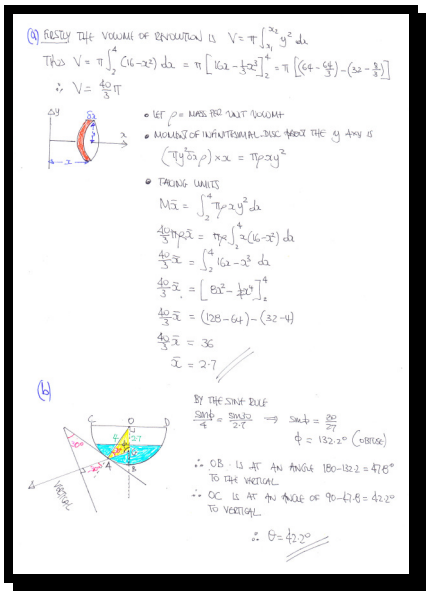
Figure 1 above shows the finite region R bounded by the circle and the line with respective equations $x^2 + y^2 = 16$ and $x = 2$. This region is fully revolved about the x axis to form a solid of revolution S

a) Use integration to determine the x coordinate of the centre of mass of S.

The solid S is carefully placed on a rough plane inclined at $60°$ to the horizontal and remains in equilibrium without slipping or toppling, as shown in figure 2.

b) Determine the angle the plane face of S makes with the upward vertical, marked as θ in figure 2.

$\boxed{\bar{x} = 2.7}$, $\boxed{\theta \approx 42.2°}$



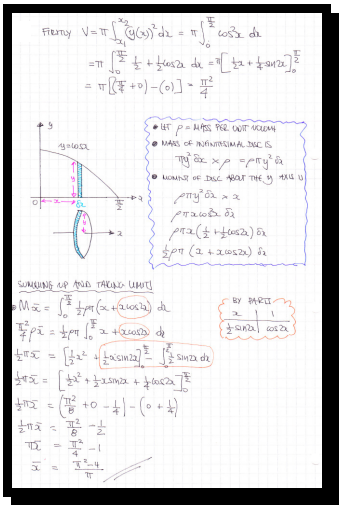
Question 16 **(****+)**

A finite region is bounded by the part of the curve with equation $y=\cos x$, the positive x axis and the positive y axis.

This region is rotated by 2π radians in the x axis forming a uniform solid S.

Use integration to find the x coordinate of the centre of mass of S.

$$\frac{\pi^2-4}{4\pi}$$



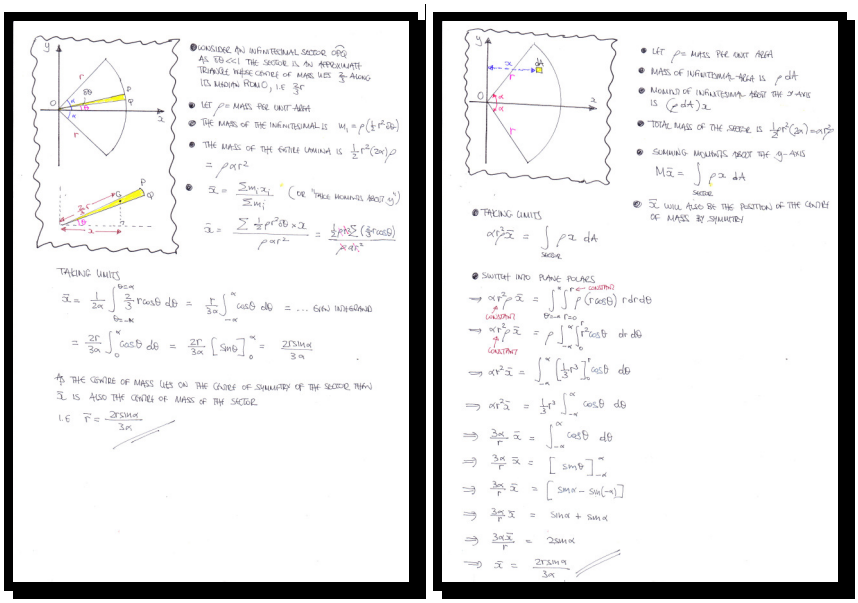
Question 17 (**+)**

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

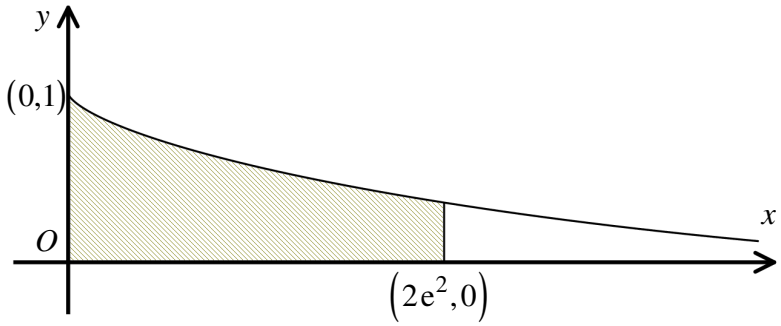
Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

proof



Question 18 (*****)



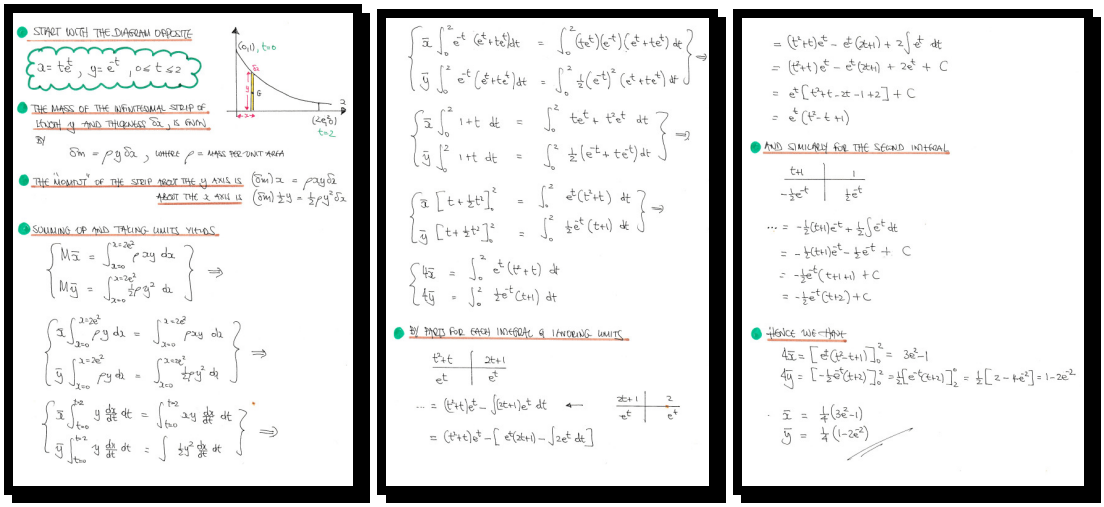
The figure above shows part of the curve with parametric equations

$$x = te^t, \quad y = e^{-t}, \quad t \in \mathbb{R}.$$

A uniform lamina occupies the finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x = 2e^2$.

Determine the exact coordinates of the centre of mass of this lamina.

$$\square, \quad \boxed{\bar{x} = \frac{1}{4}\left(3e^2 - 1\right)}, \quad \boxed{\bar{y} = \frac{1}{4}\left(1 - 2e^{-2}\right)}$$



Created by T. Madas