VECTOR

EXAM QUESTIONS

Part B

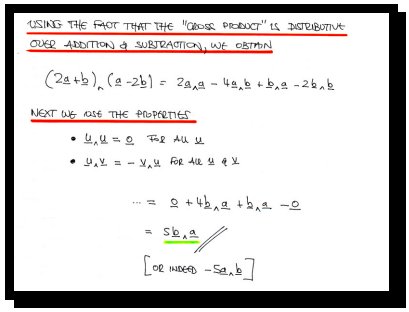
Question 1 (**)

The vectors $\mathbf{a}$ and $\mathbf{b}$, are not parallel.

Simplify fully the following expression

$$(2\mathbf{a}+\mathbf{b})\wedge(\mathbf{a}-2\mathbf{b}).$$

$\boxed{}$, $\boxed{5\mathbf{b}\wedge\mathbf{a}=-5\mathbf{a}\wedge\mathbf{b}}$



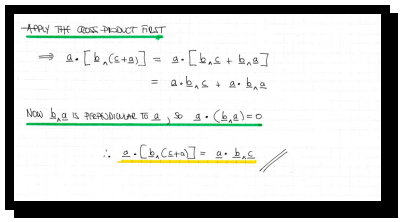
Question 2 (**)

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not parallel.

Simplify fully

$$\mathbf{a}\cdot\left[\mathbf{b}\wedge(\mathbf{c}+\mathbf{a})\right].$$

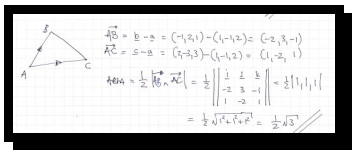
$\boxed{}$, $\boxed{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})}$



Question 3 (**)

Find the area of the triangle with vertices at $A(1,-1,2)$, $B(-1,2,1)$ and $C(2,-3,3)$.

$$\boxed{\frac{1}{2}\sqrt{3}}$$



Question 4 (**)

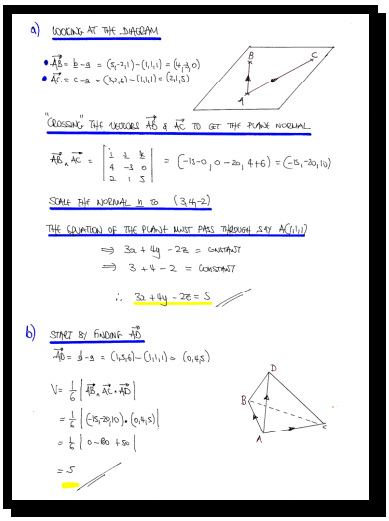
Referred to a fixed origin the coordinates of the following points are given

$$A(1,1,1),\ B(5,-2,1),\ C(3,2,6) \text{ and } D(1,5,6).$$

a) Find a Cartesian equation for the plane containing the points A, B and C.

b) Determine the volume of the tetrahedron $ABCD$.

$$\boxed{},\ \boxed{3x+4y-2z=5},\ \boxed{\text{volume}=5}$$



Question 5 (**)

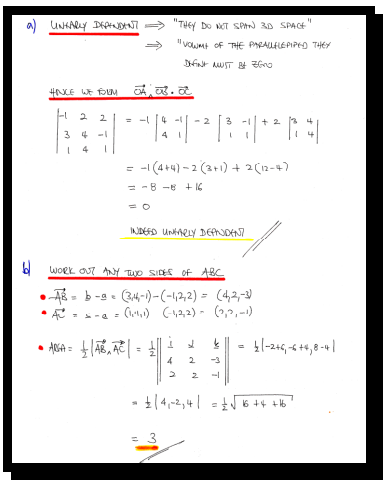
The position vectors of the points A, B and C are given below

$$\overrightarrow{OA}=-\mathbf{i}+2\mathbf{j}+2\mathbf{k}\,,\quad \overrightarrow{OB}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}\quad\text{and}\quad \overrightarrow{OC}=\mathbf{i}+4\mathbf{j}+\mathbf{k}\,.$$

a) Show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly dependent.

b) Find the area of the triangle ABC.

$\boxed{}$, $\boxed{\text{area}=3}$

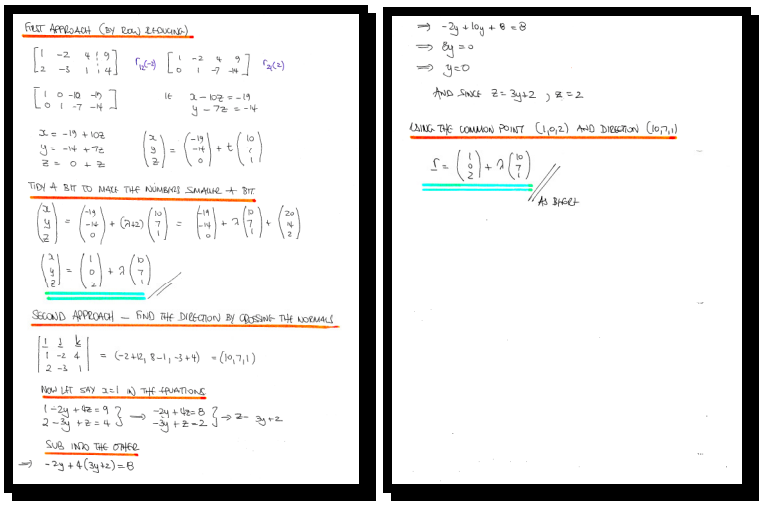


Question 6 **(**)**

Find the equation of the straight line which is common to the planes

$$x-2y+4z=9 \quad \text{and} \quad 2x-3y+z=4.$$

$\boxed{}$, $\boxed{\mathbf{r}=(\mathbf{i}+2\mathbf{k})+\lambda(10\mathbf{i}+7\mathbf{j}+\mathbf{k}) \text{ or } [\mathbf{r}-(\mathbf{i}+2\mathbf{k})]\wedge(10\mathbf{i}+7\mathbf{j}+\mathbf{k})=\mathbf{0}}$



Question 7 (+)**

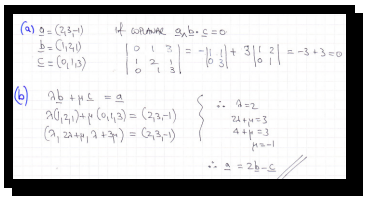
The following vectors are given.

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{j} + 3\mathbf{k}$$

a) Show the three vectors are coplanar.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$$\boxed{\mathbf{a} = 2\mathbf{b} - \mathbf{c}}$$



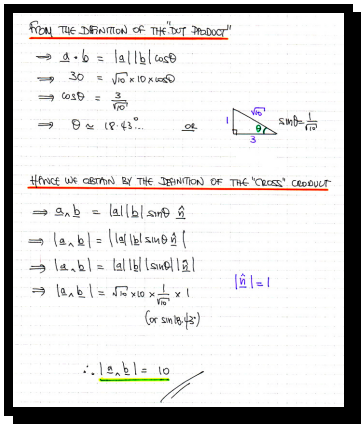
Question 8 (+)**

The vectors $\mathbf{a}$ and $\mathbf{b}$ are such so that

$$|\mathbf{a}|=\sqrt{10}\,,\quad |\mathbf{b}|=10 \quad \text{and} \quad \mathbf{a}\cdot\mathbf{b}=30\,.$$

Find the value of $|\mathbf{a}\wedge\mathbf{b}|$.

$\square$, $\boxed{|\mathbf{a}\wedge\mathbf{b}|=10}$



Question 9 (+)**

With respect to a fixed origin O, the points A and B have position vectors given by

$$\mathbf{a}=3\mathbf{i}-\mathbf{j}+2\mathbf{k} \quad \text{and} \quad \mathbf{b}=2\mathbf{i}+\mathbf{j}-\mathbf{k}\,.$$

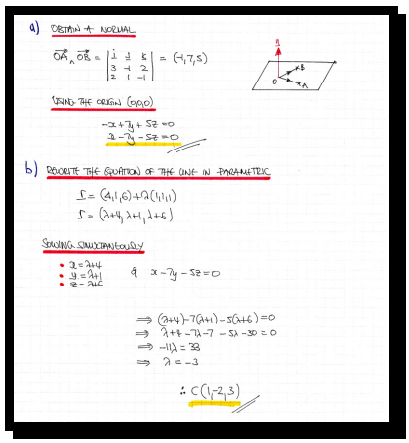
a) Find a Cartesian equation of the plane that passes through O, A and B.

A straight line has a vector equation

$$\left[\mathbf{r}-\left(4\mathbf{i}+\mathbf{j}+6\mathbf{k}\right)\right]\wedge\left(\mathbf{i}+\mathbf{j}+\mathbf{k}\right)=\mathbf{0}\,.$$

b) Determine the coordinates of the point C, where C is the intersection between the straight line and the plane.

$\boxed{}$, $\boxed{x-7y-5z=0}$, $\boxed{C\left(1,-2,3\right)}$



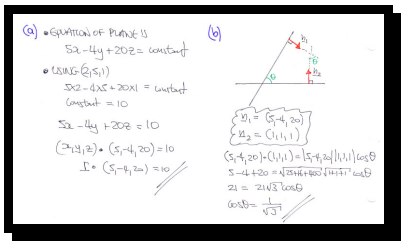
Question 10 (+)**

The plane Π_1 passes through the point with coordinates $(2,5,1)$ and is perpendicular to the vector $5\mathbf{i}-4\mathbf{j}+20\mathbf{k}$.

a) Find a vector equation of Π_1, in the form $\mathbf{r}\cdot\mathbf{n}=d$.

b) Calculate the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation $x+y+z=10$.

$$\boxed{\mathbf{r}\cdot(5\mathbf{i}-4\mathbf{j}+20\mathbf{k})=10}\,,\qquad \boxed{\cos\theta=\frac{1}{\sqrt{3}}}$$



Question 11 (+)**

The following three vectors are given

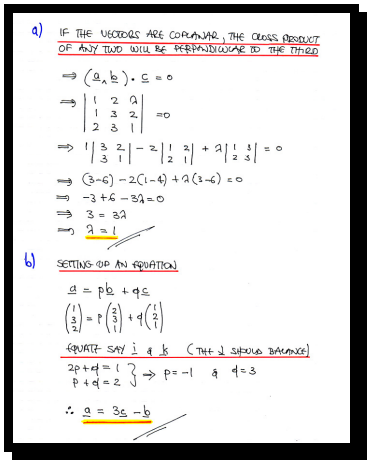
$$\mathbf{a}=\mathbf{i}+3\mathbf{j}+2\mathbf{k}$$
$$\mathbf{b}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$$
$$\mathbf{c}=\mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$$

where λ is a scalar constant.

a) If the three vectors given above are coplanar, find the value of λ.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$\boxed{}$, $\boxed{\lambda=1}$, $\boxed{\mathbf{a}=3\mathbf{c}-\mathbf{b}}$


a) IF THE VECTORS ARE COPLANAR, THE CROSS PRODUCT OF ANY TWO WILL BE PERPENDICULAR TO THE THIRD

$\Rightarrow (\underline{a}\times\underline{b})\cdot\underline{c}=0$

$\Rightarrow \begin{vmatrix} 1 & 2 & \lambda \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix}=0$

$\Rightarrow 1\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix}-2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}+\lambda\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}=0$

$\Rightarrow (3-6)-2(1-4)+\lambda(3-6)=0$

$\Rightarrow -3+6-3\lambda=0$

$\Rightarrow 3=3\lambda$

$\Rightarrow \lambda=1$

b) SETTING UP AN EQUATION

$\underline{a}=p\underline{b}+q\underline{c}$

$\begin{pmatrix}1\\3\\2\end{pmatrix}=p\begin{pmatrix}2\\3\\1\end{pmatrix}+q\begin{pmatrix}1\\2\\1\end{pmatrix}$

EQUATE - SAY $\underline{i}$ & $\underline{j}$ (THE $\underline{k}$ SHOULD BALANCE)

$2p+q=1$, $p+q=2$ $\Rightarrow p=-1$ & $q=3$

$\therefore \underline{a}=3\underline{c}-\underline{b}$

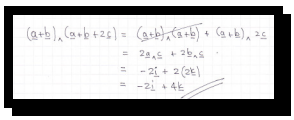
Question 12 (***)

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such so that

$$\mathbf{c}\wedge\mathbf{a}=\mathbf{i} \quad \text{and} \quad \mathbf{b}\wedge\mathbf{c}=2\mathbf{k}\,.$$

Express $(\mathbf{a}+\mathbf{b})\wedge(\mathbf{a}+\mathbf{b}+2\mathbf{c})$ in terms of $\mathbf{i}$ and $\mathbf{k}$.

$$\boxed{-2\mathbf{i}+4\mathbf{k}}$$



Question 13 (***)

Relative to a fixed origin O, the position vectors of the points A, B and C are

$$\overrightarrow{OA}=\begin{pmatrix}1\\-1\\-1\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}2\\3\\-1\end{pmatrix} \quad \text{and} \quad \overrightarrow{OC}=\begin{pmatrix}4\\-1\\5\end{pmatrix}.$$

a) Show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly independent.

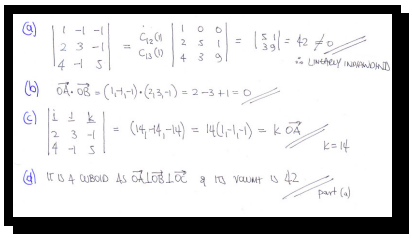
b) Evaluate $\overrightarrow{OA}\cdot\overrightarrow{OB}$.

c) Show that $\overrightarrow{OB}\wedge\overrightarrow{OC}=k\overrightarrow{OA}$, where k is a constant.

The points O, A, B and C are vertices of a solid.

d) Describe the solid geometrically and state its volume.

$$\boxed{\overrightarrow{OA}\cdot\overrightarrow{OB}=0}\,,\quad \boxed{k=14}\,,\quad \boxed{\text{cuboid, volume}=42}$$



Question 14 (***)

Relative to a fixed origin O, the plane Π_1 passes through the points A, B and C with position vectors $\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $6\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, respectively.

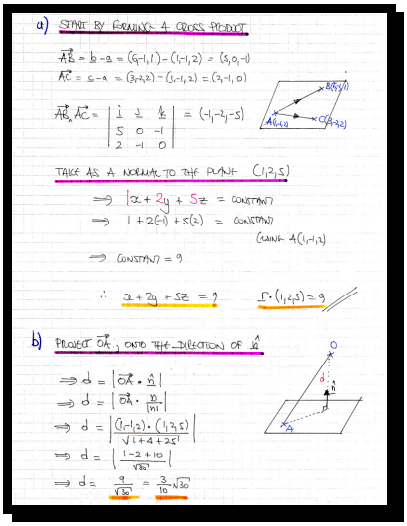
a) Determine an equation of Π_1 in the form $\mathbf{r}\cdot\mathbf{n}=c$, where $\mathbf{n}$ is the normal to Π_1 and c is a scalar constant.

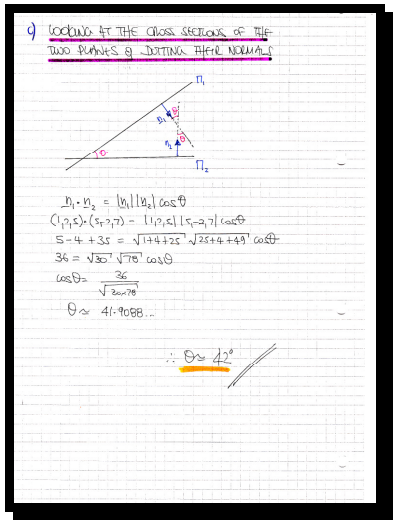
b) Find, in exact surd form, the shortest distance of Π_1 from the origin O.

The plane Π_2 passes through the point A and has normal $5\mathbf{i}-2\mathbf{j}+7\mathbf{k}$.

c) Calculate, to the nearest degree, the acute angle between Π_1 and Π_2.

$\boxed{}$, $\boxed{\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+5\mathbf{k})=9}$, $\boxed{\frac{3}{10}\sqrt{10}}$, $\boxed{42^\circ}$





Question 15 (***)

Relative to a fixed origin O, the points A, B and C have position vectors

$$\mathbf{a}=\begin{pmatrix}4\\1\\1\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}3\\2\\2\end{pmatrix}\quad \text{and}\quad \mathbf{c}=\begin{pmatrix}3-2\lambda\\ \lambda+5\\ \lambda+17\end{pmatrix},$$

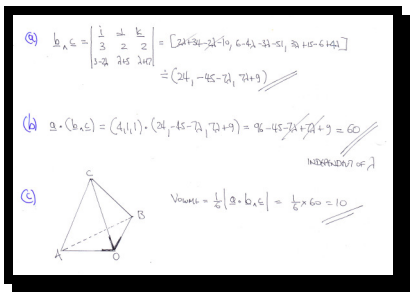
where λ is a scalar parameter.

a) Find the $\mathbf{b}\wedge\mathbf{c}$ in terms of λ.

b) Show that $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$ is independent of λ.

c) Find the volume of the tetrahedron and $OABC$.

$\boxed{24\mathbf{i}-(7\lambda+45)\mathbf{j}+(7\lambda+9)\mathbf{k}}$, $\boxed{\text{area}=10}$



Question 16 (***)

With respect to a fixed origin O, the points $A(0,1,2)$, $B(2,3,1)$ and $C(1,1,3)$ are all contained by the plane Π.

a) Calculate the area of the triangle ABC.

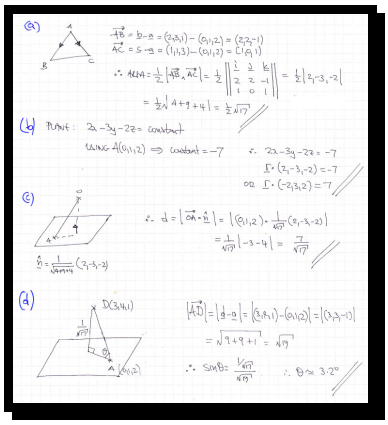
b) Determine an equation of Π, giving the answer in the form $\mathbf{r}\cdot\mathbf{n}=c$, where $\mathbf{n}$ is a normal to Π and c is a scalar constant.

c) Find the distance of Π from the origin O.

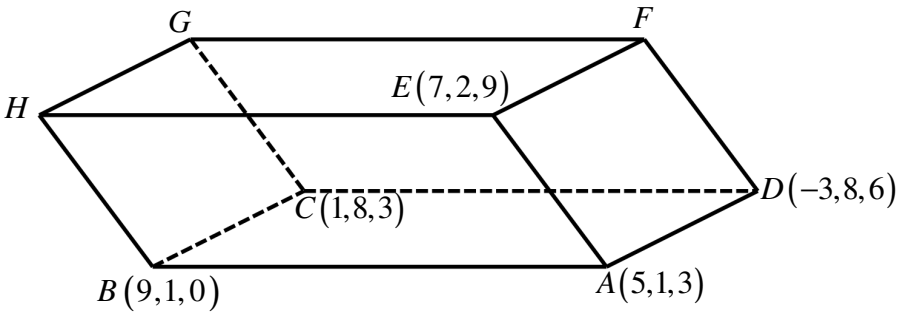
The distance of the point $D(3,4,1)$ from the plane Π is $\dfrac{1}{\sqrt{17}}$.

d) Calculate, correct to one decimal place, the acute angle between AD and Π.

$\boxed{\text{area}=\frac{1}{2}\sqrt{17}}$, $\boxed{\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}-2\mathbf{k})=-7}$, $\boxed{\text{distance}=\frac{7}{\sqrt{17}}}$, $\boxed{3.2°}$



Question 17 (***)



The figure above shows a parallelepiped.

Relative to a fixed origin O, the vertices of the parallelepiped at A, B, C, D and E have respective position vectors

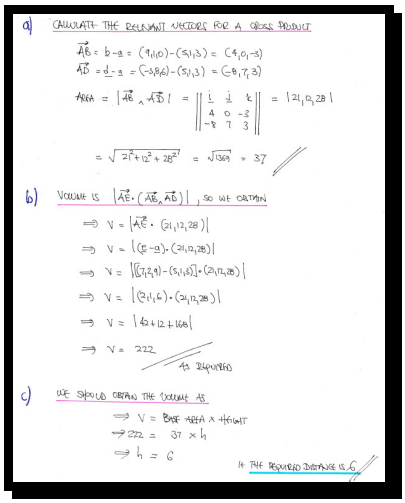
$$\mathbf{a}=5\mathbf{i}+\mathbf{j}+3\mathbf{k},$$
$$\mathbf{b}=9\mathbf{i}+\mathbf{j},$$
$$\mathbf{c}=\mathbf{i}+8\mathbf{j}+3\mathbf{k},$$
$$\mathbf{d}=-3\mathbf{i}+8\mathbf{j}+6\mathbf{k}$$
$$\mathbf{e}=7\mathbf{i}+2\mathbf{j}+9\mathbf{k}.$$

a) Calculate the area of the face $ABCD$.

b) Show that the volume of parallelepiped is 222 cubic units.

c) Hence, find the distance between the faces $ABCD$ and $EFGH$

$\boxed{}$, $\boxed{\text{area}=37}$, $\boxed{\text{distance}=6}$



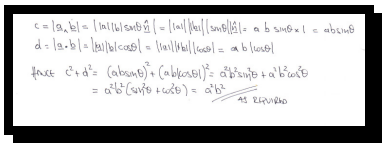
Question 18 (***)

Two non zero vectors $\mathbf{a}$ and $\mathbf{b}$ have respective magnitudes a and b, respectively.

Given that $c=|\mathbf{a}\wedge\mathbf{b}|$ and $d=|\mathbf{a}\cdot\mathbf{b}|$, show that

$$c^2+d^2=a^2b^2.$$

proof



Question 19 (***)

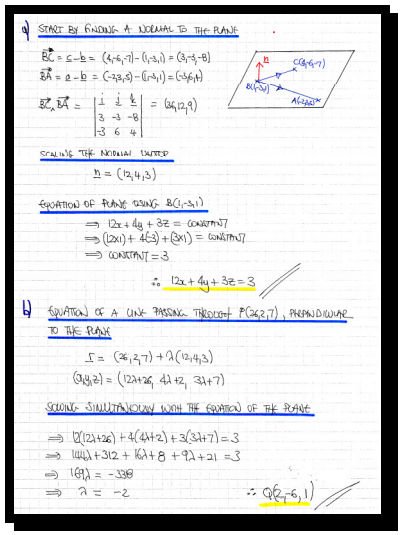
Relative to a fixed origin O, the points $A(-2,3,5)$, $B(1,-3,1)$ and $C(4,-6,-7)$ lie on the plane Π.

a) Find a Cartesian equation for Π.

The perpendicular from the point $P(26,2,7)$ meets the Π at the point Q.

b) Determine the coordinates of Q.

$\boxed{}$, $\boxed{12x+4y+3z=3}$, $\boxed{Q(2,-6,1)}$



Question 20 (***)

The points $A(3,1,0)$, $B(0,2,2)$ and $C(3,3,1)$ form a plane Π.

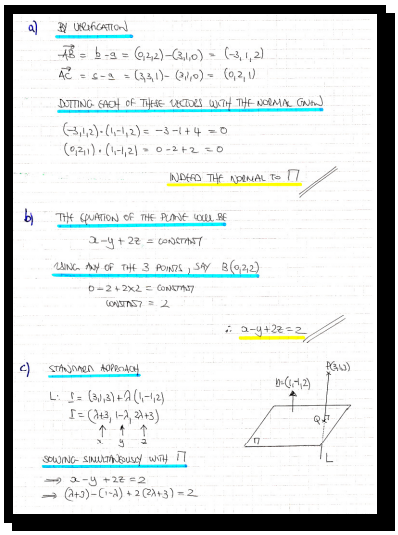
a) Show that $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is a normal to Π.

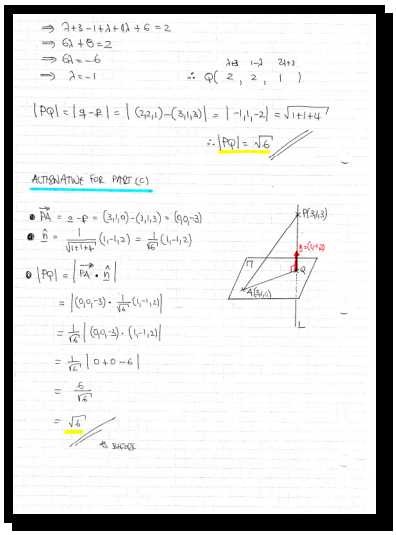
b) Find a Cartesian equation for Π.

The straight line L passes through the point $P(3,1,3)$ and meets Π at right angles at the point Q.

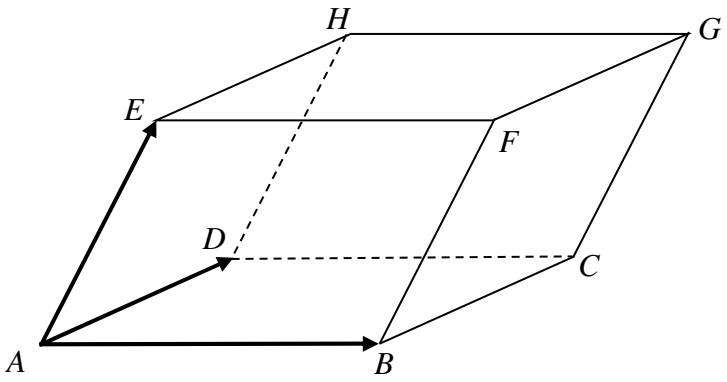
c) Determine the distance PQ.

$\square$, $\boxed{x-y+2z=2}$, $\boxed{|PQ|=\sqrt{6}}$





Question 21 (***)



The figure above shows a parallelepiped, whose vertices are located at the points $A(2,1,t)$, $B(3,3,2)$, $D(4,0,5)$ and $E(1,-2,7)$, where t is a constant.

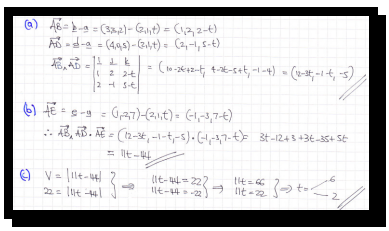
a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AD}$, in terms of t.

b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE}$

The volume of the parallelepiped is 22 cubic units.

c) Determine the possible values of t.

$\boxed{(12-3t)\mathbf{i}+(-t-1)\mathbf{j}-5\mathbf{k}}$, $\boxed{11t-44}$, $\boxed{t=2,6}$



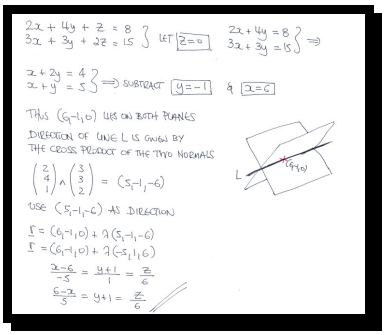
Question 22 (***)

Find in Cartesian form the equation of the intersection between the planes with the following equations

$$2x+4y+z=0$$

$$3x+3y+2z=15\,.$$

$$\boxed{\frac{6-x}{5}=y+1=\frac{z}{6}}$$



Question 23 (***)

Two planes have Cartesian equations

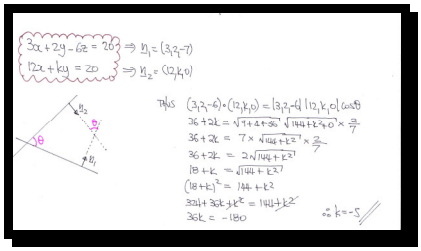
$$3x+2y-6z=20 \qquad \text{and} \qquad 12x+ky=20\,,$$

where k is a non zero constant.

The acute angle between the two planes is θ.

Given that $\cos\theta=\frac{2}{7}$, determine the value of k.

$$\boxed{k=-5}$$



Question 24 **(***)**

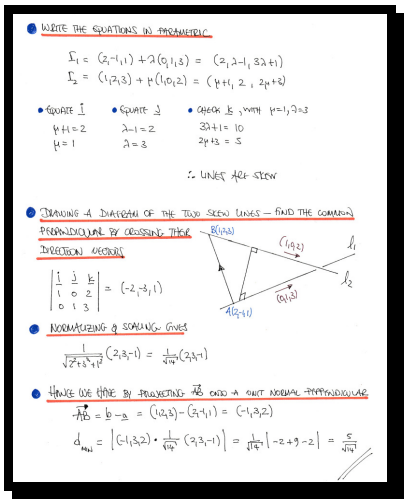
The straight lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = 2\mathbf{i}-\mathbf{j}+\mathbf{k}+\lambda(\mathbf{j}+3\mathbf{k})$$
$$\mathbf{r}_2 = \mathbf{i}+2\mathbf{j}+3\mathbf{k}+\mu(\mathbf{i}+2\mathbf{k})$$

where λ and μ are scalar parameters.

Show that l_1 and l_2 are skew and hence find the shortest distance between them.

$$\boxed{}\,,\quad \boxed{\frac{5}{\sqrt{14}}}$$



Question 25 (***)

The points $A(1,-3,1)$, $B(-1,-2,0)$ and $C(0,-1,-4)$ define a plane Π.

a) Show that $\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is a normal to Π.

b) Determine a Cartesian equation for Π.

The straight line L has equation

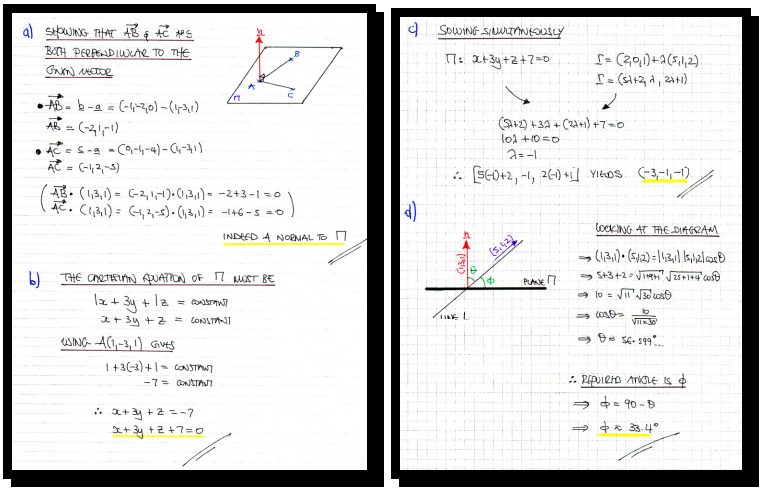
$$\mathbf{r}=2\mathbf{i}+\mathbf{k}+\lambda(5\mathbf{i}+\mathbf{j}+2\mathbf{k}),$$

where λ is a scalar parameter.

c) Find the coordinates of the point of intersection between Π and L.

d) Calculate the size of the acute angle between Π and L.

$\boxed{}$, $\boxed{x+3y+z+7=0}$, $\boxed{(-3,-1,-1)}$, $\boxed{33.4^\circ}$



Question 26 (*+)**

A tetrahedron has its four vertices at the points $A(-3,6,4)$, $B(0,11,0)$, $C(4,1,28)$ and $D(7,k,24)$, where k is a constant.

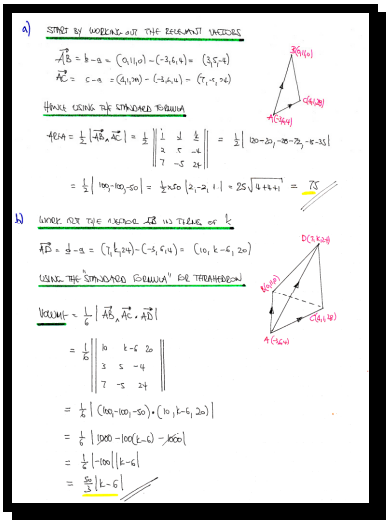
a) Calculate the area of the triangle ABC.

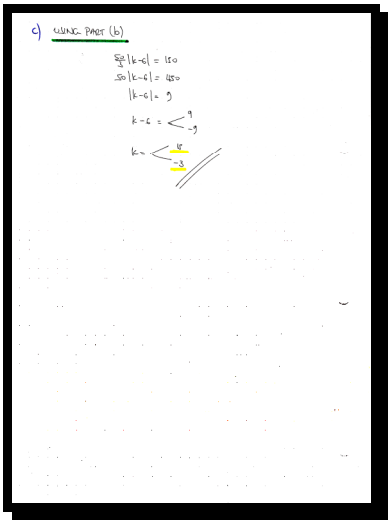
b) Find the volume of the tetrahedron $ABCD$, in terms of k.

The volume of the tetrahedron is 150 cubic units.

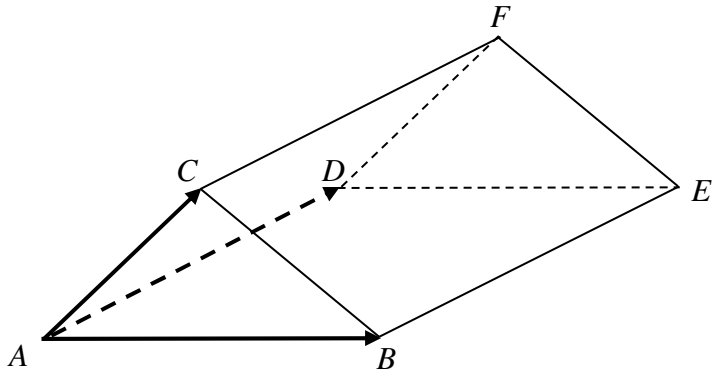
c) Determine the possible values of k.

$\boxed{}$, $\boxed{\text{area}=75}$, $\boxed{\text{volume}=\frac{50}{3}|k-6|}$, $\boxed{k=-3,\quad k=15}$





Question 27 (*+)**



A triangular prism has vertices at $A(3,3,3)$, $B(1,3,t)$, $C(5,1,5)$ and $F(8,0,10)$, where t is a constant.

The face ABC is parallel to the face DEF and the lines AD, BE and CF are parallel to each other.

a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AC}$, in terms of t.

b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}$, in terms of t.

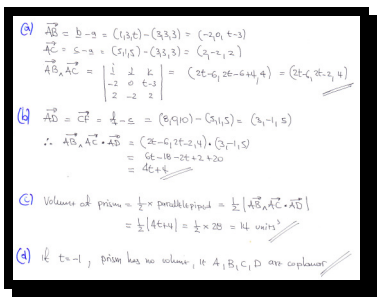
The value of t is taken to be 6.

c) Determine the volume of the prism for this value of t.

d) Explain the geometrical significance if $t=-1$.

$\boxed{(2t-6)\mathbf{i}+(2t-2)\mathbf{j}+4\mathbf{k}}$, $\boxed{4t+4}$, $\boxed{V=14 \text{ cubic units}}$,

$\boxed{A,B,C,D \text{ are coplanar, so no volume}}$



Question 28 (*+)**

Relative to a fixed origin O the point P has coordinates $(1,2,1)$.

A plane Π has Cartesian equation

$$2x+y+3z=21.$$

The straight line L passes through the point P and it is perpendicular to Π.

a) Find the coordinates of the point M, where M is the intersection of Π and L.

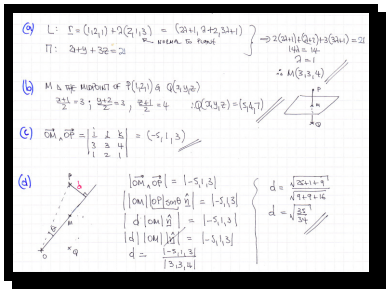
The point Q is the reflection of P about Π.

b) Find the coordinates of Q.

c) Find $\overrightarrow{OM}\wedge\overrightarrow{OP}$.

d) Hence, or otherwise, find the shortest distance from the point P to the straight line OM, giving the answer in exact form.

$\boxed{M(3,3,4)}$, $\boxed{Q(5,4,7)}$, $\boxed{5\mathbf{i}-\mathbf{j}-3\mathbf{k}}$, $\boxed{\text{distance}=\sqrt{\frac{35}{34}}}$



Question 29 (*+)**

The plane Π has an equation given by

$$\mathbf{r}=4\mathbf{i}+\mathbf{k}+\lambda(2\mathbf{j}-\mathbf{k})+\mu(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}),$$

where λ and μ are scalar parameters.

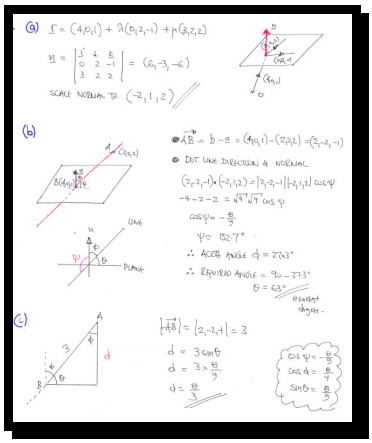
a) Find a normal vector to this plane.

The straight line L passes through the point $A(2,2,2)$ and meets Π at the point $B(4,0,1)$.

b) Calculate, to the nearest degree, the acute angle between L and Π.

c) Hence, or otherwise, find the shortest distance from A to Π.

$$\boxed{\mathbf{n}=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}}\,,\quad \boxed{63^\circ}\,,\quad \boxed{\text{distance}=\frac{8}{3}}$$



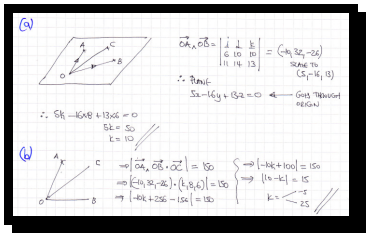
Question 30 (*+)**

With respect to a fixed origin O the points A, B and C, have respective coordinates $(6,10,10)$, $(11,14,13)$ and $(k,8,6)$, where k is a constant.

a) Given that all the three points lie on a plane which contains the origin, find the value of k.

b) Given instead that OA, OB, OC are edges of a parallelepiped of volume 150 cubic units determine the possible values of k.

$\boxed{k=10}$, $\boxed{k=-5, \quad k=25}$



Question 31 (*+)**

The straight lines L_1 and L_2 have respective Cartesian equations

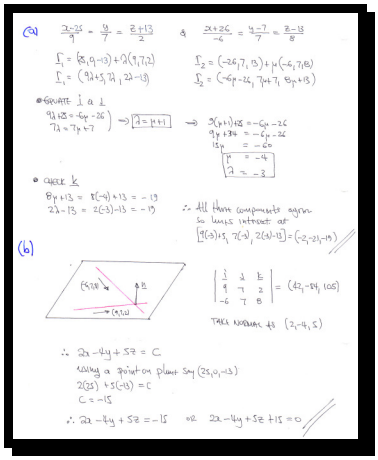
$$\frac{x-25}{9}=\frac{y}{7}=\frac{z+13}{2} \qquad \text{and} \qquad \frac{x+26}{-6}=\frac{y-7}{7}=\frac{z-13}{8}.$$

a) Show that L_1 and L_2 intersect at some point and find its coordinates.

The plane Π contains both L_1 and L_2.

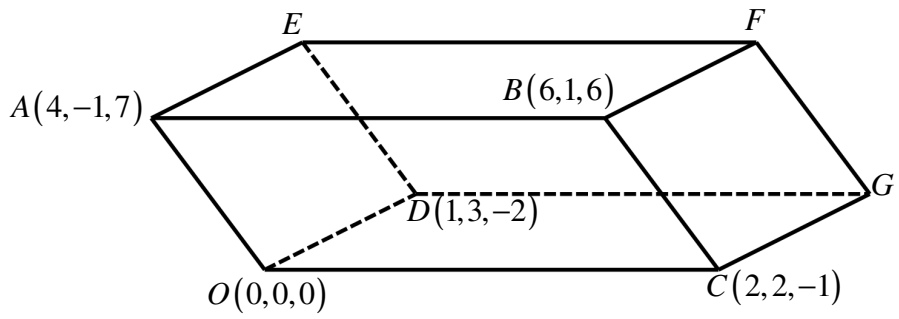
b) Find a Cartesian equation for Π.

$$\boxed{(-2,-21,-19)}, \quad \boxed{2x-4y+5z+15=0}$$



Question 32 (*+)**

The figure below shows a parallelepiped.



Relative to an origin O the points A, B, C and D have respective position vectors

$$\mathbf{a}=4\mathbf{i}-\mathbf{j}+7\mathbf{k}\,,\quad \mathbf{b}=6\mathbf{i}+\mathbf{j}+6\mathbf{k}\,,\quad \mathbf{c}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}\quad\text{and}\quad \mathbf{d}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}\,.$$

a) Find an equation of the plane $ABDG$ in the form …

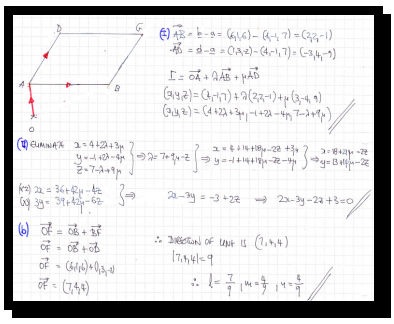
i. … $\mathbf{r}=\mathbf{u}+\lambda\mathbf{v}+\mu\mathbf{w}$.

ii. … $ax+by+cz+d=0$.

b) Hence determine the direction cosines of the straight line through O and F.

$$\boxed{\mathbf{r}=4\mathbf{i}-\mathbf{j}+7\mathbf{k}+\lambda\left(2\mathbf{i}+2\mathbf{j}-\mathbf{k}\right)+\mu\left(3\mathbf{i}-4\mathbf{j}+9\mathbf{k}\right)}\,,\quad \boxed{2x-3y-2z+3=0}\,,$$

$$\boxed{l=\frac{7}{9},m=\frac{4}{9},n=\frac{4}{9}}$$



Question 33 (*+)**

The planes Π_1 and Π_2 have the following Cartesian equations.

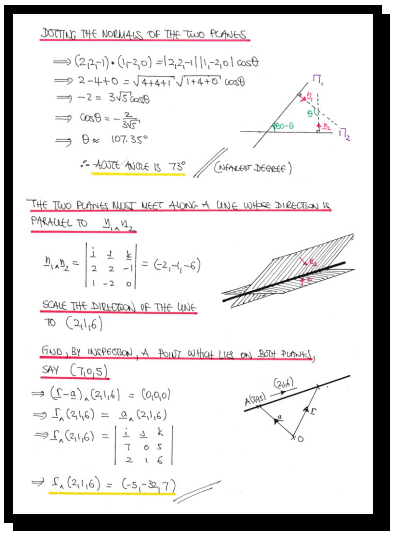
$$2x+2y-z=9$$
$$x-2y=7$$

a) Find, to the nearest degree, the acute angle between Π_1 and Π_2.

The two planes intersect along the straight line L.

b) Determine an equation of L in the form $\mathbf{r}\wedge\mathbf{a}=\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors with integer components.

$\boxed{}$, $\boxed{73^\circ}$, $\boxed{\mathbf{r}\wedge(2\mathbf{i}+\mathbf{j}+6\mathbf{k})=-5\mathbf{i}-32\mathbf{j}+7\mathbf{k}}$



Question 34 (*+)**

The straight line l has Cartesian equation

$$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{2}.$$

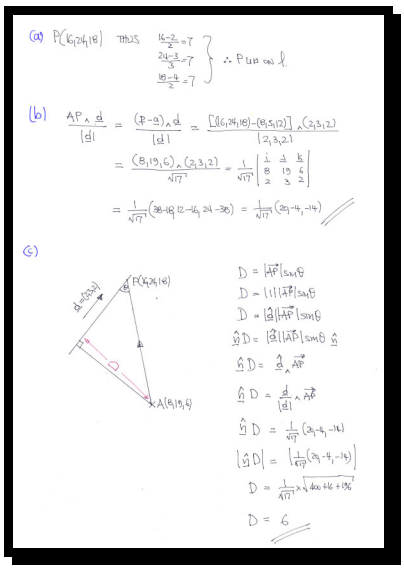
a) Show that the point P with coordinates $(16,24,18)$ lies on l.

The point A has coordinates $(8,19,6)$ and the direction vector of l is denoted by $\mathbf{d}$.

b) Calculate $\dfrac{\overrightarrow{AP}\wedge\mathbf{d}}{|\mathbf{d}|}$.

c) Hence show that the shortest distance of A from l is exactly 6 units.

$$\boxed{\frac{(20\mathbf{i}-4\mathbf{j}-14\mathbf{k})}{\sqrt{17}}}$$



Question 35 **(***+)**

The three vertices of the parallelogram $ABCD$ have coordinates

$$A(7,1,-6)\ ,\quad B(4,0,7)\quad \text{and}\quad D(-2,6,1)\ .$$

The diagonals of the parallelogram meet at the point M.

a) Determine in any order the coordinates of M and the coordinates of C.

b) Calculate in exact simplified surd form, the area of $ABCD$.

The straight line l passes through C and is perpendicular to $ABCD$.

c) Find an equation of l, giving the answer in the form $(\mathbf{r}-\mathbf{a})\wedge\mathbf{b}=\mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found.

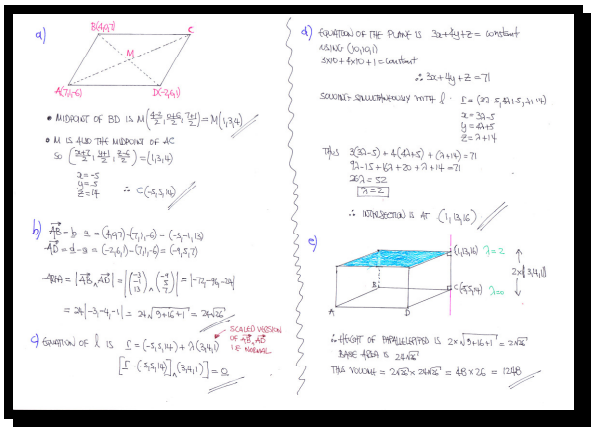
The plane Π is parallel to $ABCD$ and passes through the point with coordinates $(10,10,1)$.

d) Determine the coordinates of the point of intersection between Π and l.

The parallelogram $ABCD$ is one of the six faces of a parallelepiped whose opposite face lies in Π.

e) Calculate the volume of this parallelepiped.

$\boxed{M(1,3,4)}$, $\boxed{C(-5,5,14)}$, $\boxed{\text{area}=24\sqrt{26}}$, $\boxed{\mathbf{a}=-5\mathbf{i}+5\mathbf{j}+14\mathbf{k}}$, $\boxed{\mathbf{b}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}}$, $\boxed{(1,13,6)}$, $\boxed{\text{volume}=1248}$



Question 36 (*+)**

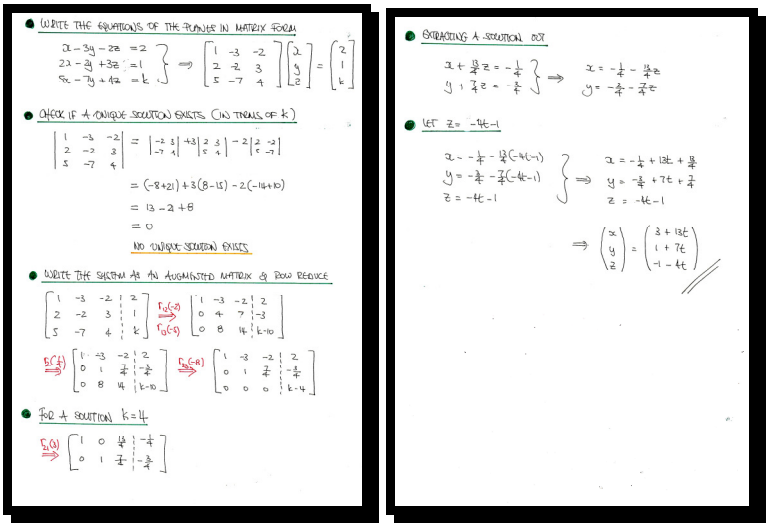
Three planes have the following Cartesian equations.

$$\begin{aligned} x-3y-2z&=2\\ 2x-2y+3z&=1\\ 5x-7y+4z&=k \end{aligned}$$

where k is a constant.

Determine the intersection of the three planes, stating any restrictions in the value of k.

$\boxed{}$, $\boxed{\mathbf{r}=3\mathbf{i}+\mathbf{j}-\mathbf{k}+t\left(13\mathbf{i}+7\mathbf{j}-4\mathbf{k}\right)}$



Question 37 (*+)**

The planes Π_1 and Π_2 have respective Cartesian equations

$$x+2y-z=1 \qquad \text{and} \qquad x+3y+z=6 .$$

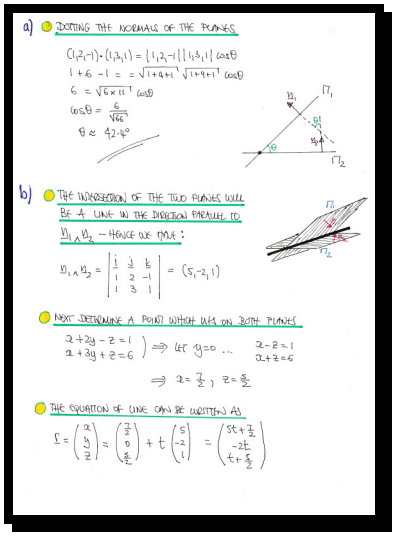
a) Find the acute angle between Π_1 and Π_2.

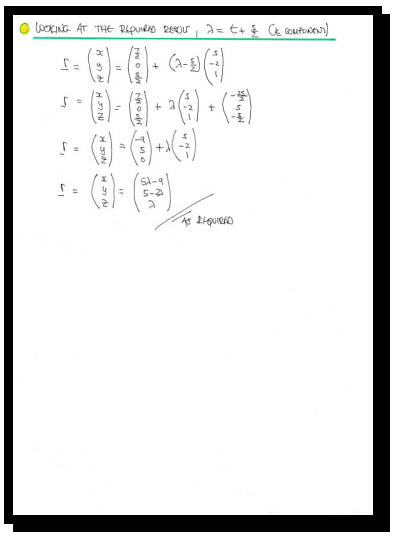
b) Show that Π_1 and Π_2 intersect along the straight line with equation

$$\mathbf{r}=(5\lambda-9)\mathbf{i}+(5-2\lambda)\mathbf{j}+\lambda\mathbf{k} ,$$

where λ is a scalar parameter.

$\boxed{}$, $\boxed{42.4°}$





Question 38 (*+)**

It is given that the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy

$$\mathbf{b}\wedge\mathbf{c}=2\mathbf{i} \qquad \text{and} \qquad \mathbf{a}\wedge\mathbf{c}=\mu\mathbf{j},$$

where μ is a scalar constant.

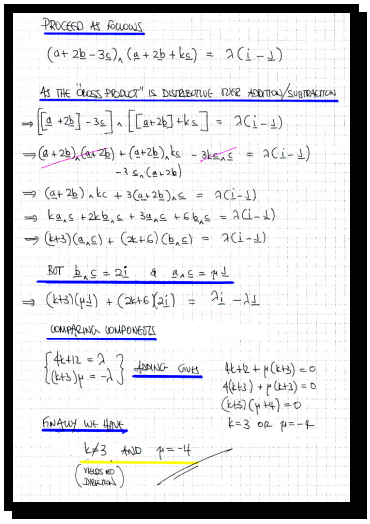
It is further given that the vector expression defined as

$$(\mathbf{a}+2\mathbf{b}-3\mathbf{c})\wedge(\mathbf{a}+2\mathbf{b}+k\mathbf{c}),$$

where k is a scalar constant, is parallel to the vector $\mathbf{i}-\mathbf{j}$.

Determine the condition that μ and k must satisfy.

$\boxed{}$, $\boxed{k\neq 3}$, $\boxed{\mu=4}$



Question 39 (*+)**

The position vector $\mathbf{r}$ of a variable point traces the plane Π with equation

$$\mathbf{r}=(4+\lambda+5\mu)\mathbf{i}+(8+2\lambda-4\mu)\mathbf{j}+(-5+\lambda+7\mu)\mathbf{k},$$

where λ and μ are parameters.

a) Express the equation of Π in the form

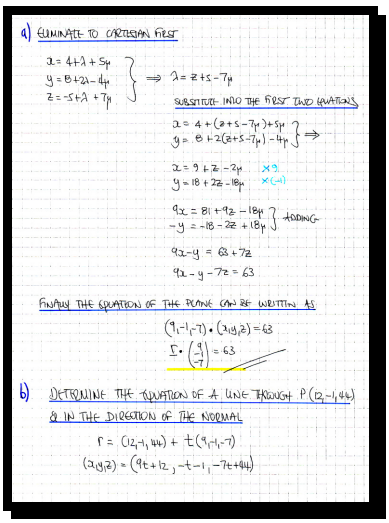
$$\mathbf{r}\cdot\mathbf{n}=c,$$

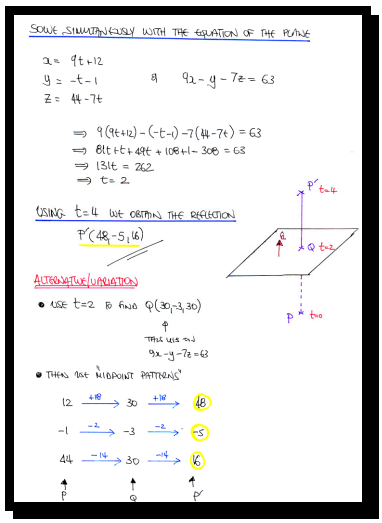
where $\mathbf{n}$ and c is a vector and scalar constant, respectively.

The point $P(12,-1,44)$ is reflected about Π onto the point P'.

b) Determine the coordinates of P'.

$\boxed{}$, $\boxed{\mathbf{r}\cdot(9\mathbf{i}-\mathbf{j}-7\mathbf{k})=63}$, $\boxed{P'(48,-5,16)}$





Question 40 (****)

The plane Π has a vector equation

$$\mathbf{r}=(1+4\lambda+3\mu)\mathbf{i}+(3+\lambda+2\mu)\mathbf{j}+(4+2\lambda-\mu)\mathbf{k},$$

where λ and μ are scalar parameters.

The straight line L has a vector equation

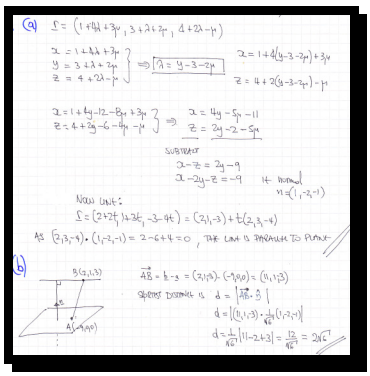
$$\mathbf{r}=(2+2t)\mathbf{i}+(1+3t)\mathbf{j}+(-3-4t)\mathbf{k},$$

where t is a scalar parameter.

a) Show that L is parallel to Π.

b) Find the shortest distance between L and Π.

$\boxed{2\sqrt{6}}$



Question 41 (****)

Relative to a fixed origin O, the following points are given.

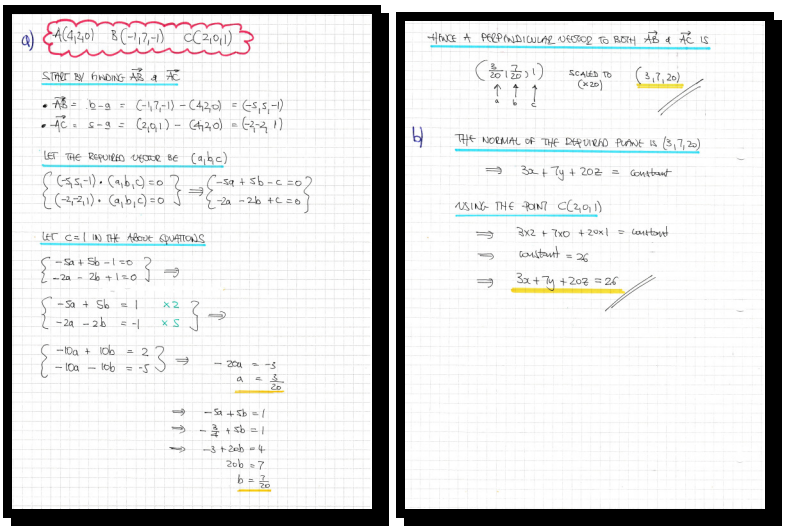
$$A(4,2,0), \quad B(-1,7,-1) \quad \text{and} \quad C(2,0,1).$$

a) Determine a vector, with integer components, which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

*You may **NOT** use the vector (cross) product for this part.*

b) Deduce a Cartesian equation of the plane, which passes through A, B and C.

$\boxed{}$, $\boxed{3\mathbf{i}+7\mathbf{j}+20\mathbf{k}}$, $\boxed{3x+7y+20z=26}$



Question 42 (****)

The straight lines L_1 and L_2 have respective Cartesian equations

$$\frac{x-2}{2}=\frac{y-3}{4}=z \quad \text{and} \quad \frac{x+2}{2}=\frac{4y}{11}=\frac{z+10}{3}.$$

a) Show that L_1 and L_2 intersect at some point P and find its coordinates.

b) Show further that the Cartesian vector $37\mathbf{i}-16\mathbf{j}-10\mathbf{k}$ is perpendicular to both L_1 and L_2.

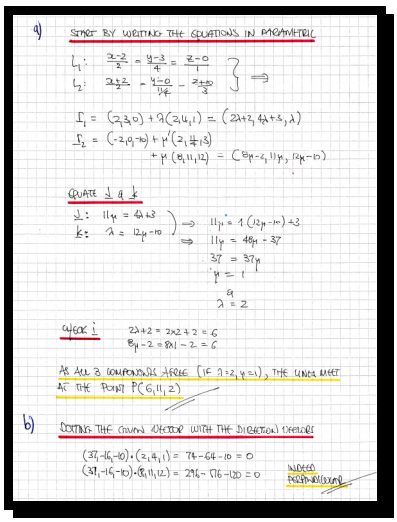
The plane Π is defined by L_1 and L_2.

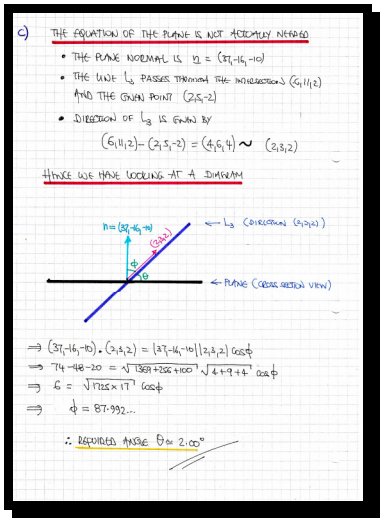
The point $Q(2,5,-2)$ does not lie on Π.

The straight line L_3 passes through Q and P.

c) Calculate the acute angle formed between L_3 and Π.

$\boxed{}$, $\boxed{P(6,11,2)}$, $\boxed{\theta \approx 2.00^\circ}$





Question 43 (****)

Relative to a fixed origin O, the following points are given.

$$A(7,2,6), \quad B(9,10,4) \quad \text{and} \quad C(-3,-2,-2).$$

a) Determine a vector, with integer components, which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence deduce a Cartesian equation of the plane Π, which passes through A, B and C.

*You may **NOT** use the vector (cross) product for this part.*

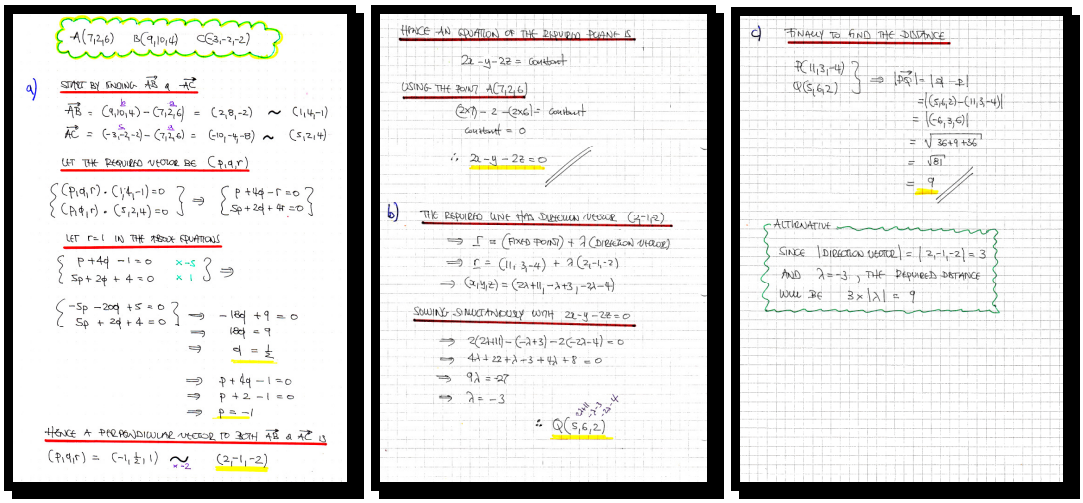
The straight line l is perpendicular to Π and passes through the point $P(11,3,-4)$.

The point Q is the intersection of l and Π.

b) Find the coordinates of Q.

c) Calculate the distance PQ.

$$\boxed{}, \quad \boxed{2\mathbf{i}-\mathbf{j}+2\mathbf{k}}, \quad \boxed{2x-y-2z=0}, \quad \boxed{Q(5,6,2)}, \quad \boxed{|PQ|=3}$$



Question 44 (****)

The straight line L and the plane Π have equations

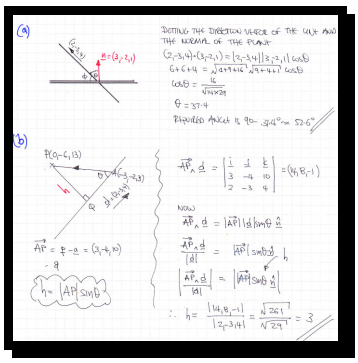
$$L \ : \ \mathbf{r}=-3\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda\left(2\mathbf{i}-3\mathbf{j}+4\mathbf{k}\right)$$

$$\Pi : \ 3x-2y+z=5$$

a) Find the size of the acute angle between L and Π.

b) Use a method involving the cross product to show that the shortest distance of the point $(0,-6,13)$ from L is 3 units.

$52.6°$



Question 45 (****)

The equations of two planes are given below

$$\mathbf{r}\cdot(6\mathbf{i}-3\mathbf{j}+2\mathbf{k})=42 \quad \text{and} \quad \mathbf{r}\cdot(17\mathbf{i}+2\mathbf{j}+\mathbf{k})=-7\,.$$

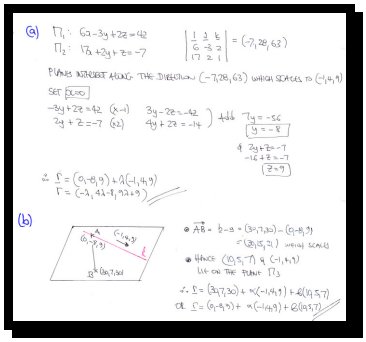
The straight line l is the intersection of the two planes.

a) Find an equation for l, in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

A third plane Π_3 contains l and the point with position vector $30\mathbf{i}+7\mathbf{j}+30\mathbf{k}$.

b) Find an equation for Π_3, in the form $\mathbf{r}=\mathbf{u}+\alpha\mathbf{v}+\beta\mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are constant vectors and α and β are scalar parameters.

$$\boxed{\mathbf{r}=-8\mathbf{j}+9\mathbf{k}+\lambda(-\mathbf{i}+4\mathbf{j}+9\mathbf{k})}\,, \quad \boxed{\mathbf{r}=(-8\mathbf{j}+9\mathbf{k})+\alpha(-\mathbf{i}+4\mathbf{j}+9\mathbf{k})+\beta(10\mathbf{i}+5\mathbf{j}+7\mathbf{k})}$$



Question 46 (****)

A triangle has vertices at $A(-2,-2,0)$, $B(6,8,6)$ and $C(-6,8,12)$.

a) Find the area of the triangle ABC.

The plane Π_1 contains the point B and is perpendicular to AB.

b) Show that an equation of Π_1 is

$$4x+5y+3z=82.$$

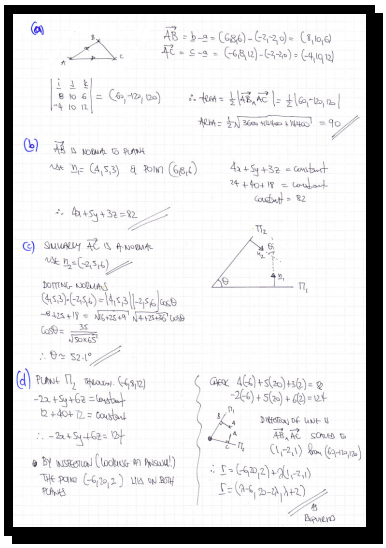
The plane Π_2 contains the point C and is perpendicular to AC.

c) Find the size of the acute angle between Π_1 and Π_2.

d) Show that the intersection of Π_1 and Π_2 is

$$(\lambda-6)\mathbf{i}+(20-2\lambda)\mathbf{j}+(2\lambda+2)\mathbf{k}.$$

$\boxed{\text{area}=90}$, $\boxed{52.1^\circ}$



Question 47 **(****)**

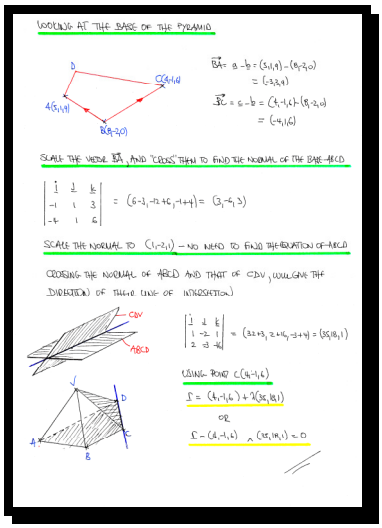
The plane quadrilateral $ABCD$ is the base of a pyramid with vertex V.

The coordinates of the points A, B and C are $(5, 1, 9)$, $(8,-2, 0)$ and $(4,-1, 6)$, respectively.

If the equation of the face CDV is $2x-3y-16z+85=0$ determine the vector equation of the line CD.

$\boxed{}$,

$$\boxed{\mathbf{r}=(4\mathbf{i}-\mathbf{j}+6\mathbf{k})+\lambda(35\mathbf{i}+18\mathbf{j}+\mathbf{k}) \text{ or } [\mathbf{r}-(4\mathbf{i}-\mathbf{j}+6\mathbf{k})]\wedge(35\mathbf{i}+18\mathbf{j}+\mathbf{k})=\mathbf{0}}$$



Question 48 (****)

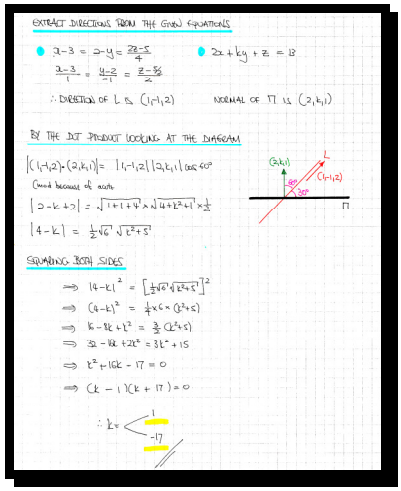
A straight line L and a plane Π have respective cartesian equations

$$L:\ x-3=2-y=\frac{1}{4}(2z-5) \quad \text{and} \quad \Pi:\ 2x+ky+z=13,$$

where k is a constant.

Given that the acute angle between L and Π is $30°$, find the possible values of k.

$\boxed{}$, $\boxed{k=1 \cup k=-17}$



Question 49 (****)

With respect to a fixed origin O the point A has position vector $\overrightarrow{OA}=-4\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

The straight line L has vector equation

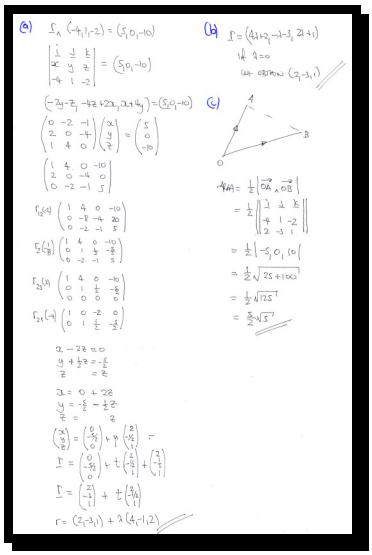
$$\mathbf{r}\wedge\overrightarrow{OA}=5\mathbf{i}-10\mathbf{k}\,.$$

a) Find, in terms of a scalar parameter λ, a vector equation of L.
Give the answer in the form $\mathbf{r}=\mathbf{p}+\lambda\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are constant vectors.

b) Verify that the point B, with position vector $\overrightarrow{OB}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$, lies on L.

c) Find the exact area of the triangle OAB.

$$\boxed{\mathbf{r}=-\frac{5}{2}\mathbf{j}+\lambda\left(4\mathbf{i}-\mathbf{j}+2\mathbf{k}\right)}\,,\quad\boxed{\text{area}=\frac{5}{2}\sqrt{5}}$$



Question 50 (****)

The planes Π_1 and Π_2 have respective Cartesian equations

$$6x+2y+9z=5 \quad \text{and} \quad 10x-y-11z=4 .$$

a) Find the acute angle between Π_1 and Π_2.

b) Show that Π_1 and Π_2 intersect along the straight line with equation

$$\mathbf{r}=\mathbf{i}-5\mathbf{j}+\mathbf{k}+t\left(\mathbf{i}-12\mathbf{j}+2\mathbf{k}\right),$$

where t is a scalar parameter.

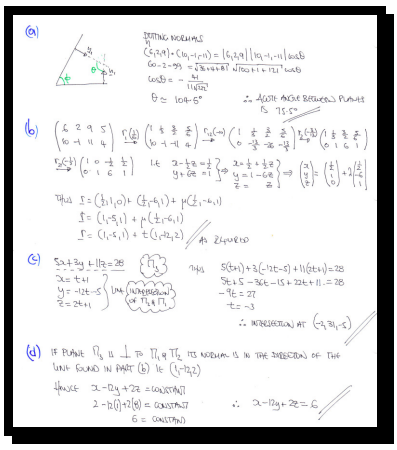
The plane Π_3 has Cartesian equation

$$5x+3y+11z=28 .$$

c) Find the coordinates of the point of intersection of all three planes.

d) Determine an equation of the plane that passes through the point $(2,1,8)$ and is perpendicular to both Π_1 and Π_2.

$$\boxed{75.5^\circ}, \quad \boxed{(-2,31,-5)}, \quad \boxed{x-12y+2z=6}$$



Question 51 (****)

The points $P(2,2,1)$ and $Q(6,-7,-1)$ lie on the plane Π with Cartesian equation

$$cx+4y-12z=k\,,$$

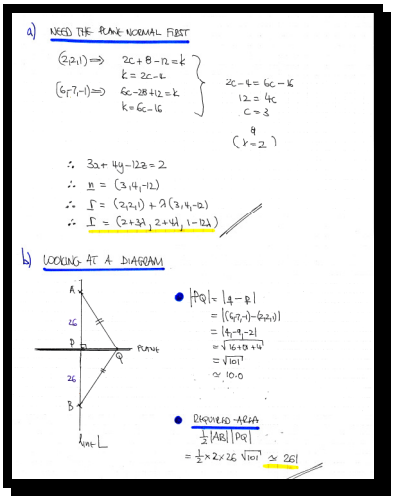
where c and k are constants.

a) Determine an equation of the straight line L, which is perpendicular to Π and passing through P.

The points A and B are both located on L and each of these points is at a distance of 26 units from Π.

b) Show that the area of the triangle ABQ is approximately 261 square units.

$\boxed{}$, $\boxed{\mathbf{r}=(3\lambda+2)\mathbf{i}+(4\lambda+2)\mathbf{j}+(1-12\lambda)\mathbf{k}}$



Question 52 (****)

The plane Π_1 contains the origin O and the points $A(2,0,-1)$ and $B(4,3,1)$.

a) Find a Cartesian equation of Π_1.

The plane Π_2 contains the point B and has normal vector $\mathbf{n}=3\mathbf{i}+\mathbf{j}-\mathbf{k}$.

b) Determine an equation of the plane in the form $\mathbf{r}\cdot\mathbf{n}=d$.

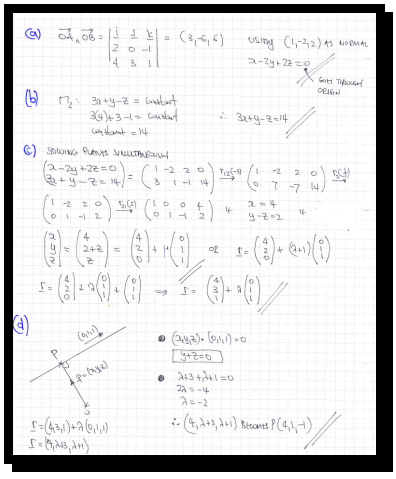
The straight line L is the intersection of Π_1 and Π_2.

The point P lies on L so that OP is perpendicular to L.

c) Find a vector equation of L.

d) Determine the coordinates of P.

$$\boxed{x-2y+2z=0}\,,\quad \boxed{\mathbf{r}\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})=14}\,,\quad \boxed{\mathbf{r}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(\mathbf{j}+\mathbf{k})}\,,\quad \boxed{P(4,1,-1)}$$



Question 53 (****)

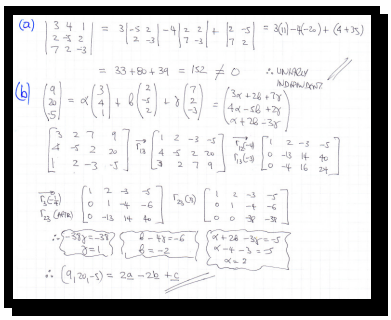
The following vectors are given

$$\mathbf{a}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}$$
$$\mathbf{b}=2\mathbf{i}-5\mathbf{j}+2\mathbf{k}$$
$$\mathbf{c}=7\mathbf{i}+2\mathbf{j}-3\mathbf{k}$$

a) Show that the vectors are linearly independent.

b) Express the vector $9\mathbf{i}+20\mathbf{j}-5\mathbf{k}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\boxed{9\mathbf{i}+20\mathbf{j}-5\mathbf{k}=2\mathbf{a}-2\mathbf{b}+\mathbf{c}}$$



Question 54 (****)

The points $A(0,2,1)$, $B(8,6,0)$ and $C(-4,1,1)$ form a plane Π_1.

a) Find a Cartesian equation for Π_1.

The point $T(1,2,t)$ lies outside Π_1.

b) Show that the shortest distance of T from Π_1 is

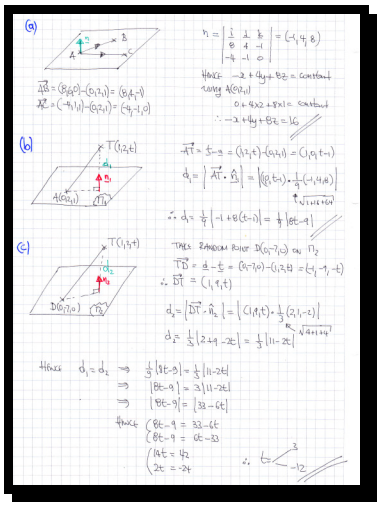
$$\left|\frac{1}{9}(8t-9)\right|.$$

The plane Π_2 has Cartesian equation

$$2x+y-2z+7=0.$$

c) Given that the T is equidistant from Π_1 and Π_2 find the possible values of t.

$\boxed{-x+4y+8z=16}$, $\boxed{t=-12,3}$



Question 55 (****)

With respect to a fixed origin O, the points $A(3,0,0)$, $B(0,2,-1)$ and $C(2,0,1)$ have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively.

a) Calculate $\overrightarrow{AC} \wedge \overrightarrow{OB}$.

The plane Π contains the point C and the straight line L with vector equation

$$(\mathbf{r}-\mathbf{a}) \wedge \mathbf{b} = \mathbf{0},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found.

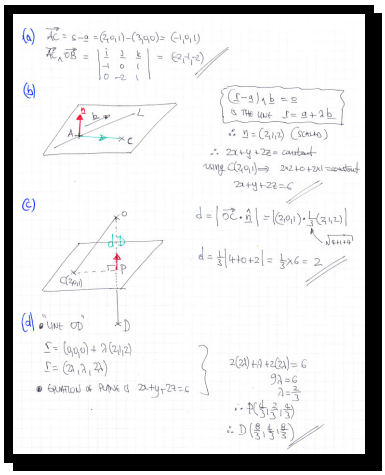
b) Find a Cartesian equation of Π.

c) Calculate the shortest distance of Π from O.

The point D is the reflection of O about Π.

d) Determine the coordinates of D.

$\boxed{-2\mathbf{i}-\mathbf{i}-2\mathbf{k}}$, $\boxed{2x+y+2z=6}$, $\boxed{\text{distance}=2}$, $\boxed{D\left(\frac{8}{3},\frac{4}{3},\frac{8}{3}\right)}$



Question 56 (****)

Relative to a fixed origin O, the point A has position vector $\mathbf{a}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

The plane Π has vector equation

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c},$$

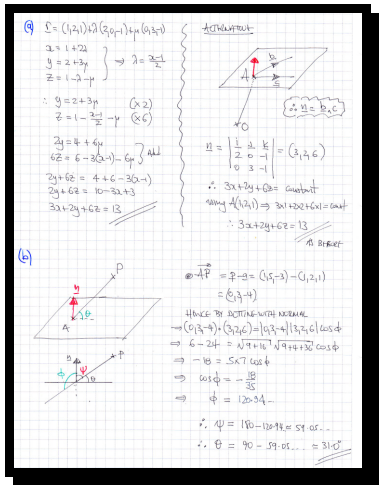
where $\mathbf{b}=2\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=3\mathbf{j}-\mathbf{k}$.

a) Find a Cartesian equation of Π.

The point P has position vector $\mathbf{i}+5\mathbf{j}-3\mathbf{k}$.

b) Calculate, to the nearest degree, the acute angle between AP and Π.

$\boxed{3x+2y+6z=13}$, $\boxed{31^\circ}$



Question 57 (****)

The system of equations below has a unique solution.

$$\begin{aligned} 5x+\ \ y+6z&=9\\ 3x+6y+2z&=8\\ 4x+2y-9z&=75 \end{aligned}$$

a) Show that $z=-5$ and find the values of x and y.

The straight line L and the plane Π have respective vector equations

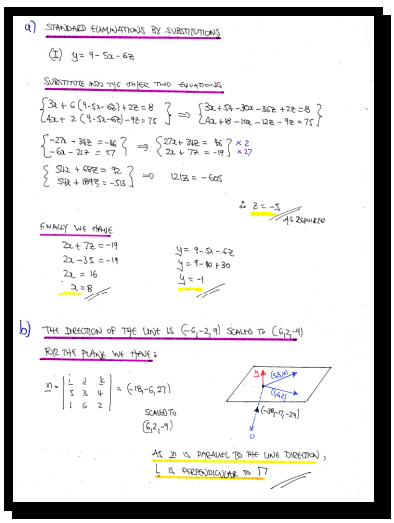
$$\mathbf{r}_1=\begin{pmatrix}-29\\-9\\46\end{pmatrix}+t\begin{pmatrix}-6\\-2\\9\end{pmatrix}\quad\text{and}\quad \mathbf{r}_2=\begin{pmatrix}-38\\-17\\-29\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\4\end{pmatrix}+\mu\begin{pmatrix}1\\6\\2\end{pmatrix},$$

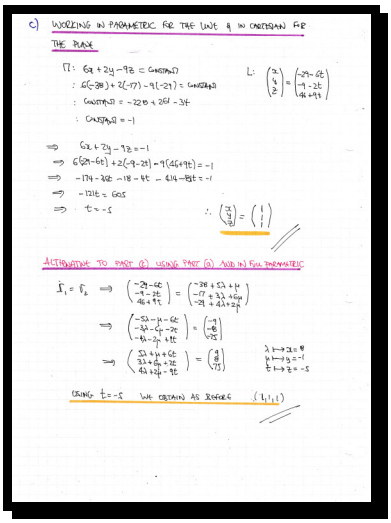
where t, λ and μ are scalar parameters.

b) Show that L is perpendicular to Π.

c) Show further that L meets Π at the point with coordinates $(1,1,1)$.

V, [ ], $\boxed{x=8,\ y=-1}$





Question 58 (****)

The straight line L has vector equation

$$\mathbf{r}=\begin{pmatrix}3\\7\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-3\end{pmatrix},$$

where λ is a scalar parameter.

The plane Π passes through the points $A(11,13,5)$ and $B(15,12,5)$.

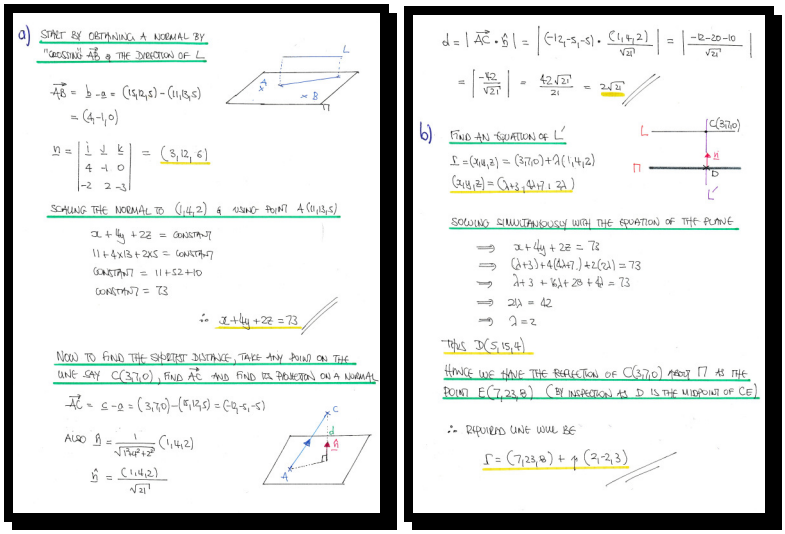
It is further given that Π is parallel to L.

a) Find a Cartesian equation for Π and hence calculate the distance between L and Π.

The straight line M is the reflection of L about Π.

b) Determine a vector equation for M.

$\boxed{}$, $\boxed{x+4y+2z=73}$, $\boxed{\text{distance}=2\sqrt{21}}$, $\boxed{\mathbf{r}=7\mathbf{i}+23\mathbf{j}+8\mathbf{k}+\mu(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})}$



Question 59 (****)

The point $P(1,3,8)$ lies on the plane Π_1.

The straight line L, whose Cartesian equation is given below also lies on Π_1.

$$x-4=\frac{y-3}{3}=\frac{2-z}{4}.$$

a) Find a Cartesian equation of Π_1.

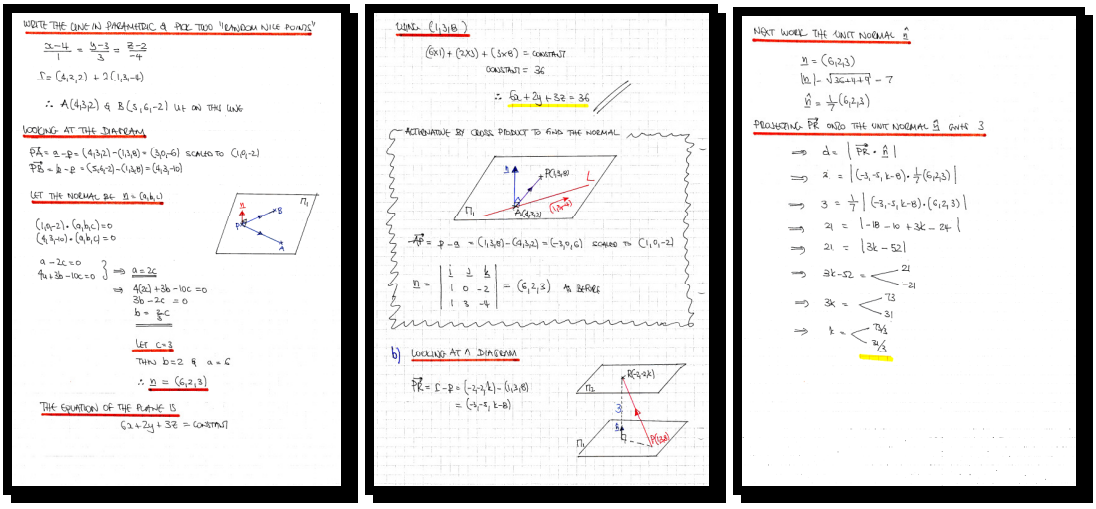
*You may not use the vector product (cross product) in part **(a)**.*

The point $R(-2,-2,k)$, where k is a constant, lies on another plane Π_2, which is parallel to Π_1.

b) Given that the distance between Π_1 and Π_2 is 3 units determine, in exact fractional form, the possible values of k.

*You may not use the standard formula which finds the distance between two parallel planes in part **(b)**.*

$\boxed{}$, $\boxed{6x+2y+3z=36}$, $\boxed{k=\frac{73}{3},\quad k=\frac{31}{3}}$



Question 60 (****)

With respect to a fixed origin O, four points have the following coordinates

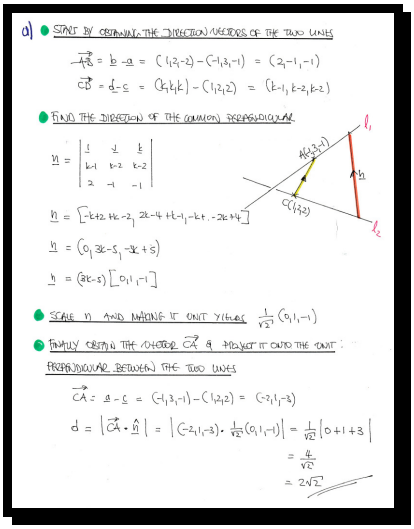
$$A(-1,3,-1),\ B(1,2,-2),\ C(1,2,2) \quad \text{and} \quad D(k,k,k),$$

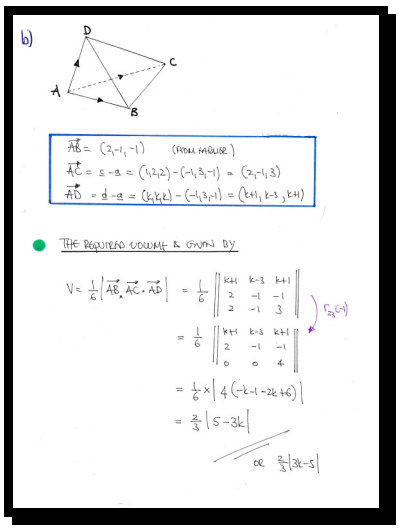
where k is a constant.

a) Determine the shortest distance between the straight lines AB and CD.

b) Find, in terms of k, the volume of the tetrahedron $ABCD$.

$\boxed{}$, $\boxed{d_{\min}=2\sqrt{2}}$, $\boxed{\text{volume}=\frac{2}{3}|3k-5|}$





Question 61 (**+)**

The straight line L has Cartesian equation

$$x-9=\frac{y-a}{2}=\frac{z-1}{b},$$

where a and b are non zero constants.

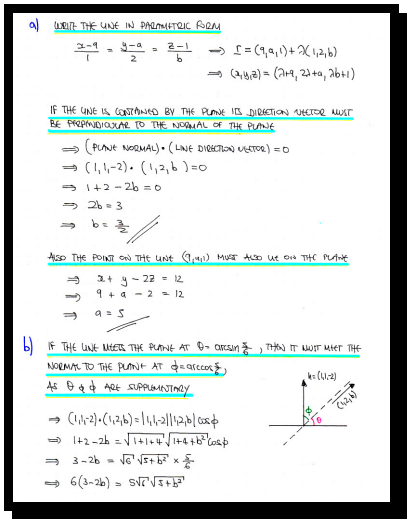
The plane Π has Cartesian equation

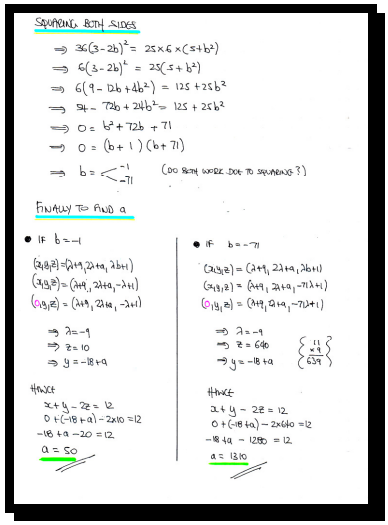
$$x+y-2z=12.$$

a) If L is contained by Π, determine the value of a and the value of b.

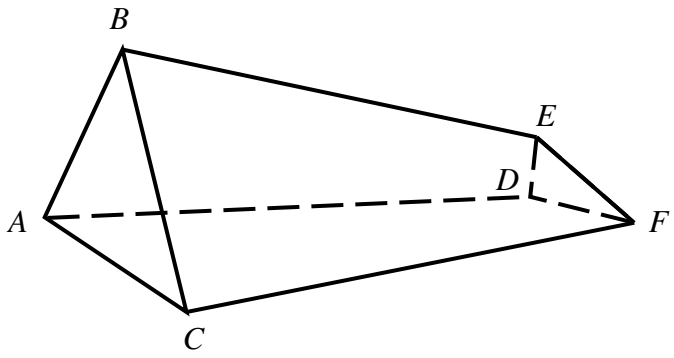
b) Given instead that L meets Π at the point where $x=0$, and is inclined at an angle $\arcsin\frac{5}{6}$ to Π, determine the value of a.

$\boxed{}$, $\boxed{a=5,\ b=\frac{2}{3}}$, $\boxed{a=50}$





Question 62 **(****+)**



The figure above shows an irregular hollow shape, consisting of two non-congruent, non-parallel triangular faces ABC and DEF, and two non-congruent quadrilateral faces $ABED$ and $BCFE$.

The respective equations of the straight lines AD and DE are

$$\mathbf{r}_1 = -5\mathbf{i}+6\mathbf{j}+\mathbf{k}+\lambda\left(2\mathbf{i}+3\mathbf{j}\right) \quad \text{and} \quad \mathbf{r}_2 = -\mathbf{i}+12\mathbf{j}+\mathbf{k}+\mu\left(-2\mathbf{i}+7\mathbf{j}-7\mathbf{k}\right),$$

where λ and μ are scalar parameters.

a) If the plane face $BCFE$ has equation $21x-14y+20z=111$, determine an equation of the straight line BE.

The straight line BC has equation

$$\mathbf{r}_3 = -\mathbf{i}-8\mathbf{j}+\mathbf{k}+\nu\left(-2\mathbf{i}+7\mathbf{j}+7\mathbf{k}\right),$$

where ν is a scalar parameter.

b) Given further that the point G has position vector $5\mathbf{i}+7\mathbf{j}$, determine the acute angle between the plane face $BCFE$ and the straight line BG.

$\boxed{}$, $\boxed{\mathbf{r}=\mathbf{i}+5\mathbf{j}+8\mathbf{k}+t\left(2\mathbf{i}+3\mathbf{j}\right)}$, $\boxed{\theta\approx 13.5^\circ}$

[solutions overleaf]

a) START BY CROSSING THE DIRECTIONS AB & BE IN ORDER TO FIND THE NORMAL TO THE PLANE ABED

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ -2 & 7 & -7 \end{vmatrix} = (-21-0,\ 0+14,\ 14+6) = (-21,\ 14,\ 20)$$

EQUATION OF ABED

$$-21x + 14y + 20z = \text{CONSTANT}$$
$$-21(-5) + 14(6) + 20(0) = \text{CONSTANT} \qquad \text{OR} \quad (5,6,1)$$
$$105 + 84 + 20 = \text{CONSTANT} \qquad \text{OR} \quad (-1,12,1)$$
$$\text{CONSTANT} = 209$$

$$\therefore -21x + 14y + 20z = 209$$

SOLVING THE EQUATIONS OF THE TWO PLANES

$$\text{ABED}: \quad -21x + 14y + 20z = 209$$
$$\text{BCFE}: \quad 21x - 14y + 20z = 111 \qquad \text{ADDING:} \quad 40z = 320$$
$$z = 8 \quad (\text{ALWAYS})$$

BOTH EQUATIONS REDUCE TO

$$-21x + 14y = 49$$
$$-3x + 2y = 7$$
$$2y = 3x + 7$$
$$y = \tfrac{3}{2}x + \tfrac{7}{2}$$

THUS WE HAVE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \frac{3}{2}x + \frac{7}{2} \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{2} \\ 8 \end{pmatrix} + x\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{2} \\ 8 \end{pmatrix} + (q+1)\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ 8 \end{pmatrix} + \eta\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ 8 \end{pmatrix} + 2t\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{r} = (1,5,8) + t(2,3,0)$$

b) NOW SOLVE THE LINES BC & BE TO FIND B

$$\text{BC}: \quad \underline{r} = (-1-2v,\ -8+7v,\ 1+7v)$$
$$\text{BE}: \quad \underline{r} = (1+2t,\ 5+3t,\ 8)$$

LINES DO INTERSECT SO NO VERIFICATION NEEDED

$$7v+1 = 8$$
$$7v = 7$$
$$v = 1$$
$$(t = -2)$$

$$\therefore B(-3,-1,8)$$

NOW $B(-3,-1,8)$ & $G(5,7,0)$

$$\Rightarrow \overrightarrow{BG} = \underline{g} - \underline{b} = (5,7,0) - (-3,-1,8) = (8,8,-8)$$

SCALE IT TO DIRECTION $(1,1,-1)$

DRAWING A DIAGRAM



FITTING NORMAL TO BCFE AND DIRECTION BG

$$(21,-14,20)\cdot(1,1,-1) = |(21,-14,20)||(1,1,-1)|\cos\theta$$
$$21 - 14 - 20 = \sqrt{1017 + 196 + 400}\ \sqrt{3}\cos\theta$$
$$-13 = \sqrt{1317 \times 3}\cos\theta$$
$$\cos\theta = \frac{-13}{\sqrt{3951}}$$
$$\theta = 103.479\ldots$$

WE REQUIRE ACUTE ANGLE (SEE DIAGRAM)

$$\theta = 180 - 103.479$$
$$\theta = 76.52°\ldots$$

FINALLY THE REQUIRED ANGLE IS ϕ

$$\phi = 90 - \theta$$
$$\phi \simeq 13.5°$$

Question 63 (**+)**

The skew straight lines L_1 and L_2 have vector equations

$$\mathbf{r}_1 = (-13\mathbf{j}+\mathbf{k})+\lambda(-3\mathbf{i}+4\mathbf{j}-7\mathbf{k}),$$

$$\mathbf{r}_2 = (5\mathbf{i}+25\mathbf{j})+\mu(2\mathbf{i}-2\mathbf{j}+3\mathbf{k}),$$

where λ and μ are scalar parameters.

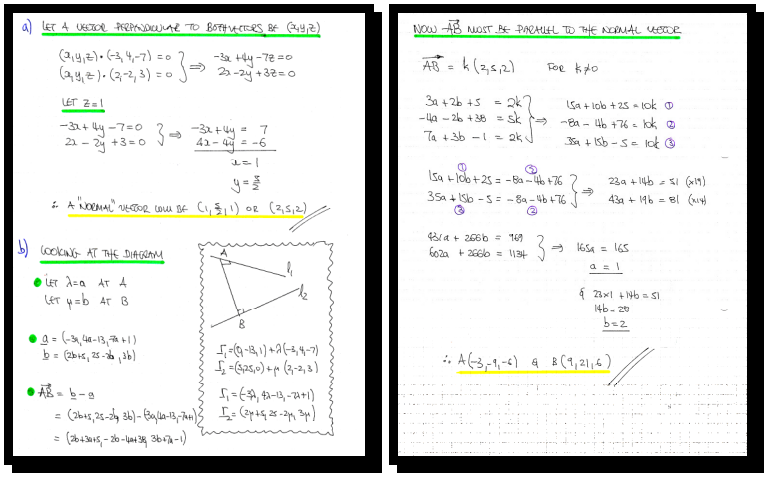
a) Find a vector which mutually perpendicular to L_1 and L_2.

*You may not use the vector (cross) product in answering part **(a)**.*

The point A lies on L_1 and the point B lies on L_2.

b) Given that the distance AB is least, determine the coordinates of A and B.

$\boxed{}$, $\boxed{A(-3,-9-6)}$, $\boxed{B(9,21,6)}$



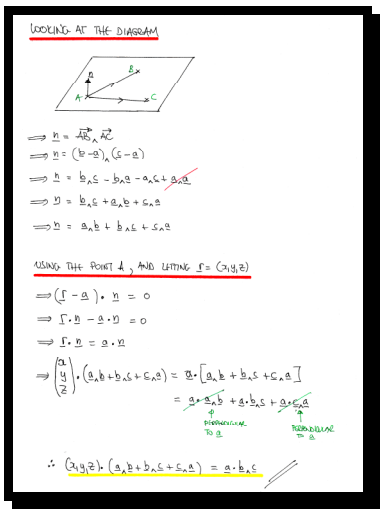
Question 64 **(*****)**

The points A, B and C have respective position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, relative to a fixed origin O.

Show that the equation of the plane through A, B and C can be written as

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a})=\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}$$

, proof



Question 65 **(*****)**

An irregular pyramid with a triangular base ABC has vertex at the point V.

The equation of the straight line VC is

$$\mathbf{r}=2\mathbf{i}+4\mathbf{k}+\lambda\left(\mathbf{i}-\mathbf{j}+4\mathbf{k}\right)$$

where λ is a scalar parameter.

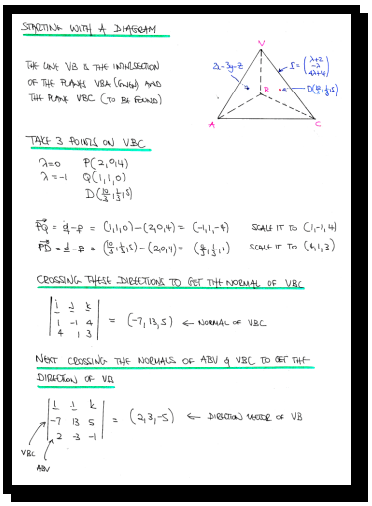
The plane face ABV has equation $2x-3y-z=1$.

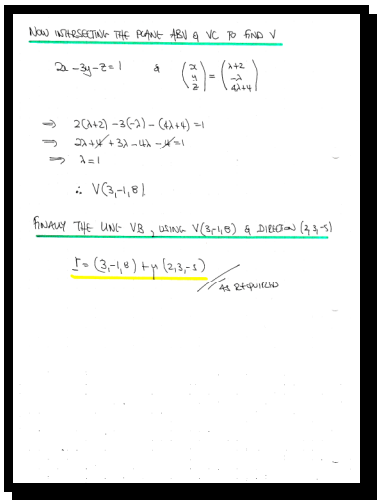
If the point D lies on the plane face VBC and has position vector $\frac{10}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+5\mathbf{k}$, show that the equation of the line VB can be written as

$$\mathbf{r}=3\mathbf{i}-\mathbf{j}+8\mathbf{k}+\mu\left(2\mathbf{i}+3\mathbf{j}-5\mathbf{k}\right),$$

where μ is a scalar parameter.

V, [ ], proof





Question 66 (*****)

The straight line L_1 has vector equation

$$\mathbf{r}=4\mathbf{i}-3\mathbf{j}+7\mathbf{k}+\lambda\left(3\mathbf{i}-4\mathbf{j}+5\mathbf{k}\right),$$

where λ is a scalar parameter.

The plane Π has vector equation

$$\mathbf{r}\cdot\left(4\mathbf{i}+3\mathbf{j}+5\mathbf{k}\right)=17.$$

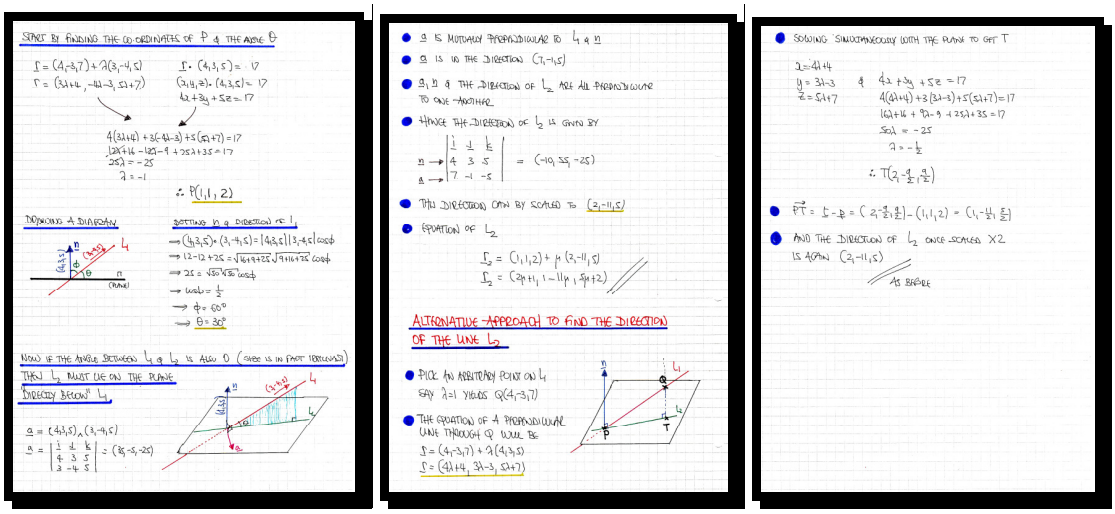
The point P is the intersection of L_1 and Π.

The acute angle θ is formed between L_1 and Π.

The straight line L_2 lies on Π, passes through P so that the acute angle between L_1 and L_2 is also θ.

Determine the value of θ and find a vector equation for L_2.

$$\boxed{},\ \boxed{\theta=30^\circ},\ \boxed{\mathbf{r}_2=\mathbf{i}+\mathbf{j}+2\mathbf{k}+\mu\left(2\mathbf{i}-11\mathbf{j}+5\mathbf{k}\right)}$$



Question 67 **(*****)**

With respect to a fixed origin O, the points A, B and C have respective position vectors

$$\mathbf{a}=3\mathbf{i}+3\mathbf{j}+3\mathbf{k}\,,\quad \mathbf{b}=6\mathbf{i}+2\mathbf{k}\quad \text{and}\quad \mathbf{c}=3\mathbf{j}+5\mathbf{k}\,,$$

so that the plane Π contains A, B and C.

The straight line L is **parallel** to Π and has vector equation

$$\mathbf{r}=(13\mathbf{i}-9\mathbf{j})+\lambda(-7\mathbf{i}+5\mathbf{j}+3\mathbf{k})\,,$$

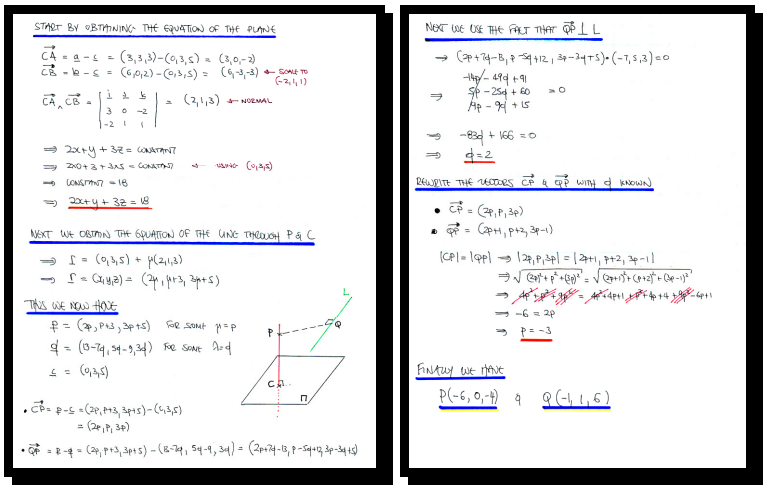
where λ is a scalar parameter.

The point P lies outside the plane so that PC is perpendicular to Π.

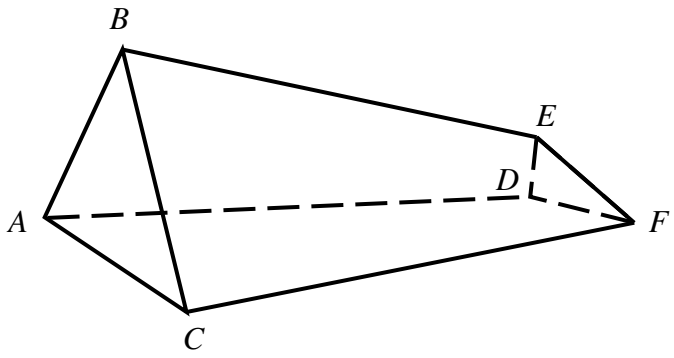
The point Q lies on L so that PQ is perpendicular to L.

Given further that P is equidistant from Π and L, find the position vector of P and the position vector of Q.

$$\boxed{}\,,\quad \boxed{\mathbf{p}=-6\mathbf{i}-4\mathbf{k}}\,,\quad \boxed{\mathbf{q}=-\mathbf{i}+\mathbf{j}+6\mathbf{k}}$$



Question 68 **(*****)**



The figure above shows an irregular hollow shape, consisting of two non-congruent, non-parallel triangular faces ABC and DEF, and two non-congruent quadrilateral faces $ABED$ and $BCFE$.

The respective equations of the straight lines AD, DE and BC are

$$\mathbf{r}_1 = -5\mathbf{i}+6\mathbf{j}+\mathbf{k}+\lambda\left(2\mathbf{i}+3\mathbf{j}\right)$$
$$\mathbf{r}_2 = -\mathbf{i}+12\mathbf{j}+\mathbf{k}+\mu\left(-2\mathbf{i}+7\mathbf{j}-7\mathbf{k}\right)$$
$$\mathbf{r}_3 = -\mathbf{i}-8\mathbf{j}+\mathbf{k}+\nu\left(-2\mathbf{i}+7\mathbf{j}+7\mathbf{k}\right)$$

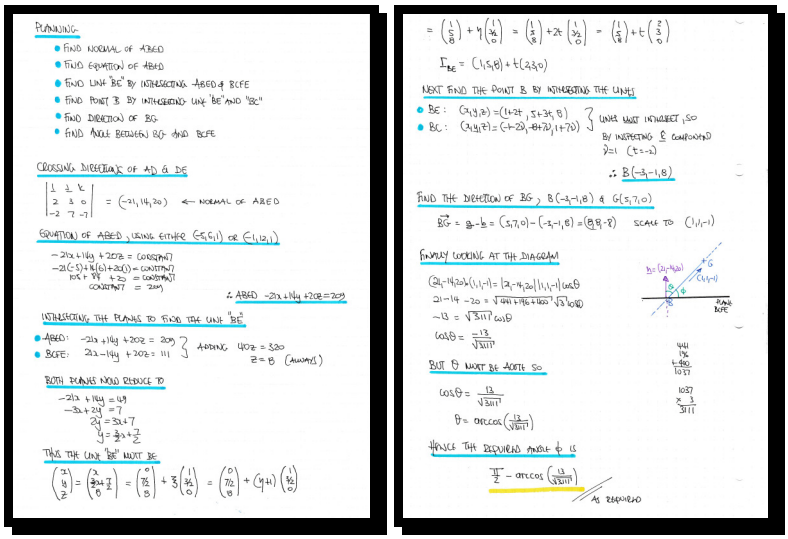
where λ, μ and ν are scalar parameters.

If the plane face $BCFE$ has equation $21x-14y+20z=111$ and the point G has position vector $5\mathbf{i}+7\mathbf{j}$, show that the acute angle between the plane face $BCFE$ and the straight line BG is

$$\frac{\pi}{2}-\arccos\left[\frac{13}{\sqrt{3111}}\right].$$

, proof

[solutions overleaf]



PLANNING

- FIND NORMAL OF ABED
- FIND EQUATION OF ABED
- FIND LINE "BE" BY INTERSECTING ABED & BCFE
- FIND POINT B BY INTERSECTING LINE "BE" AND "BC"
- FIND DIRECTION OF BG
- FIND ANGLE BETWEEN BG AND BCFE

CROSSING DIRECTIONS OF AD & DE

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -2 & 7 & -1 \end{vmatrix} = (-21, 14, 20) \quad \leftarrow \text{NORMAL OF ABED}$$

EQUATION OF ABED, USING EITHER $(-5,6,1)$ OR $(-1,12,1)$

$-21x+14y+20z = \text{CONSTANT}$
$-21(-5)+14(6)+20(1) = \text{CONSTANT}$
$105+84+20 = \text{CONSTANT}$
$\text{CONSTANT} = 209$

$\therefore$ ABED $\quad -21x+14y+20z=209$

INTERSECTING THE PLANES TO FIND THE LINE "BE"

- ABED: $-21x+14y+20z = 209$
- BCFE: $21x-14y+20z = 111$

ADDING $\quad 40z = 320$
$z = 8 \quad (\text{AWAY!})$

BOTH PLANES NOW REDUCE TO

$-21x+14y = 49$
$-3x+2y = 7$
$2y = 3x+7$
$y = \frac{3}{2}x+\frac{7}{2}$

THUS THE LINE "BE" MUST BE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2}x+\frac{7}{2} \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{2} \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{7}{2} \\ 8 \end{pmatrix} + (\lambda, \tfrac{3}{2}\lambda, 0)$$

$$= \begin{pmatrix} 1 \\ \frac{5}{2} \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$\mathbf{r}_{BE} = (1, \tfrac{5}{2}, 8) + t(2,3,0)$

NEXT FIND THE POINT B BY INTERSECTING THE LINES

- BE: $(x,y,z) = (1+2t, \tfrac{5}{2}+3t, 8)$
- BC: $(x,y,z) = (-1-2\lambda, -8+7\lambda, 1+20\lambda)$

LINES MUST INTERSECT, SO BY INSPECTING $\hat{\jmath}$ COMPONENTS $\lambda = 1 \quad (t=-2)$

$\therefore$ B$(-3,-1,8)$

FIND THE DIRECTION OF BG, B$(-3,-1,8)$ & G$(5,7,0)$

$\overrightarrow{BG} = \mathbf{g}-\mathbf{b} = (5,7,0)-(-3,-1,8) = (8,8,-8) \quad$ SCALE TO $(1,1,-1)$

FINALLY LOOKING AT THE DIAGRAM

$(21,-14,20)\cdot(1,1,-1) = |21,-14,20||1,1,-1|\cos\theta$
$21-14-20 = \sqrt{441+196+400}\sqrt{3}\cos\theta$
$-13 = \sqrt{1037}\cos\theta$
$\cos\theta = \frac{-13}{\sqrt{1037}}$

441
196
+ 400
1037

1037
× 3
3111

BUT θ MUST BE ACUTE SO

$\cos\theta = \frac{13}{\sqrt{1037}}$

$\theta = \arccos\left(\frac{13}{\sqrt{1037}}\right)$

HENCE THE REQUIRED ANGLE ϕ IS

$\frac{\pi}{2} - \arccos\left(\frac{13}{\sqrt{1037}}\right)$

AS REQUIRED