SERIES EXPANSIONS

59 QUESTIONS

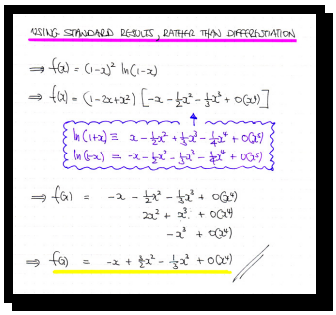
MACLAURIN EXPANSIONS 6 BASIC QUESTIONS

Question 1 **(**)**

$$f(x)=(1-x)^2\ln(1-x),\ -1\le x<1.$$

Find the Maclaurin expansion of $f(x)$ up and including the term in x^3.

$\square$, $\boxed{f(x)=-x+\frac{3}{2}x^2-\frac{1}{3}x^3+O\left(x^4\right)}$

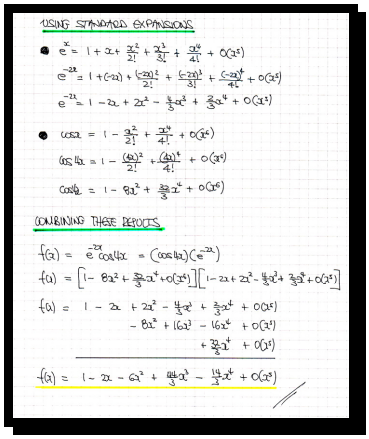


Question 2 **(**+)**

$$f(x)=\mathrm{e}^{-2x}\cos 4x.$$

Find the Maclaurin expansion of $f(x)$ up and including the term in x^4.

$\square$, $\boxed{\mathrm{e}^{-2x}\cos 4x=1-2x-6x^2+\frac{44}{3}x^3-\frac{14}{3}x^4+O\left(x^5\right)}$



Question 3 (+)**

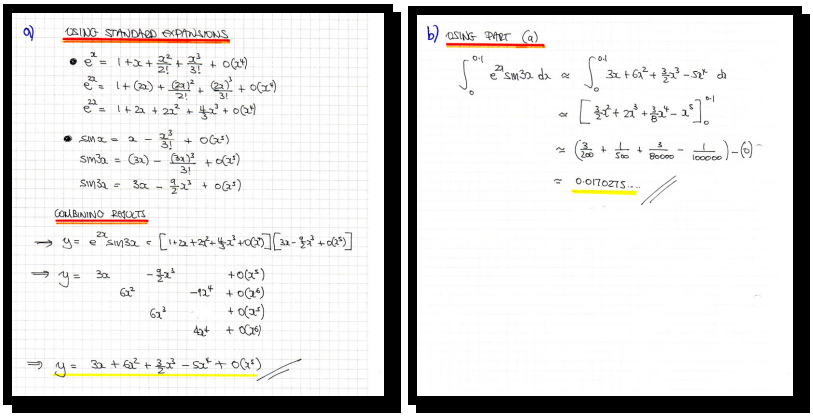
$$y = \mathrm{e}^{2x} \sin 3x \, .$$

a) Use standard results to find the series expansion of y, up and including the term in x^4.

b) Hence find an approximate value for

$$\int_0^{0.1} \mathrm{e}^{2x} \sin 3x \ \ dx \, .$$

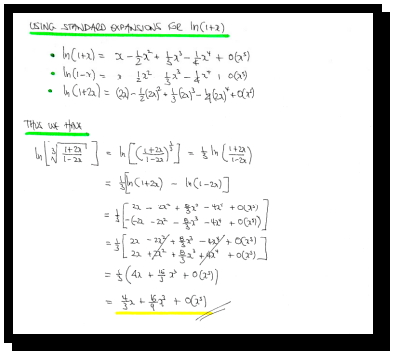
$\boxed{}$, $\boxed{\mathrm{e}^{2x} \sin 3x = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + O\left(x^5\right)}$, $\boxed{\approx 0.0170275}$



Question 4 (+)**

Find the Maclaurin's expansion of $\ln\left[\sqrt[3]{\dfrac{1+2x}{1-2x}}\right]$, up and including the term in x^3.

$\square$, $\ln\left[\sqrt[3]{\dfrac{1+2x}{1-2x}}\right]=\dfrac{4}{3}x+\dfrac{16}{9}x^3+O\left(x^5\right)$



Question 5 (***)

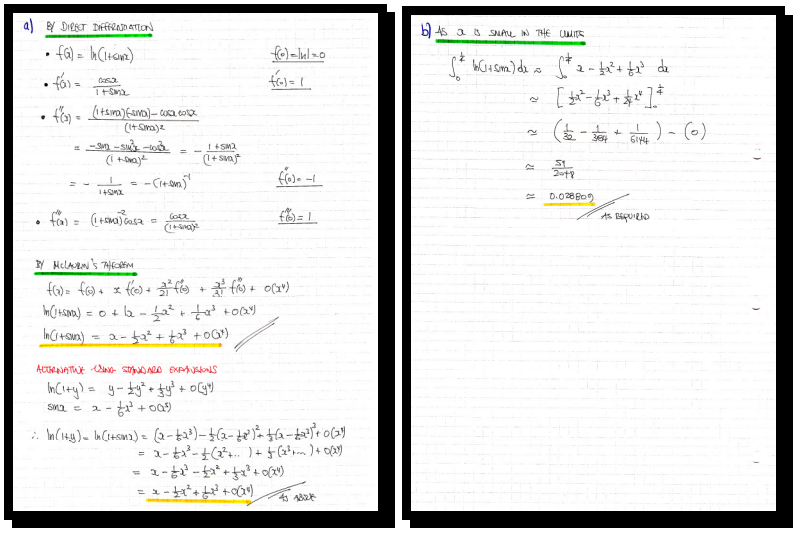
$$f\left(x\right)=\ln(1+\sin x),\ \sin x\neq -1.$$

a) Find the Maclaurin expansion of $f\left(x\right)$ up and including the term in x^3.

b) Hence show that

$$\int_0^{\frac{1}{4}}\ln(1+\sin x)\ dx\ \approx\ 0.028809.$$

$\square$, $\boxed{\ln\left(1+\sin x\right)=x-\frac{1}{2}x^2+\frac{1}{6}x^3+O\left(x^4\right)}$

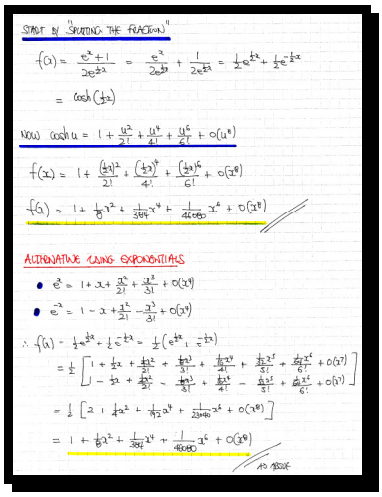


Question 6 **(***)**

$$f(x)\equiv\frac{e^x+1}{2e^{\frac{1}{2}x}},\ x\in\mathbb{R}.$$

Use standard results to determine the Maclaurin series expansion of $f(x)$, up and including the term in x^6.

$\square$, $\boxed{f(x)=1+\frac{1}{8}x^2+\frac{1}{384}x^4+\frac{1}{7680}x^6+O\left(x^8\right)}$



MACLAURIN EXPANSIONS 20 STANDARD QUESTIONS

Question 1 **(***+)**

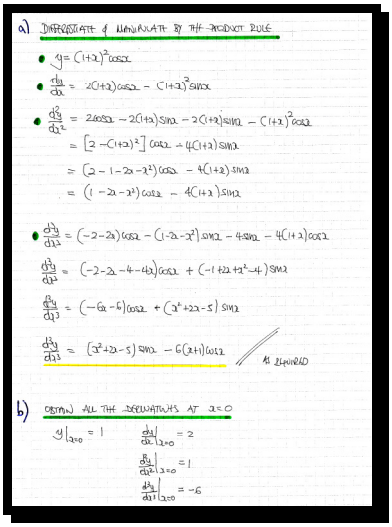
$$y=(1+x)^2\cos x\,.$$

Show clearly that …

a) … $\dfrac{d^3y}{dx^3}=\left(x^2+2x-5\right)\sin x-6\left(x+1\right)\cos x\,.$

b) … $y\approx 1+Ax+Bx^2+Cx^3$, where A, B and C are constants to be found.

$\boxed{}$, $\boxed{\text{proof}}$


a) DIFFERENTIATE BY THE PRODUCT RULE

$y=(1+x)^2\cos x$

$\frac{dy}{dx}=2(1+x)\cos x-(1+x)^2\sin x$

$\frac{d^2y}{dx^2}=2\cos x-2(1+x)\sin x-2(1+x)\sin x-(1+x)^2\cos x$

$=[2-(1+x)^2]\cos x-4(1+x)\sin x$

$=(2-1-2x-x^2)\cos x-4(1+x)\sin x$

$=(1-2x-x^2)\cos x-4(1+x)\sin x$

$\frac{d^3y}{dx^3}=(-2-2x)\cos x-(1-2x-x^2)\sin x-4\sin x-4(1+x)\cos x$

$\frac{d^3y}{dx^3}=(-2-2x-4-4x)\cos x+(-1+2x+x^2-4)\sin x$

$\frac{d^3y}{dx^3}=(-6x-6)\cos x+(x^2+2x-5)\sin x$

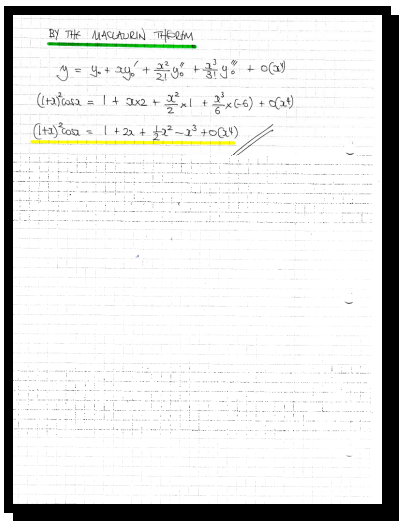
$\frac{d^3y}{dx^3}=(x^2+2x-5)\sin x-6(x+1)\cos x$ AS REQUIRED

b) OBTAIN ALL THE DERIVATIVES AT $x=0$

$y|_{x=0}=1$ $\quad \frac{dy}{dx}\Big|_{x=0}=2$

$\frac{d^2y}{dx^2}\Big|_{x=0}=1$

$\frac{d^3y}{dx^3}\Big|_{x=0}=-6$


BY THE MACLAURIN THEOREM

$y=y_0+xy_0'+\frac{x^2}{2!}y_0''+\frac{x^3}{3!}y_0'''+O(x^4)$

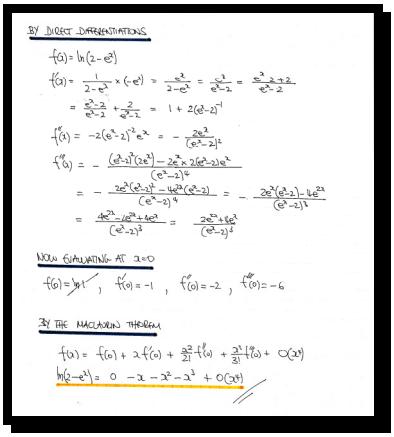
$(1+x)^2\cos x=1+2x+\frac{x^2}{2}\times 1+\frac{x^3}{6}\times(-6)+O(x^4)$

$(1+x)^2\cos x=1+2x+\frac{1}{2}x^2-x^3+O(x^4)$

Question 2 (*+)**

Find the Maclaurin expansion of $\ln(2-\mathrm{e}^x)$, up and including the term in x^3.

$\square$, $\boxed{\ln\left(2-\mathrm{e}^x\right)=-x-x^2-x^3+O\left(x^4\right)}$



Question 3 **(***+)**

$$f(x)=\ln(1+\cos 2x),\ 0\le x<\frac{\pi}{2}.$$

a) Find an expression for $f'(x)$.

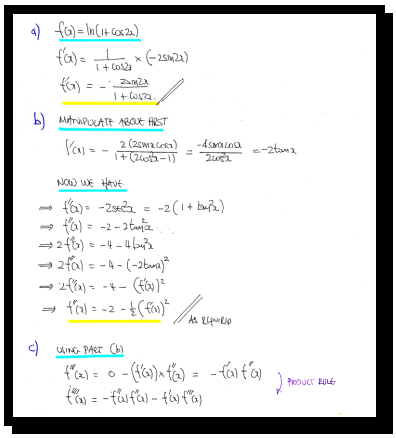
b) Show clearly that

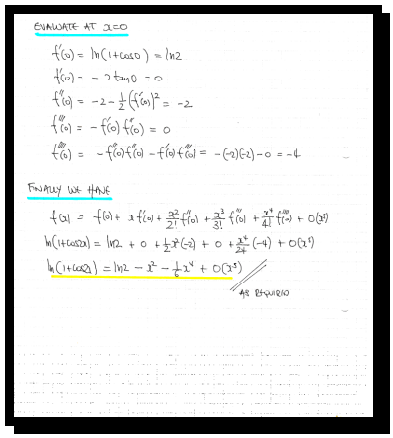
$$f''(x)=-2-\frac{1}{2}\left(f'(x)\right)^2.$$

c) Show further that the series expansion of the first three non zero terms of $f(x)$ is given by

$$\ln 2-x^2-\frac{1}{6}x^4.$$

$\square$, $\boxed{f'(x)=-\dfrac{2\sin 2x}{1+\cos 2x}}$

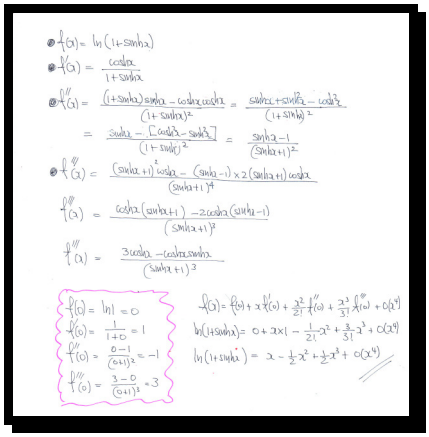




Question 4 (*+)**

Find the Maclaurin expansion of $\ln(1+\sinh x)$ up and including the term in x^3.

$$\boxed{\ln\left(1+\sinh x\right)=x-\frac{1}{2}x^2+\frac{1}{2}x^3+O\left(x^4\right)}$$

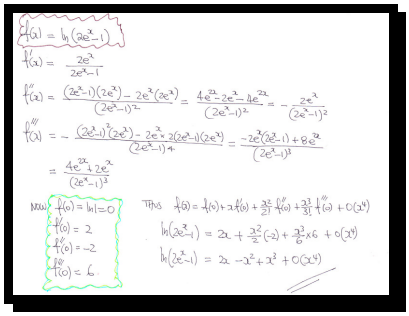


Question 5 (*+)**

$$f\left(x\right)\equiv\ln\left(2\mathrm{e}^{x}-1\right),\ x\in\mathbb{R}.$$

Find the Maclaurin expansion of $f\left(x\right)$, up and including the term in x^3.

$$\boxed{f\left(x\right)\equiv 2x-x^2+x^3+O\left(x^4\right)}$$



Question 6 **(***+)**

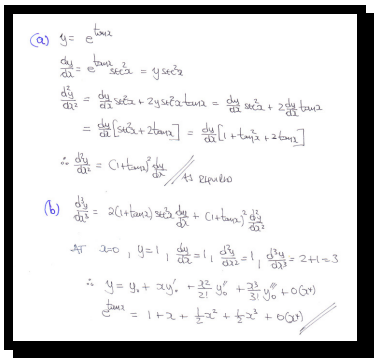
$$y = \mathrm{e}^{\tan x}, \quad x \in \mathbb{R}.$$

a) Show clearly that

$$\frac{d^2y}{dx^2} = (1+\tan x)^2 \frac{dy}{dx}.$$

b) Find a series expansion for $\mathrm{e}^{\tan x}$, up and including the term in x^3.

$$\boxed{\mathrm{e}^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + O\left(x^4\right)}$$

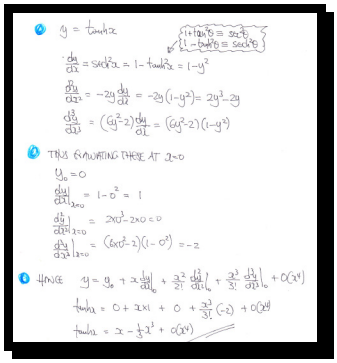


Question 7 (*+)**

$$y=\tanh x\,,\ x\in\mathbb{R}\,.$$

By expressing the derivatives of $\tanh x$ in terms of y, or otherwise find the first 2 non zero terms of a series expansion for $\tanh x$.

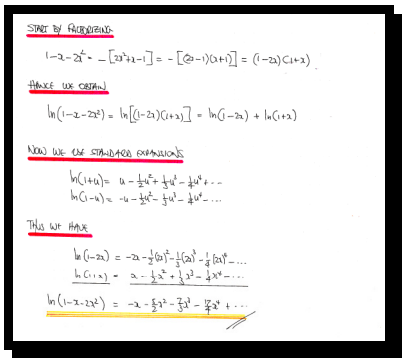
$$\boxed{y\approx x-\frac{1}{3}x^3+O\left(x^5\right)}$$



Question 8 **(***+)**

By using results for series expansions of standard functions, find the series expansion of $\ln\left(1-x-2x^2\right)$ up and including the term in x^4.

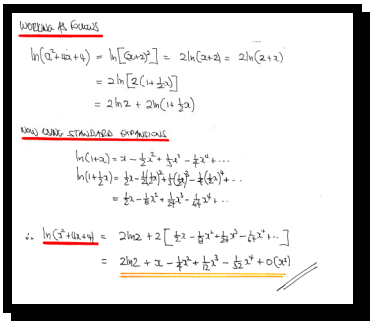
$\square$, $\boxed{\ln\left(1-x-2x^2\right)=-x-\frac{5}{2}x^2-\frac{7}{3}x^3-\frac{17}{4}x^4+O\left(x^5\right)}$



Question 9 **(***+)**

By using results for series expansions of standard functions, or otherwise, find the series expansion of $\ln\left(x^2+4x+4\right)$ up and including the term in x^4.

V, $\square$, $\boxed{\ln\left(x^2+4x+4\right)=2\ln 2+x-\frac{1}{4}x^2+\frac{1}{12}x^3-\frac{1}{32}x^4+O\left(x^5\right)}$



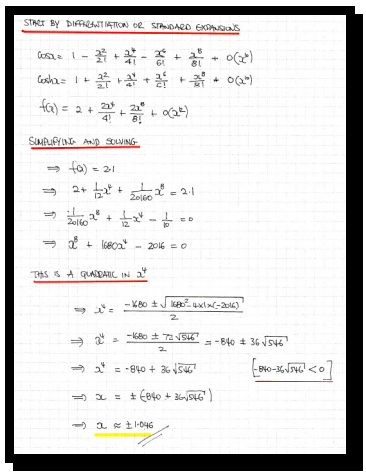
Question 10 **(***+)**

$$f(x) \equiv \cos x + \cosh x\,,\ x \in \mathbb{R}\,.$$

Use the first 3 non zero terms of the Maclaurin expansion of $f(x)$ to approximate the solutions of the equation

$$f(x) = 2.1\,.$$

$\boxed{}$, $\boxed{x \approx \pm 1.046}$



Question 11 **(****)**

$$f(x) \equiv \sin\left[\ln(1+x)\right], \quad x \in \mathbb{R}, \quad x > -1.$$

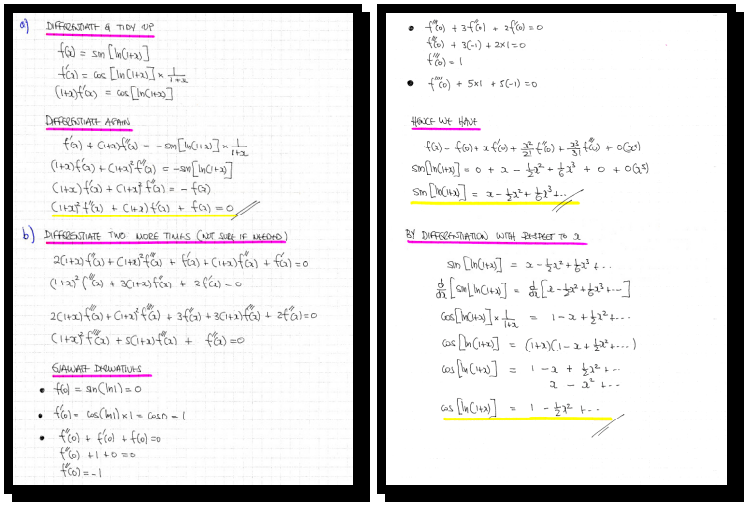
a) Show that

$$(1+x)^2 f''(x) + (1+x) f'(x) + f(x) = 0$$

b) Hence find first 3 non zero terms of the Maclaurin expansion of $f(x)$.

c) Use the result of part **(b)** to find first 2 non zero terms of the Maclaurin expansion of $\sin\left[\ln(1+x)\right]$.

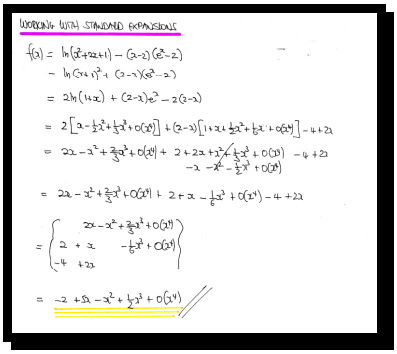
$\square$, $\boxed{\sin\left[\ln(1+x)\right] \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3}$, $\boxed{\cos\left[\ln(1+x)\right] \approx 1 - \frac{1}{2}x^2}$



Question 12 (****)

By using results for series expansions of standard functions, or otherwise, find the series expansion of $\ln\left(x^2+2x+1\right)-\left(x-2\right)\left(\mathrm{e}^x-2\right)$ up and including the term in x^3.

$\square$, $\boxed{\ln\left(x^2+2x+1\right)-\left(x-2\right)\left(\mathrm{e}^x-2\right)=-2+5x-x^2+\frac{1}{2}x^3+O\left(x^4\right)}$



Question 13 (****)

$$f(x)=\mathrm{e}^{x}\cos x\,,\ x\in\mathbb{R}\,.$$

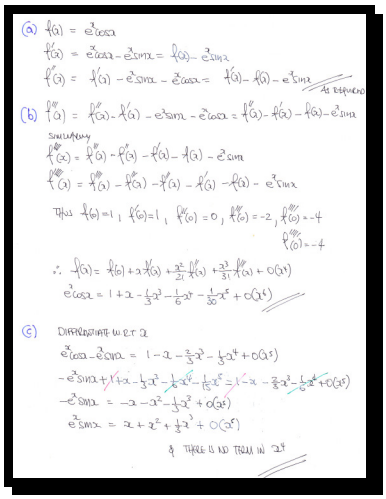
a) Show clearly that

$$f''(x)=f'(x)-f(x)-\mathrm{e}^{x}\sin x\,.$$

b) Find a series expansion for $f(x)$, up and including the term in x^5.

c) Hence find a series expansion for $\mathrm{e}^{x}\sin x$, up and including the term in x^4, showing further that the coefficient of x^4 is zero.

$$\boxed{f(x)=1+x+\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{30}x^5+O\left(x^6\right)},\quad \boxed{\mathrm{e}^{x}\sin x=x+x^2+\frac{1}{3}x^3+O\left(x^5\right)}$$



Question 14 (****)

The functions f and g are given below.

$$f(x)=\arctan\left(\frac{2}{3}x\right),\ x\in\mathbb{R}.$$

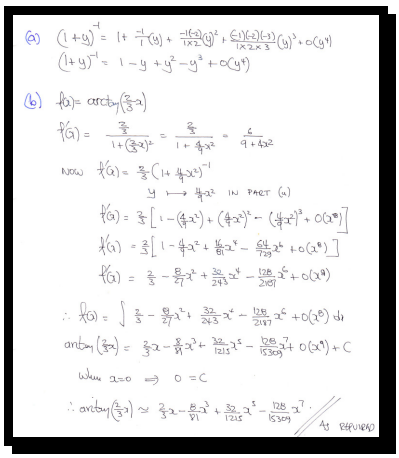
$$g(y)=\frac{1}{1+y},\ y\in\mathbb{R},\ -1<y<1.$$

a) Expand $g(y)$ as a binomial series , up and including the term in y^3.

b) Use $f'(x)$ and the answer to part (a) to show clearly that

$$\arctan\left(\frac{2}{3}x\right)\approx\frac{2}{3}x-\frac{8}{81}x^3+\frac{32}{1215}x^5-\frac{128}{15309}x^7.$$

$$\boxed{g(y)=1-y+y^2-y^3+O\left(y^4\right)}$$



Question 15 (****)

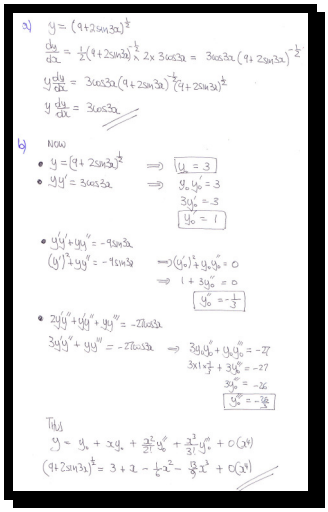
$$y=\sqrt{9+2\sin 3x}\,.$$

a) Find a simplified expression for $y\dfrac{dy}{dx}$.

b) Hence show that if x is numerically small

$$y\approx 3+x-\frac{1}{6}x^2-\frac{13}{9}x^3\,.$$

$3\cos 3x$



Question 16 **(****)**

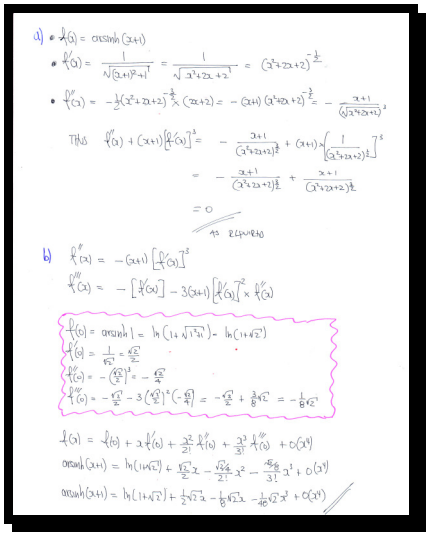
$$f(x)=\operatorname{arsinh}(x+1),\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f''(x)+(x+1)\left[f'(x)\right]^3=0$.

b) … $\operatorname{arsinh}(x+1)\approx\ln\left(1+\sqrt{2}\right)+\dfrac{\sqrt{2}}{2}x-\dfrac{\sqrt{2}}{8}x^2+\dfrac{\sqrt{2}}{48}x^3$.

proof



Question 17 **(****)**

$$y=\tan x\,,\ 0\le x<\frac{\pi}{2}\,.$$

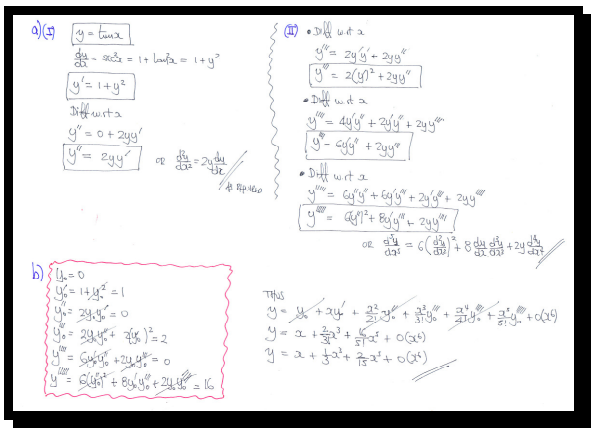
a) Show clearly that …

i. … $\dfrac{d^2y}{dx^2}=2y\dfrac{dy}{dx}$.

ii. … $\dfrac{d^5y}{dx^5}=6\left(\dfrac{d^2y}{dx^2}\right)^2+8\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}+2y\dfrac{d^4y}{dx^4}$.

b) Use these results to find the first 3 non zero terms of a series expansion for y.

$$\boxed{y\approx x+\frac{1}{3}x^3+\frac{2}{15}x^5}$$



Question 18 **(****)**

$$y=\ln(4+3x),\ x>-\frac{4}{3}.$$

a) Find simplified expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b) Hence, find the first 4 terms in the Maclaurin expansion of $y=\ln(4+3x)$.

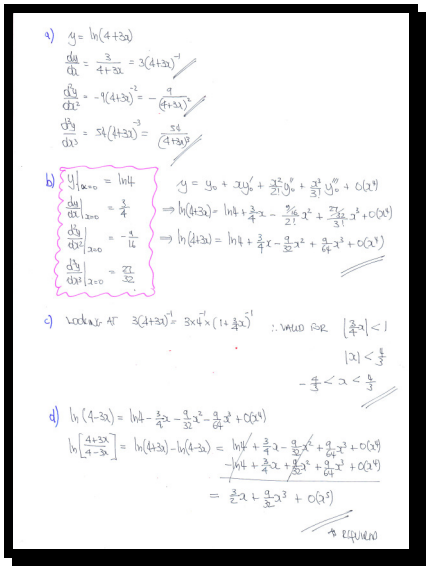
c) State the range of values of x for which the expansion is valid.

d) Show that for small values of x,

$$\ln\left(\frac{4+3x}{4-3x}\right)\approx\frac{3}{2}x+\frac{9}{32}x^3.$$

$\boxed{\frac{dy}{dx}=\frac{3}{3x+4}}$, $\boxed{\frac{d^2y}{dx^2}=-\frac{9}{(3x+4)^2}}$, $\boxed{\frac{d^3y}{dx^3}=\frac{54}{(3x+4)^2}}$, $\boxed{-\frac{4}{3}<x\leq\frac{4}{3}}$,

$$\boxed{\ln(4+3x)=\ln 4+\frac{3}{4}x-\frac{9}{32}x^2+\frac{9}{64}x^3+O\left(x^4\right)}$$



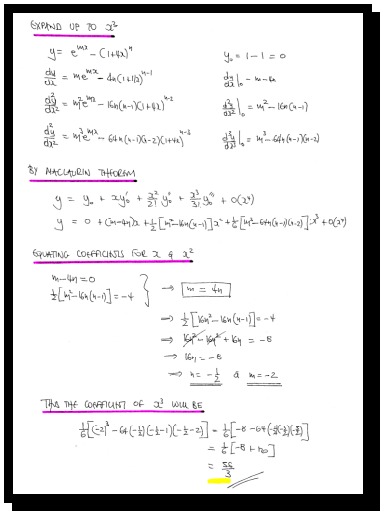
Question 19 (****)

If m and n are non zero constants, then the first non zero term in the Maclaurin expansion of $\mathrm{e}^{mx}-(1+4x)^n$ is $-4x^2$.

Find the coefficient of x^3 in this expansion.

*You may **NOT** use standard series expansions in this question.*

$\square$, $\boxed{\left[x^3\right]=\frac{56}{3}}$

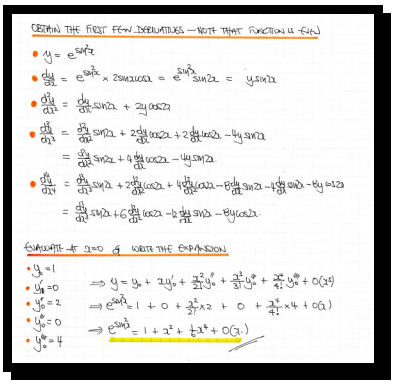


Question 20 (****)

Determine the first 3 no zero terms in the Maclaurin expansion of

$$y = \mathrm{e}^{\sin^2 x}.$$

$\boxed{}$, $\boxed{y = 1 + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)}$



MACLAURIN EXPANSIONS 7 HARD QUESTIONS

Question 1 **(****+)**

$$y=\ln\left(1+\sin x\right),\ \sin x\neq -1.$$

a) Show clearly that

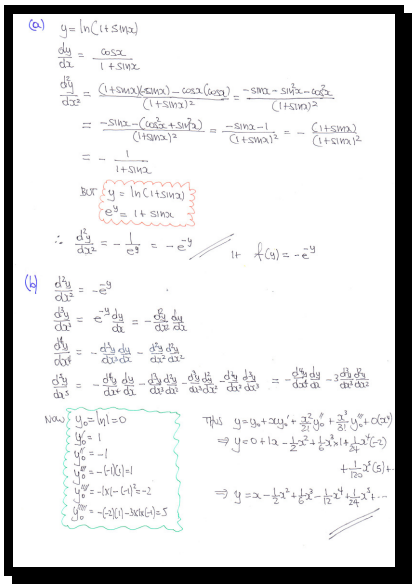
$$\frac{dy}{dx}=f\left(y\right),$$

where $f\left(y\right)$ is a function to be found.

b) Hence show further that

$$\ln\left(1+\sin x\right)\approx x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{24}x^5.$$

$\boxed{y=-\mathrm{e}^{-y}}$



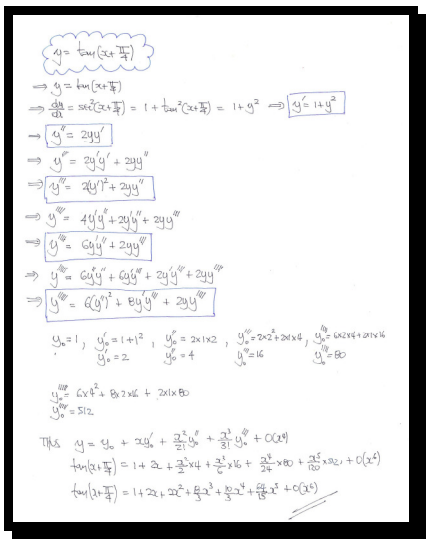
Question 2 (**+)**

$$y=\tan\left(x+\frac{\pi}{4}\right),\ -\frac{3\pi}{4}<x<\frac{\pi}{4}.$$

Use the Maclaurin theorem to show that

$$y=\tan\left(x+\frac{\pi}{4}\right)\approx 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\frac{64}{15}x^5.$$

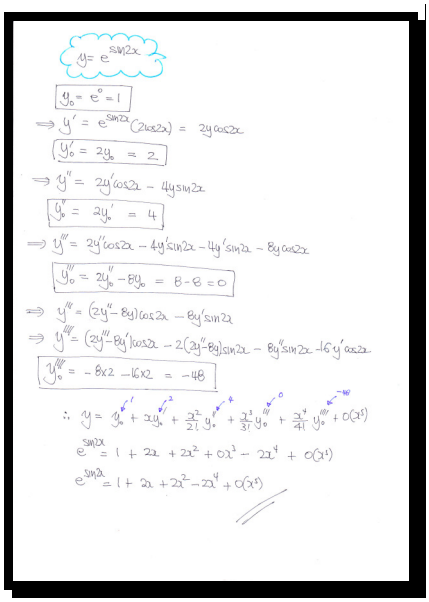
proof



Question 3 **(****+)**

Find the Maclaurin expansion, up and including the term in x^4, for $y=\mathrm{e}^{\sin 2x}$.

$$\boxed{\mathrm{e}^{\sin 2x}=1+x+2x^2-2x^4+O\left(x^5\right)}$$



$y = e^{\sin 2x}$

$y_0 = e^0 = 1$

$\Rightarrow y' = e^{\sin 2x}(2\cos 2x) = 2y\cos 2x$

$y_0' = 2y_0 = 2$

$\Rightarrow y'' = 2y'\cos 2x - 4y\sin 2x$

$y_0'' = 2y_0' = 4$

$\Rightarrow y''' = 2y''\cos 2x - 4y'\sin 2x - 4y'\sin 2x - 8y\cos 2x$

$y_0''' = 2y_0'' - 8y_0 = 8 - 8 = 0$

$\Rightarrow y'''' = (2y'' - 8y)\cos 2x - 8y'\sin 2x$

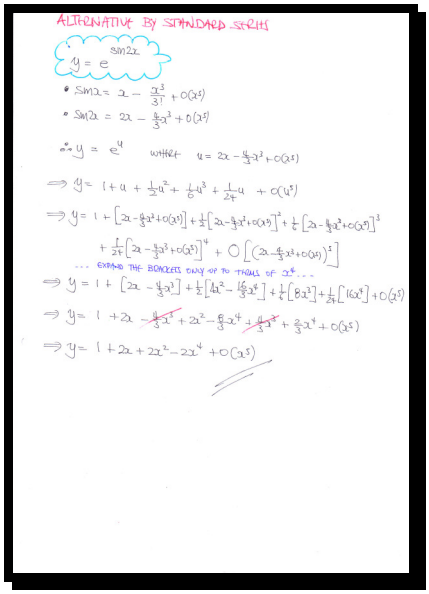
$\Rightarrow y'''' = (2y''' - 8y')\cos 2x - 2(2y'' - 8y)\sin 2x - 8y''\sin 2x - 16y'\cos 2x$

$y_0'''' = -8\times 2 - 16\times 2 = -48$

$\therefore y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0'''' + O(x^5)$

$e^{\sin 2x} = 1 + 2x + 2x^2 + 0x^3 - 2x^4 + O(x^5)$

$e^{\sin 2x} = 1 + 2x + 2x^2 - 2x^4 + O(x^5)$



ALTERNATIVE BY STANDARD SERIES

$y = e^{\sin 2x}$

$\sin 2x = 2x - \frac{(2x)^3}{3!} + O(x^5)$

$\sin 2x = 2x - \frac{4}{3}x^3 + O(x^5)$

$\therefore y = e^u$ where $u = 2x - \frac{4}{3}x^3 + O(x^5)$

$\Rightarrow y = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{24}u + O(u^5)$

$\Rightarrow y = 1 + \left[2x - \frac{4}{3}x^3 + O(x^5)\right] + \frac{1}{2}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^2 + \frac{1}{6}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^3 + \frac{1}{24}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^4 + O\left[\left(2x - \frac{4}{3}x^3 + O(x^5)\right)^5\right]$

... EXPAND THE BRACKETS ONLY UP TO TERMS OF x^4 ...

$\Rightarrow y = 1 + \left[2x - \frac{4}{3}x^3\right] + \frac{1}{2}\left[4x^2 - \frac{16}{3}x^4\right] + \frac{1}{6}\left[8x^3\right] + \frac{1}{24}\left[16x^4\right] + O(x^5)$

$\Rightarrow y = 1 + 2x - \frac{4}{3}x^3 + 2x^2 - \frac{8}{3}x^4 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)$

$\Rightarrow y = 1 + 2x + 2x^2 - 2x^4 + O(x^5)$

Question 4 **(****+)**

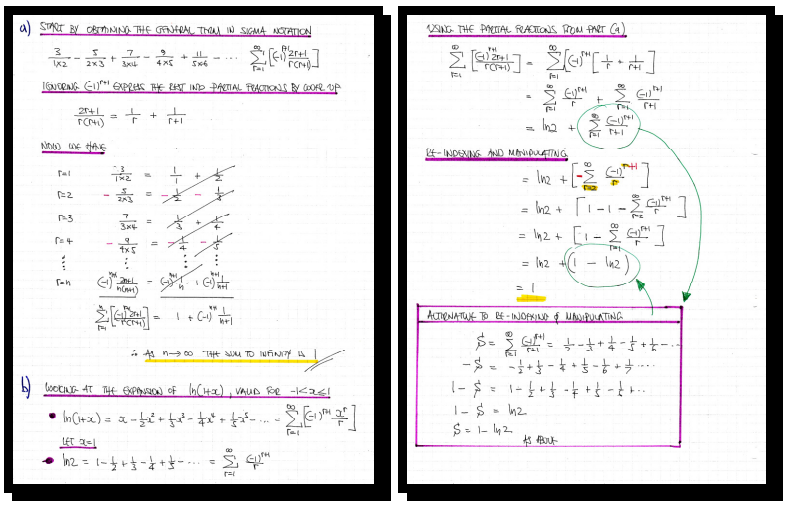
Consider the following infinite convergent series.

$$\frac{3}{1\times 2}-\frac{5}{2\times 3}+\frac{7}{3\times 4}-\frac{9}{4\times 5}+\frac{11}{5\times 6}-....$$

a) Use the method of differences, to find the sum of this series.

b) Verify the answer of part **(a)** by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, [ ], 1



Question 5 (**+)**

$$y=\ln\left(2-\mathrm{e}^{x}\right),\ x<\ln 2\,.$$

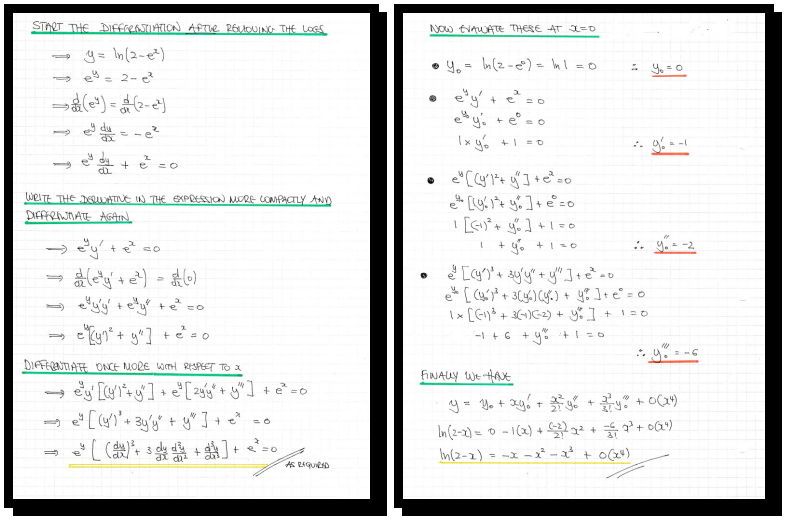
Show clearly that

$$\mathrm{e}^{y}\left[\frac{d^3y}{dx^3}+3\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right]+\mathrm{e}^{x}=0\,,$$

and hence find the first 3 non zero terms in the Maclaurin expansion of

$$y=\ln\left(2-\mathrm{e}^{x}\right),\quad x<\ln 2\ .$$

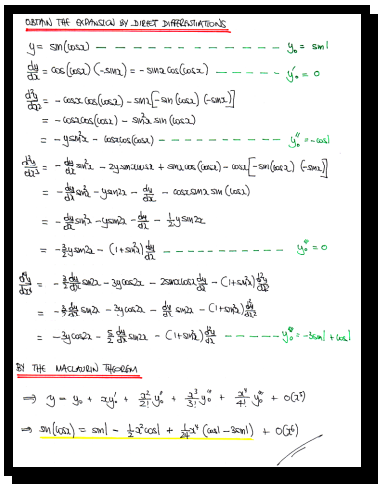
$\square$, $\boxed{y=\ln\left(2-\mathrm{e}^{x}\right)=-x-x^2-x^3+O\left(x^4\right)}$



Question 6 **(****+)**

Find the Maclaurin expansion, up and including the term in x^4, for $y=\sin\left(\cos x\right)$.

$\boxed{}$, $\boxed{\sin\left(\cos x\right)=\sin 1-\frac{1}{2}x^2\cos 1+x^4\left(\frac{1}{24}\cos 1-\frac{1}{8}\sin 1\right)+O\left(x^6\right)}$

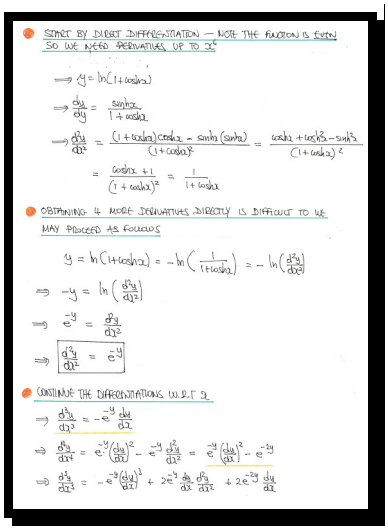


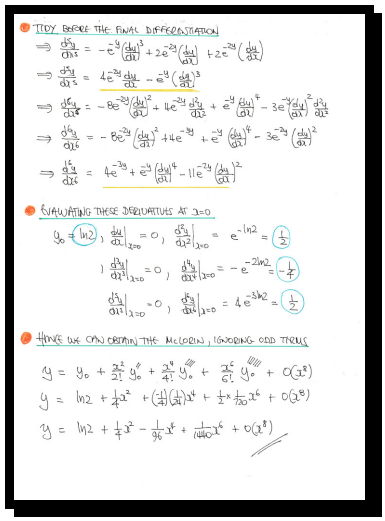
Question 7 **(****+)**

Find the first four non zero terms in the Maclaurin expansion of

$$y=\ln\left(1+\cosh x\right).$$

$\boxed{}$, $\boxed{\ln\left(1+\cosh x\right)=\ln 2+\frac{1}{4}x^2-\frac{1}{96}x^4+\frac{1}{1440}x^6+O\left(x^8\right)}$





MACLAURIN EXPANSIONS 9 ENRICHMENT QUESTIONS

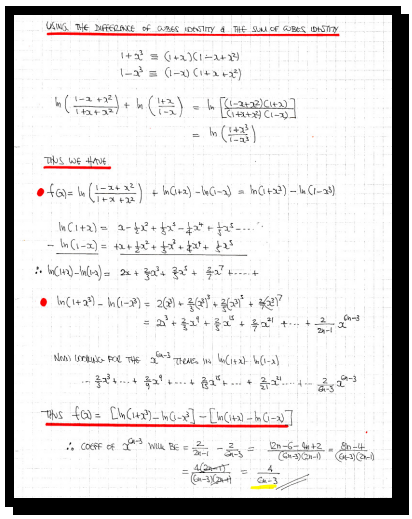
Question 1 **(*****)**

The curve with equation $y = f(x)$ is the solution of the differential equation

$$f(x) \equiv \ln\left(\frac{1-x+x^2}{1+x+x^2}\right).$$

Determine, in its simplest form, the coefficient of x^{6n-3}, $n \in \mathbb{N}$, in the Maclaurin series expansion of $f(x)$.

$\square$, $\boxed{\dfrac{4}{6n-3}}$



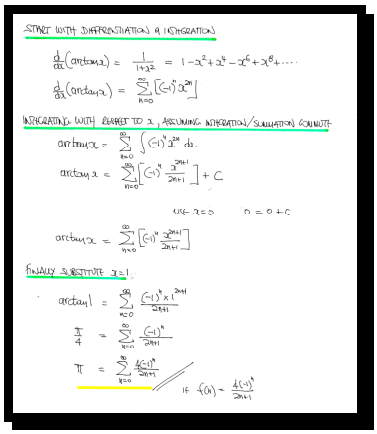
Question 2 **(*****)**

Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f.

$\square$, $\boxed{\pi = \sum_{n=0}^{\infty} \frac{4(-1)n}{2n+1}}$



Question 3 **(*****)**

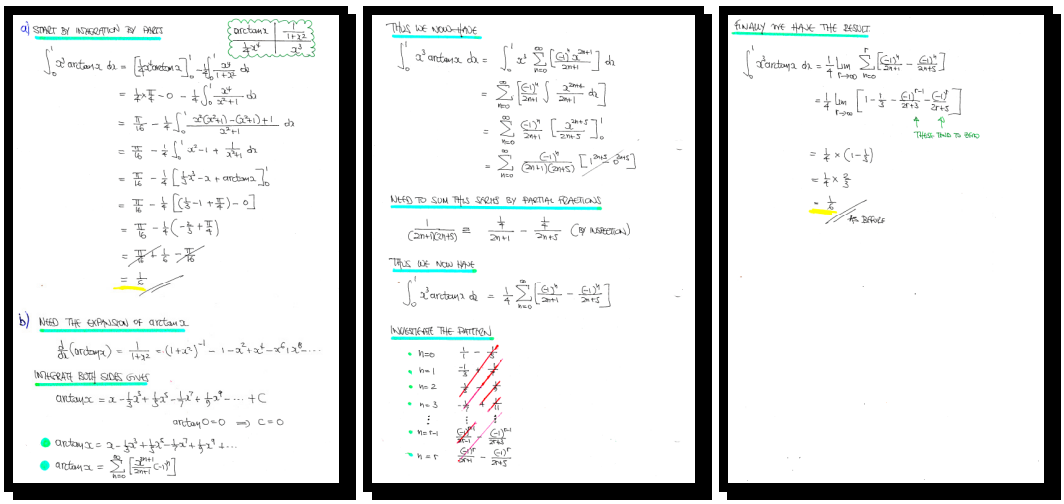
a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \ \ dx \, .$$

b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{6}}$



Question 4 **(*****)**

It is given that

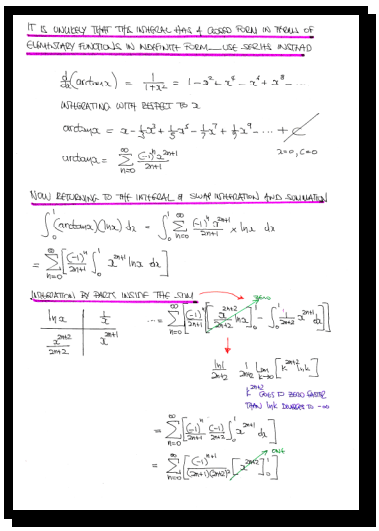
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots=\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2$

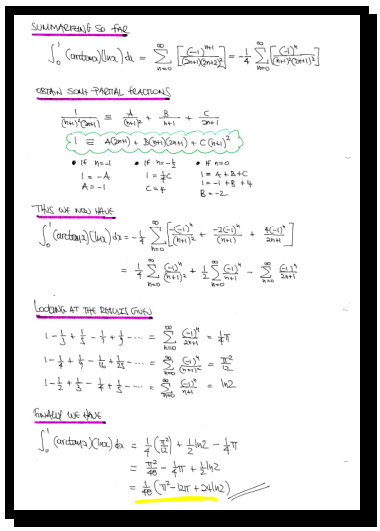
Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx\,.$$

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$



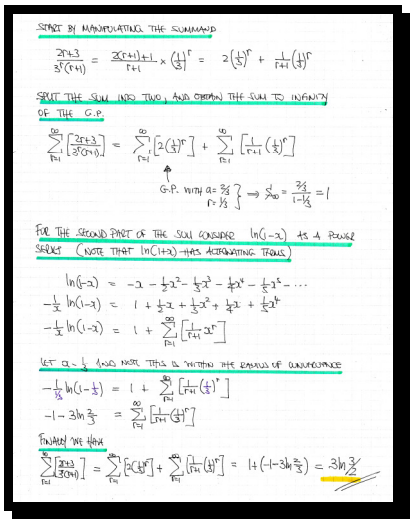


Question 5 (*****)

Show with detailed workings that

$$\sum_{r=1}^{\infty}\left[\frac{2r+3}{(r+1)3^r}\right] = 3\ln\left(\frac{3}{2}\right).$$

V, [ ], proof

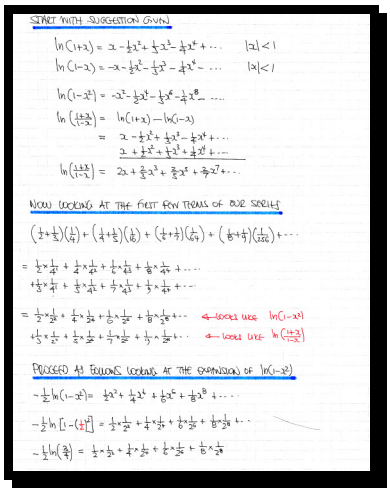


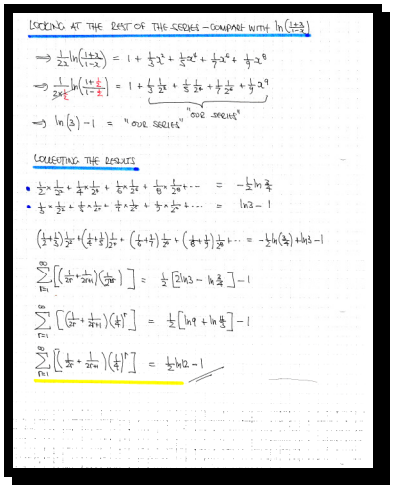
Question 6 (*****)

By considering the series expansions of $\ln\left(1-x^2\right)$ and $\ln\left(\dfrac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty}\left[\left(\frac{1}{2r}+\frac{1}{2r+1}\right)\left(\frac{1}{4}\right)^r\right].$$

$\square$, $\boxed{-1+\frac{1}{2}\ln 12}$





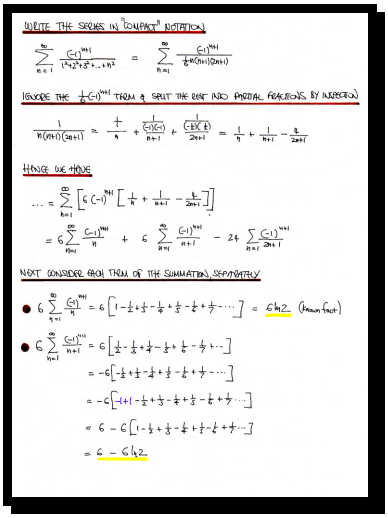
Question 7 (*****)

Find the sum to infinity of the following series.

$$\frac{1}{1}-\frac{1}{1+4}+\frac{1}{1+4+9}-\frac{1}{1+4+9+16}+\frac{1}{1+4+9+16+25}+...$$

You may find the series expansion of $\arctan x$ useful in this question.

$\square$, $\boxed{6(\pi-3)}$

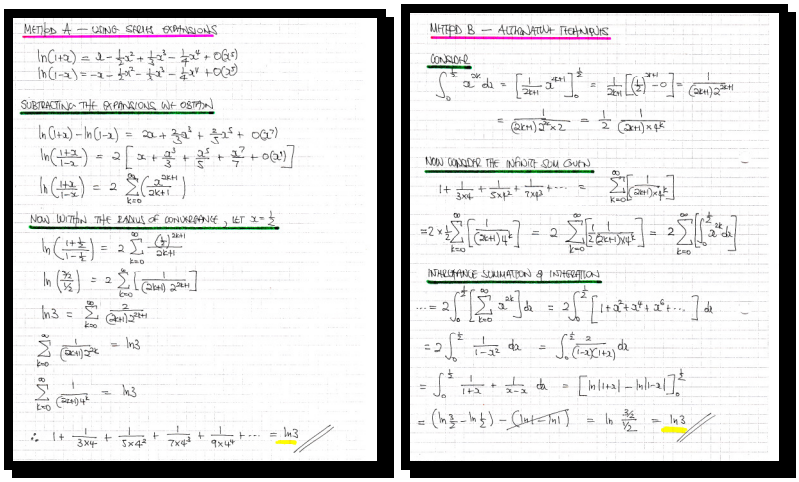


Question 8 (*****)

Find the sum to infinity of the following series.

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+...$$

$\boxed{}$, $\boxed{\ln 3}$



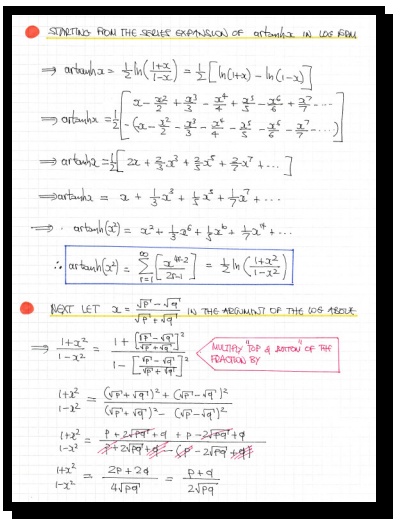
Question 9 **(*****)**

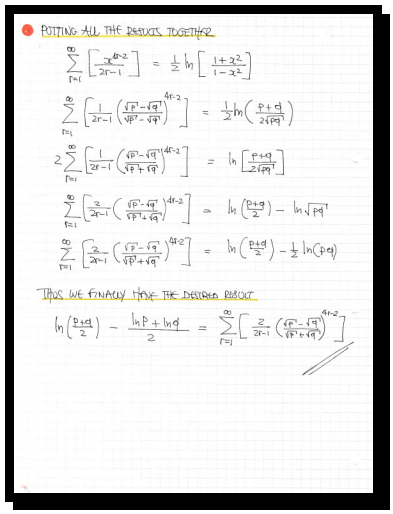
Given that p and q are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\operatorname{artanh}\left(x^2\right)$ useful in this question.

$\boxed{}$, proof





TAYLOR SERIES EXPANSIONS 4 BASIC QUESTIONS

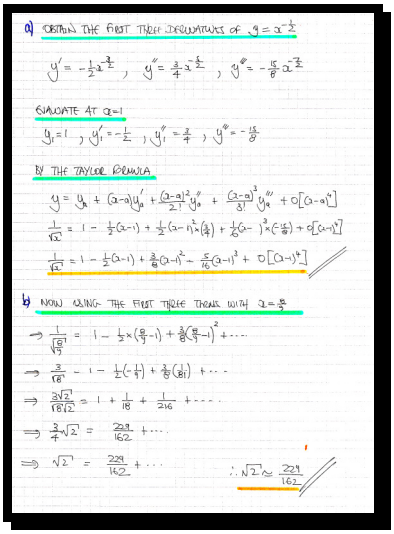
Question 1 **(***)**

$$y=\frac{1}{\sqrt{x}},\ x>0$$

a) Find the first four terms in the Taylor expansion of y about $x=1$.

b) Use the first **three** terms of the expansion found in part **(a)**, with $x=\frac{8}{9}$ to show that $\sqrt{2}\approx\frac{229}{162}$.

$\boxed{}$, $\boxed{y=1-\frac{1}{2}(x-1)+\frac{3}{8}(x-1)^2-\frac{5}{16}(x-1)^3+O\left((x-1)^4\right)}$



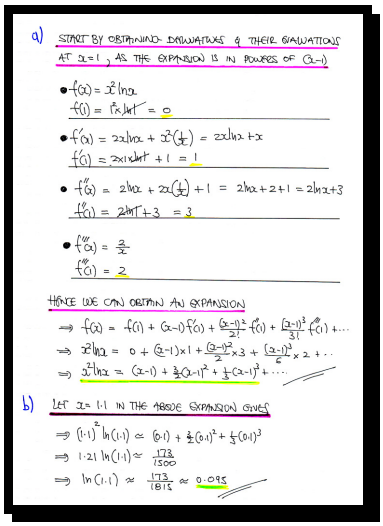
Question 2 (***)

$$f(x)=x^2\ln x,\ x>0$$

a) Find the first three non zero terms in the Taylor expansion of $f(x)$, in powers of $(x-1)$.

b) Use the first three terms of the expansion to show $\ln 1.1\approx 0.095$.

$\square$, $\boxed{f(x)=(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3+O\left((x-1)^4\right)}$



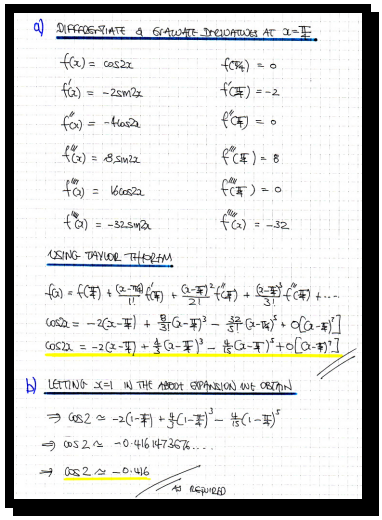
Question 3 **(***)**

$$f(x)=\cos 2x\,.$$

a) Find the first three non zero terms in the Taylor expansion of $f(x)$, in powers of $\left(x-\frac{\pi}{4}\right)$.

b) Use the first three terms of the expansion to show $\cos 2\approx -0.416$.

$\square$, $\boxed{f(x)=-2\left(x-\frac{\pi}{4}\right)+\frac{4}{3}\left(x-\frac{\pi}{4}\right)^3-\frac{4}{15}\left(x-\frac{\pi}{4}\right)^5+O\left(\left(x-\frac{\pi}{4}\right)^7\right)}$


a) DIFFERENTIATE & EVALUATE DERIVATIVES AT $x=\frac{\pi}{4}$

$f(x)=\cos 2x$ $\quad f\left(\frac{\pi}{4}\right)=0$

$f'(x)=-2\sin 2x$ $\quad f'\left(\frac{\pi}{4}\right)=-2$

$f''(x)=-4\cos 2x$ $\quad f''\left(\frac{\pi}{4}\right)=0$

$f'''(x)=8\sin 2x$ $\quad f'''\left(\frac{\pi}{4}\right)=8$

$f^{(iv)}(x)=16\cos 2x$ $\quad f^{(iv)}\left(\frac{\pi}{4}\right)=0$

$f^{(v)}(x)=-32\sin 2x$ $\quad f^{(v)}(x)=-32$

USING TAYLOR THEOREM

$f(x)=f\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)}{1!}f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots$

$\cos 2x=-2\left(x-\frac{\pi}{4}\right)+\frac{8}{3!}\left(x-\frac{\pi}{4}\right)^3-\frac{32}{5!}\left(x-\frac{\pi}{4}\right)^5+O\left[\left(x-\frac{\pi}{4}\right)^7\right]$

$\cos 2x=-2\left(x-\frac{\pi}{4}\right)+\frac{4}{3}\left(x-\frac{\pi}{4}\right)^3-\frac{4}{15}\left(x-\frac{\pi}{4}\right)^5+O\left[\left(x-\frac{\pi}{4}\right)^7\right]$

b) LETTING $x=1$ IN THE ABOVE EXPANSION WE OBTAIN

$\Rightarrow \cos 2\approx -2\left(1-\frac{\pi}{4}\right)+\frac{4}{3}\left(1-\frac{\pi}{4}\right)^3-\frac{4}{15}\left(1-\frac{\pi}{4}\right)^5$

$\Rightarrow \cos 2\approx -0.416147366\ldots$

$\Rightarrow \cos 2\approx -0.416$

AS REQUIRED

Question 4 **(***)**

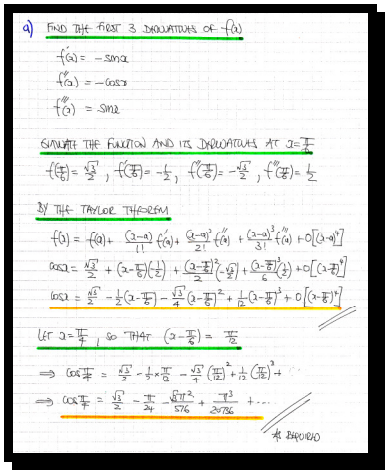
$$f(x)=\cos x\,.$$

a) Find the first four terms in the Taylor expansion of $f(x)$, in ascending powers of $\left(x-\frac{\pi}{6}\right)$.

b) Use the expansion of part **(a)** to show that

$$\cos\frac{\pi}{4}\approx\frac{\sqrt{3}}{2}-\frac{\pi}{24}-\frac{\sqrt{3}\pi^2}{576}-\frac{\pi^3}{20736}\,.$$

$\square$, $\boxed{f(x)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3+O\left(\left(x-\frac{\pi}{6}\right)^4\right)}$


a) FIND THE FIRST 3 DERIVATIVES OF $f(x)$

$f'(x)=-\sin x$

$f''(x)=-\cos x$

$f'''(x)=\sin x$

EVALUATE THE FUNCTION AND ITS DERIVATIVES AT $x=\frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$, $f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$, $f''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$, $f'''\left(\frac{\pi}{6}\right)=\frac{1}{2}$

BY THE TAYLOR THEOREM

$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+O\left[(x-a)^4\right]$

$\cos x=\frac{\sqrt{3}}{2}+\left(x-\frac{\pi}{6}\right)\left(-\frac{1}{2}\right)+\frac{\left(x-\frac{\pi}{6}\right)^2}{2}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\left(x-\frac{\pi}{6}\right)^3}{6}\left(\frac{1}{2}\right)+O\left[\left(x-\frac{\pi}{6}\right)^4\right]$

$\cos x=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3+O\left[\left(x-\frac{\pi}{6}\right)^4\right]$

LET $x=\frac{\pi}{4}$, SO THAT $\left(x-\frac{\pi}{6}\right)=\frac{\pi}{12}$

$\Rightarrow\cos\frac{\pi}{4}=\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{\pi}{12}-\frac{\sqrt{3}}{4}\left(\frac{\pi}{12}\right)^2+\frac{1}{12}\left(\frac{\pi}{12}\right)^3+\cdots$

$\Rightarrow\cos\frac{\pi}{4}=\frac{\sqrt{3}}{2}-\frac{\pi}{24}-\frac{\sqrt{3}\pi^2}{576}+\frac{\pi^3}{20736}+\cdots$

TAYLOR SERIES EXPANSIONS 3 STANDARD QUESTIONS

Question 1 **(***+)**

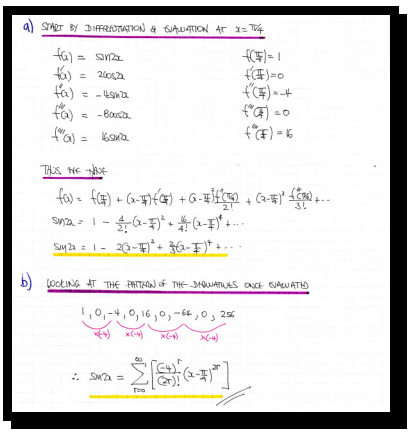
$$f(x)\equiv \sin 2x\,,\quad x\in\mathbb{R}\,.$$

a) Determine, in exact simplified form, the first 3 non zero terms, in the Taylor expansion of $f(x)$, centred at $x=\frac{1}{4}\pi$.

b) Write the **entire** expansion of $f(x)$, as a simplified expression in Σ notation.

$\square$, $\boxed{f(x)=1-2\left(x-\frac{1}{4}\pi\right)^2+\frac{2}{3}\left(x-\frac{1}{4}\pi\right)^4+\ldots}$,

$$\boxed{f(x)=\sum_{r=0}^{\infty}\left[\frac{(-4)^r}{(2r)!}\left(x-\frac{1}{4}\pi\right)^{2r}\right]}$$



Question 2 **(****)**

$$y = \tan x \, .$$

a) Show that

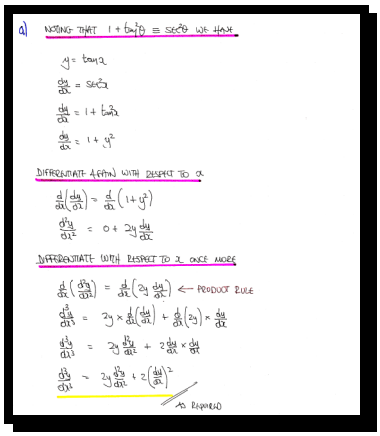
$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 .$$

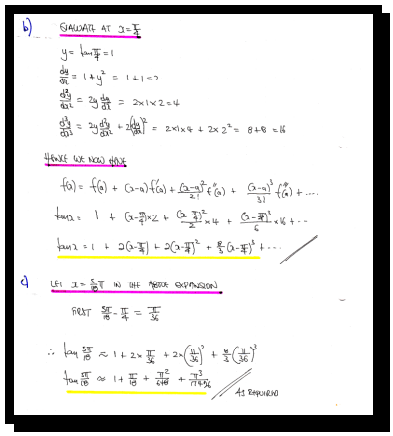
b) Determine the first four terms in the Taylor expansion of $\tan x$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$.

c) Hence deduce that

$$\tan\frac{5\pi}{18} \approx 1 + \frac{\pi}{18} + \frac{\pi^2}{648} + \frac{\pi^3}{17496} \, .$$

$\square$, $\boxed{y = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + O\left(\left(x - \frac{\pi}{4}\right)^4\right)}$





Question 3 **(****)**

$$y=\tan^2 x\,.$$

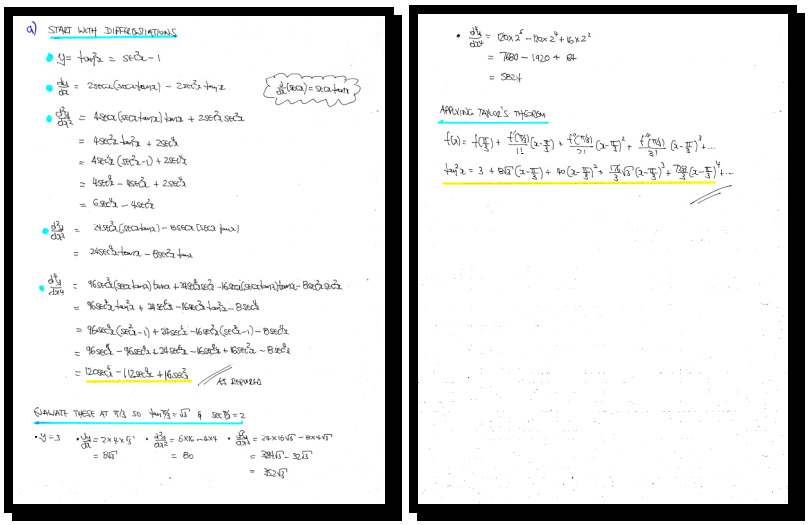
a) Show that

$$\frac{d^4y}{dx^4}=120\sec^6 x-120\sec^4 x+16\sec^2 x\,.$$

b) Determine the first 5 terms in the Taylor expansion of $\tan^2 x$, in ascending powers of $\left(x-\frac{\pi}{3}\right)$.

V, ,

$$y=3+8\sqrt{3}\left(x-\frac{\pi}{3}\right)+40\left(x-\frac{\pi}{3}\right)^2+\frac{176}{3}\left(x-\frac{\pi}{3}\right)^3+\frac{728}{3}\left(x-\frac{\pi}{3}\right)^4+O\left(\left(x-\frac{\pi}{3}\right)^5\right)$$



O.D.E.
TAYLOR SERIES EXPANSIONS
3 BASIC QUESTIONS

Question 1 (+)**

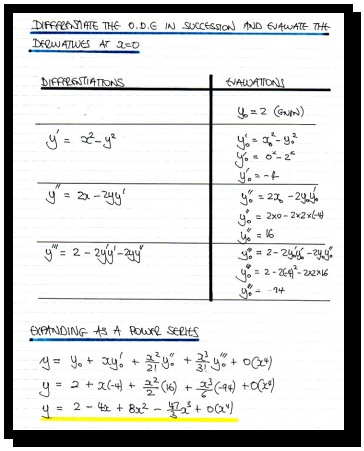
A curve has equation $y=f(x)$ which satisfies the differential equation

$$\frac{dy}{dx}=x^2-y^2,$$

subject to the condition $x=0$, $y=2$.

Determine the first 4 terms in the infinite series expansion of $y=f(x)$ in ascending powers of x.

$\boxed{}$, $\boxed{y=2-4x+8x^2-\frac{47}{3}x^3+O\left(x^4\right)}$



Question 2 (***)

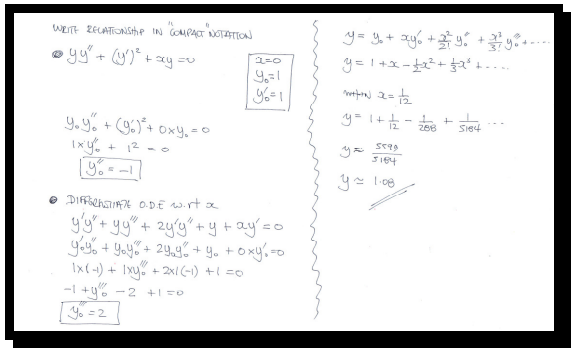
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0,$$

subject to the conditions $x=0$, $y=1$, $\frac{dy}{dx}=1$.

By using the first four terms in the expansion of $y = f(x)$ in ascending powers of x, show that $y \approx 1.08$ at $x=\frac{1}{12}$.

proof



Question 3 (***)

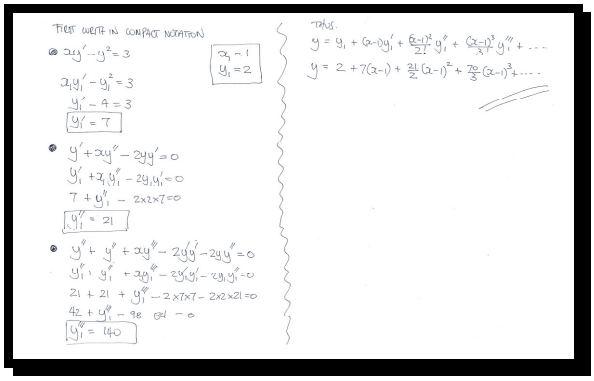
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$x\frac{dy}{dx} - y^2 = 3, \ x \neq 0,$$

subject to the condition $y = 2$ at $x = 1$.

Find the first four terms in the expansion of $y = f(x)$ as powers of $(x-1)$.

$$\boxed{y = 2 - 7(x-1) + \frac{21}{2}(x-1)^2 + \frac{70}{3}(x-1)^3 O\left((x-1)^4\right)}$$



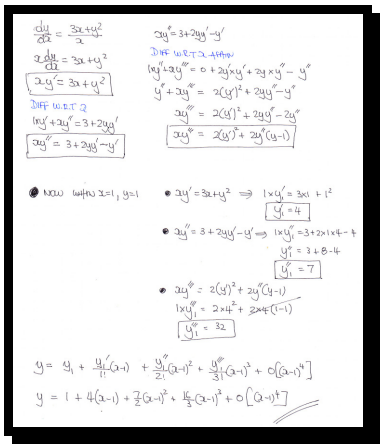
O.D.E.
TAYLOR SERIES EXPANSIONS
3 STANDARD QUESTIONS

Question 1 **(***+)**

$$\frac{dy}{dx}=\frac{3x+y^2}{x}, \quad x\neq 0 .$$

Given that $y=1$ at $x=1$, find a series solution for the above differential equation in ascending powers of $(x-1)$, up and including the terms in $(x-1)^3$.

$$\boxed{y=1+4(x-1)+\frac{7}{2}(x-1)^2+\frac{16}{3}(x-1)^3+O\left[(x-1)^4\right]}$$



Question 2 (*+)**

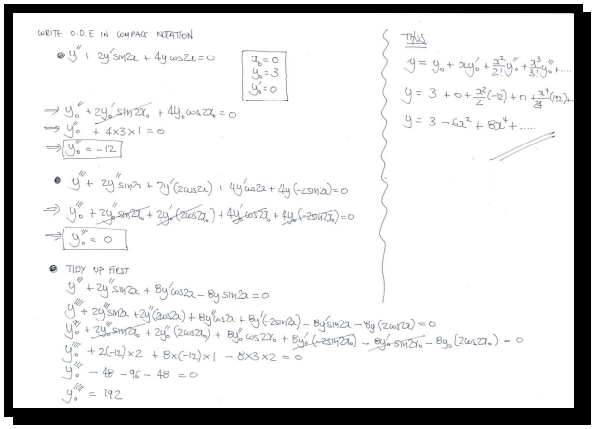
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\sin 2x + 4y\cos 2x = 0,$$

subject to the conditions $y = 3$, $\frac{dy}{dx} = 0$ at $x = 0$.

Find a series solution for $f(x)$ up and including the term in x^4.

$$\boxed{y = 3 - 6x^2 + 8x^4 + O\left(x^6\right)}$$



Question 3 **(***+)**

A curve has an equation $y=f(x)$ that satisfies the differential equation

$$\mathrm{e}^{-x}\frac{d^2y}{dx^2}=2y\frac{dy}{dx}+y^2+1$$

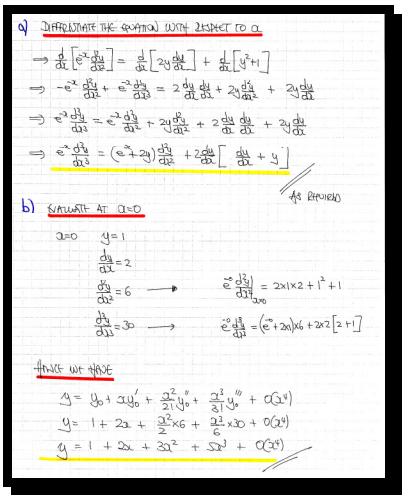
with $y=1$, $\dfrac{dy}{dx}=2$ at $x=0$.

a) Show clearly that

$$\mathrm{e}^{-x}\frac{d^3y}{dx^3}=\left(2y+\mathrm{e}^{-x}\right)\frac{d^2y}{dx^2}+2\frac{dy}{dx}\left(y+\frac{dy}{dx}\right).$$

b) Find a series solution for $f(x)$, up and including the term in x^3.

$\boxed{}$, $\boxed{y=1+2x+3x^2+5x^3+O\left(x^4\right)}$



MIXED SERIES EXPANSIONS 3 QUESTIONS

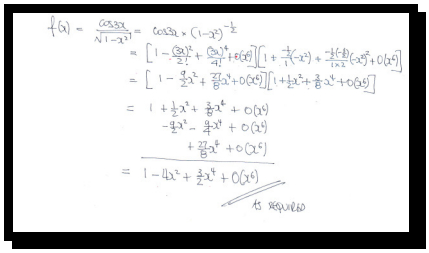
Question 1 **(***+)**

$$f(x)=\frac{\cos 3x}{\sqrt{1-x^2}}\ ,\ |x|<1.$$

Show clearly that

$$f(x)\approx 1-4x^2+\frac{3}{2}x^4.$$

proof



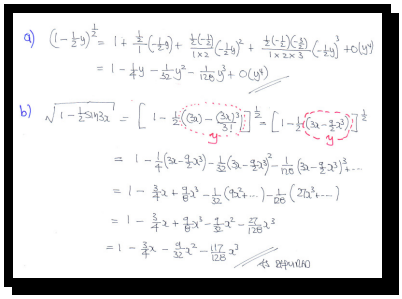
Question 2 **(***+)**

a) Find the first four terms in the series expansion of $\left(1-\frac{1}{2}y\right)^{\frac{1}{2}}$.

b) By considering the first two non zero terms in the expansion of $\sin 3x$ and the answer from part (a) , show that

$$\sqrt{1-\frac{1}{2}\sin 3x}\approx 1-\frac{3}{4}x-\frac{9}{32}x^2+\frac{117}{128}x^3.$$

$$1-\frac{1}{4}y-\frac{1}{32}y^2-\frac{1}{128}y^3+O\left(y^4\right)$$



Question 3 (*****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right].$$

, proof

