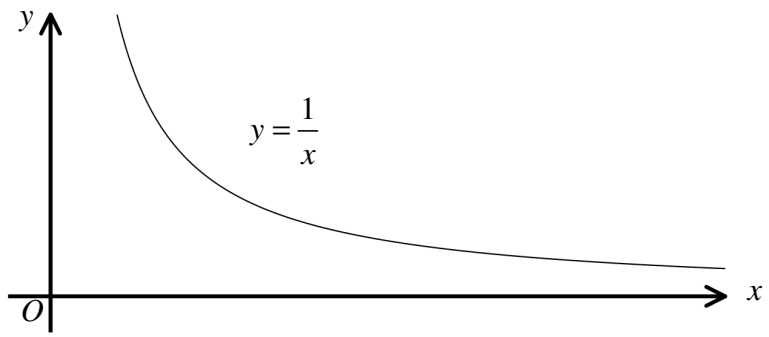
SERIES and INTEGRALS

Question 1 (***)

The figure below shows the curve C with equation $y = \mathrm{e}^{-\frac{1}{x}}$, $0 < x \leq 1$.



a) By using a two different sets of rectangles of unit width under and above the graph of C, show that

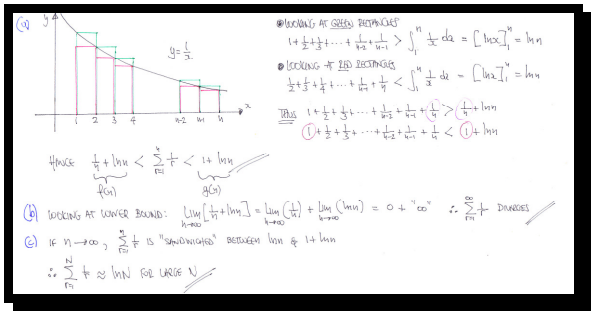
$$f(n) < \sum_{r=1}^{n} \frac{1}{r} < g(n),$$

where $f(n)$ and $g(n)$ are functions involving natural logarithms.

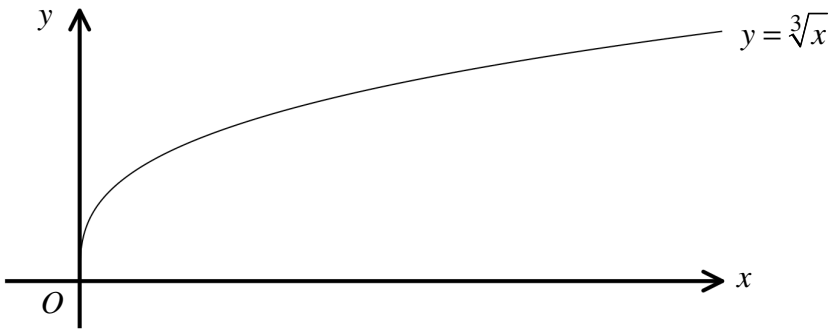
b) Determine whether $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ exists.

c) Write down an approximation for $\displaystyle\sum_{r=1}^{N} \frac{1}{r}$ if N is very large.

$\boxed{f(n) = \frac{1}{n} + \ln n}$, $\boxed{g(n) = 1 + \ln n}$, $\boxed{\displaystyle\sum_{r=1}^{\infty} \frac{1}{r} \text{ diverges}}$, $\boxed{\text{as } n \to \infty,\ \displaystyle\sum_{r=1}^{N} \frac{1}{r} \approx \ln N}$



Question 2 **(***)**



The figure above shows the curve C with equation $y=\sqrt[3]{x}$, $x\geq 0$.

a) By using two different sets of rectangles of unit width under and above the graph of C, show that

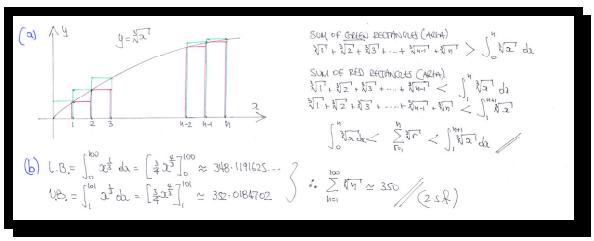
$$\int_a^b \sqrt[3]{x}\ dx \ < \ \sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{3}+...+\sqrt[3]{n} \ < \ \int_c^d \sqrt[3]{x}\ ,$$

stating the limits in the integrals.

b) Hence show that

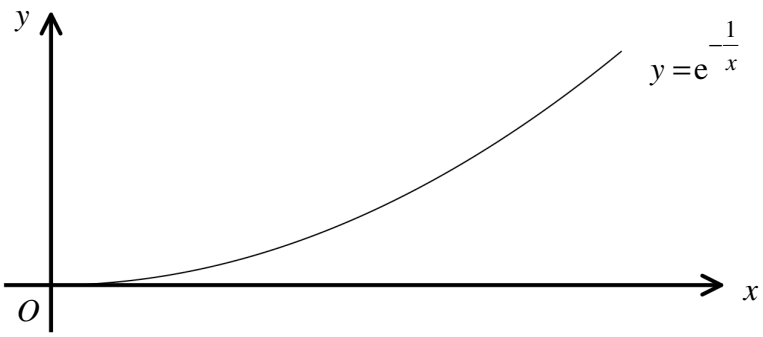
$$\sum_{n=1}^{100} \sqrt[3]{n} \approx 350 .$$

$$\boxed{a=0,\ b=n,\ c=1,\ d=n+1}$$



Question 3 (*+)**

The figure below shows the curve C with equation $y = \mathrm{e}^{-\frac{1}{x}}$, $0 < x \leq 1$.



a) By using two different sets of rectangles of width $\frac{1}{n}$ under and above the graph of C, show that

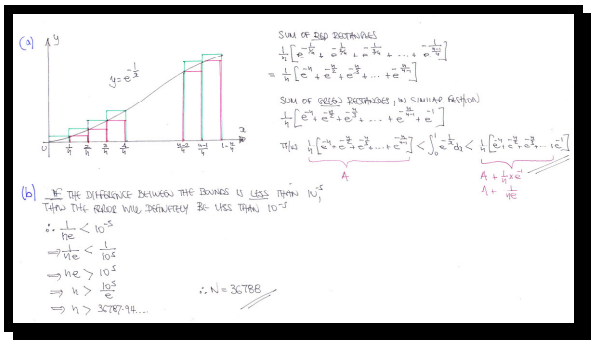
$$A < \int_0^1 \mathrm{e}^{-\frac{1}{x}}\,dx < A + \frac{1}{2\mathrm{e}},$$

where A is an exact finite series involving exponentials.

The above expression is to be used to approximate the area under C for $0 < x \leq 1$.
When $n \geq N$, the error is less than 10^{-5}.

b) Determine the least possible value of N.

$$\boxed{A = \frac{1}{n}\left[\mathrm{e}^{-n} + \mathrm{e}^{-\frac{1}{2}n} + \mathrm{e}^{-\frac{1}{3}n} + \mathrm{e}^{-\frac{1}{4}n} + \ldots + \mathrm{e}^{-\frac{n}{n-1}}\right]}, \quad \boxed{N = 36788}$$

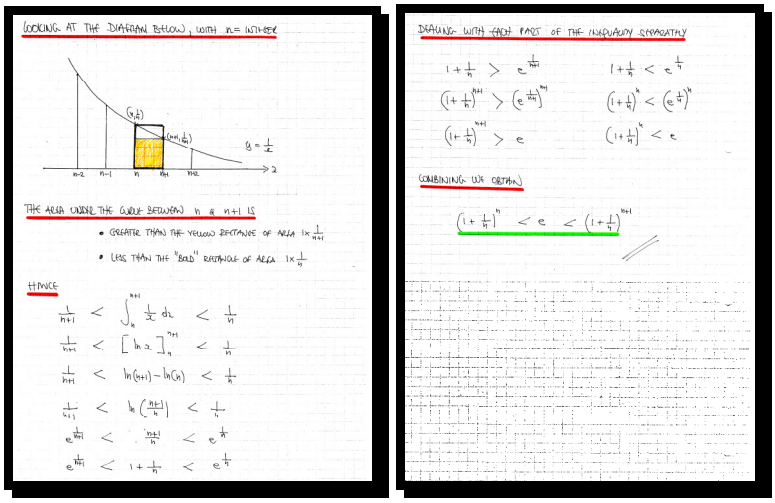


Question 4 (*+)**

By considering the area of two different rectangles of unit width under and above the graph of $y=\dfrac{1}{x}$, show that

$$\left(1+\frac{1}{n}\right)^{n} < \mathrm{e} < \left(1+\frac{1}{n}\right)^{n+1}$$

, proof



Question 5 **(*****)**

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

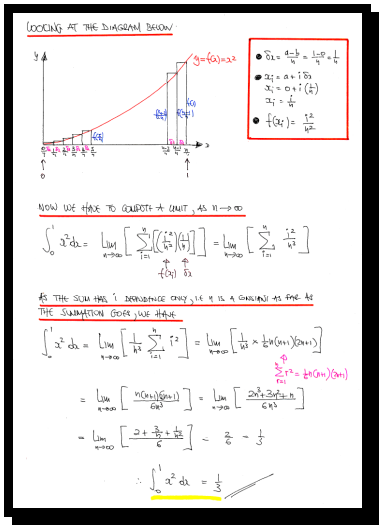
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_0^1 x^2\ dx = \frac{1}{3}.$$

, proof



Question 6 **(*****)**

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

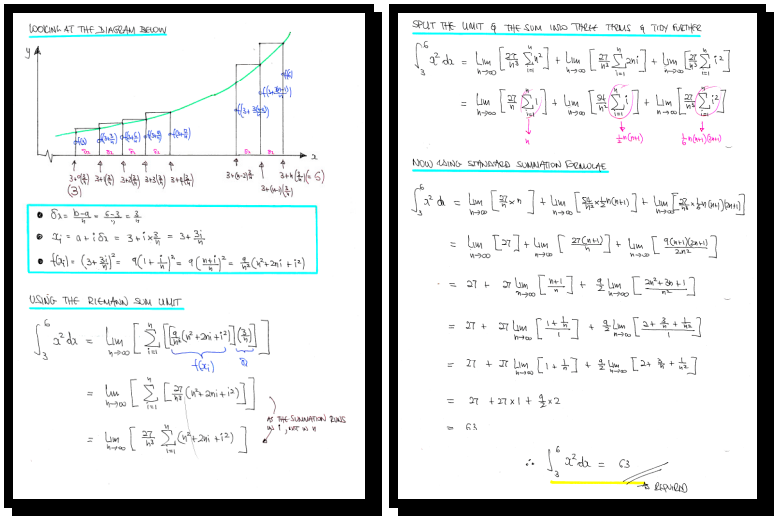
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_3^6 x^2\ dx = 63 .$$

, proof



Question 7 **(*****)**

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

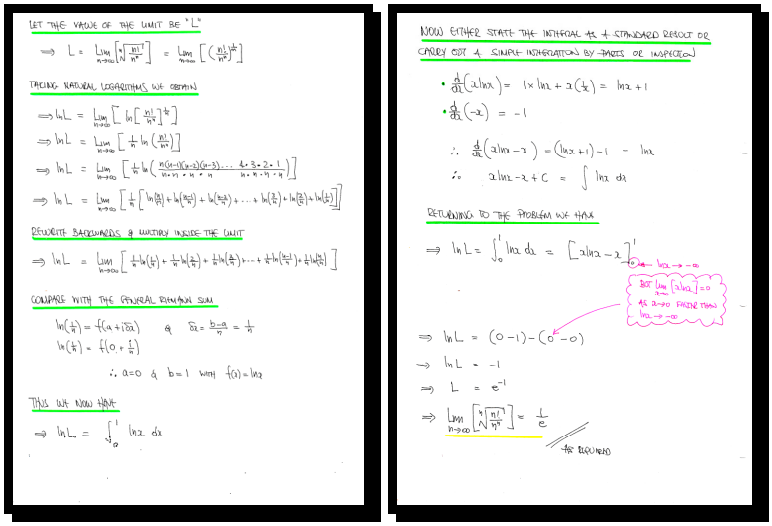
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{\mathrm{e}}.$$

[ ], [proof]



Question 8 (*****)

Determine the limit of the following series.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\ldots+\frac{1}{n+n-2}+\frac{1}{n+n-1}+\frac{1}{n+n}+\right].$$

V, [ ], ln2

