REDUCTION FORMULAS

Question 1 (**)

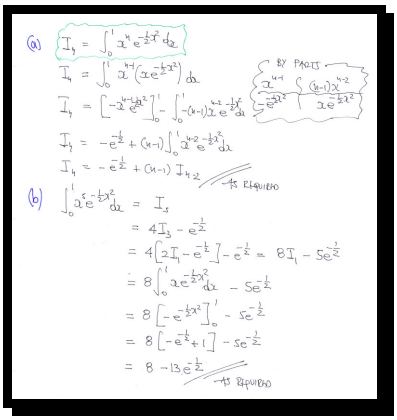
$$I_n = \int_0^1 x^n \, \mathrm{e}^{-\frac{1}{2}x^2} \, dx, \quad n \in \mathbb{N}.$$

Show clearly that …

a) … $I_n = (n-1) I_{n-2} - \mathrm{e}^{-\frac{1}{2}}, \ n \geq 2$.

b) … $\displaystyle\int_0^1 x^5 \, \mathrm{e}^{-\frac{1}{2}x^2} \, dx = 8 - 13\mathrm{e}^{-\frac{1}{2}}$.

proof



Question 2 (**)

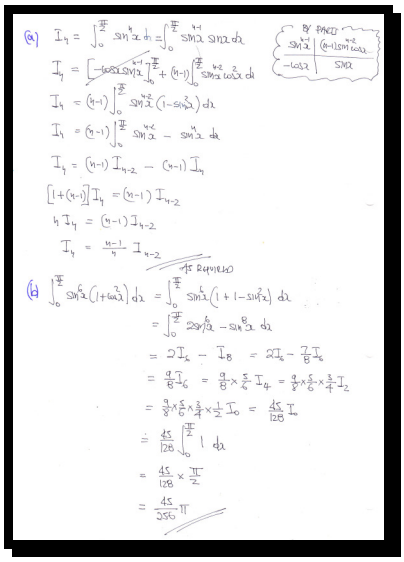
$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \ dx, \ n \in \mathbb{N}.$$

Show clearly that …

a) $\ldots I_n = \dfrac{n-1}{n} I_{n-2}, \ n \geq 2.$

b) $\ldots \displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x\left(1+\cos^2 x\right) \ dx = \frac{45\pi}{256}.$

proof



Question 3 (**)

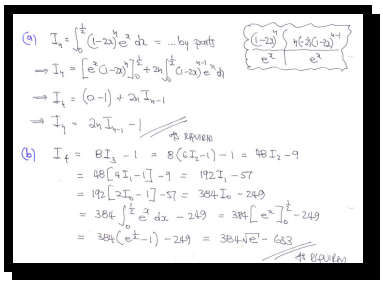
$$I_n = \int_0^{\frac{1}{2}} (1-2x)^n \, \mathrm{e}^x \, dx, \quad n \in \mathbb{N}.$$

Show clearly that …

a) …. $I_n = 2nI_{n-1} - 1$, $n \geq 1$.

b) … $I_4 = 384\sqrt{\mathrm{e}} - 633$.

proof



Question 4 (**)

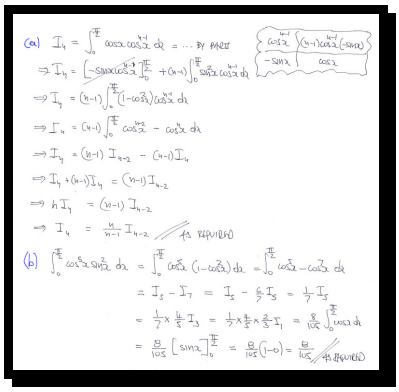
$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \ dx \,, \ n \in \mathbb{N} \,.$$

Show clearly that …

a) … $I_n = \dfrac{n-1}{n} I_{n-2} \,, \ n \geq 2 \,.$

b) … $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \ dx = \frac{8}{105} \,.$

proof



Question 5 **(**+)**

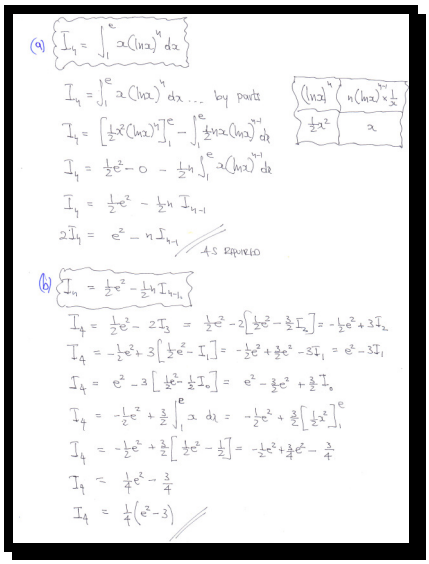
$$I_n = \int_1^{\mathrm{e}} x\left(\ln x\right)^n \, dx \,, \quad n \in \mathbb{N} \,.$$

Show clearly that …

a) … $2I_n = \mathrm{e}^2 - nI_{n-1}$.

b) … $I_4 = \frac{1}{4}\left(\mathrm{e}^2 - 3\right)$.

proof



Question 6 (+)**

$$I_n = \int_1^{\mathrm{e}} x^2 \left(\ln x\right)^n \, dx \,, \; n \in \mathbb{N} \,.$$

a) Show clearly that

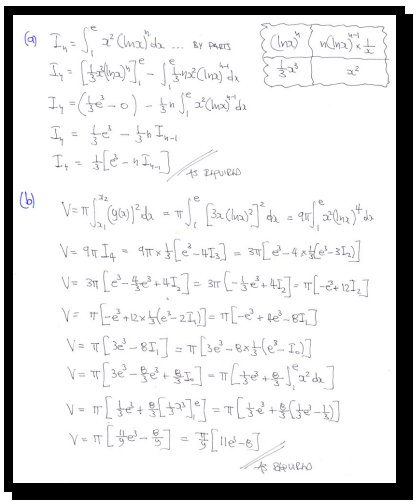
$$I_n = \frac{1}{3}\left(\mathrm{e}^3 - nI_{n-1}\right).$$

The part of the curve with equation $y = 3x\left(\ln x\right)^2$, for $1 \le x \le \mathrm{e}$, is rotated by 2π radians about the x axis.

b) Show that the volume of the solid generated is given by

$$\frac{\pi}{9}\left(11\mathrm{e}^3 - 8\right).$$

proof



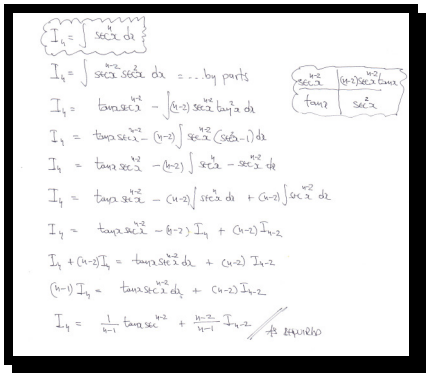
Question 7 (+)**

$$I_n = \int \sec^n x \, dx \, , \; n \in \mathbb{N} \, .$$

Show clearly that

$$I_n = \frac{1}{n-1}\big(\tan x\big)\big(\sec x\big)^{n-2} + \frac{n-2}{n-1} I_{n-2} \, , \; n \geq 2 \, .$$

proof



Question 8 **(***)**

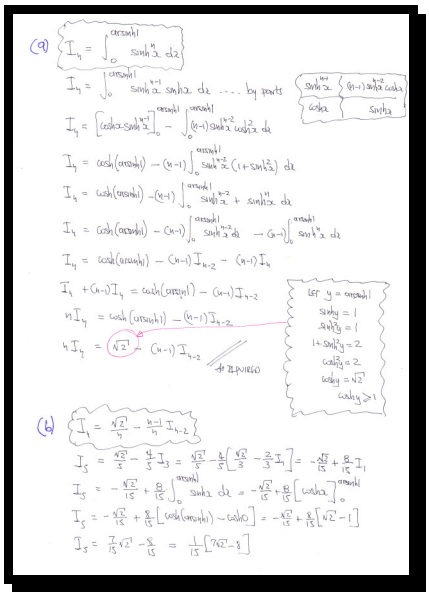
$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x \ dx \,, \ n \in \mathbb{N} \,.$$

Show clearly that …

a) … $nI_n = \sqrt{2} - (n-1) I_{n-2} \,, \ n \geq 2 \,.$

b) … $I_5 = \dfrac{1}{15}\left(7\sqrt{2} - 8\right) .$

proof



Question 9 (***)

The integral I_n is defined for $n \geq 0$ as

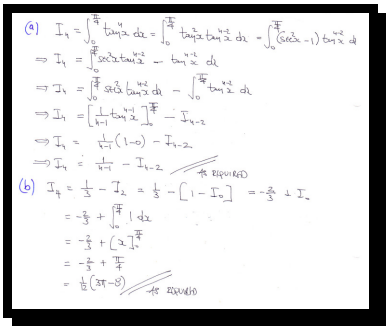
$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \, , \ n \in \mathbb{N} \, .$$

Show clearly that …

a) … $I_n = \dfrac{1}{n-1} - I_{n-2} \, , \ n \geq 1 \, .$

b) … $I_4 = \dfrac{1}{12}(3\pi - 8) \, .$

proof



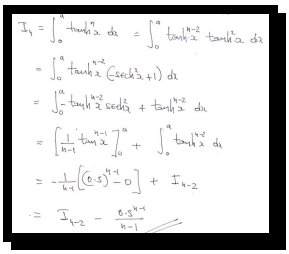
Question 10 (***)

$$I_n = \int_0^a \tanh^n x \ dx \,, \ n \in \mathbb{N} \,.$$

Given that $a = \text{artanh}\, \frac{1}{2}$, show clearly that

$$I_n = I_{n-2} - \frac{(0.5)^{n-1}}{n-1} \,, \ n \geq 2 \,.$$

proof



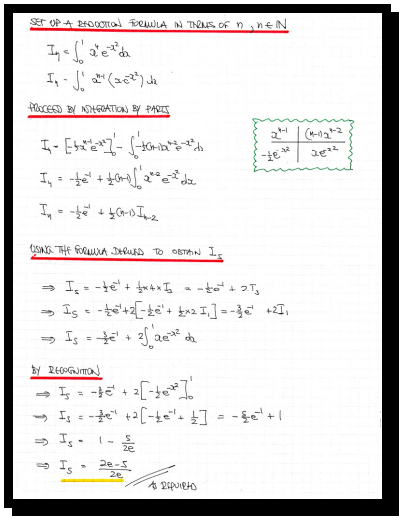
Question 11 (*+)**

By forming and using a suitable reduction formula show that

$$\int_0^1 x^5 \, \mathrm{e}^{-x^2} \, dx = \frac{2\mathrm{e}-5}{2\mathrm{e}}.$$

No credit will be given if no reduction formula is not used in this question

[ ], [proof]



Question 12 **(***+)**

The integral I_n is defined for $n \geq 0$ as

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx .$$

a) Show clearly that …

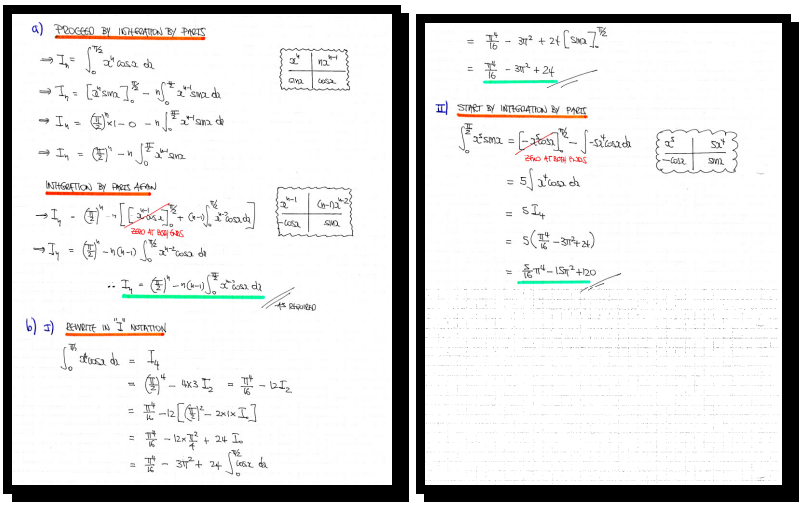
$$\ldots I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}, \quad n \geq 2 .$$

b) Hence find, in terms of π, exact expressions for …

i. … $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx .$

ii. … $\displaystyle\int_0^{\frac{\pi}{2}} x^5 \sin x \, dx .$

$\square$, $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx = \frac{\pi^4}{16} - 3\pi^2 + 24$, $\displaystyle\int_0^{\frac{\pi}{2}} x^5 \sin x \, dx = \frac{5\pi^4}{16} - 15\pi^2 + 120$



Question 13 (*+)**

The integral I_n is defined for $n \geq 0$ as

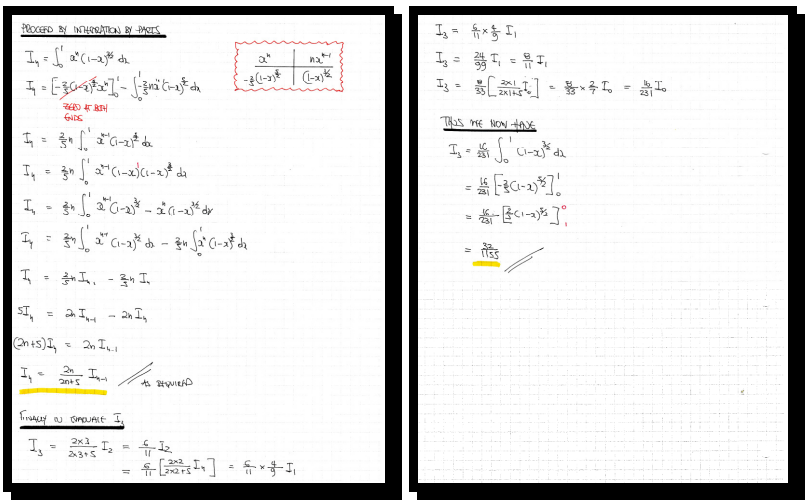
$$I_n = \int_0^1 x^n \left(1-x\right)^{\frac{3}{2}} dx \, , \, n \in \mathbb{N} \, .$$

Show that

$$I_n = \left(\frac{2n}{2n+5} \right) I_{n-1} \, , \, n \geq 1 ,$$

and use it to find as an exact fraction the value of I_3.

$\square$, $\boxed{I_3 = \frac{32}{1155}}$



Question 14 **(***+)**

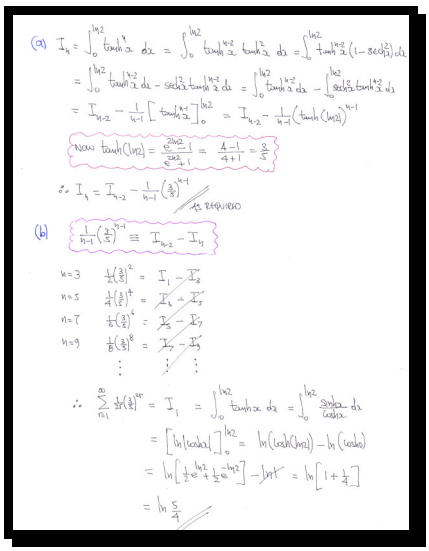
$$I_n = \int_0^{\ln 2} \tanh^n x \ dx, \ n \in \mathbb{N}.$$

Show clearly that …

a) … $I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1}$, $n \geq 2$.

b) … $\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right)$.

proof



Question 15 **(***+)**

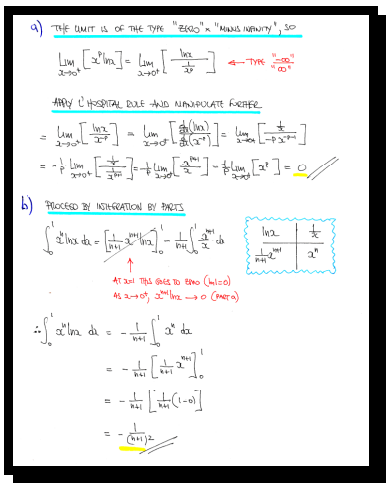
a) If $p \in (0,\infty)$, show that

$$\lim_{x\to 0^+}\left[x^p \ln x\right]=0, \quad x\in(0,\infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \ dx, \quad n\in\mathbb{N}.$$

$\square$, $\boxed{-\dfrac{1}{(n+1)^2}}$



Question 16 (***+)

$$I_n = \int \text{cosec}^n x \, dx, \quad n \in \mathbb{N}.$$

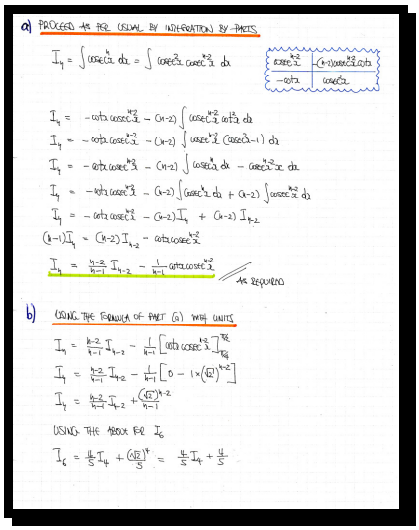
a) Show clearly that

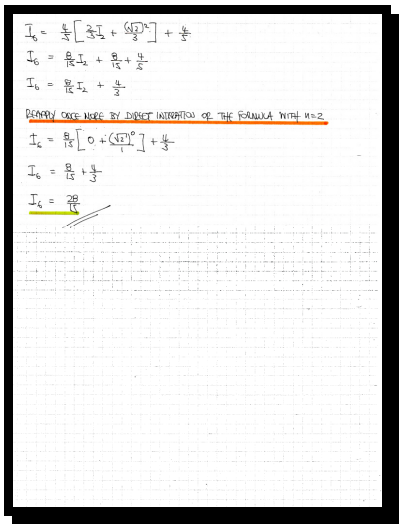
$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} (\cot x)(\text{cosec}\, x)^{n-2}, \quad n \geq 2.$$

b) Use part **(a)** to evaluate

$$I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \text{cosec}^6 x \, dx$$

$\square$, $\frac{28}{15}$





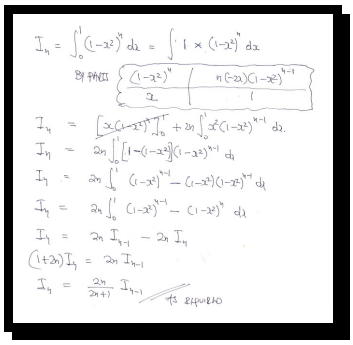
Question 17 (****)

$$I_n = \int_0^1 \left(1-x^2\right)^n \, dx \, , n \in \mathbb{N} \, .$$

Show clearly that

$$I_n = \frac{2n}{2n+1} I_{n-1} \, , \; n \geq 1 .$$

proof



$I_n = \int_0^1 (1-x^2)^n \, dx = \int 1 \times (1-x^2)^n \, dx$

By parts: $(1-x^2)^n$ | $n(-2x)(1-x^2)^{n-1}$; x | 1

$I_n = \left[x(1-x^2)^n\right]_0^1 + 2n \int_0^1 x^2 (1-x^2)^{n-1} \, dx$

$I_n = 2n \int_0^1 \left[1-(1-x^2)\right](1-x^2)^{n-1} \, dx$

$I_n = 2n \int_0^1 (1-x^2)^{n-1} - (1-x^2)(1-x^2)^{n-1} \, dx$

$I_n = 2n \int_0^1 (1-x^2)^{n-1} - (1-x^2)^n \, dx$

$I_n = 2n \, I_{n-1} - 2n \, I_n$

$(1+2n) I_n = 2n \, I_{n-1}$

$I_n = \frac{2n}{2n+1} I_{n-1}$ AS REQUIRED

Question 18 **(****)**

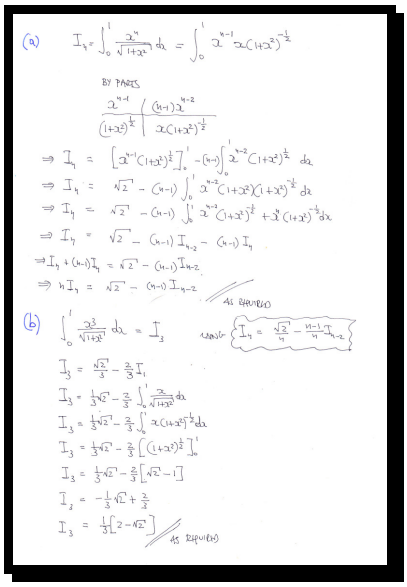
$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\, dx\,,\ n \in \mathbb{N}\,.$$

Show clearly that…

a) … $nI_n = \sqrt{2} - (n-1) I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\, dx = \frac{1}{3}\left(2-\sqrt{2}\right).$

proof



Question 19 (****)

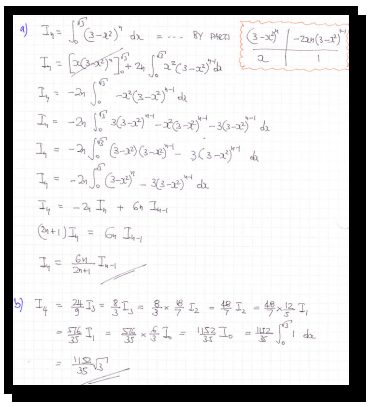
$$I_n=\int_0^{\sqrt{3}}\left(3-x^2\right)^n\,dx\,,\ n\in\mathbb{N}\,.$$

Show clearly that…

a) … $\left(2n+1\right)I_n=6nI_{n-1}\,,\ n\geq 1\,.$

b) … $\displaystyle\int_0^{\sqrt{3}}\left(3-x^2\right)^4\,dx=\frac{1152}{35}\sqrt{3}\,.$

proof



Question 20 (****)

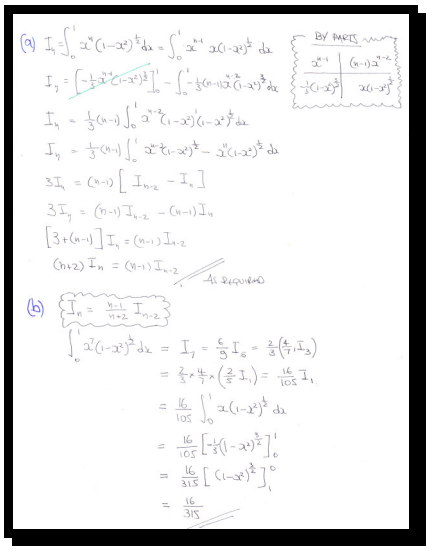
$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx\,,\ n \in \mathbb{N}\,.$$

Show clearly that…

a) … $(n+2)I_n = (n-1)I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_0^1 x^n \sqrt{1-x^2}\, dx = \frac{16}{315}\,.$

proof



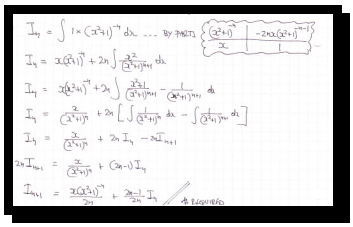
Question 21 (****)

$$I_n = \int \left(1+x^2\right)^{-n} \, dx \, , \; n \in \mathbb{N} \, .$$

Show clearly that

$$I_{n+1} = \frac{x\left(x^2+1\right)^{-n}}{2n} + \frac{2n-1}{2n} I_n \, , \; n \geq 1 .$$

proof



Question 22 (****)

The integral I_n is defined for $n \geq 0$ as

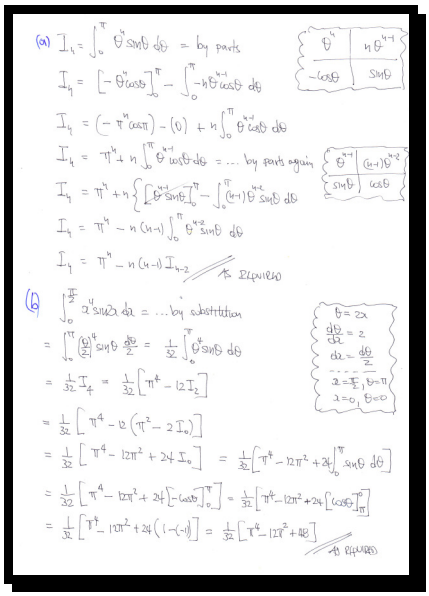
$$I_n = \int_0^{\pi} \theta^n \sin\theta \, d\theta .$$

Show clearly that …

a) … $I_n = \pi^n - n(n-1) I_{n-2}$, $n \geq 2$.

b) … $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \sin 2x \, dx = \frac{1}{32}\left(\pi^4 - 12\pi^2 + 48\right)$.

proof



Question 23 (****)

$$I_n = \int \frac{x^n}{\sqrt{1+x^2}}\,dx\,,\ n\in\mathbb{N}\,.$$

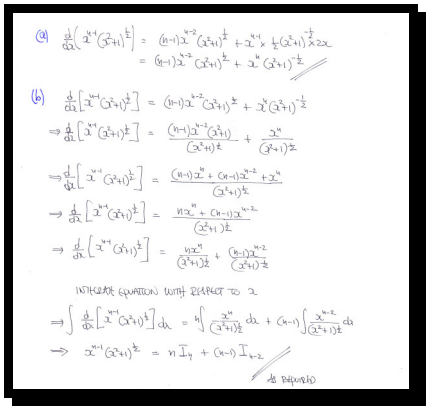
a) Find an expression for

$$\frac{d}{dx}\left[x^{n-1}\left(x^2+1\right)^{\frac{1}{2}}\right].$$

b) Use part (a) to show that

$$nI_n+(n-1)I_{n-2}=x^{n-1}\sqrt{x^2+1}\,,\ n\geq 2\,.$$

$$\boxed{\frac{d}{dx}\left[x^{n-1}\left(x^2+1\right)^{\frac{1}{2}}\right]=(n-1)x^{n-2}\left(x^2+1\right)^{\frac{1}{2}}+x^n\left(x^2+1\right)^{-\frac{1}{2}}}$$



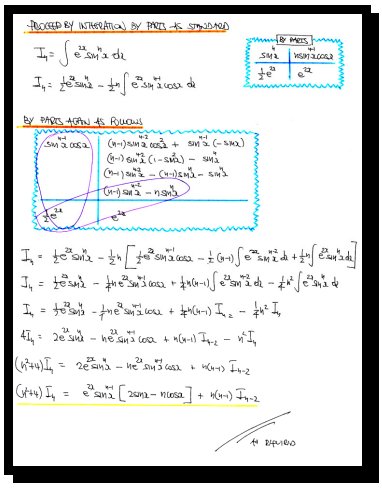
Question 24 (**+)**

$$I_n \equiv \int \mathrm{e}^{2x} \sin^n x \, dx \, , \, n \in \mathbb{N} \, , \, n \geq 2 \, .$$

Use integration by parts twice to show

$$\left(n^2+4\right) I_n = n(n-1) I_{n-2} + \left(2\sin x - n\cos x\right) \mathrm{e}^{2x} \sin^{n-1} x \, .$$

[ ], [proof]



Question 25 (**+)**

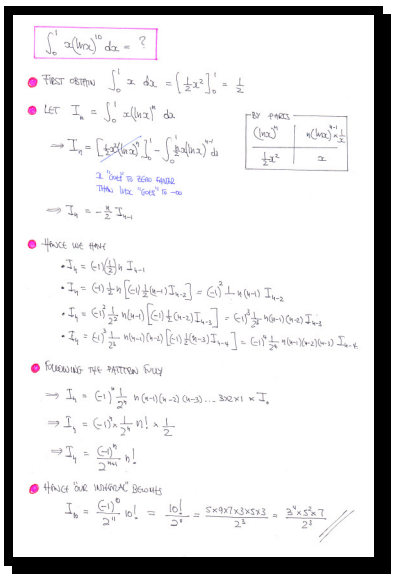
Find a suitable reduction formula and use it to find

$$\int_0^1 x\left(\ln x\right)^{10}\ dx\,.$$

You may assume that the integral converges.

Give the answer as the product of powers of prime factors.

$$\int_0^1 x\left(\ln x\right)^{10}\ dx = 2^{-3}\times 3^4\times 5^2\times 7$$



Question 26 (**+)**

$$I_n = \int \frac{\sin nx}{\sin x}\, dx \ , \ n \in \mathbb{N} .$$

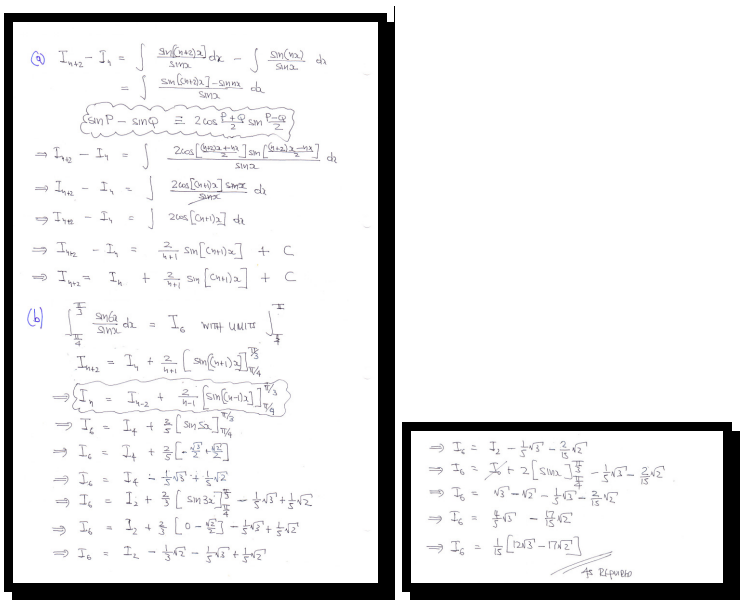
a) Show by considering $I_{n+2} - I_n$ that

$$I_{n+2} = I_n + \frac{2}{n+1}\sin\left[(n+1)x\right] + C , \ \ n \geq 0 .$$

b) Show further that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\, dx = \frac{1}{15}\left(12\sqrt{3} - 17\sqrt{2}\right).$$

proof



Question 27 **(****+)**

$$I_n = \int_0^{\pi} \frac{\sin(n\theta)}{\sin\theta}\, d\theta .$$

The integral above is defined for positive integer values n.

a) Use trigonometric identities to show that

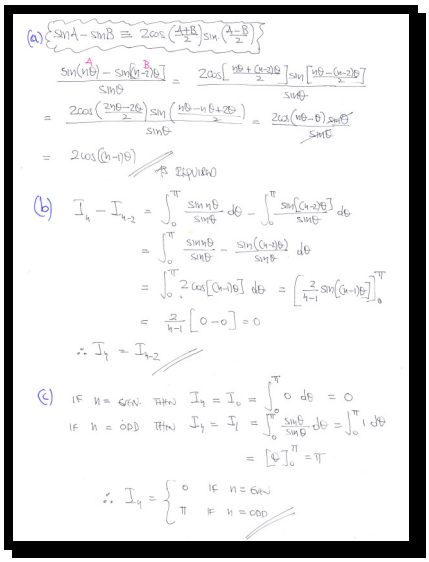
$$\frac{\sin(n\theta)-\sin\left[(n-2)\theta\right]}{\sin\theta} = 2\cos\left[(n-1)\theta\right].$$

b) Hence show that

$$I_n = I_{n-2}\,,\ n \geq 2 .$$

c) Evaluate I_n in both cases, where n is either odd or even positive integer.

$$\boxed{I_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pi & \text{if } n \text{ is odd} \end{cases}}$$



Question 28 **(****+)**

$$I_n = \int_0^{\frac{\pi}{3}} \mathrm{e}^{3x} \tan^n x \, dx \ , \ n \in \mathbb{N} .$$

a) Show clearly that…

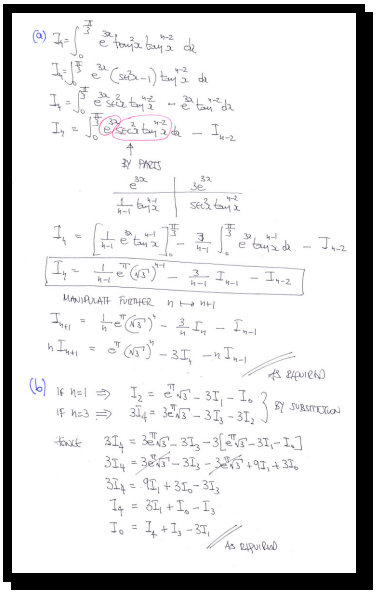
i. … $nI_{n+1} = \mathrm{e}^{\pi}\left(\sqrt{3}\right)^n - 3I_n - nI_{n-1}$, $n \geq 1$.

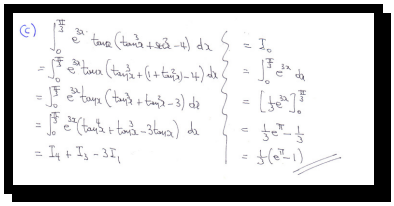
ii. … $I_0 = I_4 + I_3 - 3I_1$.

b) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} \mathrm{e}^{3x} \tan x\left(\tan^3 x + \sec^2 x - 4\right) dx .$$

proof





Question 29 **(****+)**

$$I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\ d\theta\ ,\ m\in\mathbb{N}\ ,\ n\in\mathbb{N}\ .$$

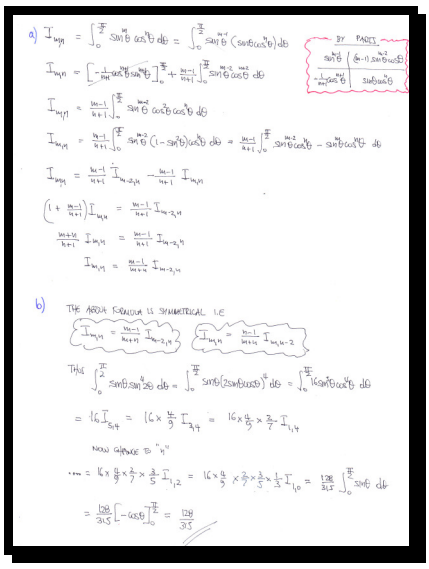
a) Show clearly that

$$I_{m,n}=\frac{m-1}{m+n}I_{m-2,n}\ .$$

b) Hence find an exact value for

$$\int_0^{\frac{\pi}{2}}\sin\theta\sin^2 2\theta\ d\theta\ .$$

$$\boxed{\frac{128}{315}}$$



Question 30 (**+)**

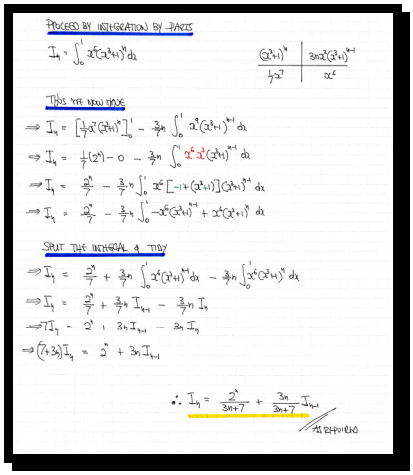
It is given that

$$I_n=\int_0^1 x^6\left(x^3+1\right)^n\ dx\,,\ n\in\mathbb{Z}\,,\ n\geq 0\,.$$

Show, with detailed workings, that

$$I_n=\frac{2^n}{3n+7}+\frac{3n}{3n+7}I_{n-1}\,,\quad n\geq 1.$$

, proof



Question 31 (**+)**

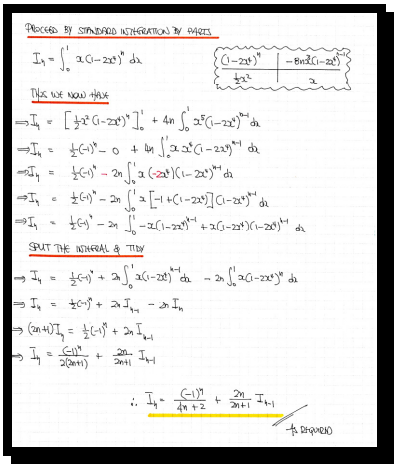
It is given that

$$I_n=\int_0^1 x\left(1-2x^4\right)^n\ dx\,,\ n\in\mathbb{Z}\,,\ n\geq 0\,.$$

Show, with detailed workings, that

$$I_n=\frac{(-1)^n}{4n+2}+\frac{2n}{2n+1}I_{n-1}\,,\ \ n\geq 1.$$

, proof



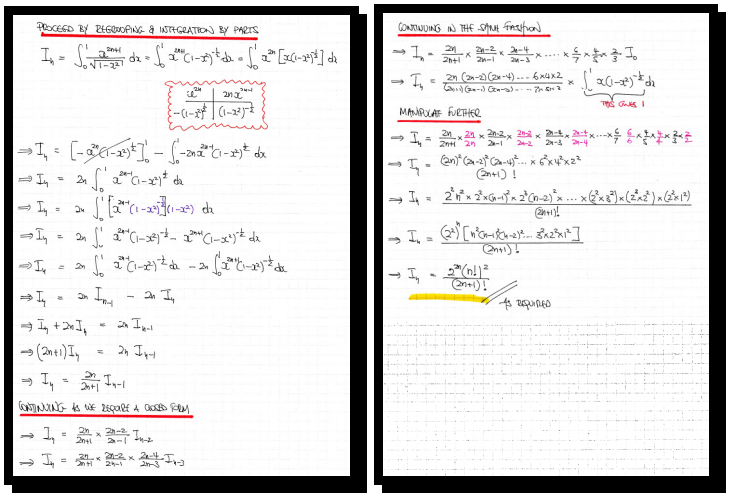
Question 32 (*****)

It is given that $I_n = \displaystyle\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$.

Show with detailed workings that

$$I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

V, [ ], proof



Question 33 **(*****)**

It is given that

$$I_n = \int_0^a x^{n+\frac{1}{2}}\sqrt{a-x}\ dx\,,\ n\in\mathbb{Z},\ n\geq 0$$

where a is a positive constant.

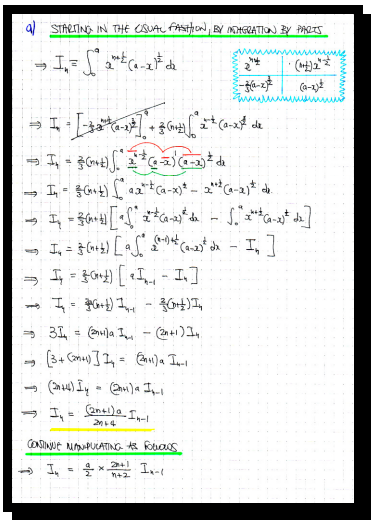
a) Use integration by parts to show

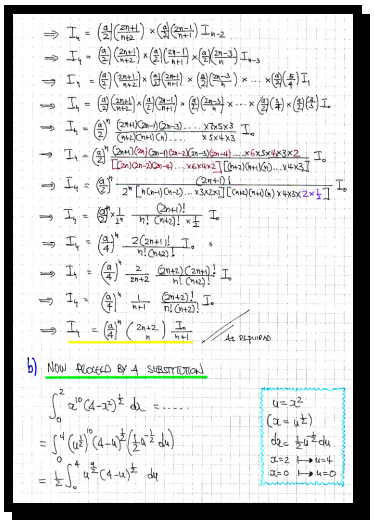
$$I_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n}\frac{I_0}{n+1},\quad n\geq 1.$$

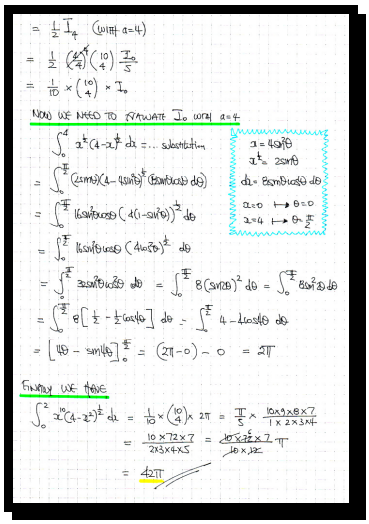
b) Determine the value of

$$\int_0^2 x^{10}\sqrt{4-x^2}\ dx\,.$$

$\boxed{}$, $\boxed{42\pi}$







Question 34 **(*****)**

It is given that

$$I_{n,m}=\int_0^1 (1-x)^n x^m \, dx,$$

where $n,m\in\mathbb{Z}$, with $n,m\geq 0$.

a) Show that …

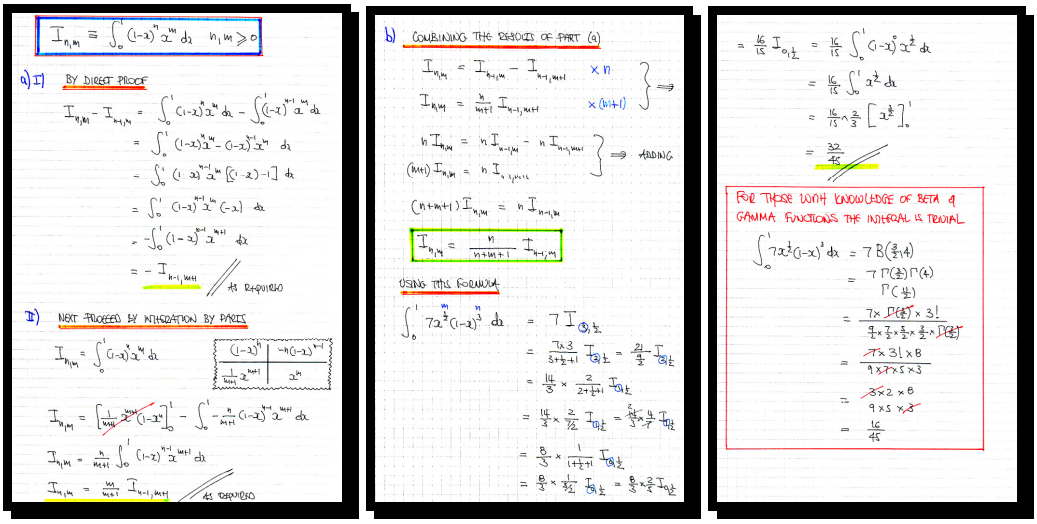
i. … $I_{n,m}-I_{n-1,m}=-I_{n-1,m+1}$.

ii. … $I_{n,m}=\dfrac{n}{m}I_{n-1,m+1}$

b) Hence derive an expression of $I_{n,m}$ and use it to find

$$\int_0^1 7x^{\frac{1}{2}}(1-x)^3 \, dx.$$

$\square$, $\displaystyle\int_0^1 7x^{\frac{1}{2}}(1-x)^3 \, dx=\frac{32}{45}$



Question 35 **(*****)**

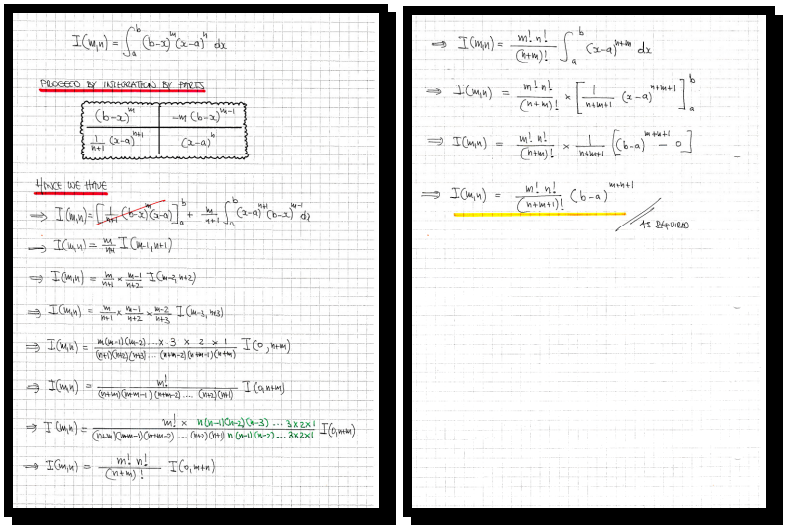
$$I(m,n)=\int_a^b (b-x)^m (x-a)^n \, dx, \quad m\in\mathbb{N},\ n\in\mathbb{N}.$$

Show that

$$I(m,n)=\frac{m!\,n!}{(m+n+1)!}(b-a)^{m+n+1},$$

where a and b are real constants such that $b>a$

, proof



Question 36 **(*****)**

$$I_n = \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}}\, dx\,,\ n \in \mathbb{N}\,,\ a > 0\,.$$

Show clearly that…

a) … $I_n = \dfrac{a^2(n-1)}{n} I_{n-2}\,,\ n \geq 2\,.$

b) … $\displaystyle\int_2^4 \frac{3x^3 - 18x^2 + 36x - 18}{\sqrt{4x - x^2}}\, dx = 3\pi - 16\,.$

[ ], [proof]

