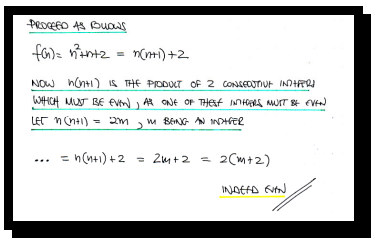
GENERAL PROOF

Question 1 **(**)**

$$f(n)=n^2+n+2,\ n\in\mathbb{N}.$$

Show that $f(n)$ is always even.

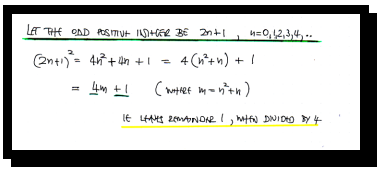
$\boxed{}$, proof



Question 2 **(**)**

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

$\boxed{}$, proof



Question 3 (**)

Show that $a^3 - a + 1$ is odd for all positive integer values of a.

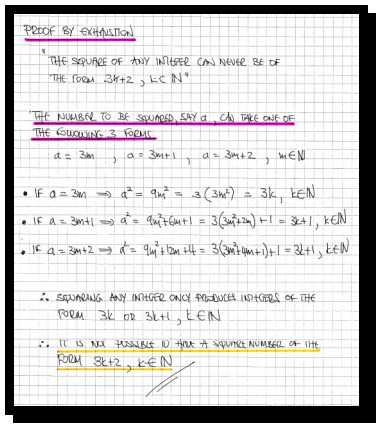
[ ], proof



Question 4 (**)

Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$.

[ ], proof



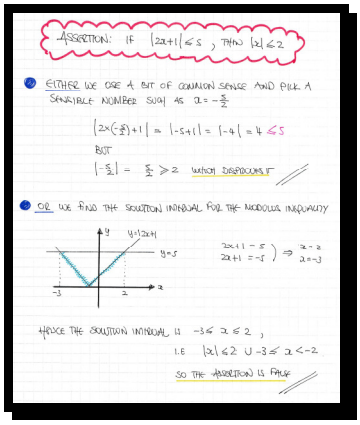
Question 5 **(**+)**

It is asserted that

$$|2x+1|\le 5 \quad \Rightarrow \quad |x|\le 2\,.$$

Disprove this assertion by a **counter-example**.

[ ], [proof]

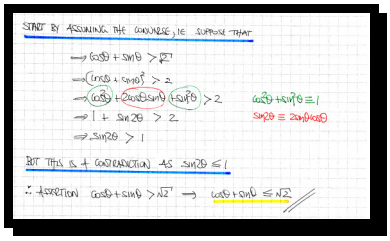


Question 6 **(**+)**

Prove by **contradiction** that for all real θ

$$\cos\theta+\sin\theta\le\sqrt{2}\,.$$

[ ], [proof]

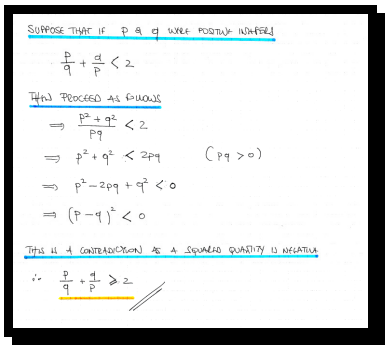


Question 7 (+)**

Prove by contradiction that if p and q are positive integers, then

$$\frac{p}{q}+\frac{q}{p}\geq 2\,.$$

, proof

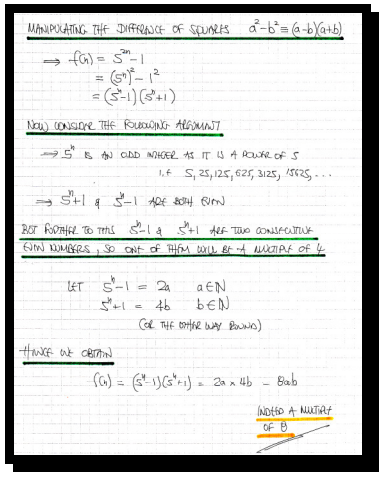


Question 8 (***)

$$f(n)=5^{2n}-1,\ n\in\mathbb{N}\,.$$

Without using proof by induction, show that $f(n)$ is a multiple of 8.

, proof

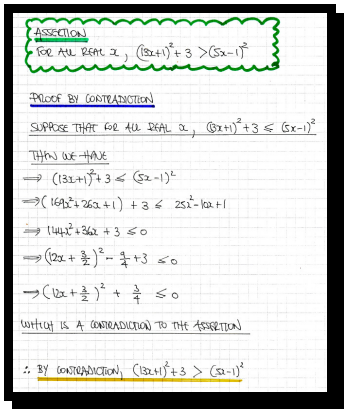


Question 9 **(***)**

Prove by **contradiction** that for all real x

$$(13x+1)^2+3>(5x-1)^2 .$$

$\boxed{}$, $\boxed{\text{proof}}$



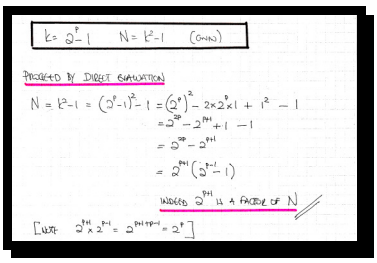
Question 10 **(***)**

It is given that

$$N=k^2-1 \quad \text{and} \quad k=2^p-1, \qquad p\in\mathbb{N} .$$

Use direct proof to show that 2^{p+1} is a factor of N.

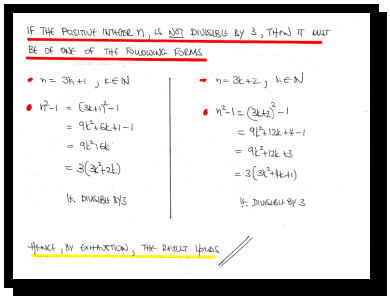
$\boxed{}$, $\boxed{\text{proof}}$



Question 11 (***)

Prove by exhaustion that if n is a positive integer that is **not** divisible by 3, then n^2-1 is divisible by 3.

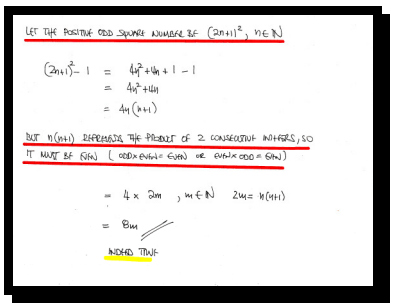
$\boxed{}$, proof



Question 12 (***)

Prove that if we subtract 1 from a positive odd square number, the answer is always divisible by 8.

$\boxed{}$, proof

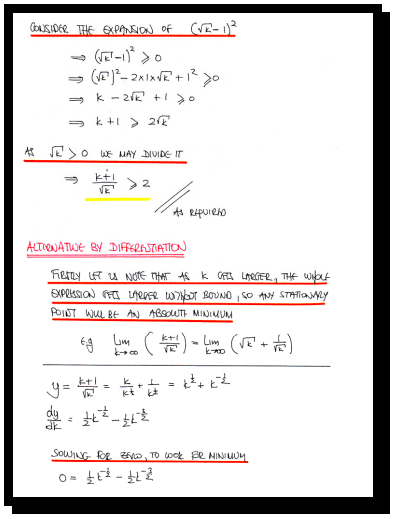


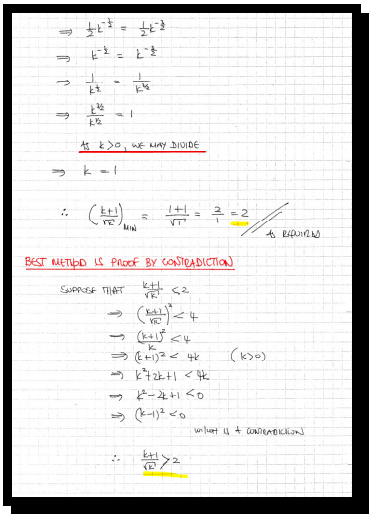
Question 13 **(***+)**

Given that $k>0$, use algebra to show that

$$\frac{k+1}{\sqrt{k}}\geq 2.$$

$\boxed{}$, proof



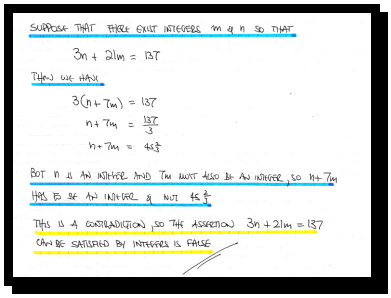


Question 14 **(***)**

Prove by the method of **contradiction** that there are no integers n and m which satisfy the following equation.

$$3n+21m=137$$

$\boxed{}$, proof

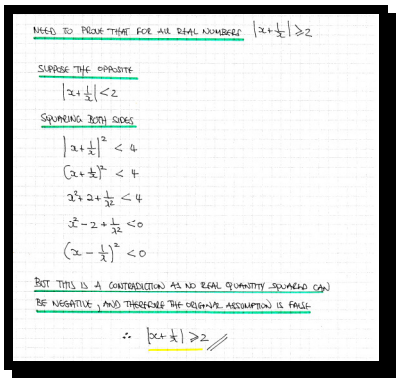


Question 15 (***)

Use the method of **proof by contradiction** to show that if x then

$$\left|x+\frac{1}{x}\right|\geq 2\,.$$

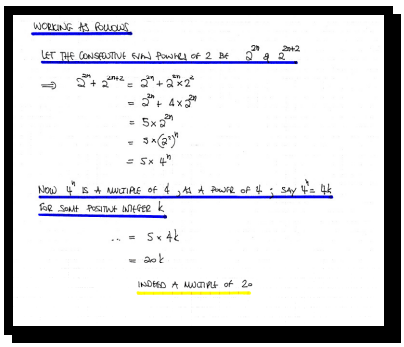
$\boxed{}$, $\boxed{\text{proof}}$



Question 16 (***)

Prove that the sum of two even consecutive powers of 2 is always a multiple of 20.

$\boxed{}$, $\boxed{\text{proof}}$

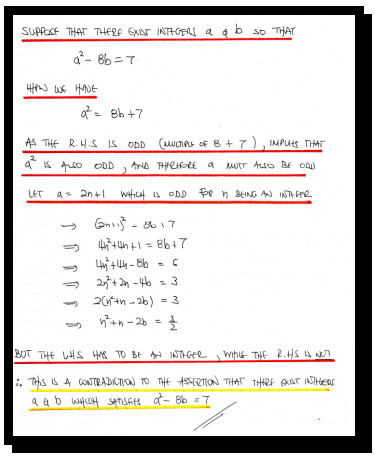


Question 17 (*+)**

Prove by the method of **contradiction** that there are no integers a and b which satisfy the following equation.

$$a^2 - 8b = 7$$

[ ], [proof]

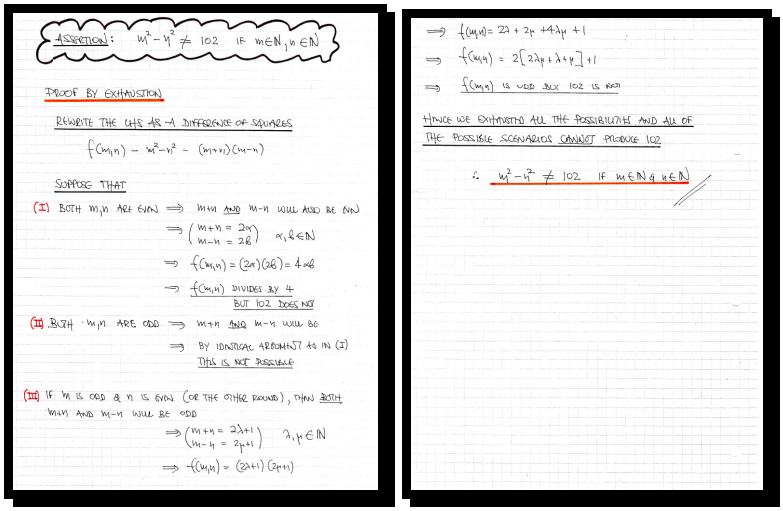


Question 18 (*+)**

Use **proof by exhaustion** to show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then

$$m^2 - n^2 \neq 102 .$$

[ ], [proof]

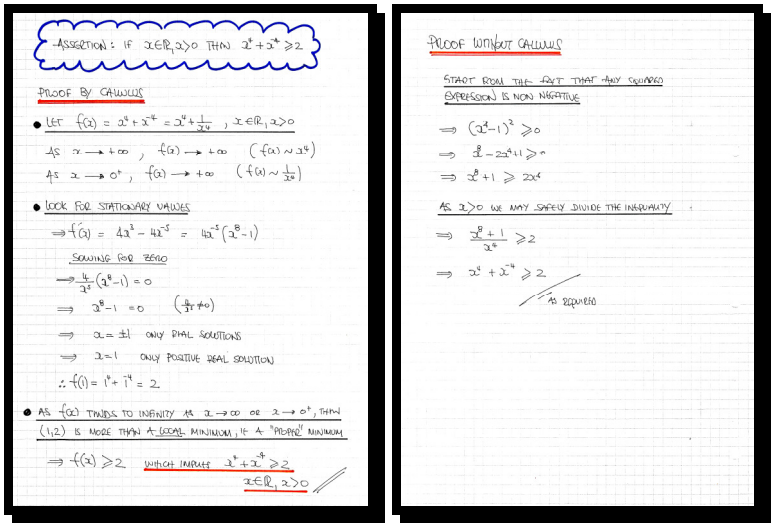


Question 19 (*+)**

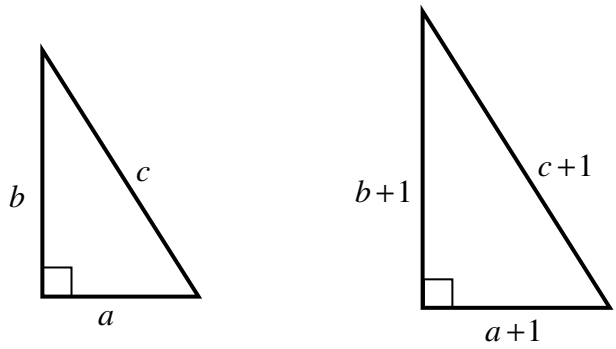
Use a **calculus method** to prove that if $x \in \mathbb{R}$, $x > 0$, then

$$x^4 + x^{-4} \geq 2 .$$

[ ], [proof]



Question 20 **(***+)**



The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of

$$a\,,\ b \text{ and } c\,,$$

where c is the length of its hypotenuse.

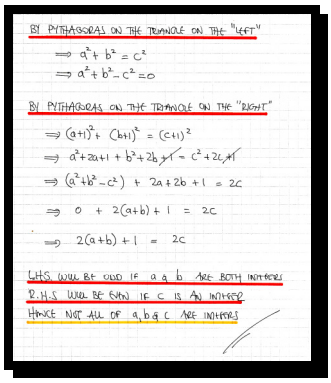
- The triangle, on the right section of the figure, has side lengths of

$$a+1\,,\quad b+1 \quad \text{and} \quad c+1\,,$$

where $c+1$ is the length of its hypotenuse.

Show that a, b and c cannot all be integers.

[ ], [proof]


BY PYTHAGORAS ON THE TRIANGLE ON THE "LEFT"

$\Rightarrow a^2+b^2=c^2$

$\Rightarrow a^2+b^2-c^2=0$

BY PYTHAGORAS ON THE TRIANGLE ON THE "RIGHT"

$\Rightarrow (a+1)^2+(b+1)^2=(c+1)^2$

$\Rightarrow a^2+2a+1+b^2+2b+1=c^2+2c+1$

$\Rightarrow (a^2+b^2-c^2)+2a+2b+1=2c$

$\Rightarrow 0+2(a+b)+1=2c$

$\Rightarrow 2(a+b)+1=2c$

LHS WILL BE ODD IF a & b ARE BOTH INTEGERS

R.H.S WILL BE EVEN IF c IS AN INTEGER

HENCE NOT ALL OF a, b & c ARE INTEGERS

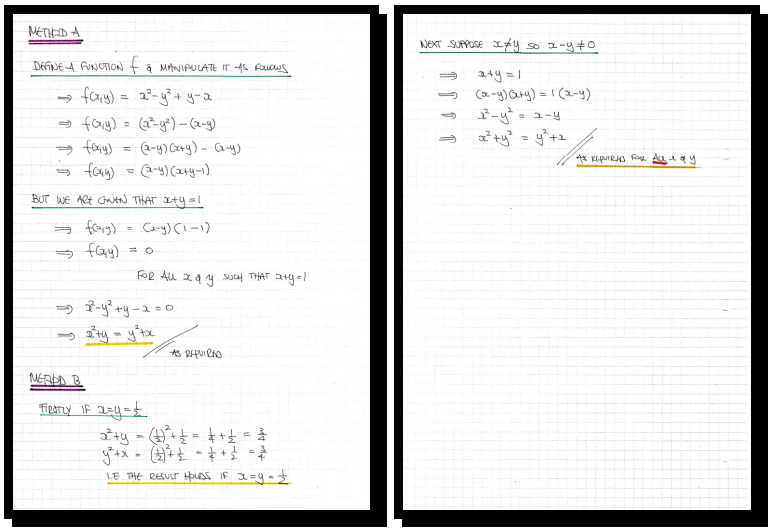
Question 21 (*+)**

It is given that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x + y = 1$.

Prove that

$$x^2 + y = y^2 + x \, .$$

[ ], [proof]

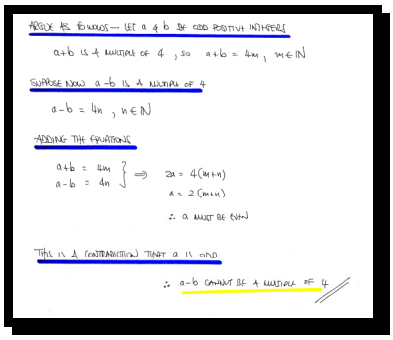


Question 22 (*+)**

It is given that a and b are positive odd integers, with $a > b$.

Use **proof by contradiction** to show that if $a+b$ is a multiple of 4, then $a-b$ **cannot** be a multiple of 4.

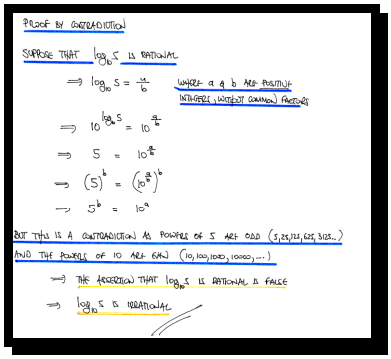
$\boxed{}$, $\boxed{\text{proof}}$



Question 23 (*+)**

Prove by **contradiction** that $\log_{10} 5$ is an irrational number.

$\boxed{}$, $\boxed{\text{proof}}$



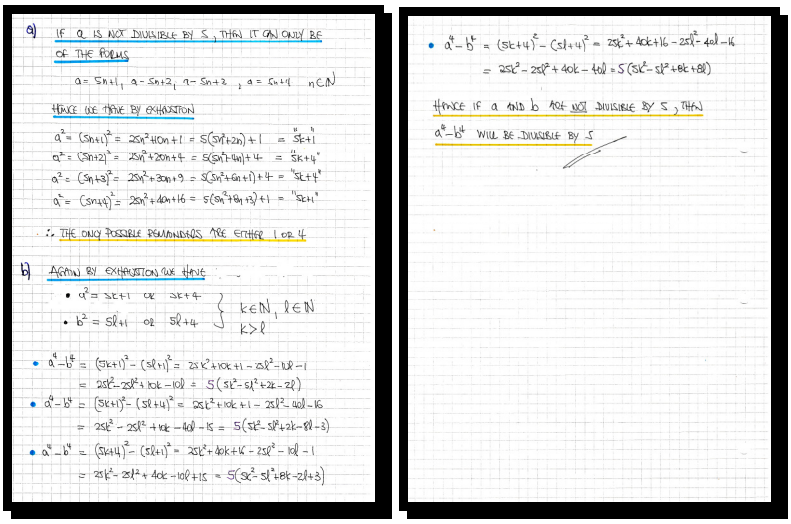
Question 24 **(****)**

Let $a \in \mathbb{N}$ with $\frac{1}{5}a \notin \mathbb{N}$.

a) Show that the remainder of the division of a^2 by 5 is either 1 or 4.

b) Given further that $b \in \mathbb{N}$ with $\frac{1}{5}b \notin \mathbb{N}$, deduce that $\frac{1}{5}\left(a^4 - b^4\right) \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 25 (****)

It is asserted that

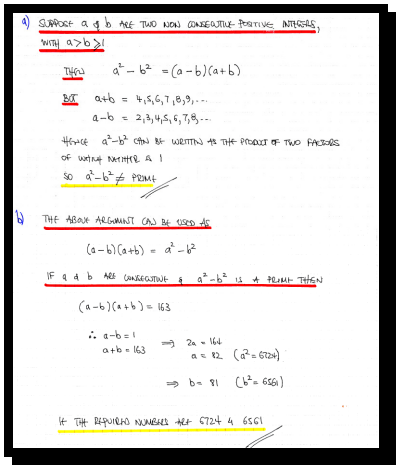
" The difference of the squares of two non consecutive positive integers can never be a prime number ".

a) Prove the validity of the above assertion.

The difference between two consecutive square numbers is 163.

b) Given further that 163 is a prime number find the above mentioned consecutive square numbers.

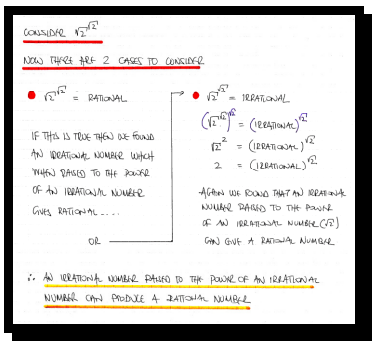
$\square$, $6561,\ 6724$



Question 26 (****)

By considering $\left(\sqrt{2}\right)^{\sqrt{2}}$, or otherwise, prove that an irrational number raised to the power of an irrational number **can be** a rational number.

[ ], [proof]



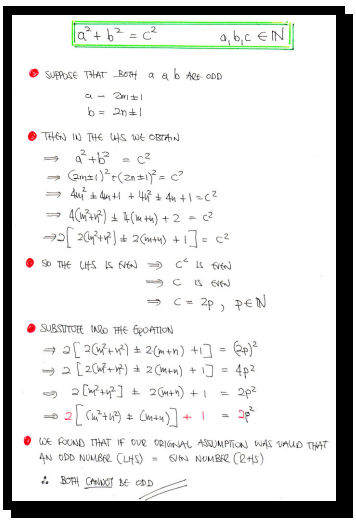
Question 27 (****)

It is given that

$$a^2+b^2=c^2, \quad a\in\mathbb{N}, \quad b\in\mathbb{N}.$$

Show that a and b cannot both be odd.

[ ], [proof]

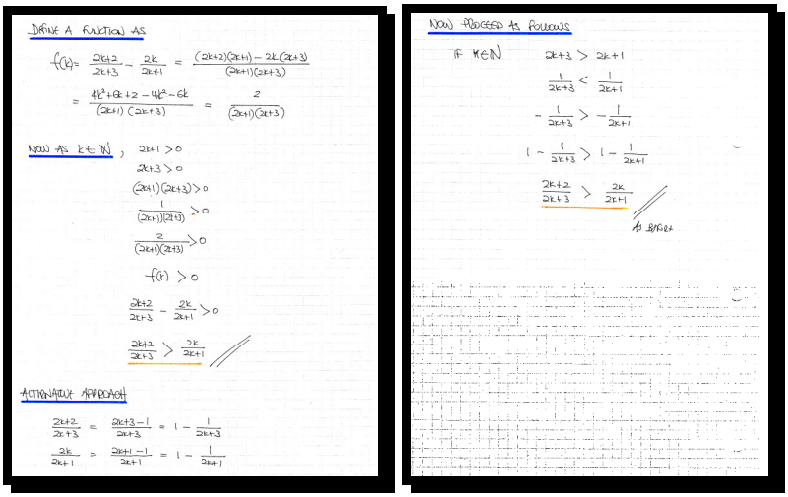


Question 28 (****)

Given that $k \in \mathbb{N}$, use algebra to prove that

$$\frac{2k+2}{2k+3} > \frac{2k}{2k+1}.$$

$\boxed{}$, $\boxed{\text{proof}}$

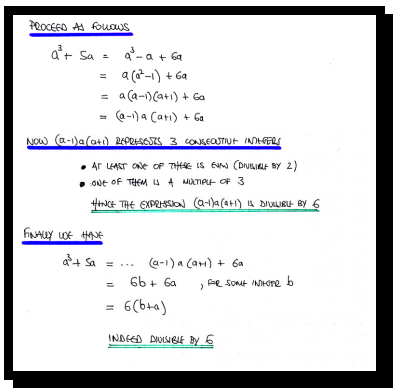


Question 29 (****)

$$f(a) = a^3 + 5a, \ a \in \mathbb{N}.$$

Without using proof by induction, show that $f(a)$ is a multiple of 6.

$\boxed{}$, $\boxed{\text{proof}}$

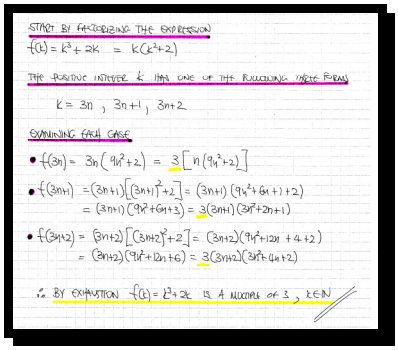


Question 30 (****)

$$f(k)=k^3+2k\,,\quad k\in\mathbb{N}\,.$$

Without using proof by induction, show that $f(k)$ is always a multiple of 3.

$\boxed{}$, proof



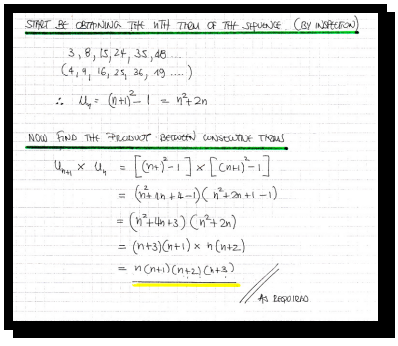
Question 31 (****)

Consider the following sequence

$$3,\ 8,\ 15,\ 24,\ 35,\ 48,\ \ldots$$

Prove that the product of any two consecutive terms of the above sequence can be written as the product of 4 consecutive integers.

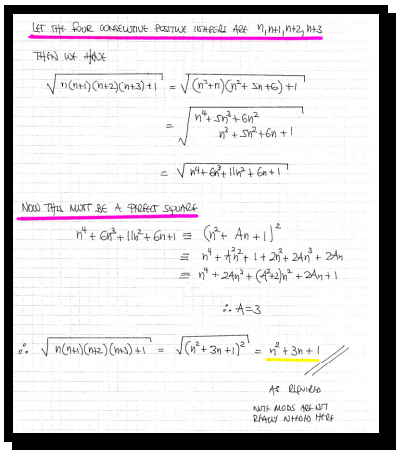
$\boxed{}$, proof



Question 32 (****)

Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

, proof

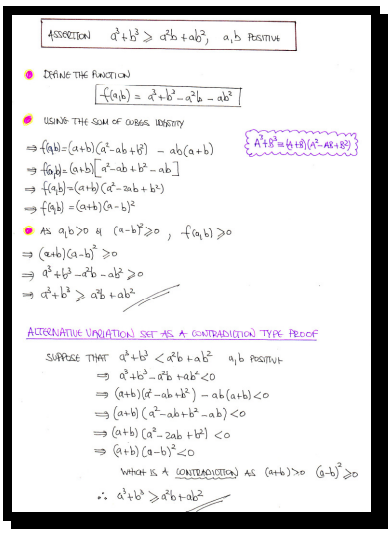


Question 33 (**+)**

Show that for all positive real numbers a and b

$$a^3+b^3 \geq a^2b+ab^2 .$$

, proof

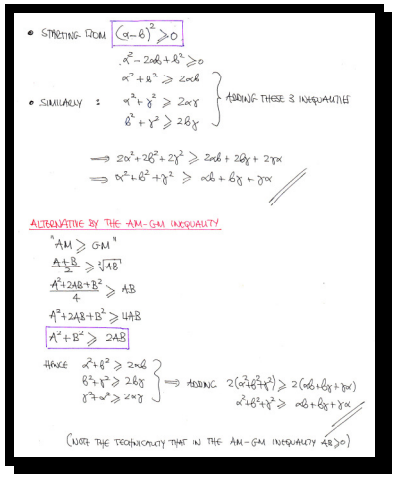


Question 34 **(****+)**

Show clearly that for all real numbers α, β and γ

$$\alpha^2+\beta^2+\gamma^2 \geq \alpha\beta+\beta\gamma+\gamma\alpha\,.$$

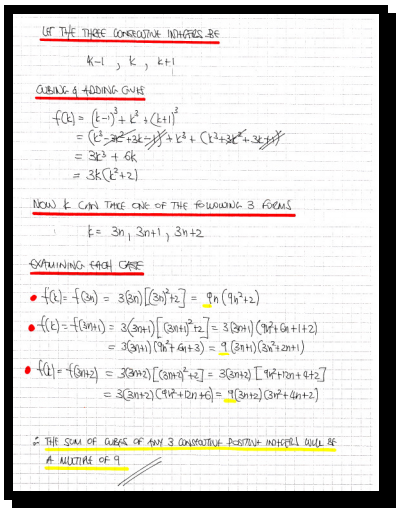
[ ], [proof]



Question 35 **(****+)**

Show, without using proof by induction, that the sum of cubes of any 3 consecutive positive integers is a multiple of 9.

[ ], [proof]



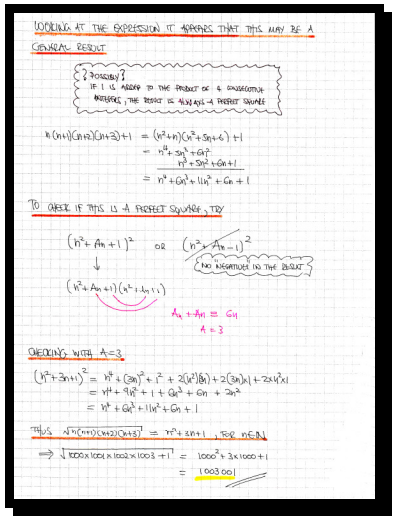
Question 36 (**+)**

Use a **detailed method** to show that

$$\sqrt{1000\times1001\times1002\times1003+1} \;=\; 1003001$$

*You may **NOT** use a calculating aid in this question.*

$\boxed{}$, $\boxed{\text{proof}}$



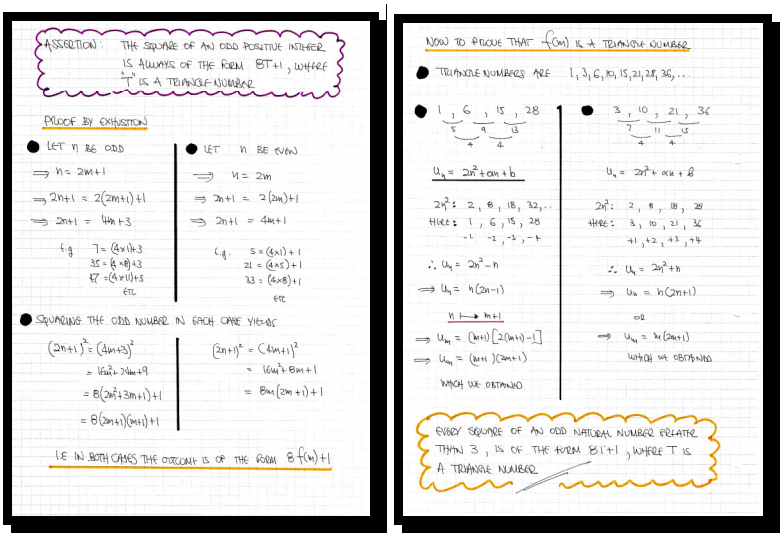
Question 37 (*****)

Show that the square of an odd positive integer greater than 1 is of the form

$$8T+1,$$

where T is a triangular number.

[ ], [proof]



Question 38 **(*****)**

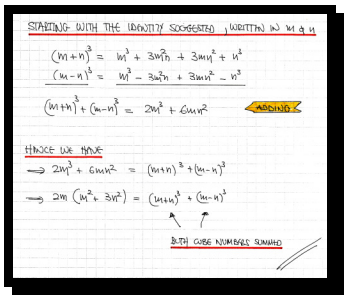
It is given that

$$f(m,n) \equiv 2m\left(m^2+3n^2\right),$$

where m and n are distinct positive integers, with $m>n$.

By using the expansion of $(A\pm B)^3$, prove that $f(m,n)$ can always be written as the sum of two cubes.

[ ], [proof]



Question 39 **(*****)**

It is given that

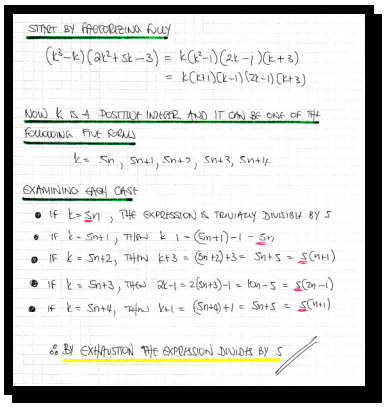
$$f(k) \equiv \left(k^3 - k\right)\left(2k^2 + 5k - 3\right),$$

where k is a positive integer.

Prove that $f(k)$ is divisible by 5.

*You may **not** use proof by induction in this question.*

[ ], [proof]

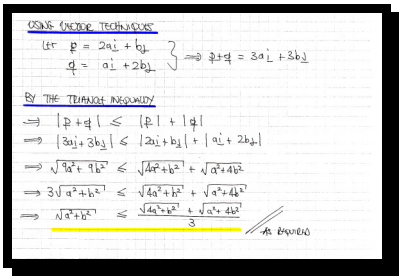


Question 40 (*****)

Prove that for all real numbers, a and b,

$$\sqrt{a^2+b^2} \leq \frac{\sqrt{4a^2+b^2}+\sqrt{a^2+4b^2}}{3}.$$

[ ], [proof]

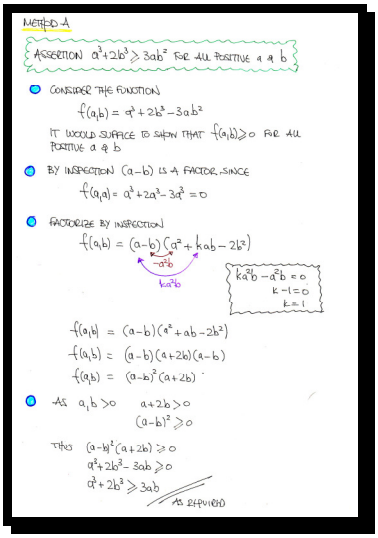


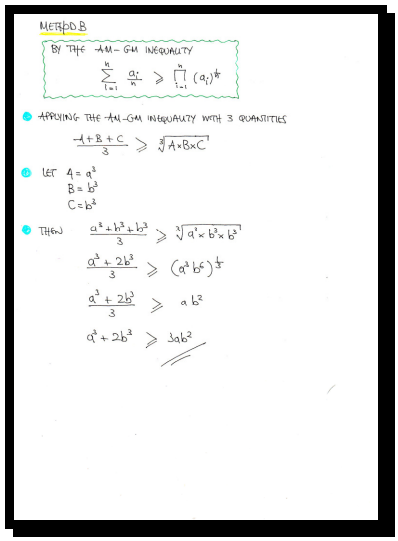
Question 41 (*****)

Show that for all positive real numbers a and b

$$a^3+2b^3\geq 3ab^2\,.$$

, proof





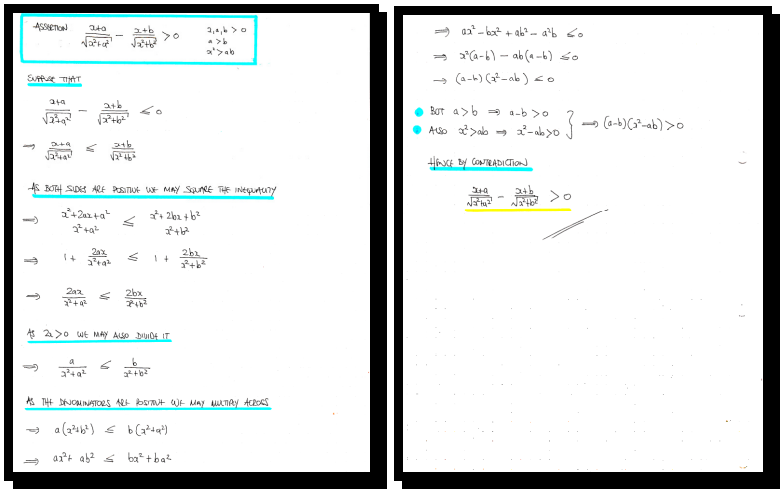
Question 42 (*****)

It is given that x, a and b are positive real numbers, with $a > b$ and $x^2 > ab$.

Use proof by contradiction to show that

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0 .$$

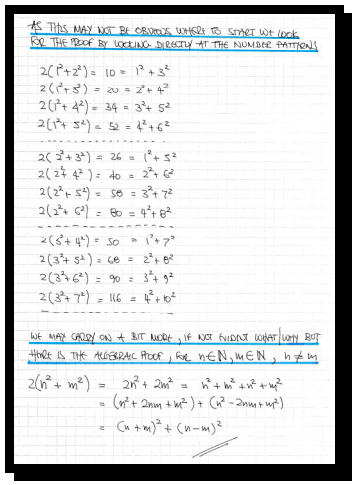
[ ], [proof]



Question 43 (*****)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers

[ ], [proof]



Question 44 **(*****)**

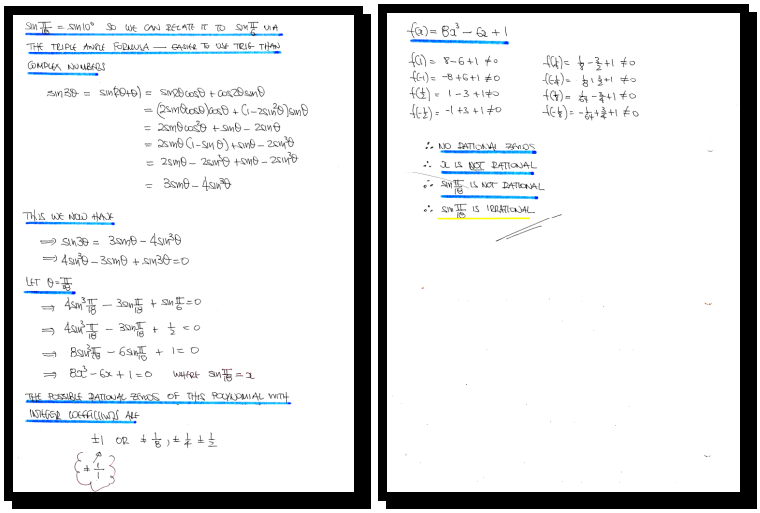
The *Rational Zero Theorem* asserts that if the polynomial

$$f(x) \equiv a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

has integer coefficients, then **every rational zero** of $f(x)$ has the form pq, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Use this result to show that $\sin\left(\frac{\pi}{18}\right)$ is irrational.

[ ], [proof]



Question 45 (*****)

By using the definition of e as an infinite convergent series, prove **by contradiction** that e is irrational.

[ ], [proof]

