PROOF BY INDUCTION

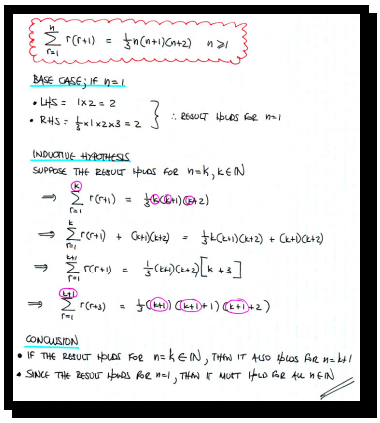
SUMMATION RESULTS

Question 1 **(**)**

Prove by induction that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2), \quad n \geq 1, \ n \in \mathbb{N}.$$

, proof

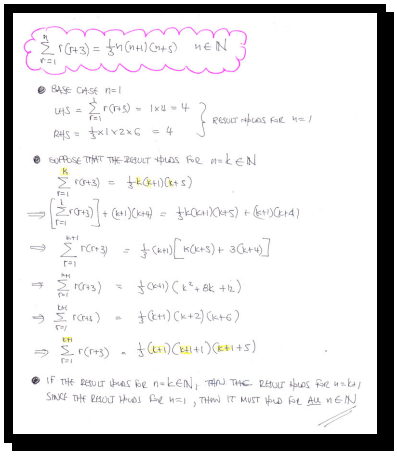


Question 2 **(**)**

Prove by induction that

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5), \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

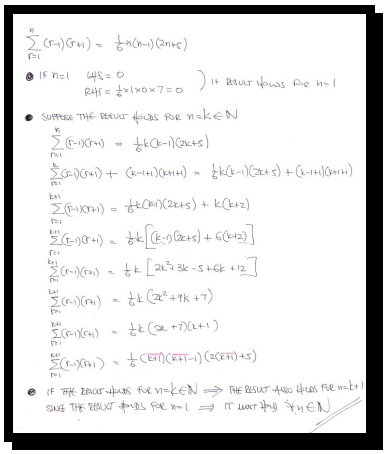


Question 3 (+)**

Prove by induction that

$$\sum_{r=1}^{n}(r-1)(r+1)=\frac{1}{6}n(n-1)(2n+5),\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

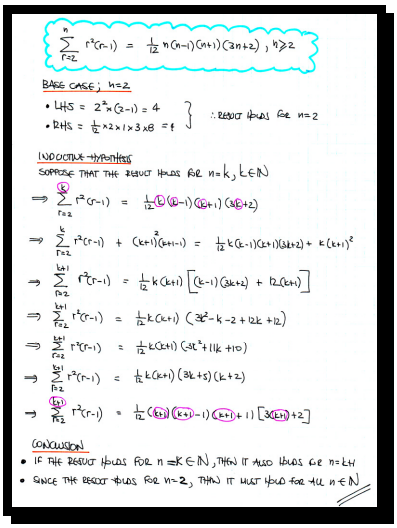


Question 4 (+)**

Prove by induction that

$$\sum_{r=2}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad n \geq 2, \ n \in \mathbb{N}.$$

, proof

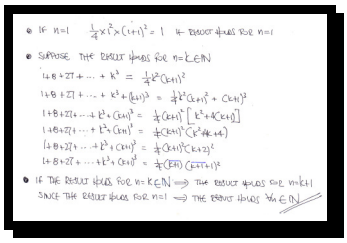


Question 5 (+)**

Prove by induction that

$$1+8+27+64+...+n^3=\frac{1}{4}n^2\left(n+1\right)^2,\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

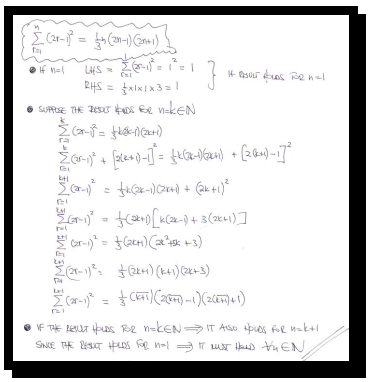


Question 6 (***)

Prove by induction that

$$\sum_{r=1}^{n}\left(2r-1\right)^2=\frac{1}{3}n\left(2n-1\right)\left(2n+1\right),\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

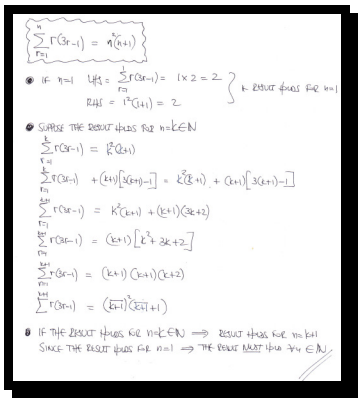


Question 7 (***)

Prove by induction that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

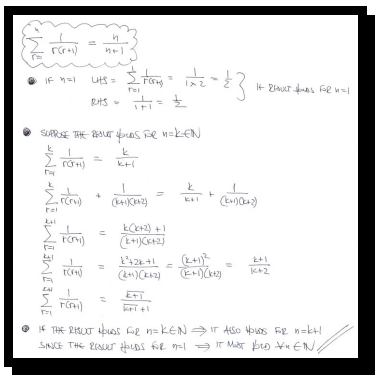


Question 8 (***)

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}, \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

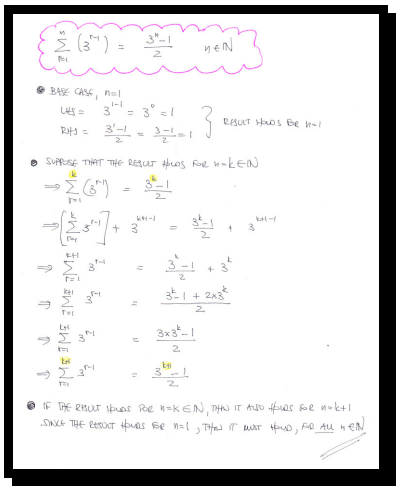


Question 9 (***)

Prove by induction that

$$\sum_{r=1}^{n}\left(3^{r-1}\right)=\frac{3^n-1}{2},\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

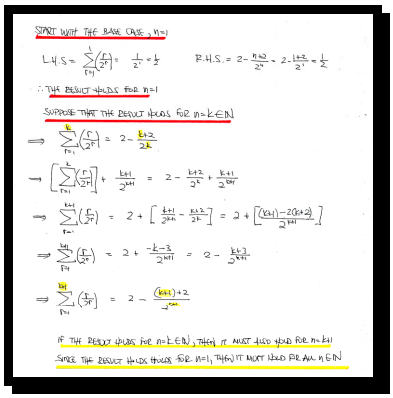


Question 10 (*+)**

Prove by induction that

$$\sum_{r=1}^{n}\frac{r}{2^r}=2-\frac{n+2}{2^n},\ \ n\geq 1,\ n\in\mathbb{N}.$$

[ ], [proof]

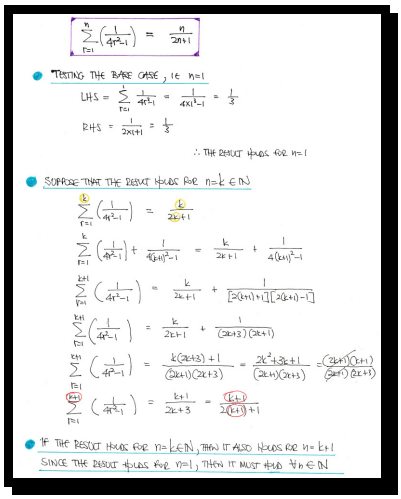


Question 11 (*+)**

Prove by induction that

$$\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1},\ \ n\geq 1,\ n\in\mathbb{N}.$$

[ ], [proof]

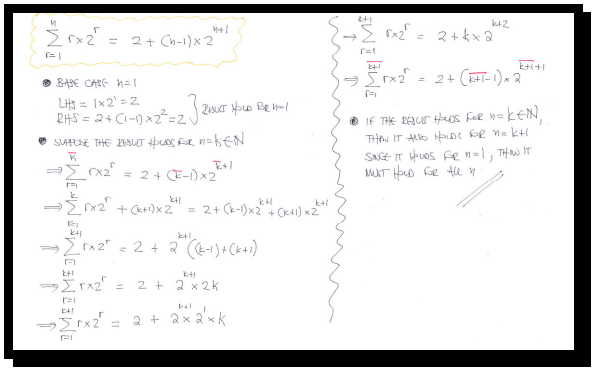


Question 12 (*+)**

Prove by induction that

$$\sum_{r=1}^{n}r\times 2^r=2+(n-1)2^{n+1},\ \ n\geq 1,\ n\in\mathbb{N}.$$

[proof]

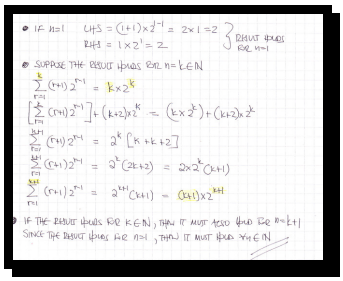


Question 13 (*+)**

Prove by induction that

$$\sum_{r=1}^{n}\left[(r+1)\times 2^r\right]=n\times 2^n,\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

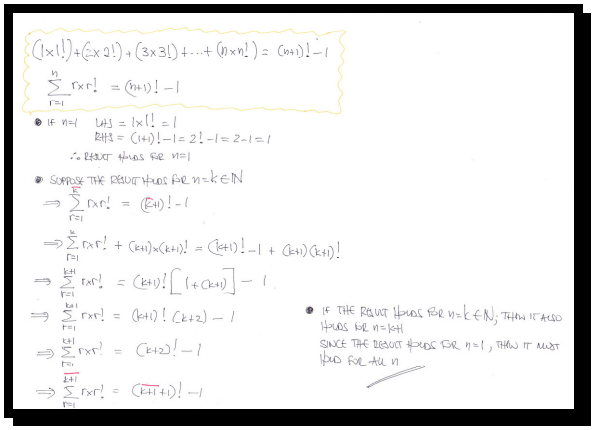


Question 14 (*+)**

If $n\geq 1,\ n\in\mathbb{N}$, prove by induction that

$$1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n!=(n+1)!-1.$$

proof

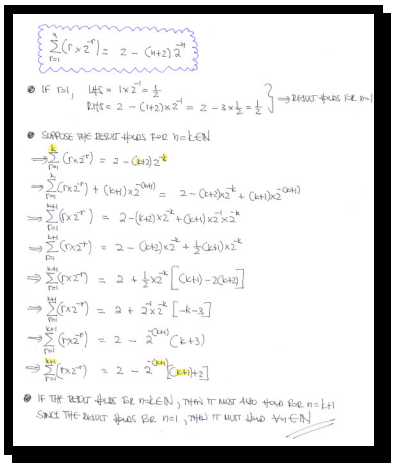


Question 15 (*+)**

Prove by induction that

$$\sum_{r=1}^{n} r\times 2^{-r} = 2-(n+2)2^{-n}, \quad n\geq 1,\ n\in\mathbb{N}.$$

proof

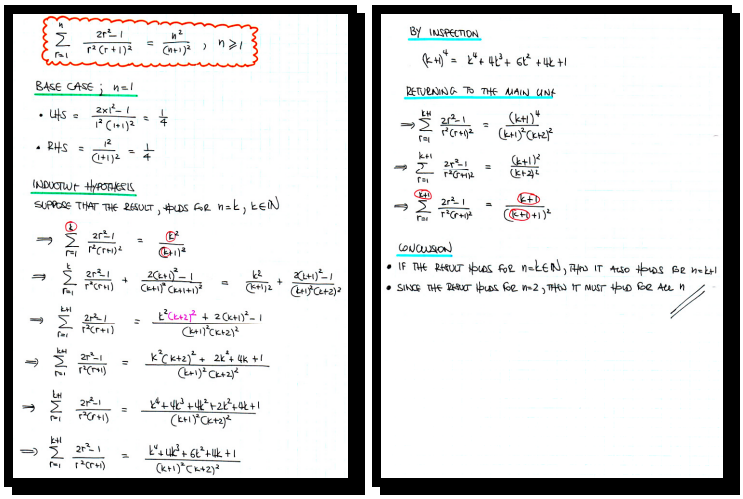


Question 16 (****)

Prove by induction that

$$\sum_{r=1}^{n} \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geq 1, \ n \in \mathbb{N}.$$

[ ], [proof]

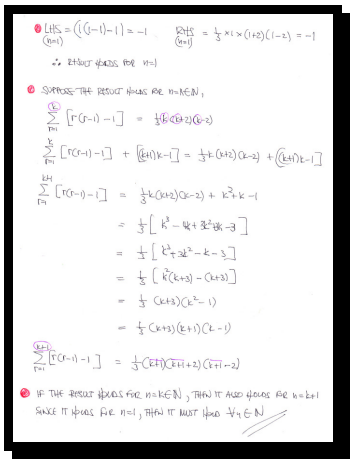


Question 17 (****)

Prove by induction that

$$\sum_{r=1}^{n}\left[r(r-1)-1\right]=\frac{1}{3}n(n+2)(n-2),\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof

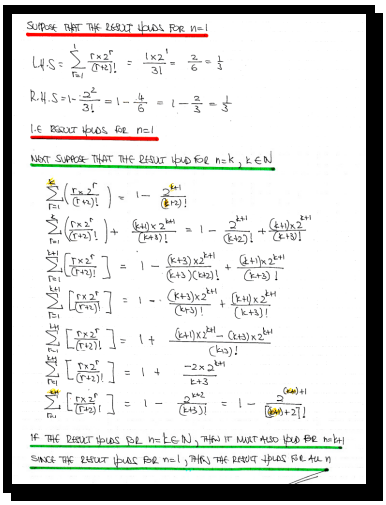


Question 18 (****)

Prove by induction that

$$\sum_{r=1}^{n}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!},\quad n\geq 1,\ n\in\mathbb{N}.$$

, proof

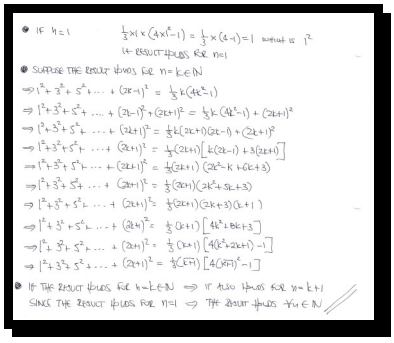


Question 19 (****)

Prove by induction that

$$1^2+3^2+5^2+7^2+\ldots+(2n-1)^2\equiv\frac{1}{3}n\left(4n^2-1\right),\quad n\geq 1,\ n\in\mathbb{N}.$$

proof



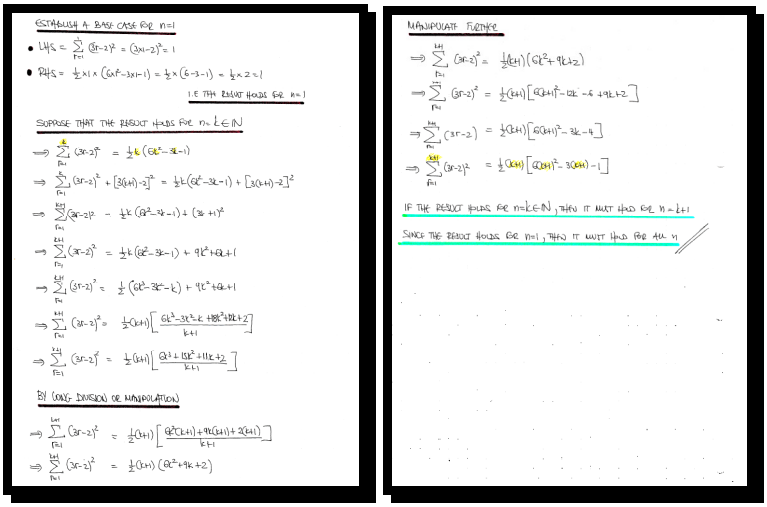
Question 20 (****)

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left(6n^2-3n-1\right).$$

You may not use other methods of proof in this question.

, proof



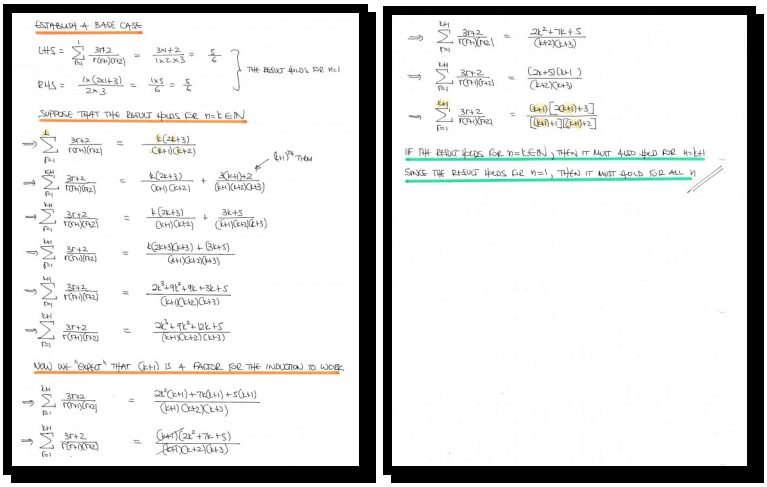
Question 21 (****)

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^{n} \frac{3r+2}{r(r+1)(r+2)} = \frac{n(2n+3)}{(n+1)(n+2)}.$$

You may not use other methods of proof in this question.

, proof

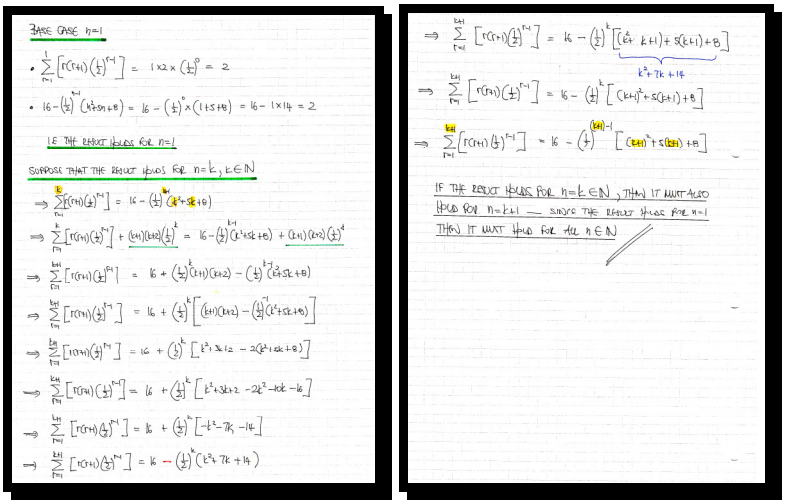


Question 22 (****)

Prove by induction that

$$\sum_{r=1}^{n}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right]=16-\left(\frac{1}{2}\right)^{n-1}\left(n^2+5n+8\right),\ \ n\geq 1,\ n\in\mathbb{N}.$$

[ ], [proof]



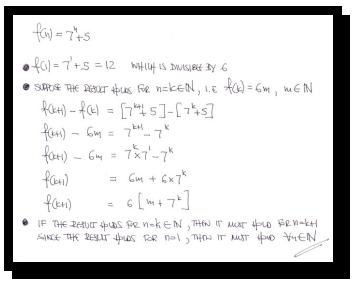
DIVISIBILITY RESULTS

Question 1 (**)

$$f(n)=7^n+5,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 6, for all $n\in\mathbb{N}$.

proof

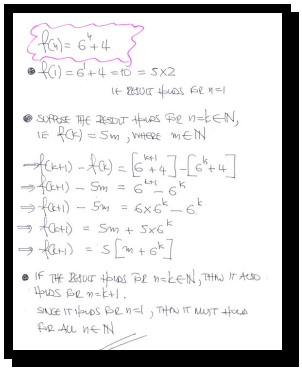


Question 2 (**)

$$f(n)=6^n+4,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 5, for all $n\in\mathbb{N}$.

proof

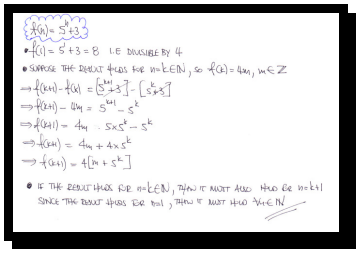


Question 3 **(**)**

$$f(n)=5^n+3,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n\in\mathbb{N}$.

proof



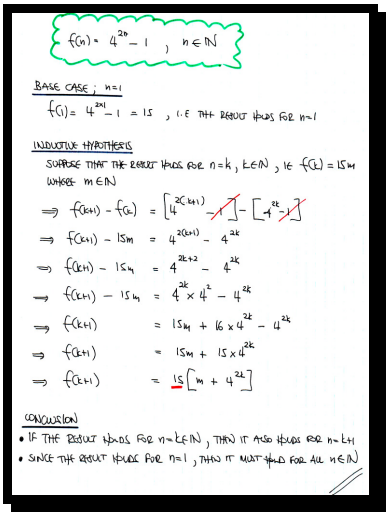
Question 4 **(**)**

Prove by induction that for all natural numbers n,

$$4^{2n}-1$$

is divisible by 15.

, proof



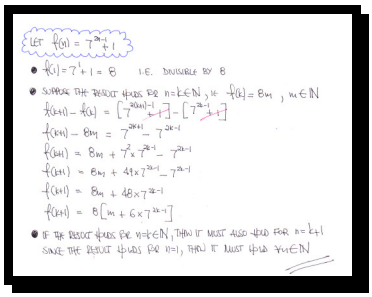
Question 5 (**)

Prove by induction that for all natural numbers n,

$$7^{2n-1}+1$$

is divisible by 8.

proof

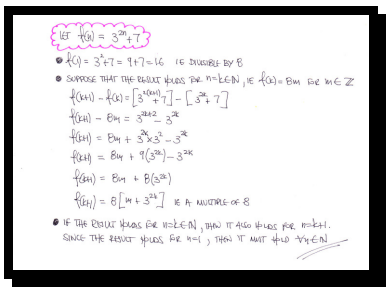


Question 6 (**)

Prove by induction that for all natural numbers n,

$$3^{2n}+7 \text{ is divisible by } 8.$$

proof

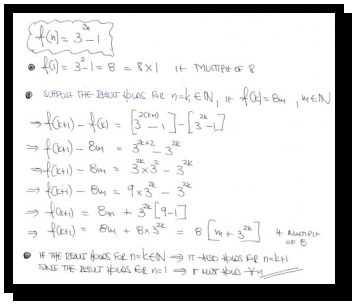


Question 7 (**)

$$f(n)=3^{2n}-1,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is a multiple of 8, for all $n\in\mathbb{N}$.

proof

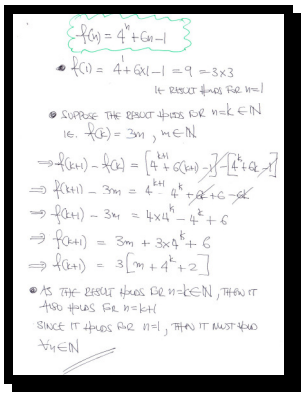


Question 8 (+)**

$$f(n)=4^n+6n-1,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 3, for all $n\in\mathbb{N}$.

proof

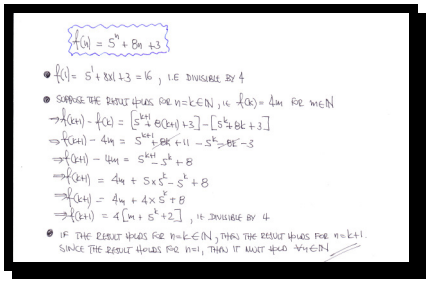


Question 9 (+)**

$$f(n)=5^n+8n+3\ ,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n\in\mathbb{N}$.

proof

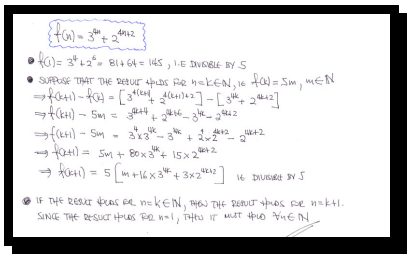


Question 10 (+)**

$$f(n)=3^{4n}+2^{4n+2}\ ,\ n\in\mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n\in\mathbb{N}$.

proof



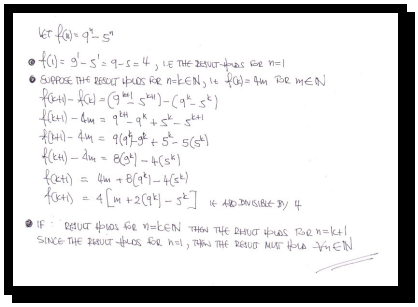
Question 11 (+)**

Prove by induction that for all natural numbers n,

$$9^n - 5^n$$

is divisible by 4.

proof

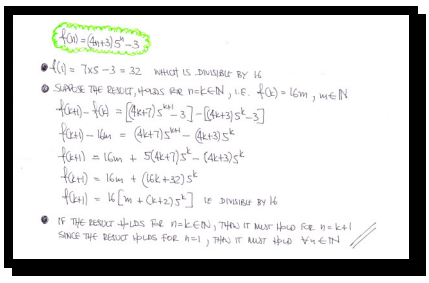


Question 12 (+)**

$$f(n) = (4n+3)5^n - 3\,,\ n \in \mathbb{N}\,.$$

Prove by induction that $f(n)$ is divisible by 16, for all $n \in \mathbb{N}$.

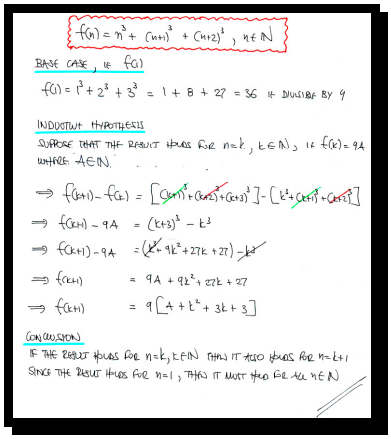
proof



Question 13 **(***)**

Prove by induction that the sum of the cubes of any three consecutive positive integers is always divisible by 9.

$\boxed{}$, proof



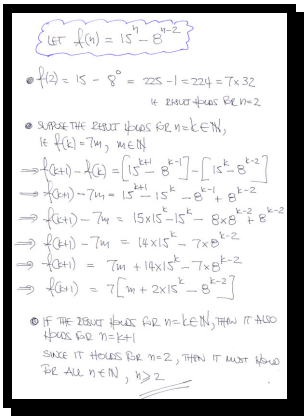
Question 14 **(***)**

Prove by induction that for all natural numbers n, such that $n \geq 2$,

$$15^n - 8^{n-2},$$

is divisible by 7.

proof



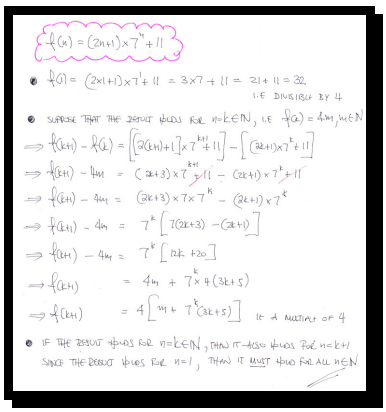
Question 15 (***)

Prove by induction that for all natural numbers n,

$$(2n+1)7^n+11\ ,$$

is divisible by 4.

proof

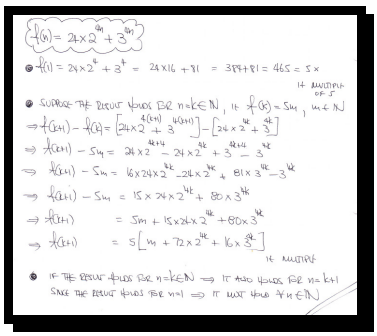


Question 16 (***)

$$f(n)=24\times 2^{4n}+3^{4n}\ ,\ n\in\mathbb{N}\ .$$

Prove by induction that $f(n)$ is divisible by 5, for all $n\in\mathbb{N}$.

proof

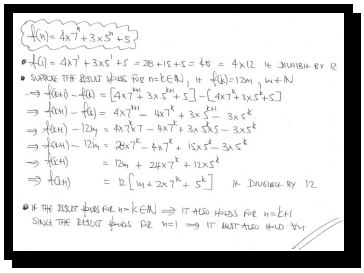


Question 17 (***)

$$f(n)=4\times 7^n+3\times 5^n+5,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 12, for all $n\in\mathbb{N}$.

proof

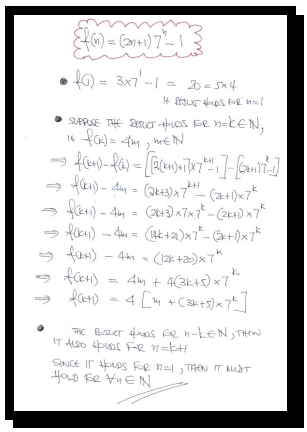


Question 18 (***)

$$f(n)=(2n+1)7^n-1,\ n\in\mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 4, for all $n\in\mathbb{N}$.

proof



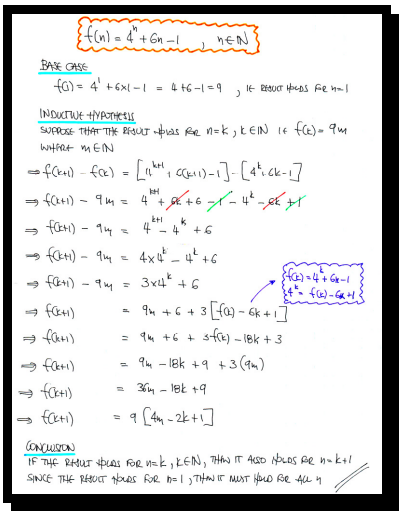
Question 19 (*+)**

Prove by induction that for all natural numbers n,

$$4^n + 6n - 1$$

is divisible by 9.

$\square$, proof



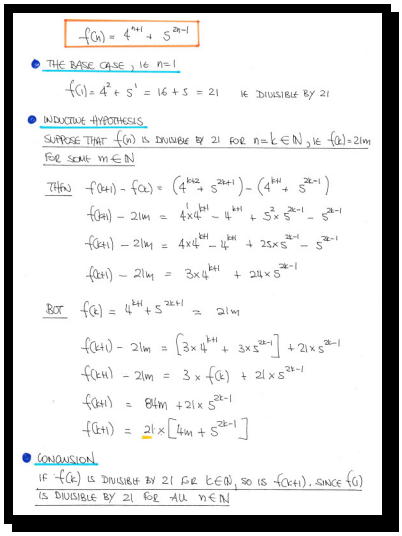
Question 20 (*+)**

Prove by induction that for all natural numbers n,

$$4^{n+1}+5^{2n-1}$$

is divisible by 21.

$\boxed{}$, proof

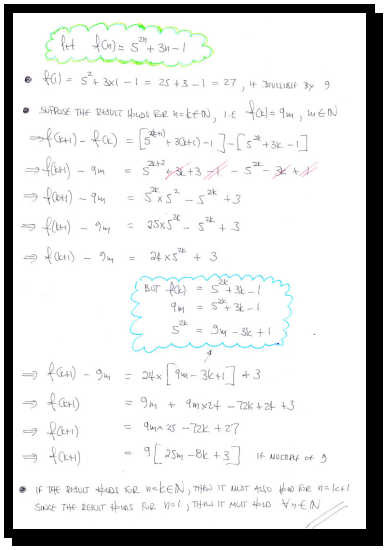


Question 21 (*+)**

$$f(n)=5^{2n}+3n-1,\ n\in\mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 9, for all $n\in\mathbb{N}$.

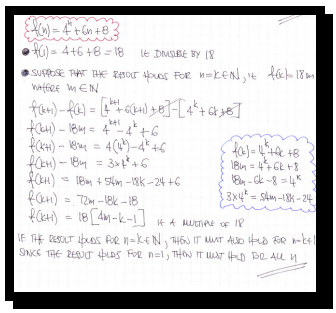
proof



Question 22 (****)

Prove by induction that 18 is a factor of 4^n+6n+8, for all $n\in\mathbb{N}$.

proof



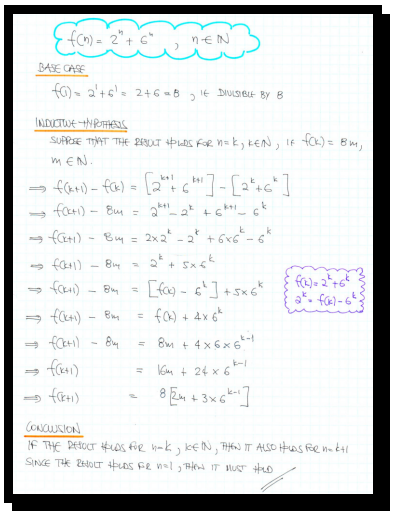
Question 23 (****)

Prove by induction that for all natural numbers n,

$$2^n + 6^n$$

is divisible by 8.

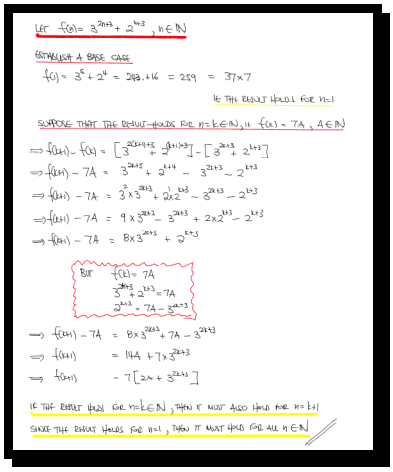
$\boxed{}$, $\boxed{\text{proof}}$



Question 24 **(***+)**

Prove by mathematical induction that if n is a positive integer then $3^{2n+3}+2^{n+3}$ is always divisible by 7.

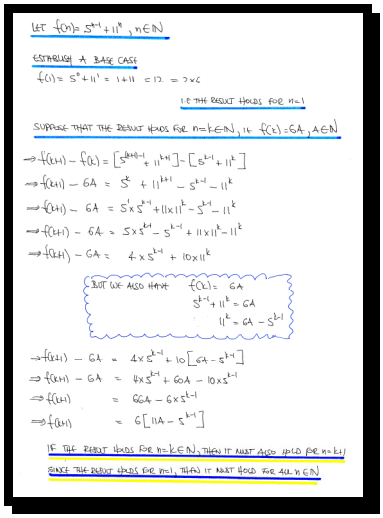
$\boxed{}$, $\boxed{\text{proof}}$



Question 25 (*+)**

Prove by mathematical induction that if n is a positive integer then $5^{n-1}+11^n$ is always divisible by 6.

$\boxed{}$, $\boxed{\text{proof}}$



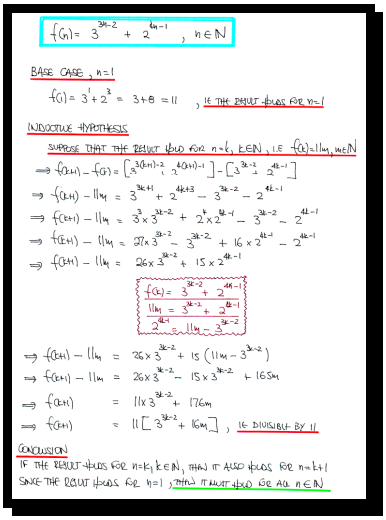
Question 26 (*+)**

Prove by the method of induction that

$$3^{3n-2}+2^{4n-1}\ ,\ n\in\mathbb{N}\,,$$

is divisible by 11.

$\boxed{}$, $\boxed{\text{proof}}$

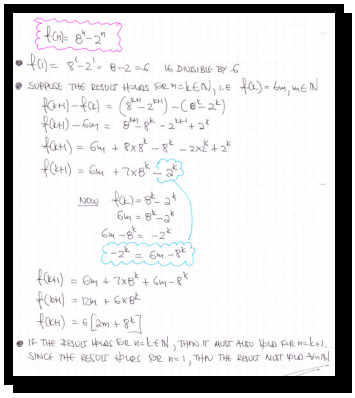


Question 27 (*+)**

$$f(n)=8^n-2^n,\ n\in\mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 6, for all $n\in\mathbb{N}$.

proof

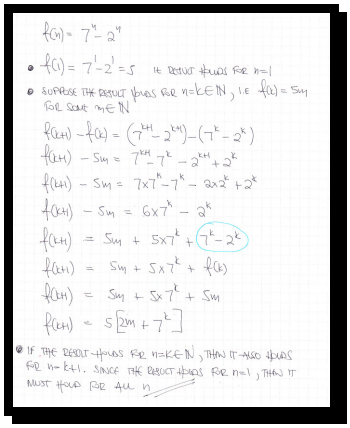


Question 28 (*+)**

$$f(n)=7^n-2^n,\ n\in\mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n\in\mathbb{N}$.

proof



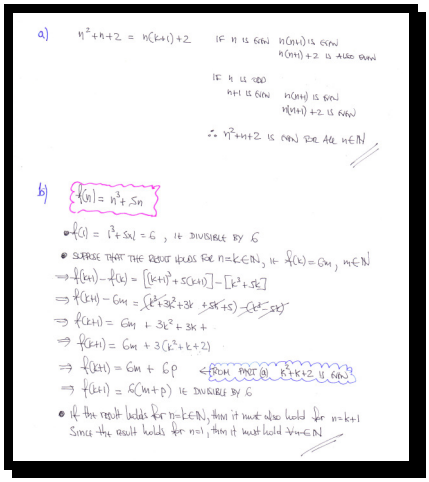
Question 29 (*+)**

$$f(n)=n^3+5n\,,\ n\in\mathbb{N}\,.$$

a) Show that n^2+n+2 is always even for all $n\in\mathbb{N}$.

b) Hence, prove by induction that $f(n)$ is divisible by 6, for all $n\in\mathbb{N}$.

proof



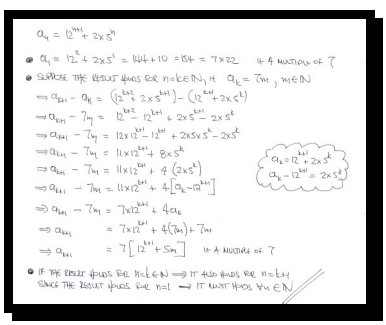
Question 30 (*+)**

A sequence of positive numbers is given by

$$a_n=12^{n+1}+2\times 5^n\,,\ n\in\mathbb{N}\,.$$

Prove by induction that every term of the sequence is a multiple of 7

proof

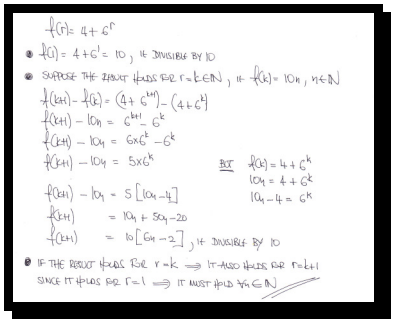


Question 31 (*+)**

$$f(r)=4+6^r,\ r\in\mathbb{N}.$$

Prove by induction that $f(r)$ is divisible by 10

proof



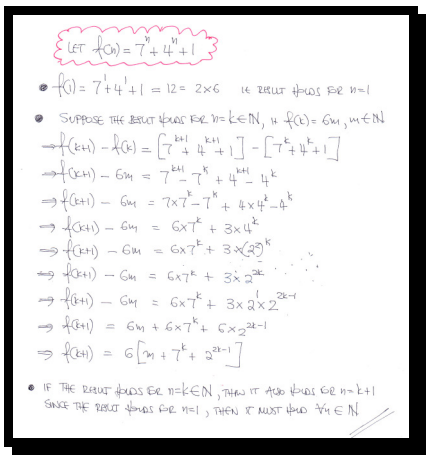
Question 32 (*+)**

Prove by induction that for all natural numbers n, the following expression

$$7^n+4^n+1$$

is divisible by 6.

proof



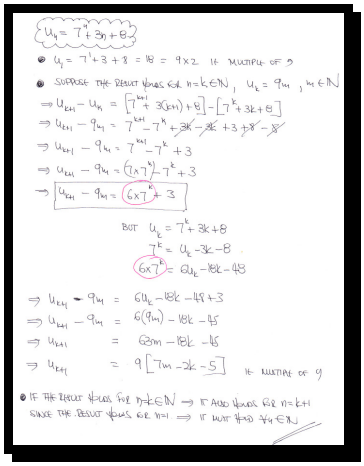
Question 33 (*+)**

A sequence of positive numbers is given by

$$u_n = 7^n + 3n + 8,\ n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 9

proof

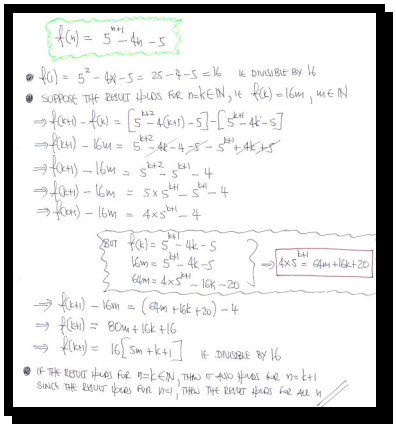


Question 34 (*+)**

$$f(n) = 5^{n+1} - 4n - 5,\ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 16

proof



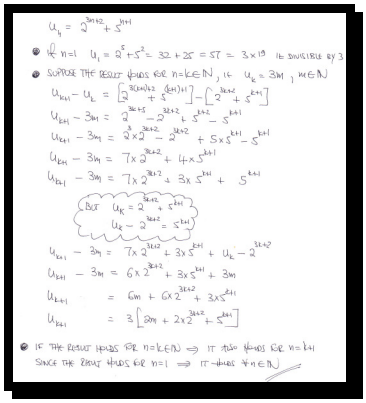
Question 35 (*+)**

A sequence of positive numbers is given by

$$u_n = 2^{3n+2} + 5^{n+1}, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 3.

proof

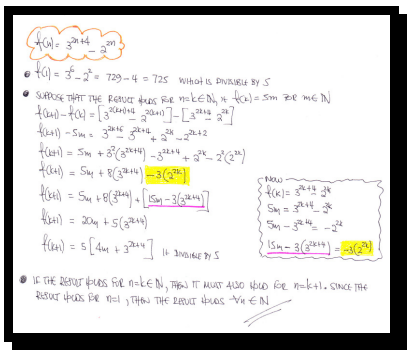


Question 36 (*+)**

$$f(n) = 3^{2n+4} - 2^{2n}, \ n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



RECURRENCE RELATIONS

Question 1 (**)

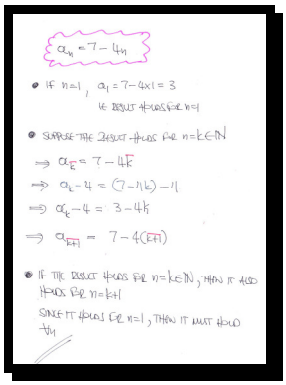
A sequence of integers is defined recursively by the relation

$$a_{n+1}=a_n-4,\ a_1=3,\ n=1,2,3,...$$

Prove by induction that its n^{th} term is given by

$$a_n=7-4n,\ n=1,2,3,...$$

proof



Question 2 (**)

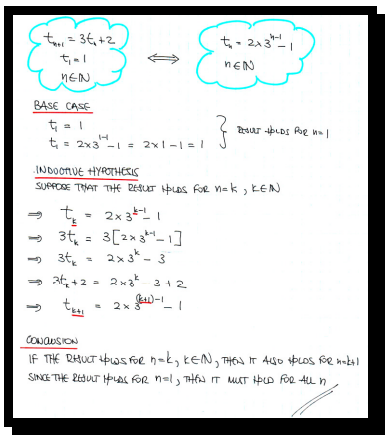
A sequence of integers t_1, t_2, t_3, … is given by the recurrence relation

$$t_{n+1} = 3t_n + 2, \quad t_1 = 1, \quad n \in \mathbb{N}.$$

Prove by induction that its n^{th} term of the sequence is given by

$$t_n = 2 \times 3^{n-1} - 1, \quad n \in \mathbb{N}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 3 (**)

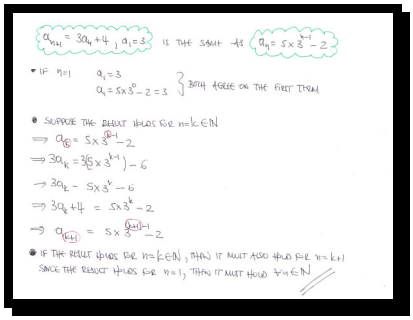
A sequence of integers is defined inductively by the relation

$$a_{n+1}=3a_n+4\,,\quad a_1=3\,,\quad n=1,2,3,...$$

Prove by induction that its n^{th} term is given by

$$a_n=5\times 3^{n-1}-2\,,\quad n=1,2,3,...$$

proof



Question 4 (**)

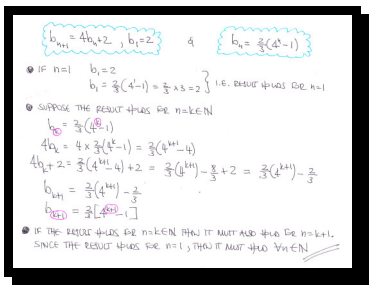
The terms of a sequence can be generated by the recurrence relation

$$b_{n+1}=4b_n+2\,,\quad b_1=2\,,\quad n=1,2,3,...$$

Prove by induction that the n^{th} term of the sequence is given by

$$b_n=\frac{2}{3}\left(4^n-1\right),\quad n=1,2,3,...$$

proof



Question 5 (**)

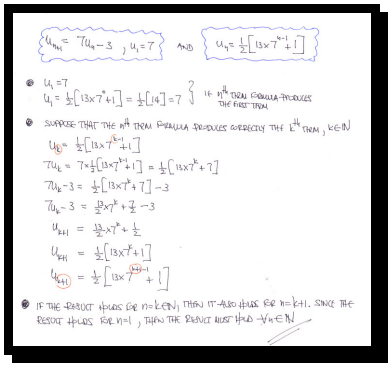
A sequence is defined by the recurrence relation

$$u_{n+1}=7u_n-3,\ u_1=7,\ n=1,2,3,...$$

Prove by induction that its n^{th} term is given by

$$u_n=\frac{1}{2}\left(13\times 7^{n-1}+1\right),\ n=1,2,3,...$$

proof



Question 6 (**)

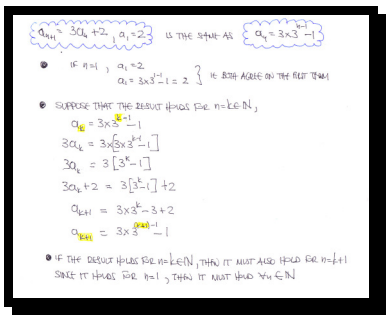
A sequence of integers a_1, a_2, a_3, a_4, … is given by

$$a_{n+1}=3a_n+2\,,\ a_1=2\,,\ n=1,2,3,\ ...$$

Prove by induction that its n^{th} term is given by

$$a_n=3\times 3^{n-1}-1\,,\ n=1,2,3,\ ...$$

proof



Question 7 (+)**

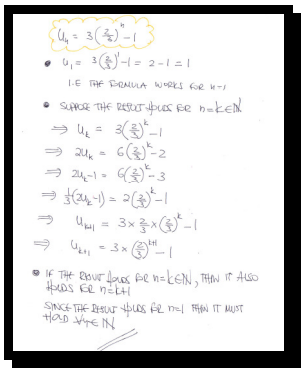
A certain sequence can be generated by the recurrence relation

$$u_{n+1}=\frac{1}{3}\left(2u_n-1\right),\quad u_1=1,\quad n=1,2,3,...$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n=3\left(\frac{2}{3}\right)^n-1,\quad n=1,2,3,...$$

proof



Question 8 **(***+)**

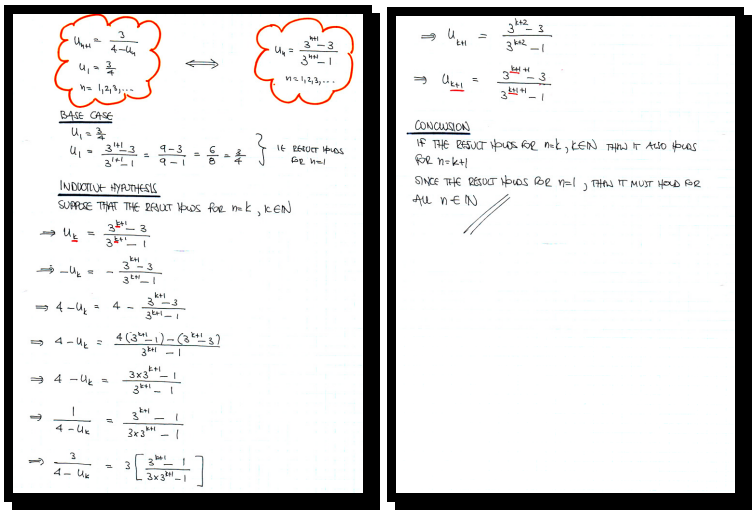
A sequence is defined recursively by

$$u_{n+1}=\frac{3}{4-u_n},\ u_1=\frac{3}{4},\ n=1,2,3,\ldots$$

Prove by induction that

$$u_n=\frac{3^{n+1}-3}{3^{n+1}-1},\ n=1,2,3,\ldots$$

$\boxed{}$, proof



Question 9 (*+)**

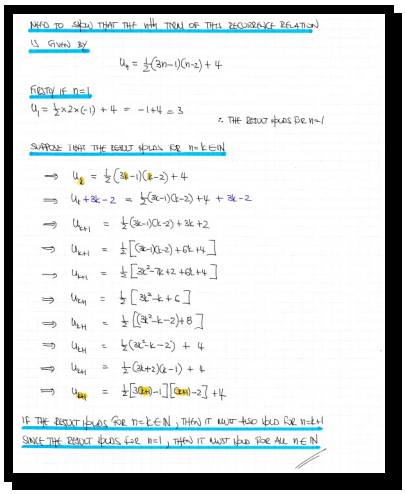
A sequence is defined recursively by

$$u_{n+1}=u_n+3k-2\,,\ u_1=3\,,\quad n=1,2,3,...$$

Prove by induction that

$$u_n=\frac{1}{2}(3n-1)(n-2)+4\,,\ n=1,2,3,...$$

[ ] , [proof]



Question 10 (*+)**

A sequence is defined recursively by

$$u_{n+1}=\frac{u_n}{u_n+1},\ u_1=2,\ n\in\mathbb{N}\ .$$

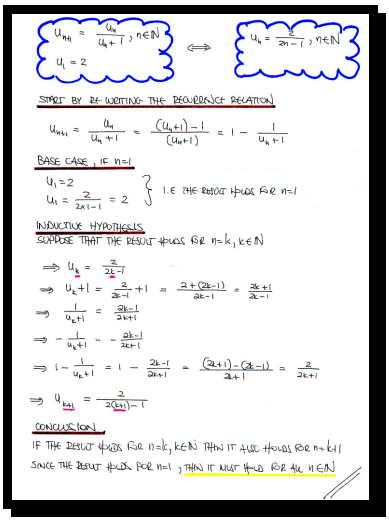
By writing the above recurrence relation in the form

$$u_{n+1}=A+\frac{B}{u_n+1},$$

where A and B are integers, use proof by induction to show that

$$u_n=\frac{2}{2n-1},\ n\in\mathbb{N}\ .$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 11 (*+)**

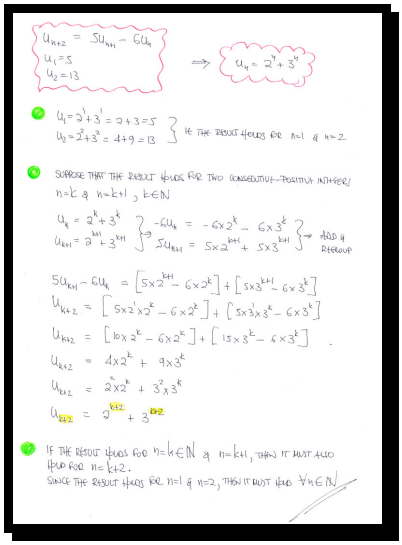
A sequence is generated by the recurrence relation

$$u_{n+2}=5u_{n+1}-6u_n,\ u_1=5,\ u_2=13,\ \ n=1,2,3,...$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n=2^n+3^n,\ n=1,2,3,...$$

proof



Question 12 (****)

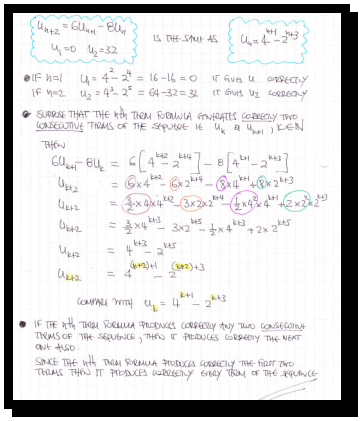
A sequence is generated by the recurrence relation

$$u_{n+2}=6u_{n+1}-8u_n\,,\ u_1=0\,,\ u_2=32\,,\ \ n=1,2,3,...$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n=4^{n+1}-2^{n+3}\,,\ n=1,2,3,...$$

proof



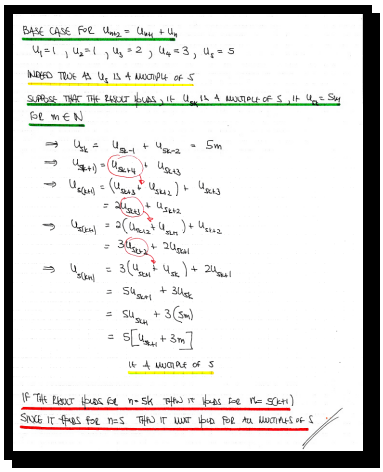
Question 13 (****)

A sequence is generated by the recurrence relation

$$u_{n+2}=u_{n+1}+u_n, \qquad u_1=0,\ u_2=1, \qquad n=1,2,3,...$$

Prove by induction that u_{5m} is a multiple of 5, for all $m\in\mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 14 **(****+)**

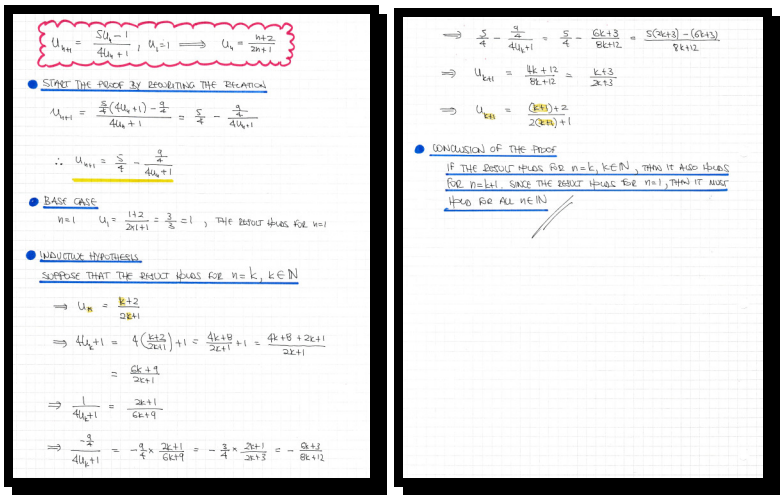
A sequence of numbers is given by the recurrence relation

$$u_{n+1}=\frac{5u_n-1}{4u_n+1},\ u_1=1,\ n\in\mathbb{N},\ n\geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n=\frac{n+2}{2n+1}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 15 (**+)**

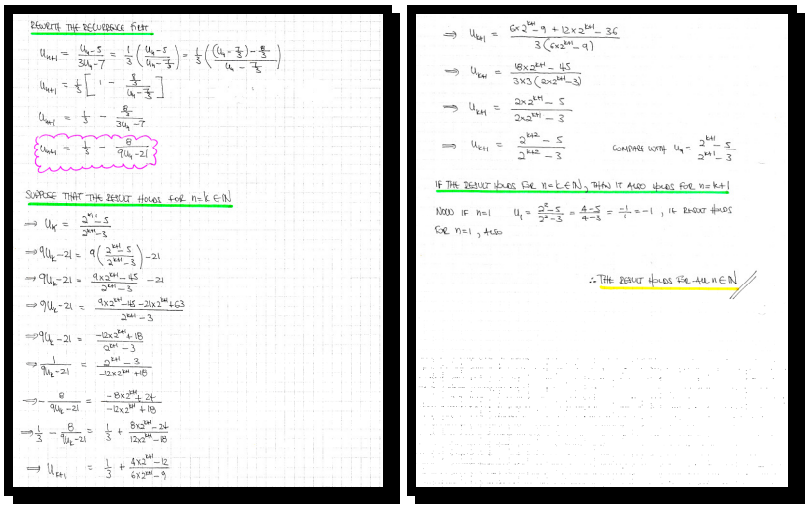
A sequence of numbers is given by the recurrence relation

$$u_{n+1}=\frac{u_n-5}{3u_n-7},\ u_1=-1,\ n\in\mathbb{N},\ n\geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n=\frac{2^{n+1}-5}{2^{n+1}-3}.$$

[ ], [proof]



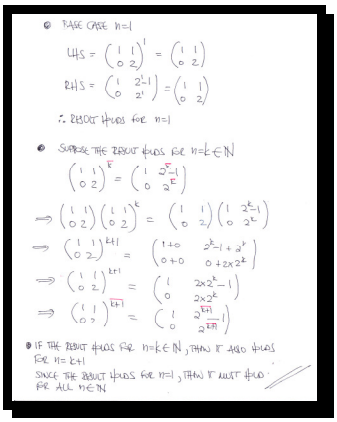
POWERS OF MATRICES

Question 1 (**)

Prove by induction that

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N} .$$

proof



Question 2 (**)

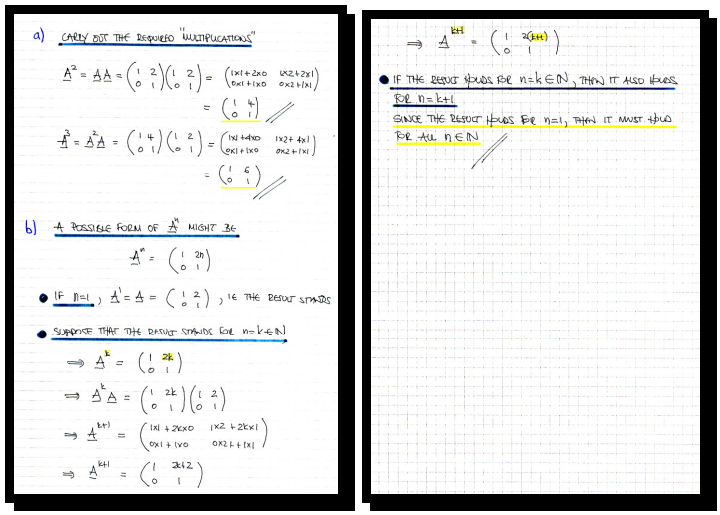
A transformation where $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

a) Find the elements of the matrices, $\mathbf{A}^2$ and $\mathbf{A}^3$.

b) Write down a suitable form for $\mathbf{A}^n$ and use the method of proof by induction to prove it.

$$\boxed{}, \quad \boxed{\mathbf{A}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}}, \quad \boxed{\mathbf{A}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}}, \quad \boxed{\mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}}$$

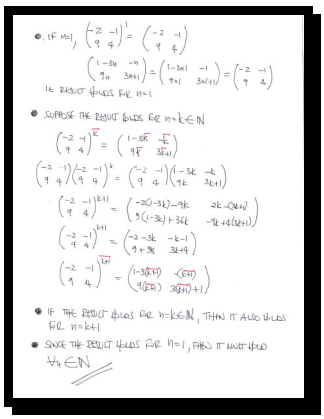


Question 3 (**)

Prove by induction that if $n \geq 1$, $n \in \mathbb{N}$, then

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix}.$$

proof



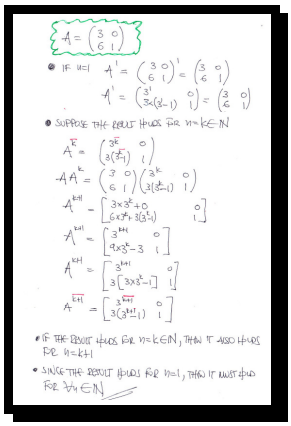
Question 4 (**)

$$\mathbf{A}=\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

Prove by induction that if $n \geq 1$, $n \in \mathbb{N}$, then

$$\mathbf{A}^n=\begin{pmatrix} 3^n & 0 \\ 3\left(3^n-1\right) & 1 \end{pmatrix}.$$

proof

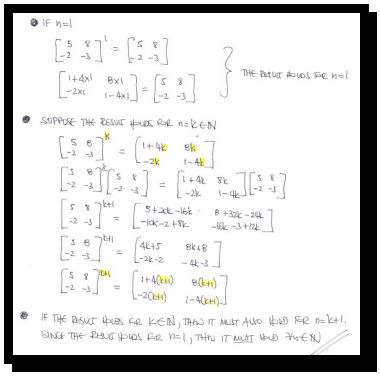


Question 5 (**)

Prove by induction that

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & 8n \\ -2n & 1-4n \end{pmatrix}, \quad n \geq 1, \ n \in \mathbb{N} .$$

proof



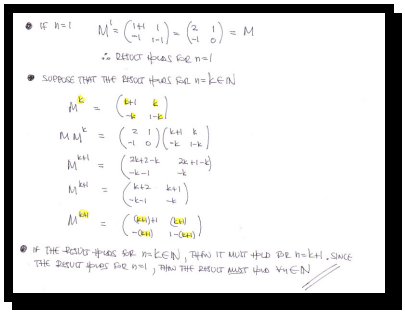
Question 6 (**)

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{M}^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}, \ n \geq 1, \ n \in \mathbb{N} .$$

proof



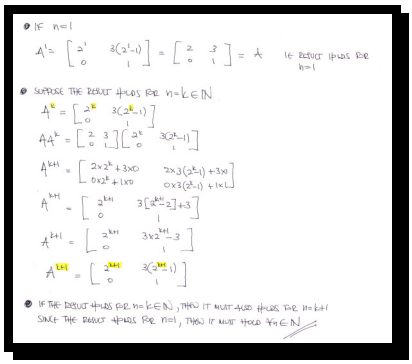
Question 7 (+)**

$$\mathbf{A}=\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n=\begin{pmatrix} 2^n & 3\left(2^n-1\right) \\ 0 & 1 \end{pmatrix},\ n\geq 1,\ n\in\mathbb{N}\ .$$

proof

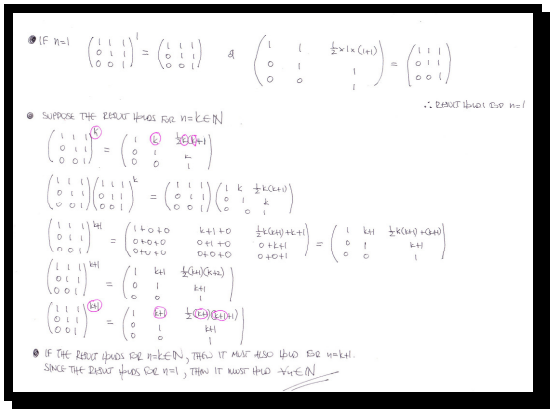


Question 8 (+)**

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1, \ n \in \mathbb{N} .$$

proof

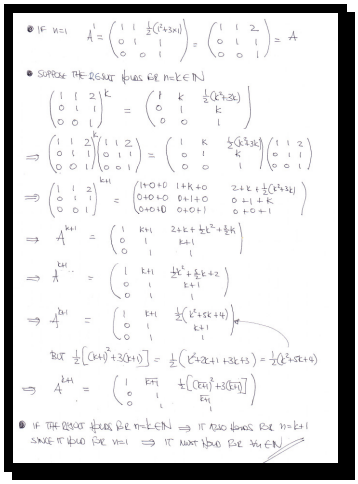


Question 9 (+)**

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}\left(n^2+3n\right) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N} .$$

proof



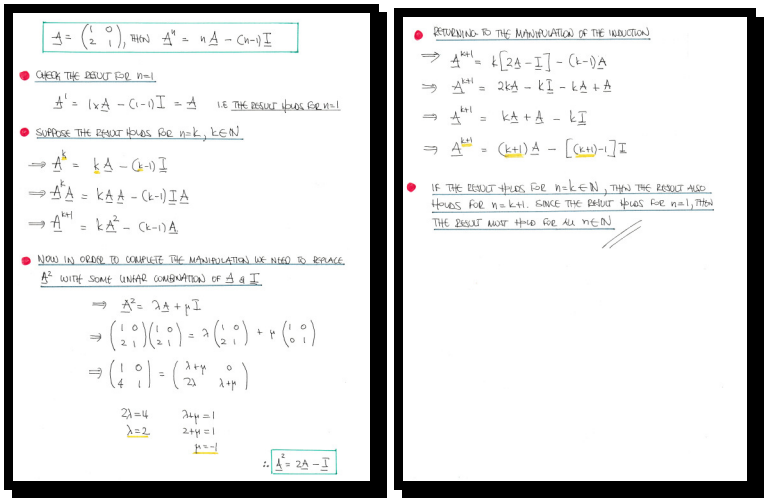
Question 10 (*****)

$$\mathbf{A}=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n=n\mathbf{A}-(n-1)\mathbf{I}\,,\ n\geq 1,\ n\in\mathbb{N}\ .$$

[ ], [proof]



MISCELLANEOUS RESULTS

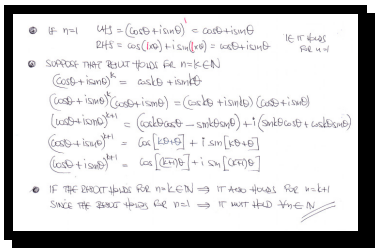
Question 1 (+)**

De Moivre's theorem states

$$\left(\cos\theta+\mathrm{i}\sin\theta\right)^n\equiv\cos n\theta+\mathrm{i}\sin n\theta\,,\ n\in\mathbb{N}\,.$$

Prove this theorem by induction.

proof

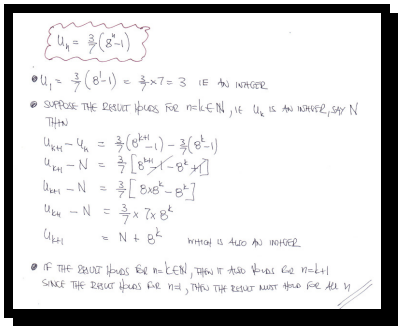


Question 2 (+)**

$$u_n=\frac{3}{7}\left(8^n-1\right),\ n\in\mathbb{N}\,.$$

Prove by induction that every term of this sequence is an integer.

proof



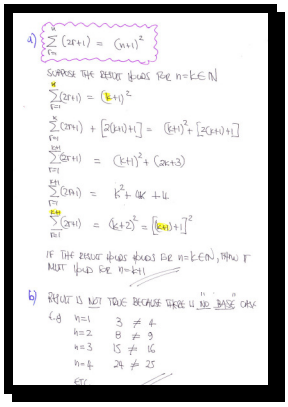
Question 3 (+)**

$$\sum_{r=1}^{n}(2r+1)=(n+1)^2,\ n\in\mathbb{N}$$

a) Show that if the above result holds for $n=k$, then it also holds for $n=k+1$.

b) Explain why the result is **not** true.

proof



Question 4 (+)**

The distinct square matrices $\mathbf{A}$ and $\mathbf{B}$ have the properties

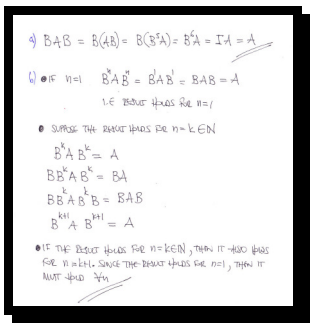
- $\mathbf{AB} = \mathbf{B}^5\mathbf{A}$
- $\mathbf{B}^6 = \mathbf{I}$

where $\mathbf{I}$ is the identity matrix.

a) Show that $\mathbf{BAB} = \mathbf{A}$.

b) Hence prove by induction that $\mathbf{B}^n\mathbf{AB}^n = \mathbf{A}$, for all $n \in \mathbb{N}$.

proof



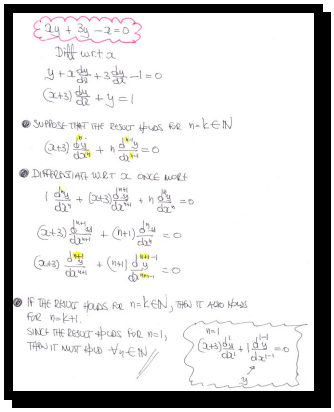
Question 5 (***)

$$xy+3y=x\,.$$

Prove by induction

$$(x+3)\frac{d^n}{dx^n}(y)+(n+1)\frac{d^{n-1}}{dx^{n-1}}(y)=0\,.$$

proof



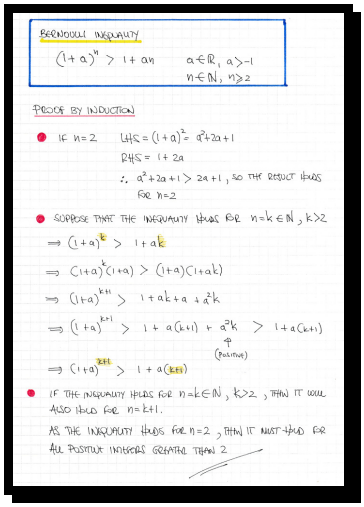
Question 6 (***)

Bernoulli's inequality asserts that if $a \in \mathbb{R}$, $a > -1$ and $n \in \mathbb{N}$, $n \geq 2$, then

$$(1+a)^n > 1+an .$$

Prove, by induction, the validity of Bernoulli's identity.

[ ], proof

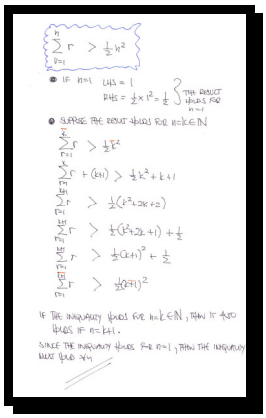


Question 7 (*+)**

Prove by induction that

$$\sum_{r=1}^{n} r > \frac{1}{2}n^2, \quad \text{for} \quad n \geq 1,\ n \in \mathbb{N}\ .$$

proof

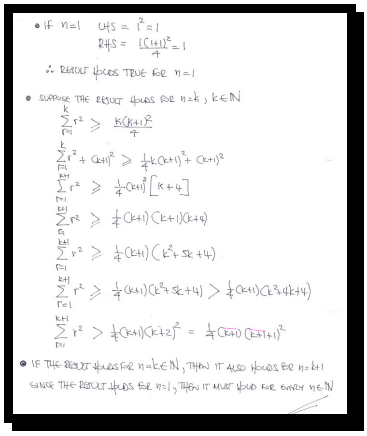


Question 8 (*+)**

Prove by induction that

$$\sum_{r=1}^{n} r^2 \geq \frac{1}{4}n(n+1)^2, \quad \text{for} \quad n \geq 1,\ n \in \mathbb{N}\ .$$

proof

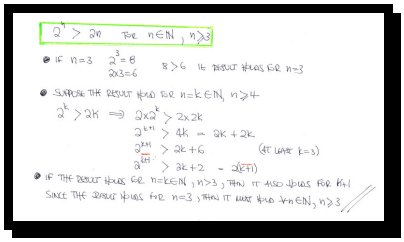


Question 9 (****)

Prove by induction that

$$2^n > 2n, \quad \text{for} \quad n \geq 3, \, n \in \mathbb{N}.$$

proof

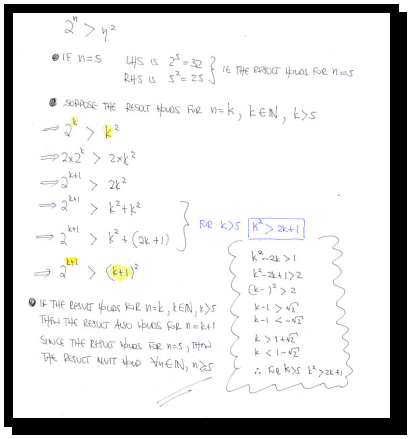


Question 10 (****)

Prove by induction that

$$2^n > n^2, \quad \text{for} \quad n \geq 5, \, n \in \mathbb{N}.$$

proof

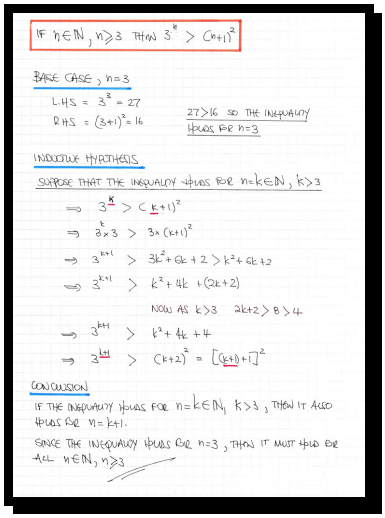


Question 11 (****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2 .$$

[ ], proof

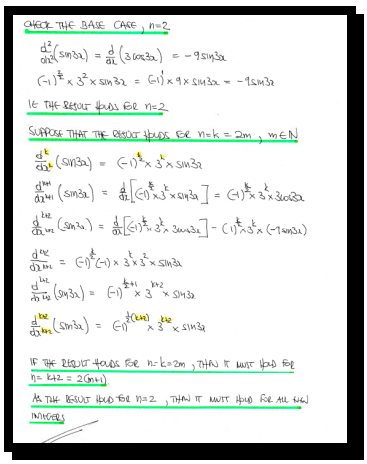


Question 12 (****)

Prove by induction that for all even natural numbers n

$$\frac{d^n}{dx^n}(\sin 3x) = (-1)^{\frac{n}{2}} \times 3^n \times \sin 3x .$$

[ ], proof

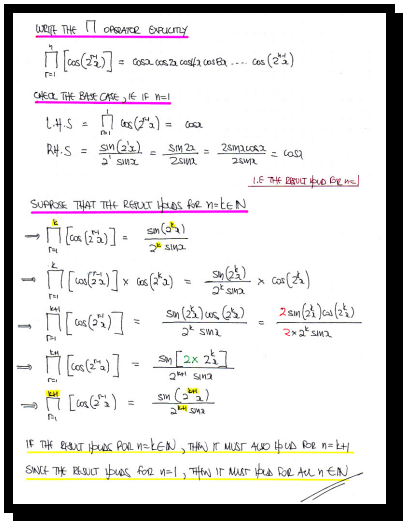


Question 13 (**+)**

Prove by induction that for $n \geq 1,\ n \in \mathbb{N}$

$$\prod_{r=1}^{n}\left(\cos\left(2^{r-1}x\right)\right)=\frac{\sin\left(2^{n}x\right)}{2^{n}\sin x}.$$

, proof



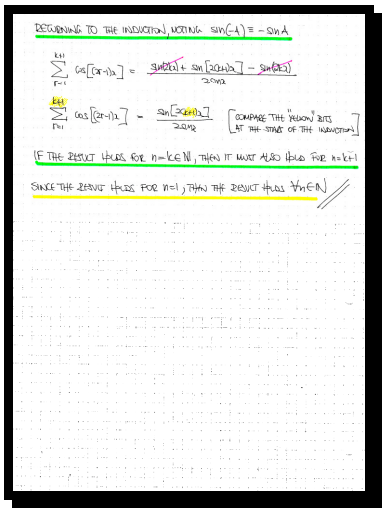
Question 14 **(****+)**

Prove by induction that

$$\cos x+\cos 3x+\cos 5x+...+\cos\left[(2n-1)x\right]\equiv\frac{\sin(2nx)}{2\sin x}.$$

[ ], proof

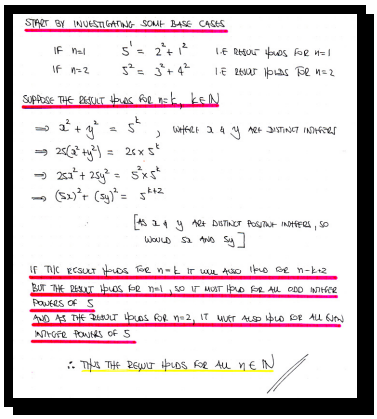




Question 15 **(****+)**

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.

[ ], proof

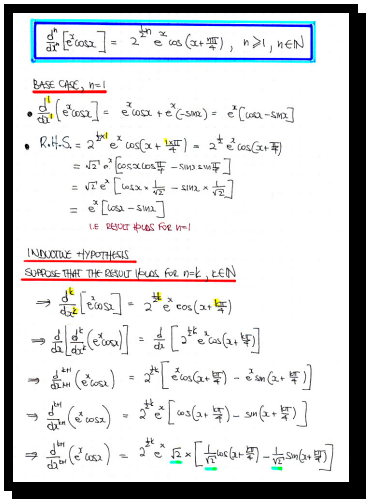


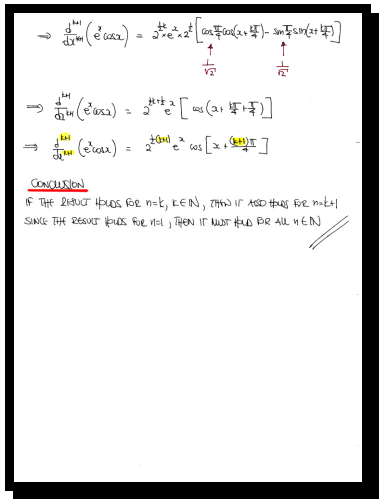
Question 16 (*****)

Prove by induction that

$$\frac{d^n}{dx^n}\left(\mathrm{e}^x\cos x\right)=2^{\frac{1}{2}n}\,\mathrm{e}^x\cos\left(x+\frac{n\pi}{4}\right),\quad n\geqslant 1,\ n\in\mathbb{N}\ .$$

, proof





Question 17 (*****)

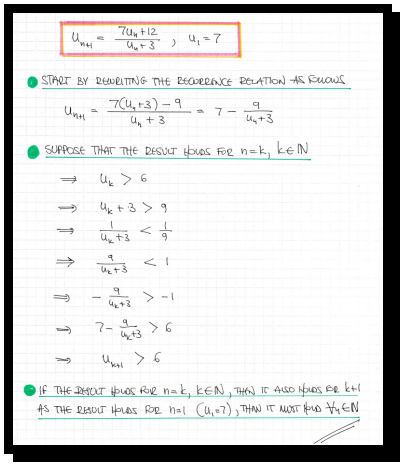
It is given that for $n \in \mathbb{N}$

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7.$$

Prove by induction that

$$u_n > 6.$$

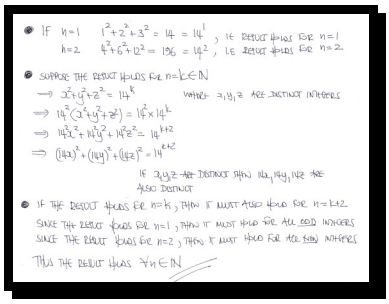
[ ], proof



Question 18 (*****)

Prove by induction that every positive integer power of 14 can be written as the sum of squares of three distinct positive integers.

[ ], proof



Question 19 (*****)

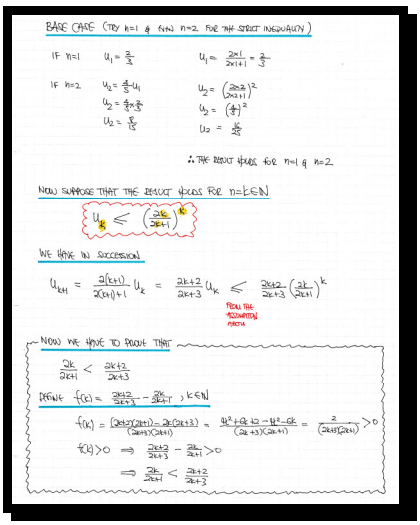
It is given that for $n \in \mathbb{N}$

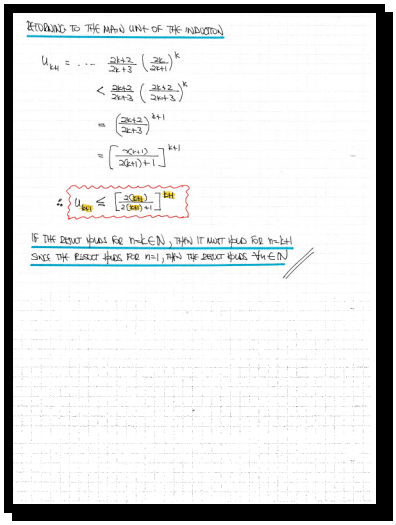
$$U_n = \frac{2n}{2n+1} U_{n-1}, \quad U_1 = \frac{2}{3}.$$

Prove by induction that

$$U_n \leq \left(\frac{2n}{2n+1} \right)^n.$$

[ ], proof



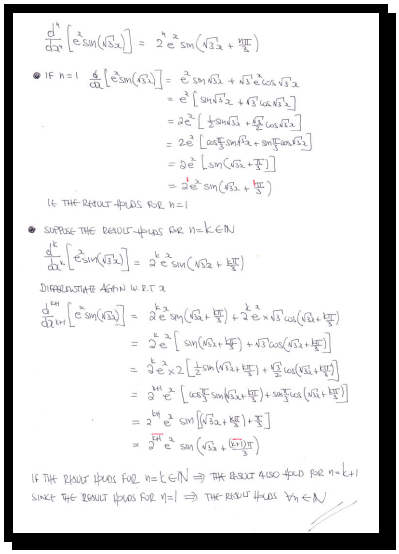


Question 20 (*****)

Prove by induction that

$$\frac{d^n}{dx^n}\left(\mathrm{e}^x \sin\left(\sqrt{3}x\right)\right) = 2^n \mathrm{e}^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right), \quad n \geq 1,\ n \in \mathbb{N}\ .$$

[ ], [proof]



Question 21 (*****)

The function $f(x)$ is defined by

$$f(x)=2-\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

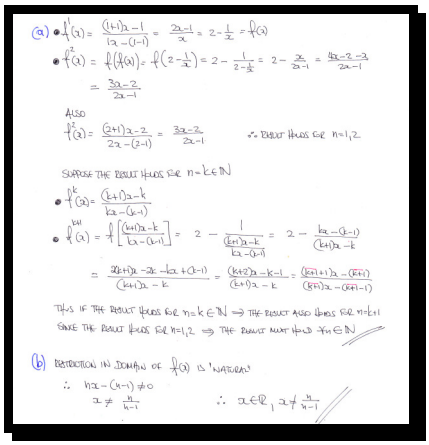
a) Prove that

$$f^n(x)=\frac{(n+1)x-n}{nx-(n-1)},\ n\geq 1,$$

where $f^n(x)$ denotes the n^{th} composition of $f(x)$ by itself.

b) State an expression for the domain of $f^n(x)$.

$\square$, $\boxed{x\in\mathbb{R},\ x\neq\frac{n-1}{n}}$



(a) • $f^1(x)=\frac{(1+1)x-1}{1x-(1-1)}=\frac{2x-1}{x}=2-\frac{1}{x}=f(x)$

• $f^2(x)=f\left(f(x)\right)=f\left(2-\frac{1}{x}\right)=2-\frac{1}{2-\frac{1}{x}}=2-\frac{x}{2x-1}=\frac{4x-2-x}{2x-1}$

$=\frac{3x-2}{2x-1}$

ALSO

$f^2(x)=\frac{(2+1)x-2}{2x-(2-1)}=\frac{3x-2}{2x-1}$ ∴ RESULT HOLDS FOR $n=1,2$

SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

• $f^k(x)=\frac{(k+1)x-k}{kx-(k-1)}$

• $f^{k+1}(x)=f\left[f^k(x)\right]=2-\frac{1}{\frac{(k+1)x-k}{kx-(k-1)}}=2-\frac{kx-(k-1)}{(k+1)x-k}$

$=\frac{2(k+1)x-2k-kx+(k-1)}{(k+1)x-k}=\frac{(k+2)x-k-1}{(k+1)x-k}=\frac{(k+1+1)x-(k+1)}{(k+1)x-(k+1-1)}$

THUS IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}\Rightarrow$ THE RESULT ALSO HOLDS FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1,2\Rightarrow$ THE RESULT MUST HOLD $\forall n\in\mathbb{N}$

(b) RESTRICTION IN DOMAIN OF $f^n(x)$ IS 'WHATEVER'

$\therefore\ nx-(n-1)\neq 0$

$x\neq\frac{n-1}{n}$ $\therefore\ x\in\mathbb{R},\ x\neq\frac{n-1}{n}$

Question 22 (*****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

[ ], proof

