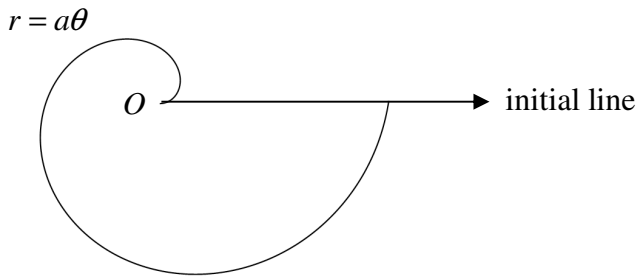
POLAR COORDINATES
54 EXAM QUESTIONS

8 BASIC QUESTIONS

Question 1 (**)



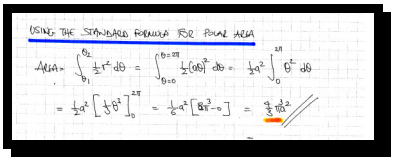
The figure above shows a spiral curve with polar equation

$$r = a\theta \, , \ 0 \le \theta \le 2\pi \, ,$$

where a is a positive constant.

Find the area of the finite region bounded by the spiral and the initial line.

$\boxed{}$, $\boxed{\text{area} = \frac{4}{3}a^2\pi^3}$



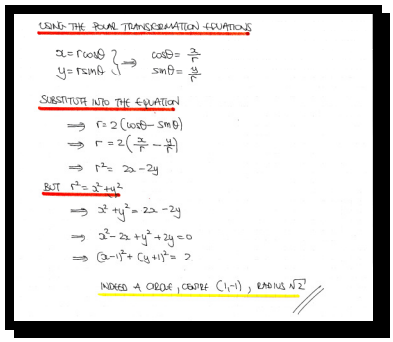
Question 2 (**)

The polar curve C has equation

$$r=2\left(\cos\theta-\sin\theta\right),\ 0\leq\theta<2\pi\,.$$

Find a Cartesian equation for C and show it represents a circle, indicating its radius and the Cartesian coordinates of its centre.

$\boxed{}$, $\boxed{(x-1)^2+(y+1)^2=2}$, $\boxed{r=\sqrt{2}}$, $\boxed{(1,-1)}$



Question 3 (**)

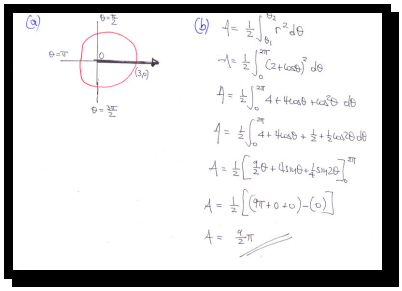
The polar curve C has equation

$$r=2+\cos\theta\,,\ 0\leq\theta<2\pi\,.$$

a) Sketch the graph of C.

b) Show that the area enclosed by the curve is $\dfrac{9}{2}\pi$.

proof



Question 4 (+)**

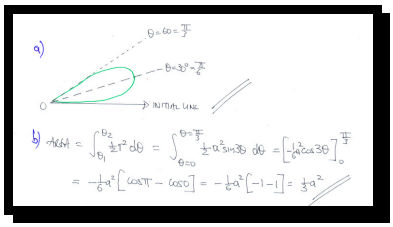
The curve C has polar equation

$$r^2 = a^2 \sin 3\theta, \ 0 \le \theta \le \frac{\pi}{3}.$$

a) Sketch the graph of C.

b) Find the exact value of area enclosed by the C.

$$\boxed{\text{area} = \frac{1}{3}a^2}$$



Question 5 (+)**

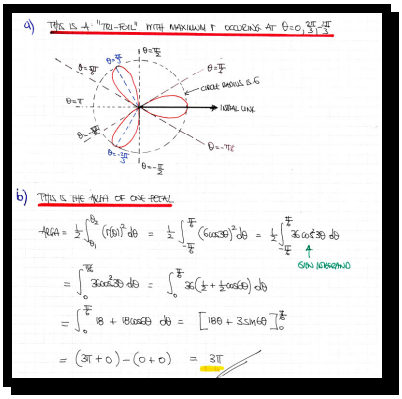
The curve C has polar equation

$$r = 6\cos 3\theta\,,\ -\pi < \theta \leq \pi\,.$$

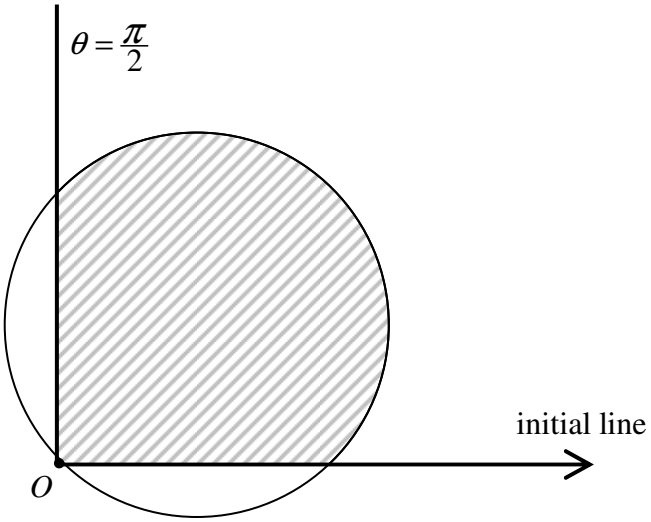
a) Sketch the graph of C.

b) Find the exact value of area enclosed by the C, for $-\dfrac{\pi}{6} < \theta \leq \dfrac{\pi}{6}$.

$\boxed{}$, $\boxed{\text{area} = 3\pi}$



Question 6 **(**+)**



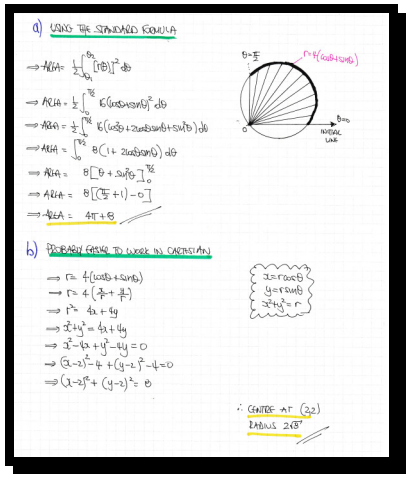
The figure above shows a circle with polar equation

$$r = 4\left(\cos\theta + \sin\theta\right) \quad 0 \le \theta < 2\pi \, .$$

a) Find the exact area of the shaded region bounded by the circle, the initial line and the half line $\theta = \frac{\pi}{2}$.

b) Determine the Cartesian coordinates of the centre of the circle and the length of its radius.

$\boxed{}$, $\boxed{\text{area} = 4\pi + 8}$, $\boxed{(2,2),\ \text{radius} = \sqrt{8}}$



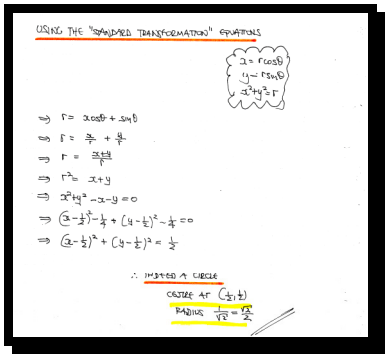
Question 7 (***)

Write the polar equation

$$r=\cos\theta+\sin\theta\,,\ 0\le\theta<2\pi$$

in Cartesian form, and hence show that it represents a circle, further determining the coordinates of its centre and the size of its radius.

$\square$, $\boxed{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}}$



Question 8 (***)

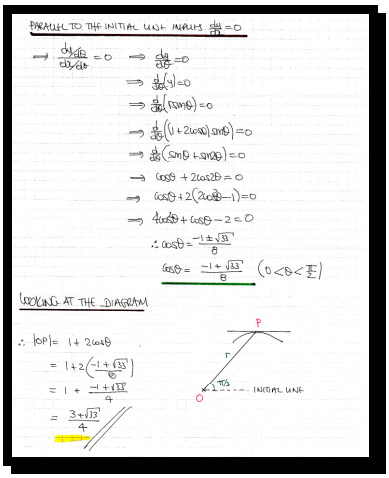
A Cardioid has polar equation

$$r = 1 + 2\cos\theta,\ 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the Cardioid so that the tangent to the Cardioid at P is parallel to the initial line.

Determine the exact length of OP, where O is the pole.

$\square$, $\boxed{\frac{1}{4}\left(3+\sqrt{33}\right)}$



26 STANDARD QUESTIONS

Question 1 (*+)**

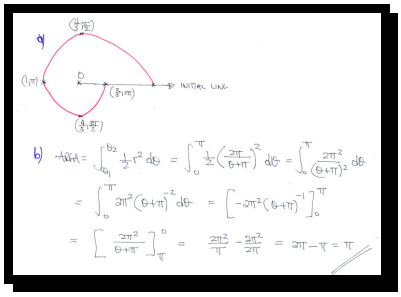
A curve has polar equation

$$r=\frac{2\pi}{\theta+\pi},\ 0\le\theta<2\pi\ .$$

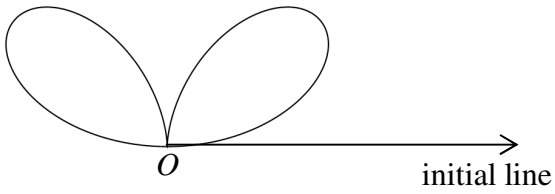
a) Sketch the curve.

b) Find the exact value of area enclosed by the curve, the initial line and the half line with equation $\theta=\pi$.

$$\boxed{\text{area}=\pi}$$



Question 2 (*+)**

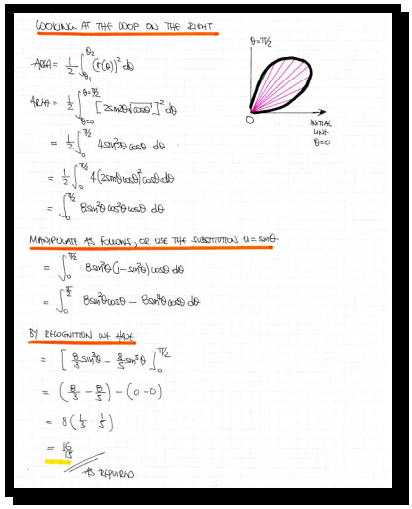


The figure above shows the polar curve C with equation

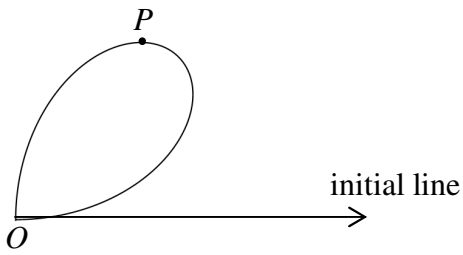
$$r = 2\sin 2\theta\sqrt{\cos\theta}\,,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

Show that the area enclosed by one of the two identical loops of the curve is $\dfrac{16}{15}$.

$\boxed{}$, proof



Question 3 **(***+)**



The figure above shows the polar curve with equation

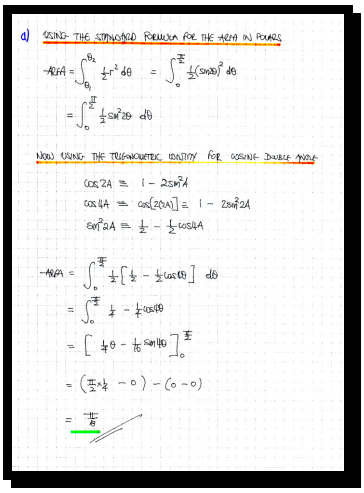
$$r = \sin 2\theta\,,\ 0 \le \theta \le \frac{\pi}{2}.$$

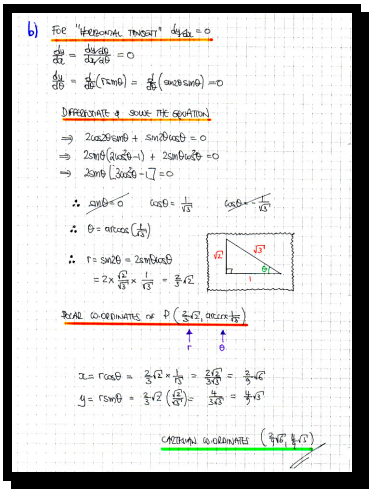
a) Find the exact value of the area enclosed by the curve.

The point P lies on the curve so that the tangent at P is parallel to the initial line.

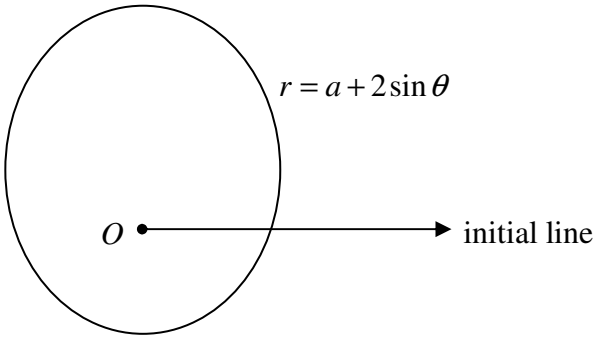
b) Find the **Cartesian** coordinates of P.

$\square$, $\boxed{\text{area} = \frac{\pi}{8}}$, $\boxed{\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)}$





Question 4 (*+)**



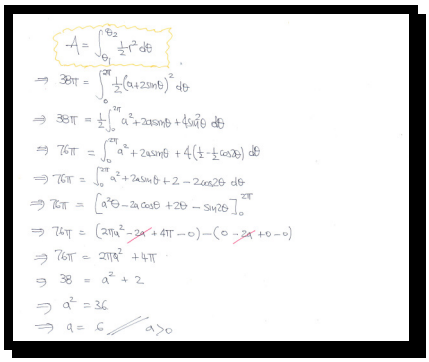
The diagram above shows the curve with polar equation

$$r = a + 2\sin\theta\,,\ 0 \le \theta < 2\pi\,,$$

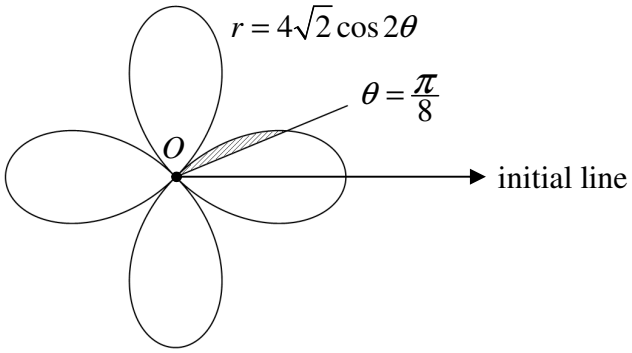
where a is a positive constant.

Determine the value of a given that the area bounded by the curve is 38π.

$\boxed{a = 6}$



Question 5 **(***+)**

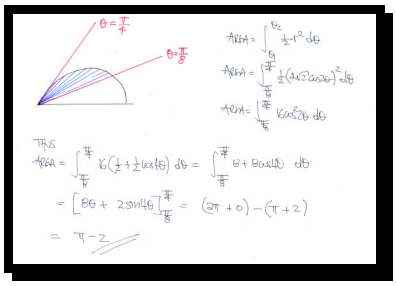


The figure above shows the curve with polar equation

$$r = 4\sqrt{2}\cos 2\theta\,,\ 0 \le \theta < 2\pi\,.$$

Find in exact form the area of the finite region bounded by the curve and the line with polar equation $\theta = \frac{\pi}{8}$, which is shown shaded in the above figure.

$$\boxed{\text{area} = \pi - 2}$$



Question 6 (*+)**

A curve C_1 has polar equation

$$r = 2\sin\theta\,,\ 0 \le \theta < 2\pi\,.$$

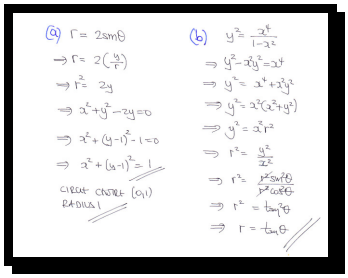
a) Find a Cartesian equation for C_1, and describe it geometrically.

A different curve C_2 has Cartesian equation

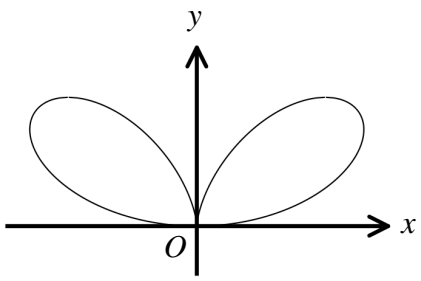
$$y^2 = \frac{x^4}{1-x^2}\,,\ x \neq \pm 1\,.$$

b) Find a polar equation for C_2, in the form $r = f(\theta)$.

$\boxed{x^2+(y-1)^2=1}$, $\boxed{r=\tan\theta}$



Question 7 (*+)**



The figure above shows the curve C with Cartesian equation

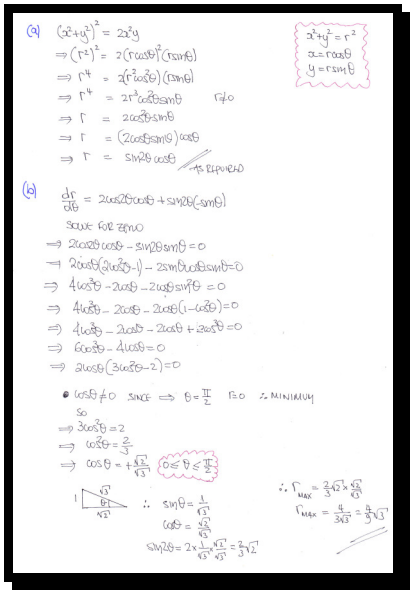
$$\left(x^2+y^2\right)^2=2x^2y\,.$$

a) Show that a polar equation for C can be written as

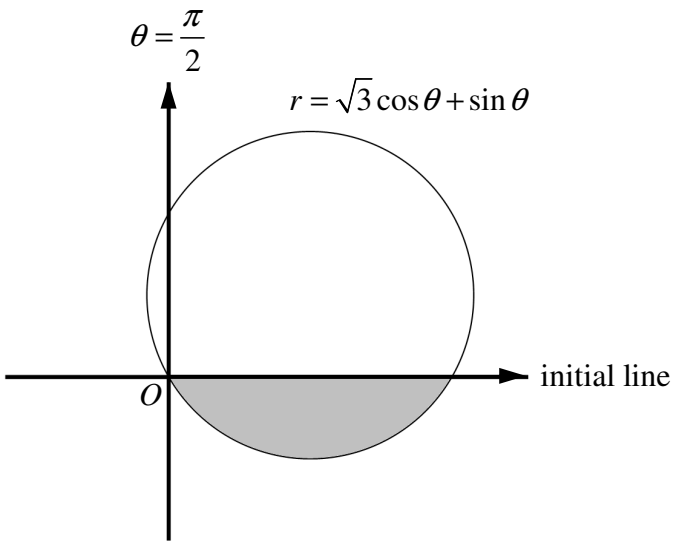
$$r=\sin 2\theta\cos\theta\,.$$

b) Determine in exact surd form the maximum value of r.

$$\boxed{r_{\max}=\frac{4}{9}\sqrt{3}}$$



Question 8 **(***+)**



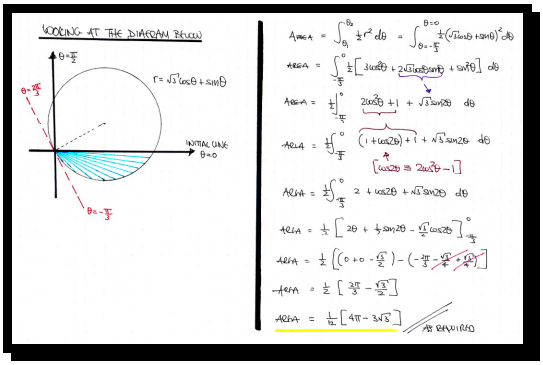
The diagram above shows the curve with polar equation

$$r=\sqrt{3}\cos\theta+\sin\theta\,,\ -\frac{\pi}{3}\le\theta<\frac{2\pi}{3}.$$

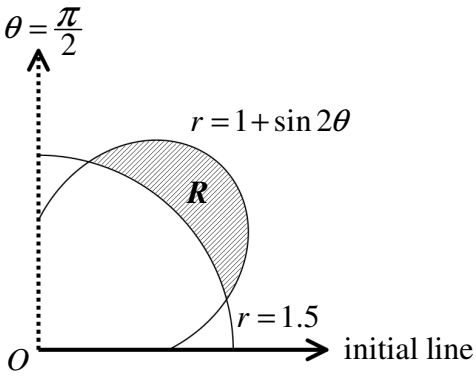
By using a method involving integration in polar coordinates, show that the area of the shaded region is

$$\frac{1}{12}\left(4\pi-3\sqrt{3}\right).$$

[ ], [proof]



Question 9 **(****)**



The diagram above shows the curves with polar equations

$$r=1+\sin 2\theta\,,\ 0\le\theta\le\frac{1}{2}\pi\,,$$

$$r=1.5\,,\ 0\le\theta\le\frac{1}{2}\pi\,.$$

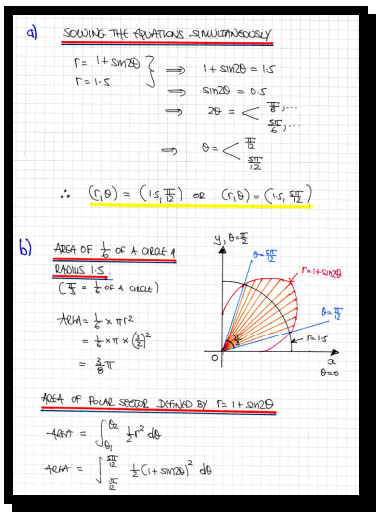
a) Find the polar coordinates of the points of intersection between the two curves.

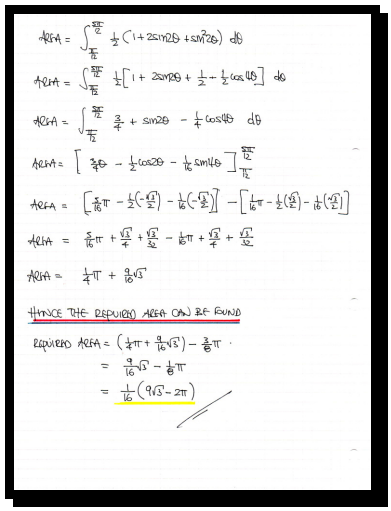
The finite region R, is bounded by the two curves and is shown shaded in the figure.

b) Show that the area of R is

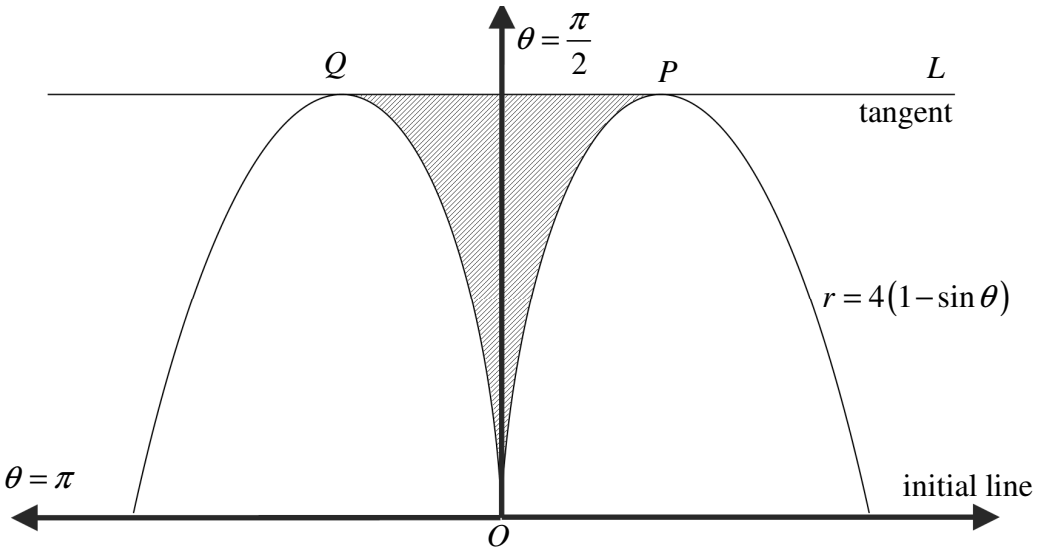
$$\frac{1}{16}\left(9\sqrt{3}-2\pi\right).$$

$\square$, $\boxed{\left(\frac{3}{2},\frac{\pi}{12}\right),\ \left(\frac{3}{2},\frac{5\pi}{12}\right)}$





Question 10 **(****)**



The figure above shows the graph of the curve with polar equation

$$r=4(1-\sin\theta),\ 0\le\theta\le\pi.$$

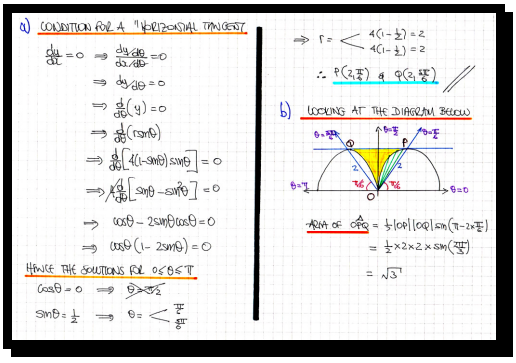
The straight line L is a tangent to the curve parallel to the initial line, touching the curve at the points P and Q.

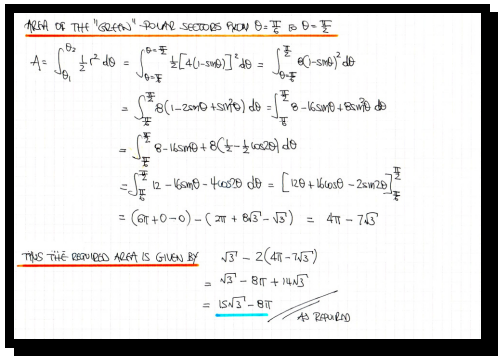
a) Find the polar coordinates of P and the polar coordinates of Q.

b) Show that the area of the shaded region is exactly

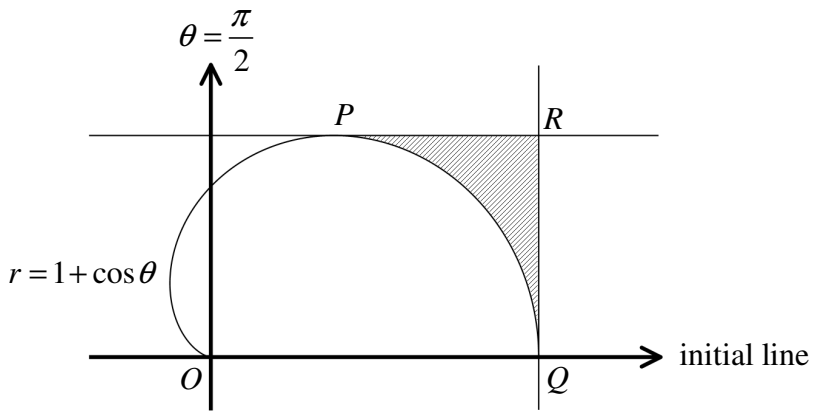
$$15\sqrt{3}-8\pi.$$

$\square$, $\boxed{P\left(2,\frac{1}{6}\pi\right),\ Q\left(2,\frac{5}{6}\pi\right)}$





Question 11 **(****)**



The diagram above shows the curve with polar equation

$$r=1+\cos\theta\,,\quad 0\le\theta\le\pi\,.$$

The curve meets the initial line at the origin O and at the point Q. The point P lies on the curve so that the tangent to the curve at P is parallel to the initial line.

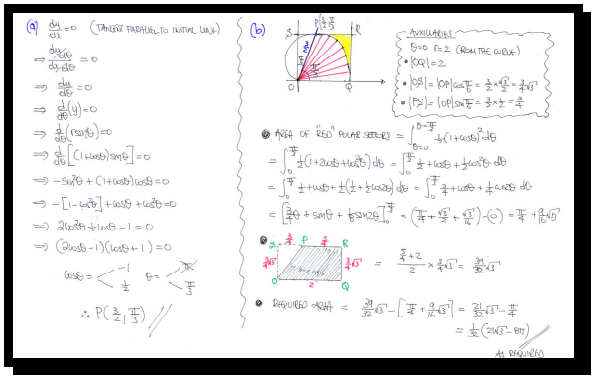
a) Determine the polar coordinates of P.

The tangent to the curve at Q is perpendicular to the initial line and meets the tangent to the curve at P, at the point R.

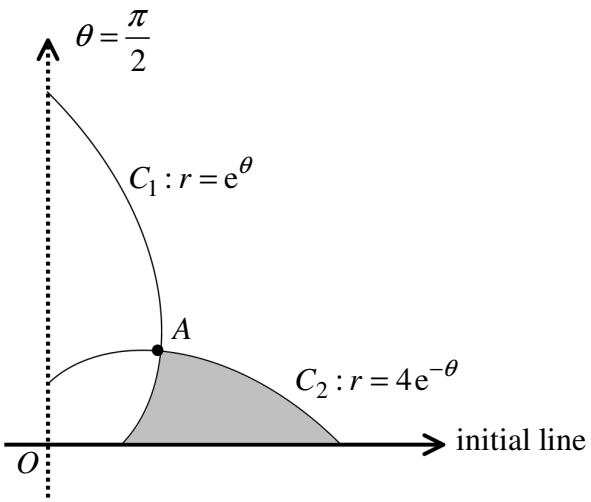
b) Show that the area of the finite region bounded by the line segments PR, QR and the arc PQ is

$$\frac{1}{32}\left(21\sqrt{3}-8\pi\right).$$

$$\boxed{P\left(\frac{3}{2},\frac{\pi}{3}\right)}$$



Question 12 (****)



The diagram below shows the curves with polar equations

$$C_1 : r = \mathrm{e}^{\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

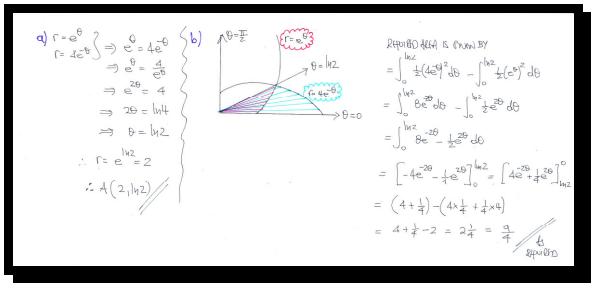
$$C_2 : r = 4\mathrm{e}^{-\theta}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The curves intersect at the point A.

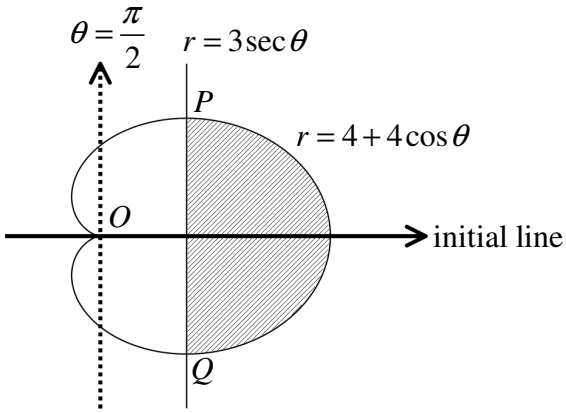
a) Find the exact polar coordinates of A.

b) Show that area of the shaded region is $\frac{9}{4}$.

$\boxed{A\left(2, \ln 2\right)}$



Question 13 (****)



The figure above shows a curve and a straight line with respective polar equations

$$r = 4 + 4\cos\theta\,,\ -\pi < \theta \le \pi \quad \text{and} \quad r = 3\sec\theta\,,\ -\frac{\pi}{2} < \theta \le \frac{\pi}{2}\,.$$

The straight line meets the curve at two points, P and Q.

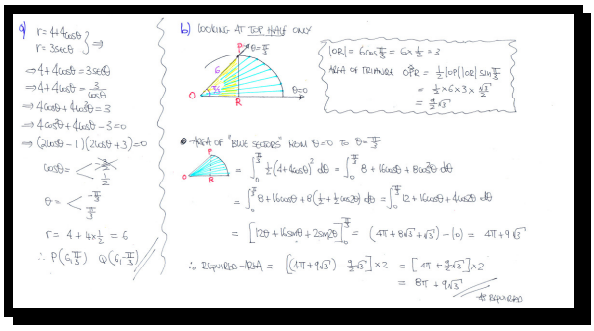
a) Determine the polar coordinates of P and Q.

The finite region, shown shaded in the figure, is bounded by the curve and the straight line.

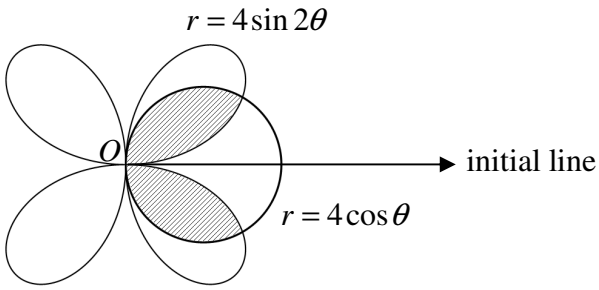
b) Show that the area of this finite region is

$$8\pi + 9\sqrt{3}\,.$$

$$\boxed{P\left(6, \frac{\pi}{3}\right), Q\left(6, -\frac{\pi}{3}\right)}$$



Question 14 **(****)**



The figure above shows the curves with polar equations

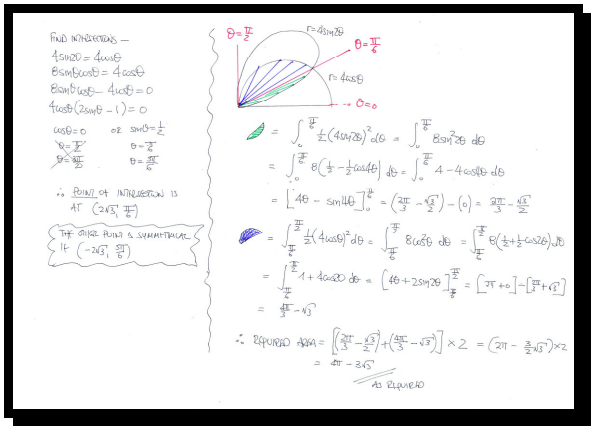
$$r = 4\cos\theta,\ 0 \le \theta \le 2\pi,$$

$$r = 4\sin 2\theta,\ 0 \le \theta \le 2\pi.$$

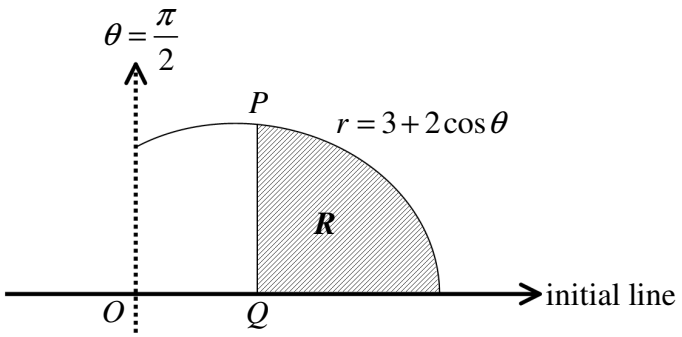
Show that the area of the shaded region which consists of all the points which are bounded by **both** curves is

$$4\pi - 3\sqrt{3}.$$

proof



Question 15 **(****)**



The figure above shows the cardioid with polar equation

$$r = 3 + 2\cos\theta,\ 0 < \theta \leq \frac{\pi}{2}\ .$$

The point P lies on the cardioid and its distance from the pole O is 4 units.

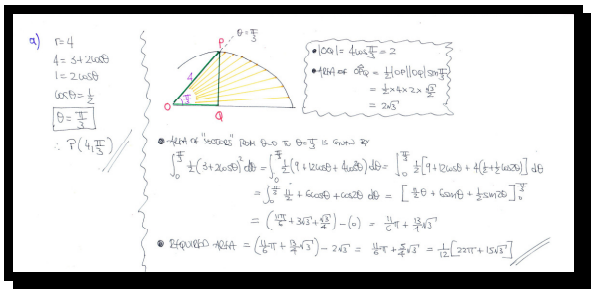
a) Determine the polar coordinates of P.

The point Q lies on the initial line so that the line segment PQ is perpendicular to the initial line. The finite region R, shown shaded in the figure, is bounded by the curve, the initial line and the line segment PQ.

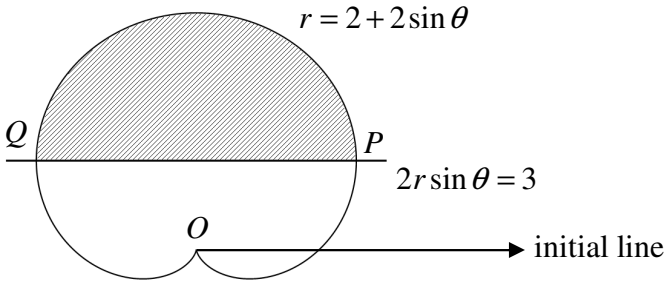
b) Show that the area of R is

$$\frac{1}{12}\left(22\pi + 15\sqrt{3}\right).$$

$$\boxed{P\left(4, \frac{\pi}{3}\right)}$$



Question 16 **(****)**



The figure above shows the curve with polar equation

$$r=2+2\sin\theta,\ 0\le\theta\le 2\pi,$$

intersected by the straight line with polar equation

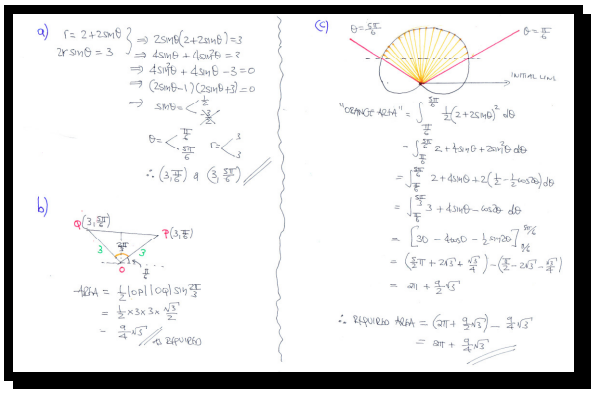
$$2r\sin\theta=3,\ 0<\theta<\pi.$$

a) Find the coordinates of the points P and Q, where the line meets the curve.

b) Show that the area of the triangle OPQ is $\frac{9}{4}\sqrt{3}$.

c) Hence find the exact area of the **shaded** region bounded by the curve and the straight line.

$$\boxed{P\left(3,\frac{5\pi}{6}\right)},\ \boxed{Q\left(3,\frac{\pi}{6}\right)},\ \boxed{\text{area}=2\pi+\frac{9}{4}\sqrt{3}}$$



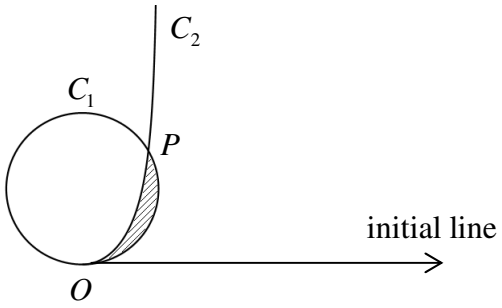
Question 17 **(****)**

The curves C_1 and C_2 have respective polar equations

$$C_1: \ r = 2\sin\theta, \ 0 \le \theta < 2\pi$$

$$C_2: \ r = \tan\theta, \ 0 \le \theta < \frac{\pi}{2}.$$

a) Find a Cartesian equation for C_1 and a Cartesian equation for C_2.



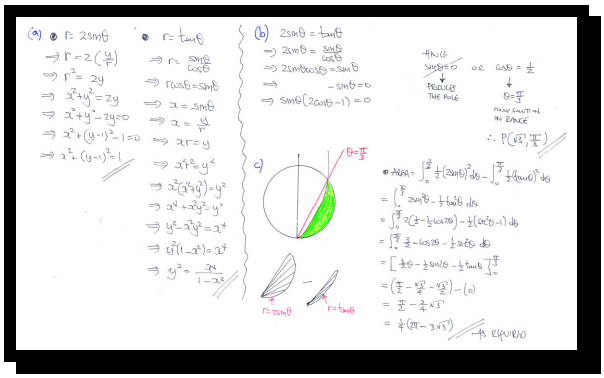
The figure above shows the two curves intersecting at the pole and at the point P.

The finite region, shown shaded in the figure, is bounded by the two curves.

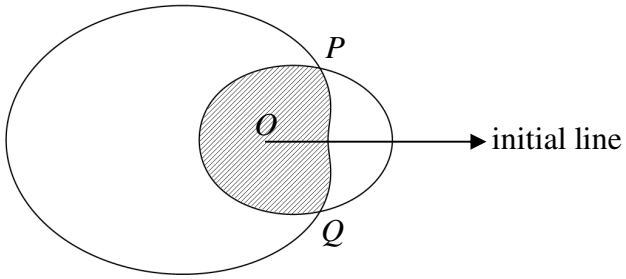
b) Determine the exact polar coordinates of P.

c) Show that the area of the shaded region is $\frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$.

$$\boxed{C_1: x^2 + (y-1)^2 = 1}, \ \boxed{C_1: x^2 + (y-1)^2 = 1}, \ \boxed{P\left(\sqrt{3}, \frac{\pi}{3}\right)}$$



Question 18 **(****)**



The figure above shows two overlapping closed curves C_1 and C_2, with respective polar equations

$$C_1: \ r = 3 + \cos\theta, \ 0 \le \theta < 2\pi$$

$$C_2: \ r = 5 - 3\cos\theta, \ 0 \le \theta < 2\pi .$$

The curves meet at two points, P and Q.

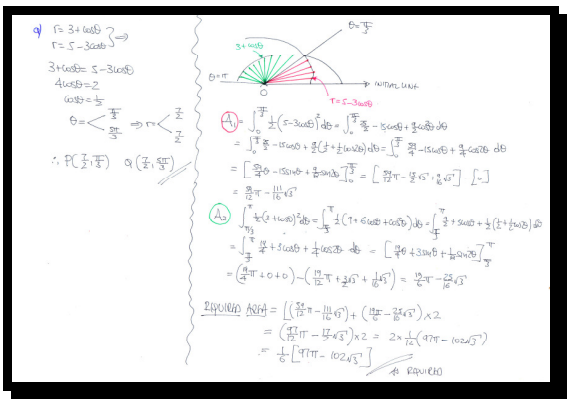
a) Determine the polar coordinates of P and Q.

The finite region R, shown shaded in the figure, consists of all the points which lie **inside both** C_1 and C_2.

b) Show that the area of R is

$$\frac{1}{6}\left(97\pi - 102\sqrt{3}\right).$$

$$\boxed{P\left(\frac{7}{2}, \frac{\pi}{3}\right), Q\left(\frac{7}{2}, \frac{5\pi}{3}\right)},$$



Question 19 (****)

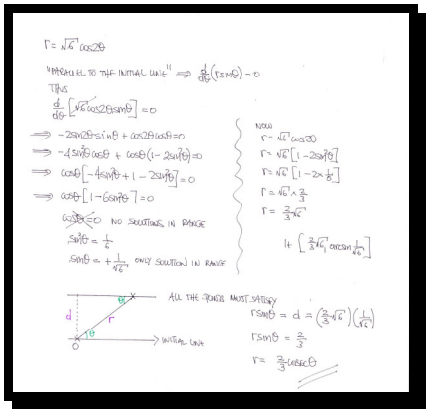
The curve C with polar equation

$$r=\sqrt{6}\cos 2\theta\,,\ 0\le\theta\le\frac{\pi}{4}.$$

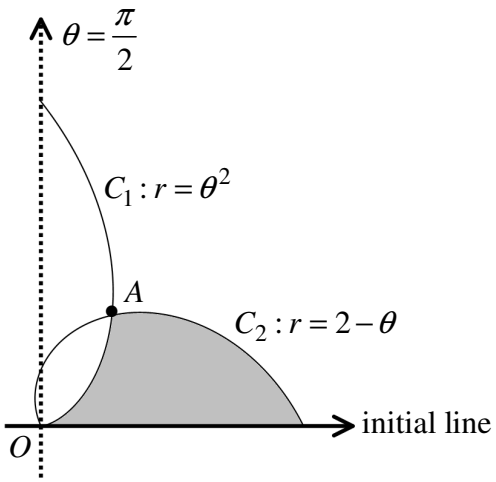
The straight line l is parallel to the initial line and is a tangent to C.

Find an equation of l, giving the answer in the form $r=f(\theta)$.

$$\boxed{r=\frac{2}{3}\operatorname{cosec}\theta}$$



Question 20 (****)



The diagram above shows the curves with polar equations

$$C_1 : r = \theta^2 , \ 0 \le \theta \le \frac{\pi}{2}$$

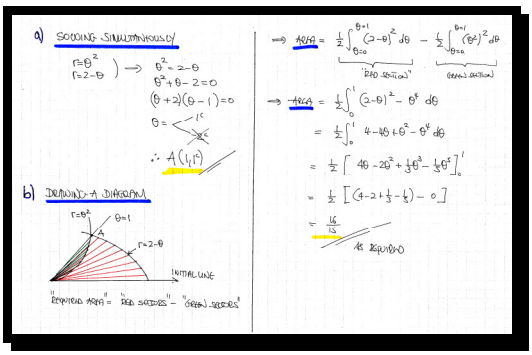
$$C_2 : r = 2 - \theta , \ 0 \le \theta \le 2 .$$

The curves intersect at the point A.

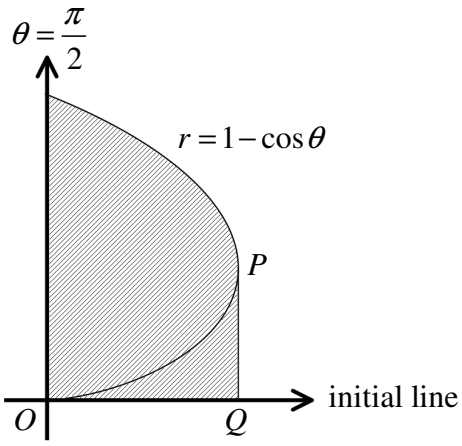
a) Find the polar coordinates of A.

b) Show that the area of the shaded region is $\frac{16}{15}$.

$\boxed{}$, $\boxed{A(1,1)}$



Question 21 **(****)**



The figure above shows the curve C with polar equation

$$r=1-\cos\theta,\ 0\le\theta<\frac{\pi}{2}.$$

The point P lies on C so that tangent to C is perpendicular to the initial line.

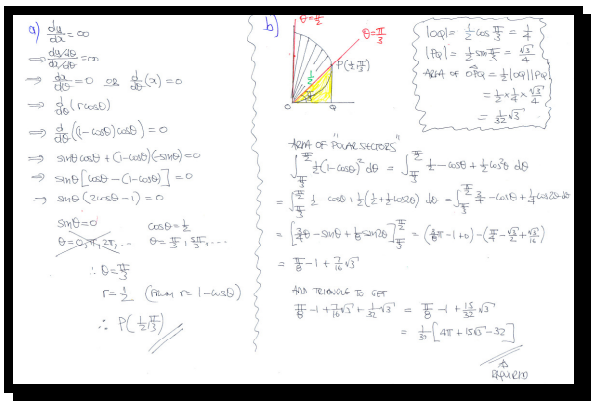
a) Determine the polar coordinates of P.

The finite region R consists of all the points which are bounded by C, the straight line segment PQ, the initial line and the line with equation $\theta=\frac{\pi}{2}$.

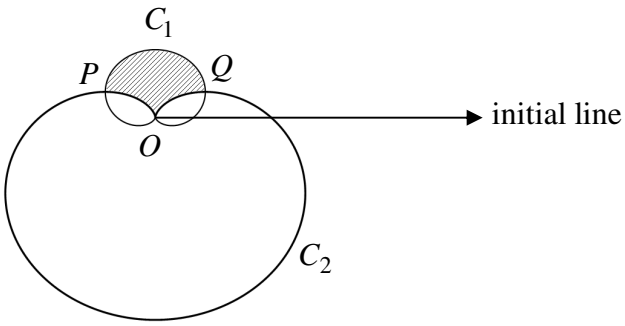
b) Show that the area of R, shown shaded in the figure above, is exactly

$$\frac{1}{32}\left(4\pi+15\sqrt{3}-32\right).$$

$$\boxed{P\left(\frac{1}{2},\frac{\pi}{3}\right)}$$



Question 22 **(****)**



The figure above shows two closed curves with polar equations

$$C_1:\ r=a(1+\sin\theta),\ 0\le\theta\le 2\pi \quad \text{and} \quad C_2:\ r=3a(1-\sin\theta),\ 0\le\theta\le 2\pi,$$

intersecting each other at the pole O and at the points P and Q.

a) Find the polar coordinates of the points P and Q.

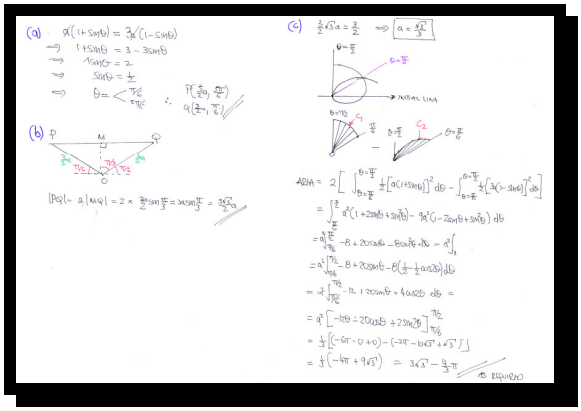
b) Show that the distance PQ is $\frac{3\sqrt{3}}{2}a$.

The finite region shown shaded in the above figure consists of all the points inside C_1 but outside C_2.

c) Given that the distance PQ is $\frac{3}{2}$, show that the area of the **shaded** region is

$$3\sqrt{3}-\frac{4}{3}\pi.$$

$$\boxed{P\left(\tfrac{3}{2}a,\tfrac{5\pi}{6}\right)},\ \boxed{Q\left(\tfrac{3}{2}a,\tfrac{\pi}{6}\right)}$$



Question 23 (****)

The points A and B have respective coordinates $(-1,0)$ and $(1,0)$.

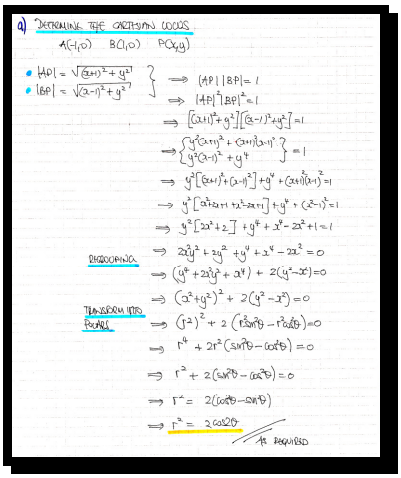
The locus of the point $P(x,y)$ traces a curve in such a way so that $|AP||BP|=1$.

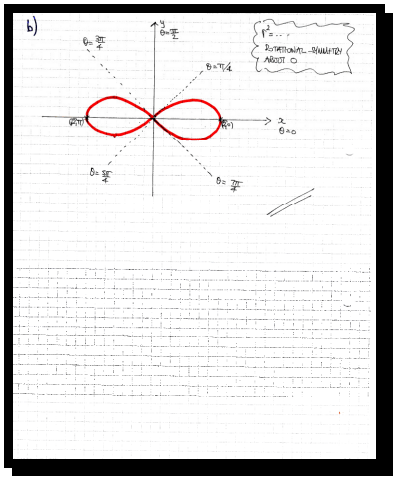
a) By forming a Cartesian equation of the locus of P, show that the polar equation of the curve is

$$r^2=2\cos 2\theta,\ 0\le\theta<2\pi.$$

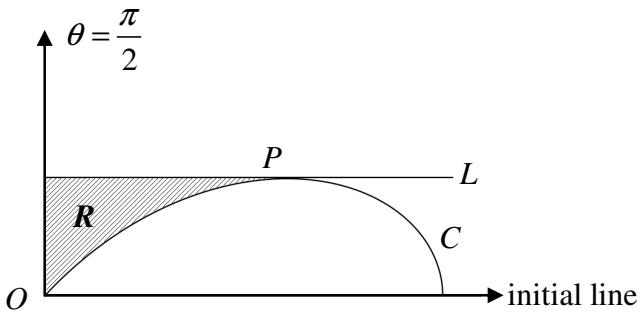
b) Sketch the curve.

[ ], [proof]





Question 24 (****)



The figure above shows a curve C with polar equation

$$r^2 = 2\cos 2\theta, \ 0 \le \theta < \frac{\pi}{4}.$$

The straight line L is parallel to the initial line and is a tangent to C at the point P.

a) Show that the polar coordinates of P are $\left(1, \frac{\pi}{6}\right)$.

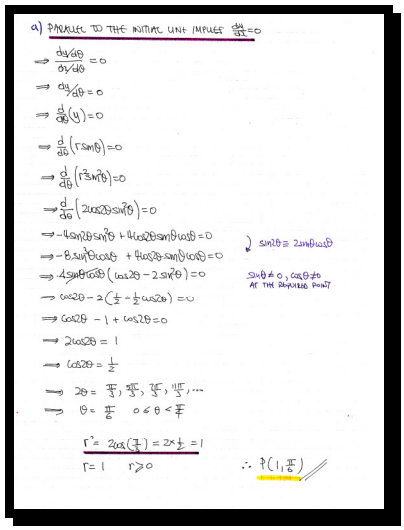
The finite region R, shown shaded in the figure above, is bounded by C, L and the half line with equation $\theta = \frac{\pi}{2}$.

b) Show that the area of R is

$$\frac{1}{8}\left(3\sqrt{3} - 4\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

[solution overleaf]



a) PARALLEL TO THE INITIAL LINE IMPLIES $\frac{dy}{d\theta}=0$

$\Rightarrow \frac{dy/d\theta}{dx/d\theta}=0$

$\Rightarrow \frac{dy}{dx}=0$

$\Rightarrow \frac{d}{dx}(y)=0$

$\Rightarrow \frac{d}{d\theta}\left(r\sin\theta\right)=0$

$\Rightarrow \frac{d}{d\theta}\left(r^2\sin\theta\right)=0$

$\Rightarrow \frac{d}{d\theta}\left(2\cos2\theta\sin\theta\right)=0$

$\Rightarrow -4\sin2\theta\sin\theta+4\cos2\theta\cos\theta=0$ $\quad$ $\sin2\theta\equiv 2\sin\theta\cos\theta$

$\Rightarrow -8\sin^2\theta\cos\theta+4\cos2\theta\sin\theta\cos\theta=0$

$\Rightarrow 4\sin\theta\cos\theta\left(\cos2\theta-2\sin^2\theta\right)=0$ $\quad$ $\sin\theta\neq 0,\ \cos\theta\neq 0$ AT THE REQUIRED POINT

$\Rightarrow \cos2\theta-2\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)=0$

$\Rightarrow \cos2\theta-1+\cos2\theta=0$

$\Rightarrow 2\cos2\theta=1$

$\Rightarrow \cos2\theta=\frac{1}{2}$

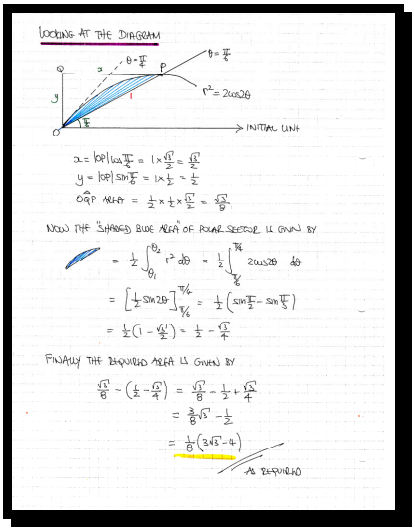
$\Rightarrow 2\theta=\frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{11\pi}{3},\ \cdots$

$\Rightarrow \theta=\frac{\pi}{6}\quad 0<\theta<\frac{\pi}{4}$

$r^2=2\cos\left(\frac{\pi}{3}\right)=2\times\frac{1}{2}=1$

$r=1\quad r>0$

$\therefore P\left(1,\frac{\pi}{6}\right)$



LOOKING AT THE DIAGRAM

$x=|OP|\cos\frac{\pi}{6}=1\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$

$y=|OP|\sin\frac{\pi}{6}=1\times\frac{1}{2}=\frac{1}{2}$

$\text{OQP AREA}=\frac{1}{2}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{8}$

NOW THE "SHADED BLUE AREA" OF POLAR SECTOR IS GIVEN BY

$=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\cos2\theta\,d\theta$

$=\left[\frac{1}{2}\sin2\theta\right]_{\pi/6}^{\pi/4}=\frac{1}{2}\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{3}\right)$

$=\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)=\frac{1}{2}-\frac{\sqrt{3}}{4}$

FINALLY THE REQUIRED AREA IS GIVEN BY

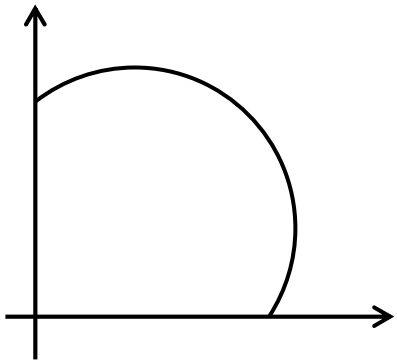
$\frac{\sqrt{3}}{8}-\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)=\frac{\sqrt{3}}{8}-\frac{1}{2}+\frac{\sqrt{3}}{4}$

$=\frac{3}{8}\sqrt{3}-\frac{1}{2}$

$=\frac{1}{8}\left(3\sqrt{3}-4\right)$

AS REQUIRED

Question 25 **(****)**



The figure above shows the curve C, with Cartesian equation

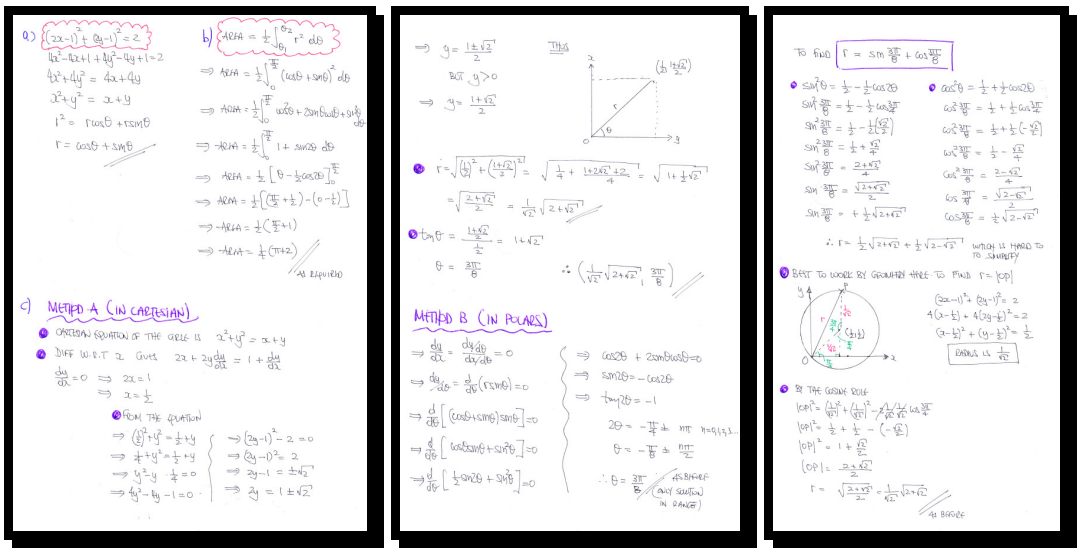
$$(2x-1)^2+(2y-1)^2=2,\ x\geq 0,\ y\geq 0$$

a) Find a polar equation for C, in the form $r=f(\theta)$.

b) Show that the area bounded by C and the coordinate axes is $\frac{1}{4}(\pi+2)$.

c) Determine, in exact simplified form, the polar coordinates of the point on C, where the tangent to C is parallel to the x axis.

$$\boxed{s=\frac{1}{4}\sqrt{5}+\ln\left[\frac{1}{4}\left(1+\sqrt{5}\right)\right]}$$



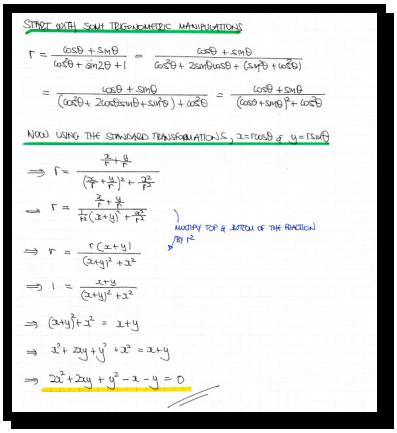
Question 26 (****)

A curve has polar equation

$$r=\frac{\cos\theta+\sin\theta}{\cos^2\theta+\sin 2\theta+1},\quad 0\le\theta<2\pi\,.$$

Find a Cartesian equation of the curve giving the answer in the form $f(x,y)=0$.

V, [ ], $\boxed{2x^2+2xy+y^2-x-y=0}$



10 HARD QUESTIONS

Question 1 (**+)**

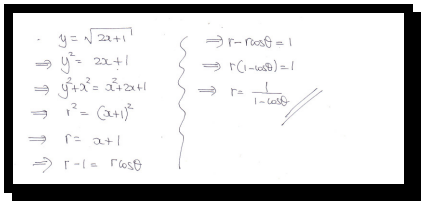
Show that the polar equation of the top half of the parabola with Cartesian equation

$$y=\sqrt{2x+1}\,,\ x\geq -\frac{1}{2}\,,$$

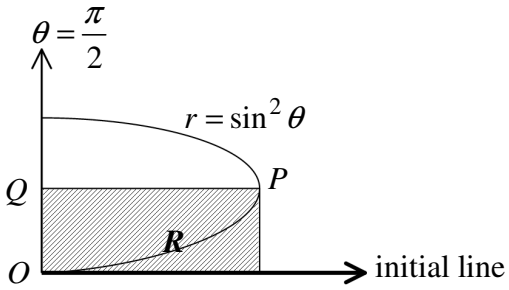
is given by the polar equation

$$r=\frac{1}{1-\cos\theta}\,,\ r\geq 0\,.$$

proof



Question 2 (**+)**



The figure above shows the curve with polar equation

$$r = \sin^2\theta, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the curve so that the tangent to the curve at P is perpendicular to the initial line.

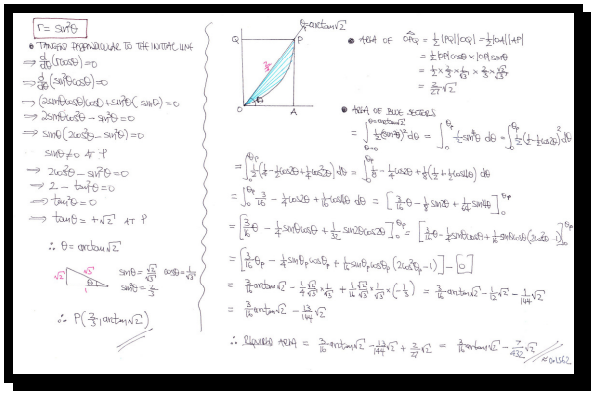
a) Find, in exact form, the polar coordinates of P

The point Q lies on the half line $\theta = \frac{\pi}{2}$, so that PQ is parallel to the initial line.

The finite region R, shown shaded in the above figure, is bounded by the curve and the straight line segments PQ and OQ, where O is the pole.

b) Determine the area of R, in exact simplified form.

$$\boxed{P\left(\frac{2}{3}, \arctan\sqrt{2}\right)}, \quad \boxed{\text{area} = \frac{1}{2}\arctan\sqrt{2} - \frac{7}{432}\sqrt{2} \approx 0.1562}$$



Question 3 (**+)**

A curve C has polar equation

$$r=\frac{2}{1+\cos\theta},\ 0\le\theta<2\pi\,.$$

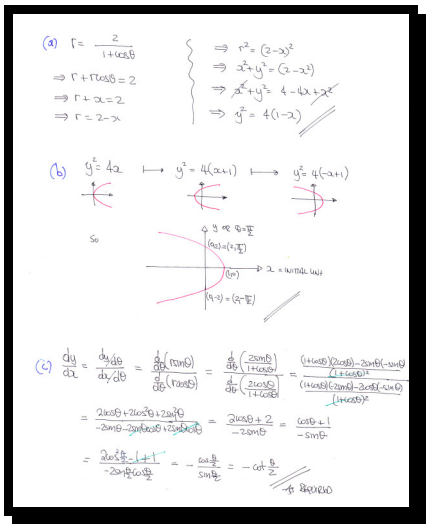
a) Find a Cartesian equation for C.

b) Sketch the graph of C.

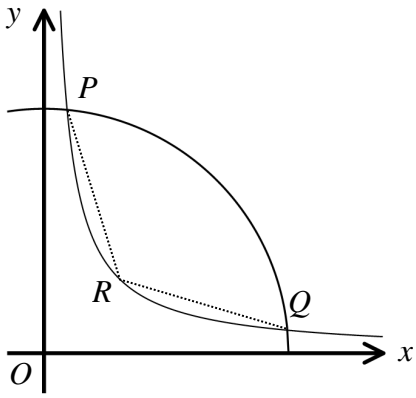
c) Show that on any point on C with coordinates (r,θ)

$$\frac{dy}{dx}=-\cot\frac{\theta}{2}.$$

$$\boxed{y^2=4(1-x)}$$



Question 4 **(****+)**



The figure above shows a hyperbola and a circle with respective Cartesian equations

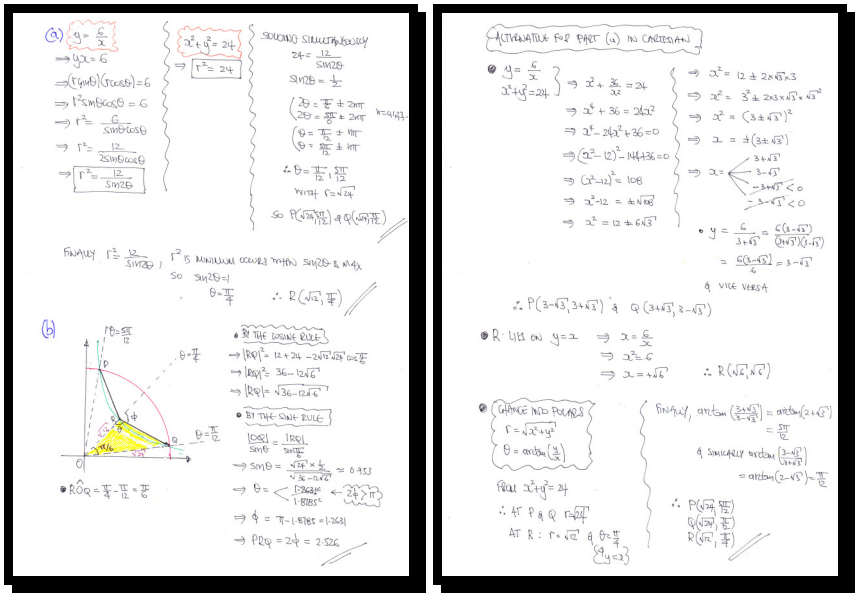
$$y=\frac{6}{x},\ x>0 \qquad \text{and} \qquad x^2+y^2=8,\ x>0,\ y>0.$$

The points P and Q are the points of intersection between the hyperbola and the circle, and the point R lies on the hyperbola so that the distance OR is least.

a) Determine the **polar** coordinates of P, Q and R.

b) Calculate in radians the angle PRQ, correct to 3 decimal places.

$$\boxed{P\left(\sqrt{24},\frac{5\pi}{12}\right)},\ \boxed{Q\left(\sqrt{24},\frac{\pi}{12}\right)},\ \boxed{R\left(\sqrt{12},\frac{\pi}{4}\right)},\ \boxed{\angle ABC\approx 2.526^{\text{c}}}$$



Question 5 **(****+)**

The curve C has Cartesian equation

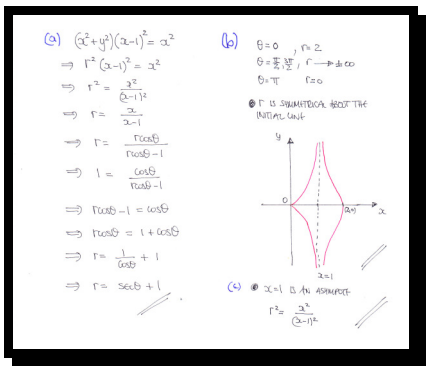
$$\left(x^2+y^2\right)(x-1)^2=x^2 .$$

a) Find a polar equation of C in the form $r=f\left(\theta\right)$.

b) Sketch the curve in the Cartesian plane.

c) State the equation of the asymptote of the curve.

$\boxed{r=1+\sec\theta}$, $\boxed{x=1}$



Question 6 (**+)**

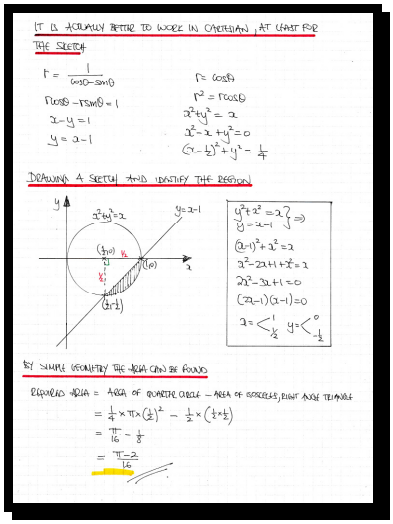
The following polar equations are given.

$$r_1 = \cos\theta, \quad 0 \le \theta \le \pi .$$

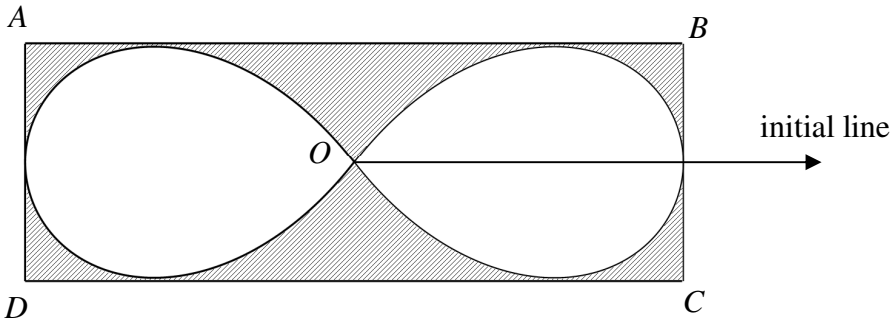
$$r_2 = \frac{1}{\cos\theta - \sin\theta}, \quad -\frac{1}{4}\pi \le \theta \le \frac{5}{4}\pi .$$

Find, in exact simplified form, the area of the **smaller** of the two finite regions, bounded by r_1 and r_2.

V, , $\boxed{\text{area} = \dfrac{3\pi + 2}{16}}$



Question 7 **(****+)**



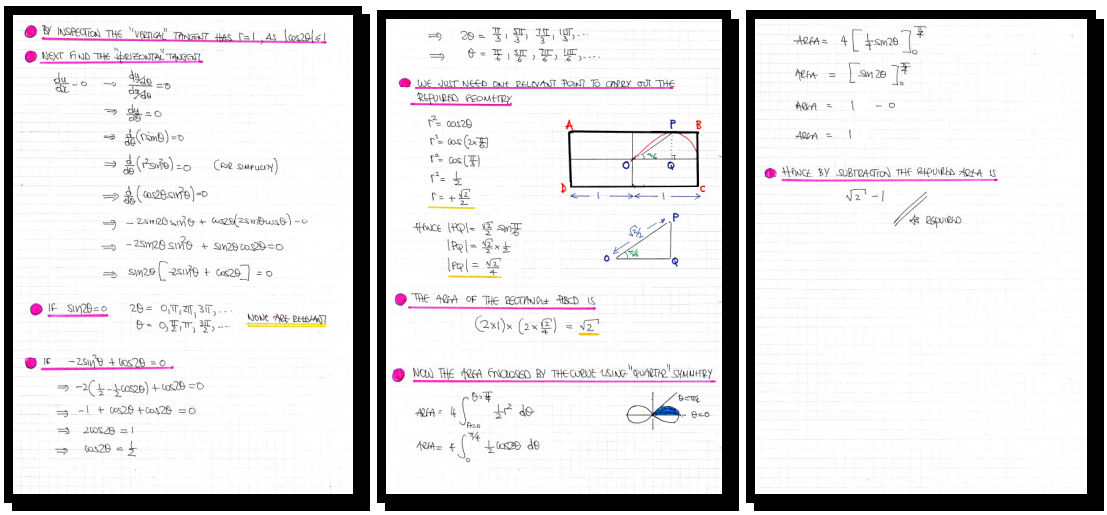
The figure above shows the rectangle $ABCD$ enclosing the curve with polar equation

$$r^2=\cos 2\theta\,, \qquad \theta\in\left[0,\frac{1}{4}\pi\right]\cup\left[\frac{3}{4}\pi,\frac{5}{4}\pi\right]\cup\left[\frac{7}{4}\pi,2\pi\right).$$

Each of the straight line segments AB and CD is a tangent to the curve parallel to the initial line, while each of the straight line segments AD and BC is a tangent to the curve perpendicular to the initial line.

Show with detailed calculations that the total area enclosed between the curve and the rectangle $ABCD$ is $\sqrt{2}-1$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 8 (**+)**

The curves C_1 and C_2 have polar equations

$$C_1: \ r = 2\cos\theta - \sin\theta, \quad 0 < \theta \leq \frac{\pi}{3}$$

$$C_2: \ r = \sqrt{2} + \sin\theta, \quad 0 \leq \theta < 2\pi .$$

The point P lies on C_1 so that the tangent at P is parallel to the initial line.

a) Show clearly that at P

$$\tan 2\theta = 2$$

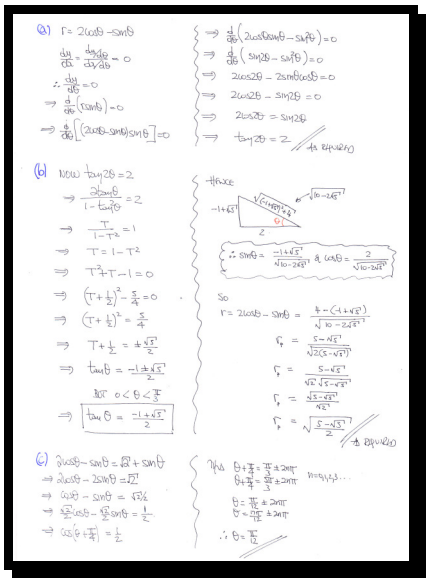
b) Hence show further that the exact distance of P from the origin O is

$$\sqrt{\frac{5-\sqrt{5}}{2}} .$$

The point Q is the point of intersection between C_1 and C_2.

c) Find the value of θ at Q.

$$\boxed{\theta = \frac{\pi}{12}}$$



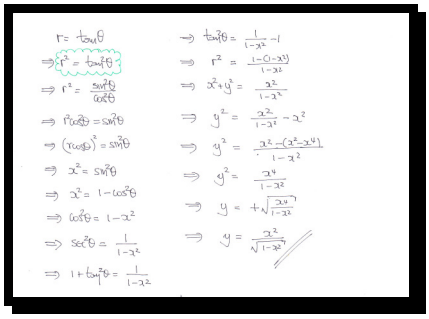
Question 9 (**+)**

The curve C has polar equation

$$r = \tan\theta\,,\ 0 \le \theta < \frac{\pi}{2}.$$

Find a Cartesian equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{\sqrt{1-x^2}}}$$



Question 10 (**+)**

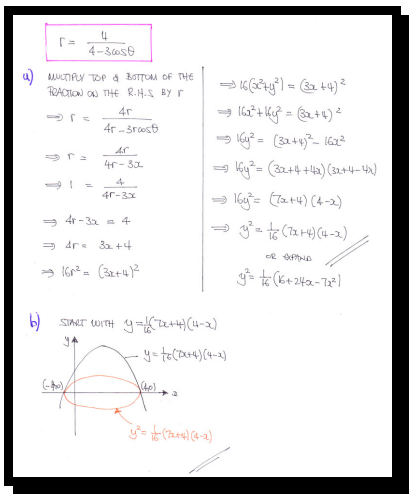
The curve C has polar equation

$$r=\frac{4}{4-3\cos\theta},\ 0\le\theta<2\pi\ .$$

a) Find a Cartesian equation of C in the form $y^2=f(x)$.

b) Sketch the graph of C.

$$\boxed{y^2=\frac{1}{16}\left(16+24x-7x^2\right)}$$



10 ENRICHMENT QUESTIONS

Question 1 **(*****)**

Two curves, C_1 and C_2, have polar equations

$$C_1: \ r = 12\cos\theta, \ -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$$

$$C_2: \ r = 4 + 4\cos\theta, \ -\pi < \theta \leq \pi.$$

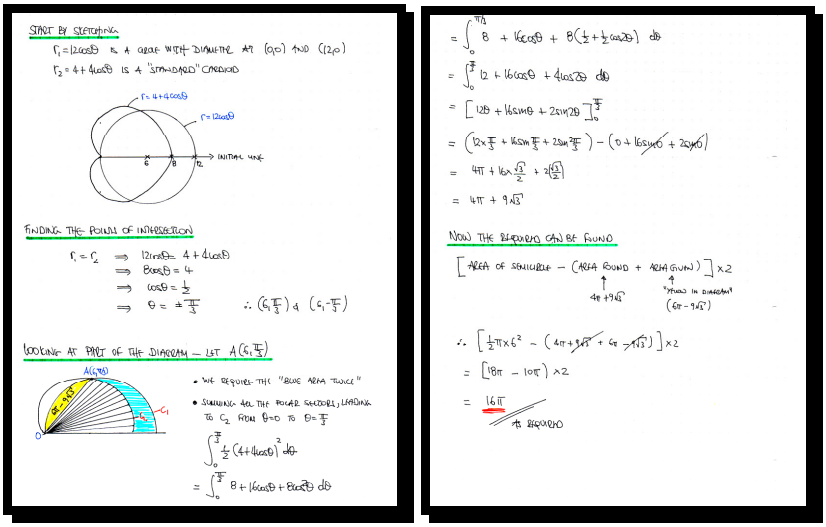
One of the points of intersection between the graphs of C_1 and C_2 is denoted by A. The area of the **smallest** of the two regions bounded by C_1 and the straight line segment OA is

$$6\pi - 9\sqrt{3}.$$

The finite region R represents points which lie inside C_1 but outside C_2.

Show that the area of R is 16π.

[ ], [proof]



Question 2 **(*****)**

A curve has polar equation

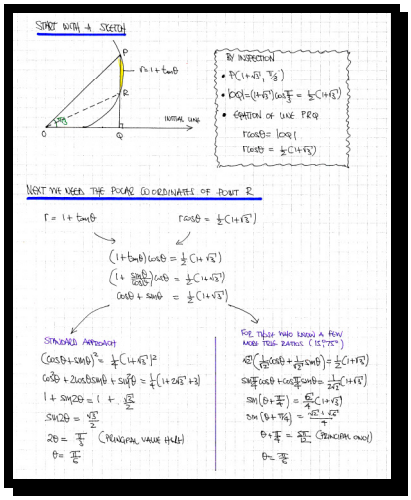
$$r = 1 + \tan\theta \ , \quad 0 \le \theta \le \frac{1}{2}\pi \ .$$

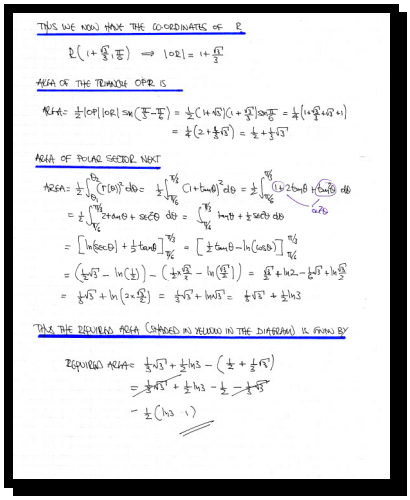
The point P lies on the curve where $\theta = \frac{1}{3}\pi$.

The point Q lies on the initial line so that the straight line L, which passes through P and Q meets the initial line at right angles.

Determine, in exact simplified form, the area of the finite region bounded by the curve and L.

$\square$, $\boxed{\frac{1}{2}[\ln 3 - 1]}$





Question 3 (*****)

A set of cartesian axes is superimposed over a set of polar axes, so that both set of axes have a common origin O, and the positive x axis coincides with the initial line.

A parabola P has Cartesian equation

$$y^2 = 8(2-x) \ , \quad x \leq 2 \ .$$

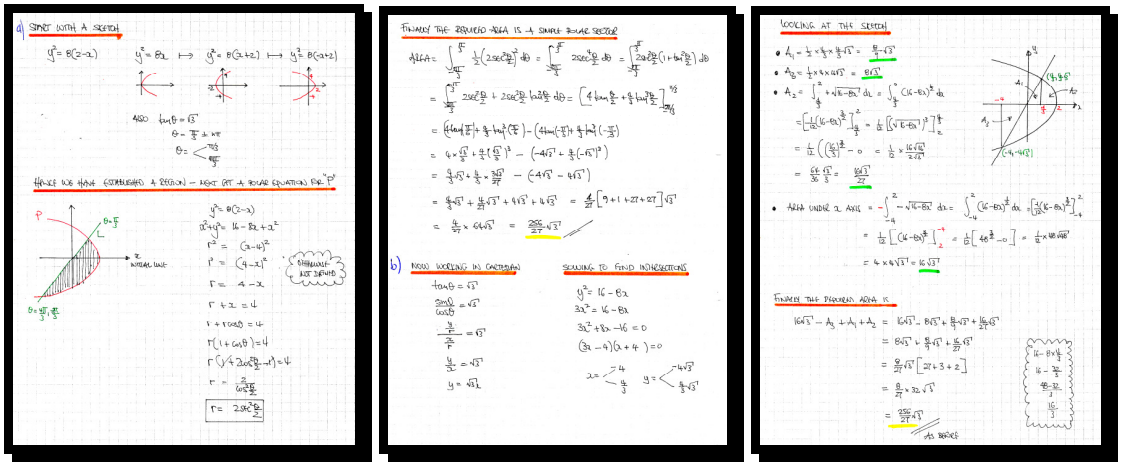
A straight line L has polar equation

$$\tan\theta = \sqrt{3} \ , \ -\pi < \theta \Leftrightarrow \pi \ .$$

a) Use polar coordinates to determine, in exact simplified form, the area of the finite region bounded by P and L.

b) Verify the answer of part **(a)** by using calculus in cartesian coordinates

$\boxed{}$, $\boxed{\dfrac{256}{27}\sqrt{3}}$



Question 4 (*****)

A curve has polar equation

$$r = 1 + \tan\theta \ , \quad 0 \le \theta \le \frac{1}{2}\pi \ ,$$

meets the initial line at the point P.

Another curve has polar equation

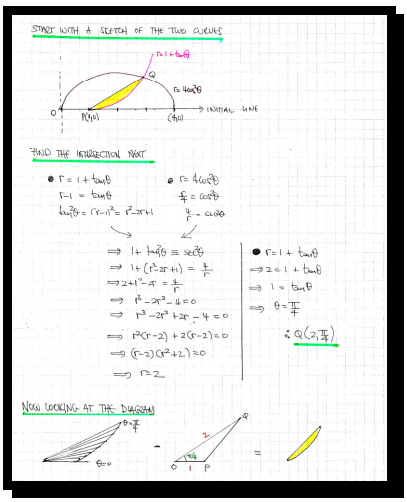
$$r = 4\cos^2\theta \ , \quad 0 \le \theta \le \frac{1}{2}\pi \ .$$

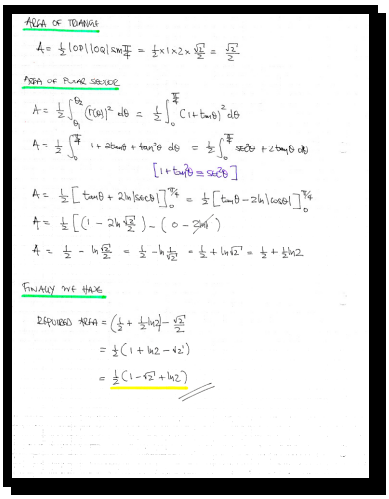
The two curves meet at the point Q.

Determine, in exact simplified form, the area of the finite region bounded by the straight line through P and Q, and the curve with equation $r = 1 + \tan\theta$.

Give the answer in the form $\frac{1}{k}\left[1 - \sqrt{k} + \ln k\right]$, where k is a positive integer.

$\square$, $\boxed{\frac{1}{2}\left[1 - \sqrt{2} + \ln 2\right]}$





Question 5 (*****)

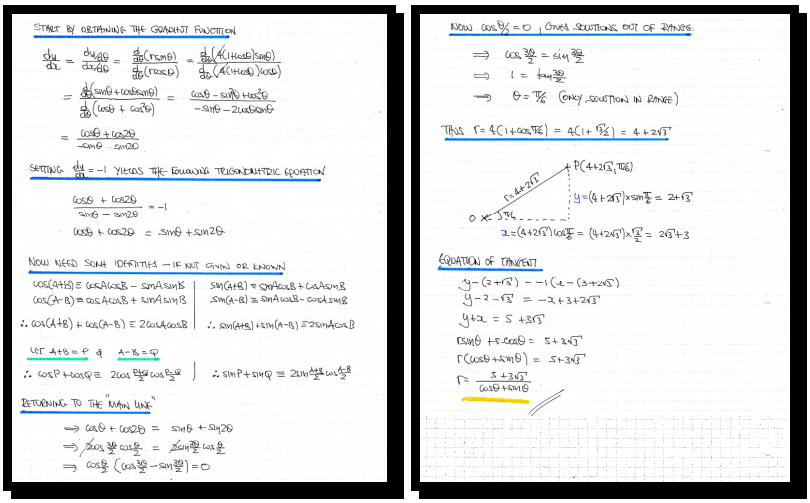
A cardioid has polar equation

$$r = 4(1+\cos\theta), \quad 0 \le \theta \le \frac{1}{2}\pi .$$

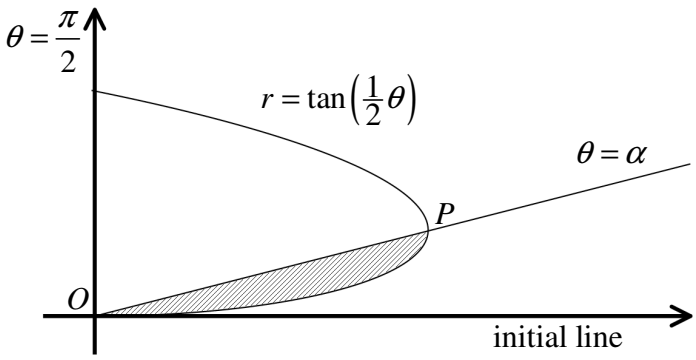
A tangent to the curve at some point P has gradient -1.

Find, in the form $r = f(\theta)$, the polar equation of this tangent.

V, $\square$, $\boxed{r = \dfrac{5+3\sqrt{3}}{\cos\theta+\sin\theta}}$



Question 6 **(*****)**



The figure above shows the curve C with polar equation

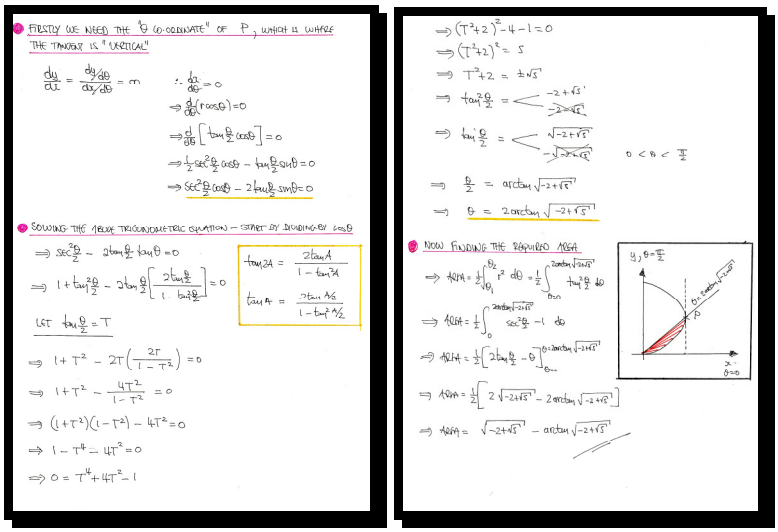
$$r = \tan\left(\frac{1}{2}\theta\right), \ 0 \le \theta < \frac{\pi}{2}.$$

The point P lies on C so that tangent to C is perpendicular to the initial line.

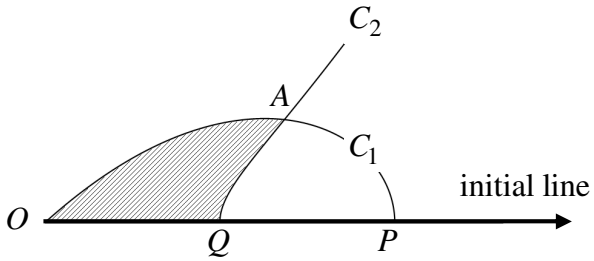
The half line with equation $\theta = \alpha$ passes through P.

Find, in exact simplified form, the area of the finite region bounded by C and the above mentioned half line.

$\square$, $\boxed{\text{area} = \sqrt{-2+\sqrt{5}} - \arctan\sqrt{-2+\sqrt{5}}}$



Question 7 (*****)



The figure above shows the curves C_1 and C_2 with respective polar equations

$$r_1 = \sec\theta\left(1-\tan^2\theta\right) \quad \text{and} \quad r_2 = \frac{1}{2}\sec^3\theta, \quad 0 \le \theta < \frac{1}{4}\pi .$$

The points P and Q are the respective points where C_1 and C_2 meet the initial line, and the point A is the intersection of C_1 and C_2 .

a) Find the exact area of the curvilinear triangle OAQ, where O is the pole.

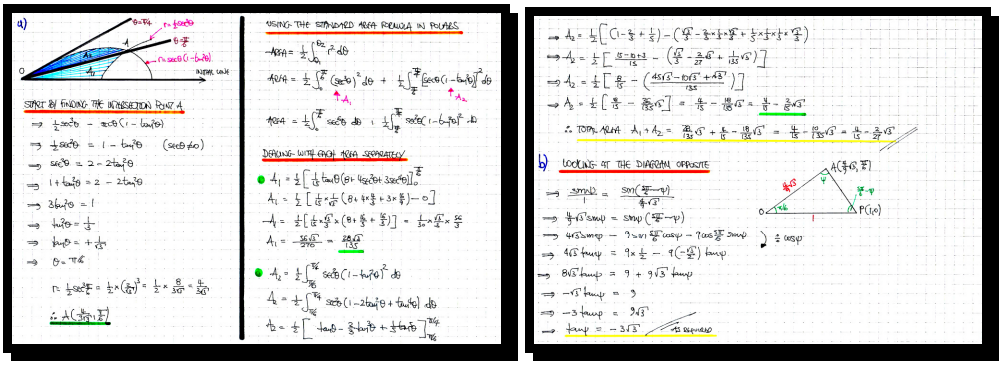
The angle OAP is denoted by ψ.

b) Show that $\tan\psi = -3\sqrt{3}$.

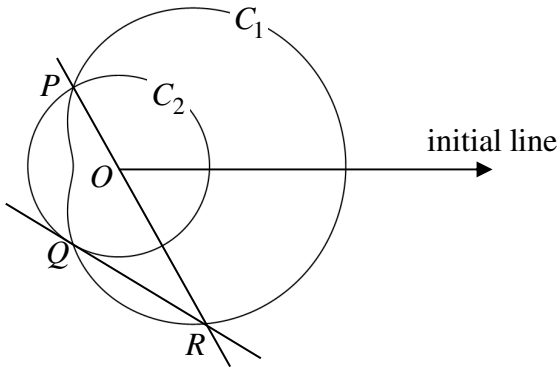
You may assume without proof

$$\int \sec^6 x \ dx = \frac{1}{15}\left(8+4\sec^2 x+3\sec^4 x\right)\tan x + C$$

$\square$, $\dfrac{2\left(18-5\sqrt{3}\right)}{135}$



Question 8 **(*****)**



The figure above shows the curves C_1 and C_2 with respective polar equations

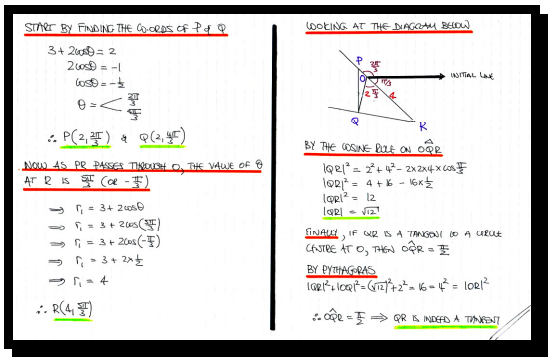
$$r_1 = 3 + 2\cos\theta\,,\ 0 \le \theta < 2\pi \qquad \text{and} \qquad r_2 = 2\,.$$

The two curves intersect at the points P and Q.

A straight line passing through P and the pole O intersects C_1 again at the point R.

Show that RQ is a tangent of C_1 at Q.

[ ], [proof]



Question 9 **(*****)**

The curves C_1 and C_2 have respective polar equations

$$r = 1 + \sin\theta\,,\ 0 < \theta < \frac{1}{2}\pi \qquad \text{and} \qquad r = 1 + \cos 2\theta\,,\ 0 < \theta < \frac{1}{2}\pi\,.$$

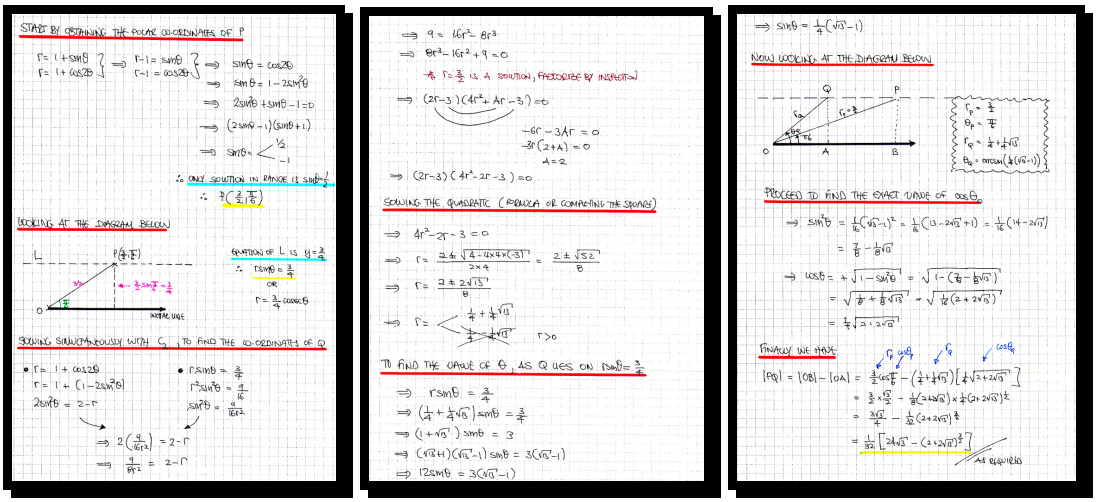
The point P is the point of intersection of C_1 and C_2.

A straight line, which is parallel to the initial line, passes through P and intersects C_2 at the point Q.

Show that

$$|PQ| = \frac{1}{32}\left[24\sqrt{3} - \left(2 + 2\sqrt{13}\right)^{\frac{3}{2}}\right].$$

, proof



Question 10 (*****)

A straight line L, whose gradient is $-\frac{3}{11}$, is a tangent to the curve with polar equation

$$r = 25\cos 2\theta\,,\ 0 \le \theta \le \frac{1}{2}\pi$$

Show that the area of the finite region bounded by the curve, the straight line L and the initial line is

$$\frac{25}{12}\left[46 - 75\arctan\frac{1}{3}\right].$$

[ ], [proof]

