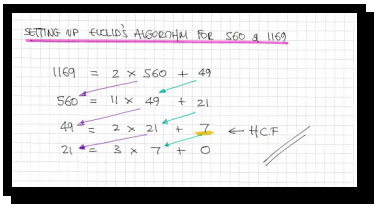
NUMBER THEORY

Question 1 (**)

Use Euclid's algorithm to find the Highest common factor of 560 and 1169.

$\boxed{}$, $\boxed{7}$

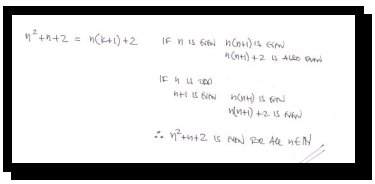


Question 2 (**)

$$f(n)=n^2+n+2,\ n\in\mathbb{N}.$$

Show that $f(n)$ is always even.

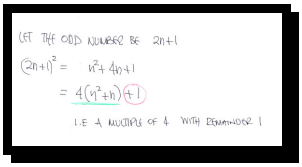
$\boxed{}$, $\boxed{\text{proof}}$



Question 3 (**)

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

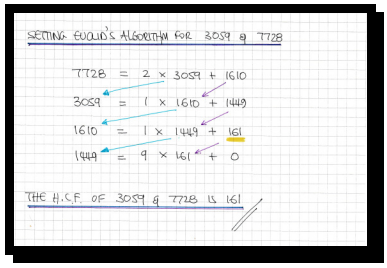
$\boxed{}$, $\boxed{\text{proof}}$



Question 4 (**)

Use Euclid's algorithm to find the Highest common factor of 3059 and 7728.

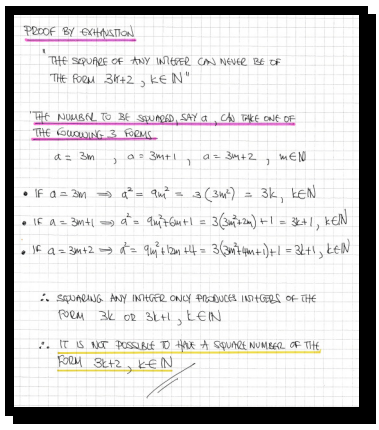
$\boxed{}$, $\boxed{161}$



Question 5 (**)

Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 6 (**)

Show that $a^3 - a + 1$ is odd for all positive integer values of a.

[ ], [proof]



Question 7 (+)**

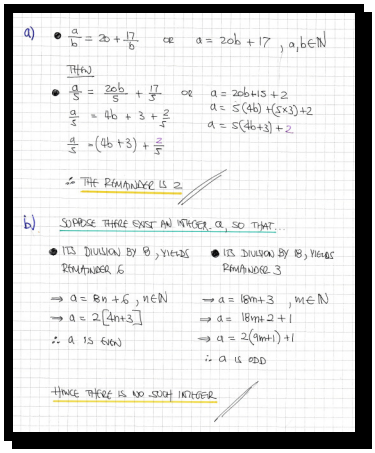
When a, $a \in \mathbb{N}$, is divided by b, $b \in \mathbb{N}$, the quotient is 20 and the remainder is 17.

a) Find the remainder when a is divided by 5.

Suppose that when a positive integer is divided by 8 the remainder is 6, and when the same positive integer is divided by 18 the remainder is 3.

b) Determine whether the positive integer of part **(b)** exists.

$\boxed{}$, $\boxed{2}$, $\boxed{\text{no such integer}}$



Question 8 **(**+)**

$$f(n) \equiv n^2 + 4n + 3 \, , \; n \in \mathbb{N} \, .$$

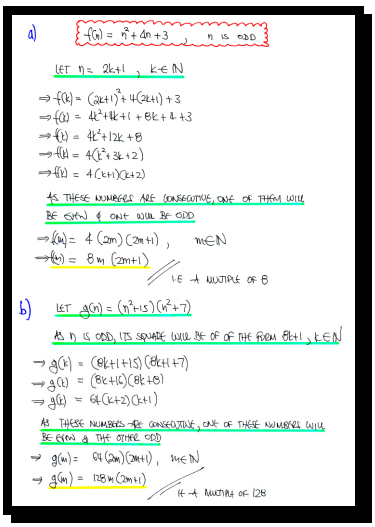
a) Given that n is odd show that $f(n)$ is a multiple of 8.

$$g(n) \equiv \left(n^2 + 15\right)\left(n^2 + 7\right) \, , \; n \in \mathbb{N} \, .$$

b) Given that n is odd show that $g(n)$ is a multiple of 128.

You may assume that the square of an odd integer is of the form $8k+1$, $k \in \mathbb{N}$.

proof

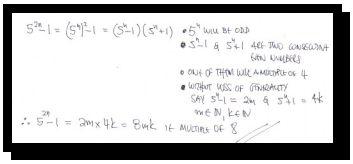


Question 9 **(***)**

$$f(n)=5^{2n}-1,\ n\in\mathbb{N}.$$

Without using proof by induction, show that $f(n)$ is a multiple of 8.

$\boxed{}$, proof



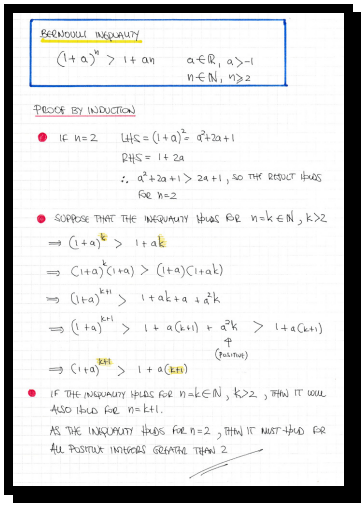
Question 10 **(***)**

Bernoulli's inequality asserts that if $a\in\mathbb{R}$, $a>-1$ and $n\in\mathbb{N}$, $n\geq 2$, then

$$(1+a)^n>1+an.$$

Prove, by induction, the validity of Bernoulli's identity.

$\boxed{}$, proof

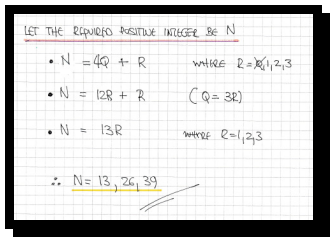


Question 11 (***)

When some positive integer N is divided by 4, the quotient is 3 times as large as the remainder.

Determine the possible values of N.

$\boxed{}$, $\boxed{N=\{13,\ 26,\ 39\}}$

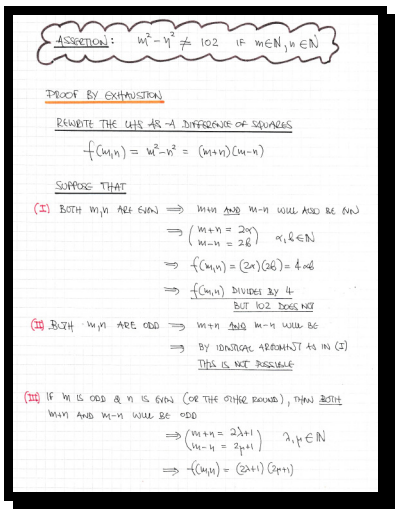


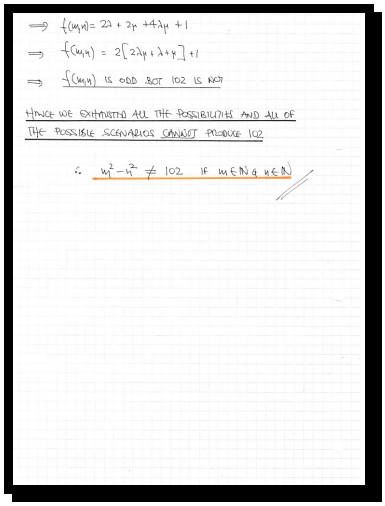
Question 12 (*+)**

Use **proof by exhaustion** to show that if $m\in\mathbb{N}$ and $n\in\mathbb{N}$, then

$$m^2-n^2\neq 102\,.$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 13 (+)**

It is given that for $a\in\mathbb{N}$, $b\in\mathbb{N}$, $c\in\mathbb{N}$,

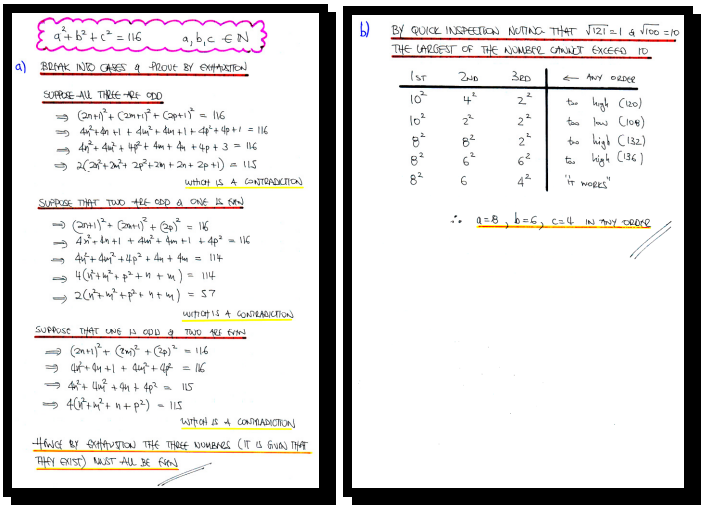
$$a^2+b^2+c^2=116\,.$$

a) Prove that a, b and c are all even.

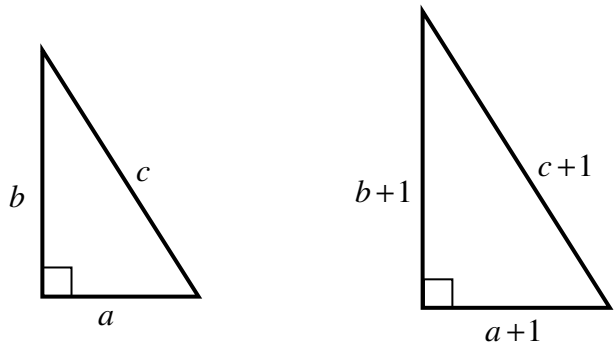
You may assume that the square of an odd integer is of the form $8k+1$, $k\in\mathbb{N}$.

b) Determine the values of a, b and c.

$$\boxed{a=8,\quad b=6,\quad c=4\quad \text{in any order}}$$



Question 14 (*+)**



The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of

  $$a\,,\ b \text{ and } c\,,$$

  where c is the length of its hypotenuse.

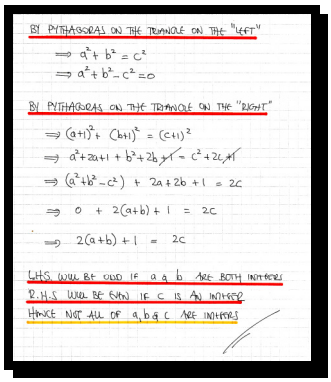
- The triangle, on the right section of the figure, has side lengths of

  $$a+1\,,\quad b+1 \quad \text{and} \quad c+1\,,$$

  where $c+1$ is the length of its hypotenuse.

Show that a, b and c cannot all be integers.

[ ], [proof]


BY PYTHAGORAS ON THE TRIANGLE ON THE "LEFT"

$\Rightarrow a^2+b^2=c^2$

$\Rightarrow a^2+b^2-c^2=0$

BY PYTHAGORAS ON THE TRIANGLE ON THE "RIGHT"

$\Rightarrow (a+1)^2+(b+1)^2=(c+1)^2$

$\Rightarrow a^2+2a+1+b^2+2b+1=c^2+2c+1$

$\Rightarrow (a^2+b^2-c^2)+2a+2b+1=2c$

$\Rightarrow 0+2(a+b)+1=2c$

$\Rightarrow 2(a+b)+1=2c$

LHS WILL BE ODD IF a & b ARE BOTH INTEGERS

R.H.S WILL BE EVEN IF c IS AN INTEGER

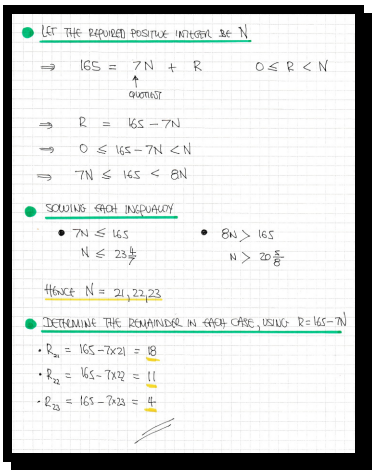
HENCE NOT ALL OF a, b & c ARE INTEGERS

Question 15 (*+)**

When 165 is divided by some integer the quotient is 7 and the remainder is R.

Determine the possible values of R.

$\boxed{}$, $\boxed{R=\{4,11,18\}}$

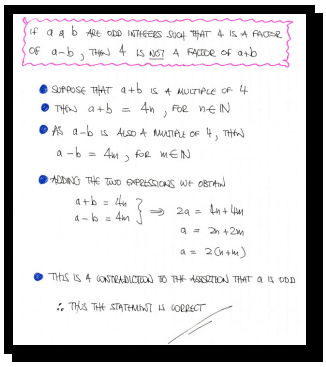


Question 16 (*+)**

It is given that a and b are positive integers, with $a > b$.

Use **proof by contradiction** to show that if $a+b$ is a multiple of 4, then $a-b$ **cannot** be a multiple of 4.

, proof



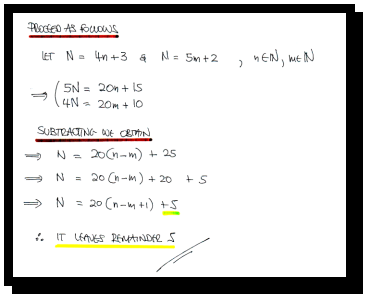
Question 17 (*+)**

When a positive integer N is divided by 4 the remainder is 3.

When N is divided by 5 the remainder is 2.

Show that the remainder of the division of N by 20 is 5.

proof

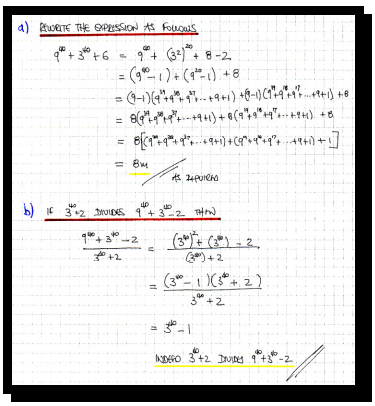


Question 18 (*+)**

a) Show that $9^{40}+3^{40}+6$ is a multiple of 8.

b) Show further that $3^{40}+2$ divides $9^{40}+3^{40}-2$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 19 (*+)**

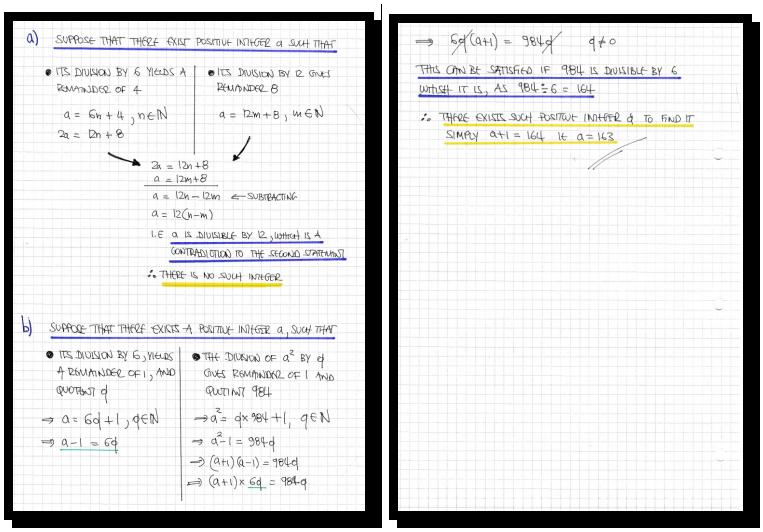
Suppose that when a positive integer is divided by 6 the remainder is 4, and when the same positive integer is divided by 12 the remainder is 8.

a) Determine whether such positive exists.

Suppose next that when a positive integer is divided by 6, the quotient is q and the remainder is 1. When the square of the same positive integer is divided by q, the quotient is 984 and the remainder is 1.

b) Determine whether the positive integer of part **(b)** exists.

[ ], [no such integer], [163]



Question 20 (*+)**

In the following question A, B and C are positive odd integers.

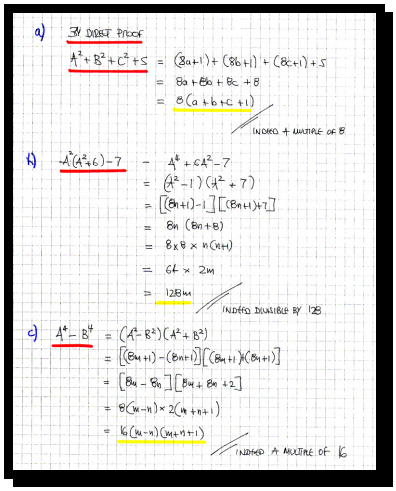
Show, using a clear method, that …

a) … $A^2+B^2+C^2+5$ is a multiple of 8.

b) … $A^2\left(A^2+6\right)-7$ is divisible by 128.

c) … A^4-B^4 is a multiple of 16.

[ ], [proof]



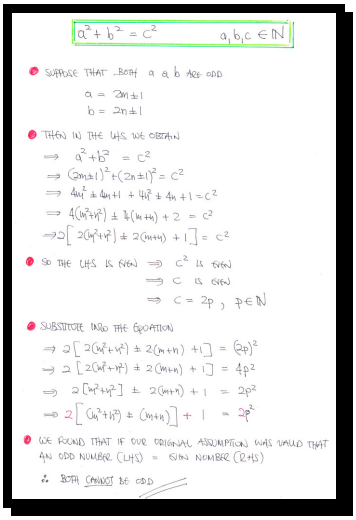
Question 21 (****)

It is given that

$$a^2+b^2=c^2,\ \ a\in\mathbb{N}\ ,\ b\in\mathbb{N}\,.$$

Show that a and b cannot both be odd.

, proof



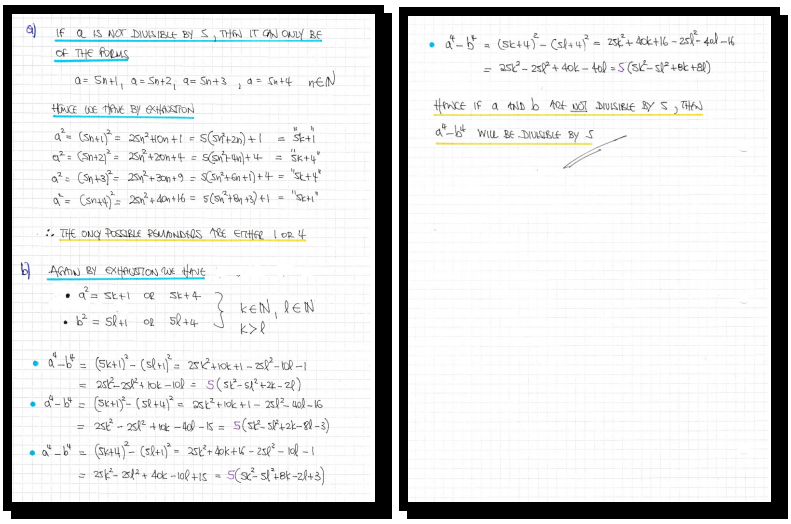
Question 22 (****)

Let $a \in \mathbb{N}$ with $\frac{1}{5}a \notin \mathbb{N}$.

a) Show that the remainder of the division of a^2 by 5 is either 1 or 4.

b) Given further that $b \in \mathbb{N}$ with $\frac{1}{5}b \notin \mathbb{N}$, deduce that $\frac{1}{5}\left(a^4 - b^4\right) \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 23 (****)

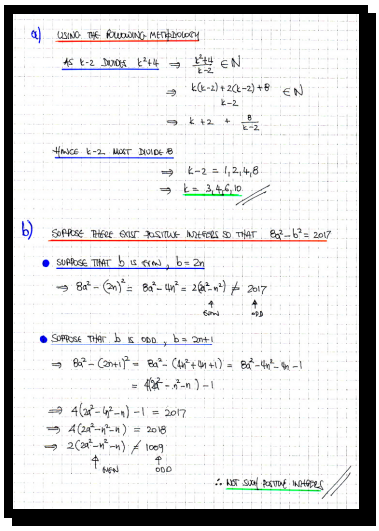
It is given that k is a positive integer.

a) If $k-2$ divides k^2+4, determine the possible values of k.

It is further given that a and b are positive integers.

b) Show that $8a^2-b^2$ **cannot** equal 2017.

$\boxed{}$, $\boxed{k=3,\ 4,\ 6,\ 10}$

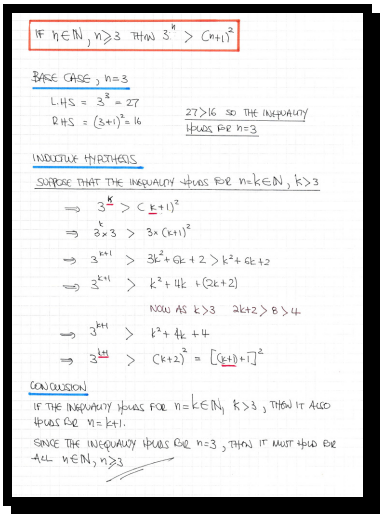


Question 24 **(****)**

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2 .$$

[ ], proof



Question 25 (****)

i. The function f is defined as

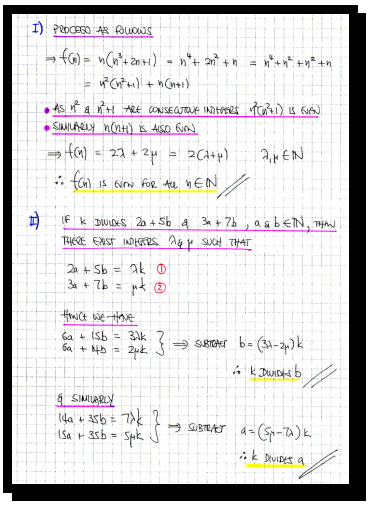
$$f(n)\equiv n\left(n^3+2n+1\right),\ n\in\mathbb{N}.$$

Show that f is even for all $n\in\mathbb{N}$.

ii. The positive integer k divides both $2a+5b$ and $3a+7b$, where $a\in\mathbb{N}$, $b\in\mathbb{N}$.

Show that k must then divide both a and b.

[ ], [proof]



Question 26 (****)

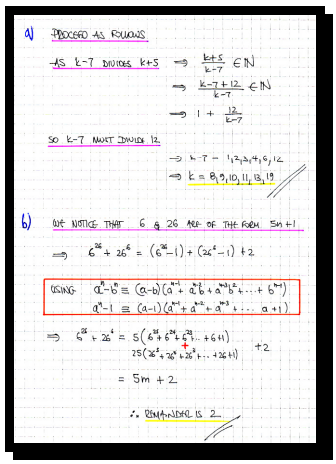
It is given that k is a positive integer.

a) If $k-7$ divides $k+5$, determine the possible values of k.

The second part of this question is unrelated to the first part.

b) By showing a detailed method, find the remainder of the division of $6^{26}+26^6$ by 5.

$\boxed{}$, $\boxed{k=8,\ 9,\ 10,\ 11,\ 13,\ 19}$, $\boxed{2}$



Question 27 (****)

i. The function f is defined as

$$f(n)\equiv\left(n^2+n\right)(n+5),\ n\in\mathbb{N}.$$

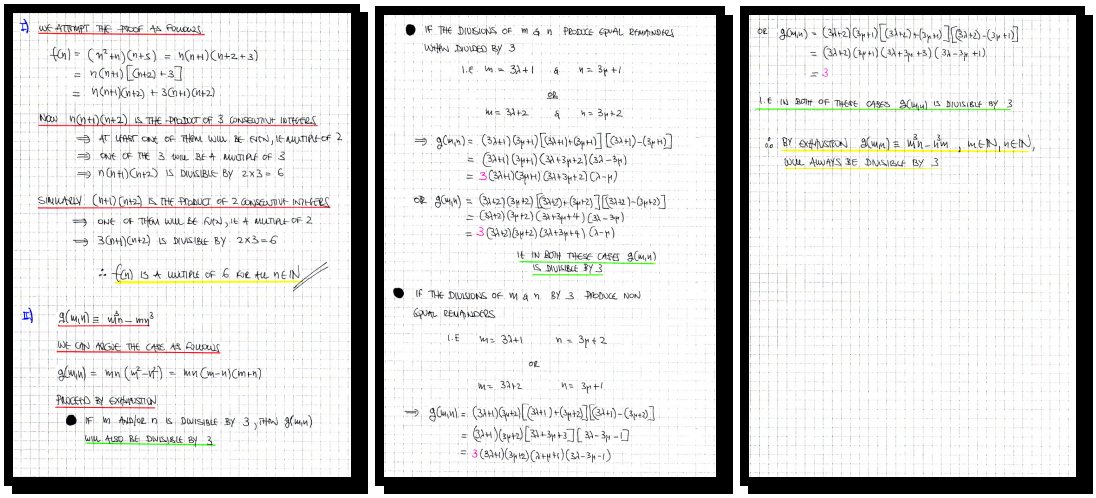
Show that f is multiple of 6 for all $n\in\mathbb{N}$.

ii. The function g is defined as

$$g(m,n)\equiv m^3n-mn^3,\ m\in\mathbb{N},\ n\in\mathbb{N}.$$

Show that g is divisible by 3 for all $m\in\mathbb{N}$, $n\in\mathbb{N}$.

[ ], [proof]



Question 28 **(*****)**

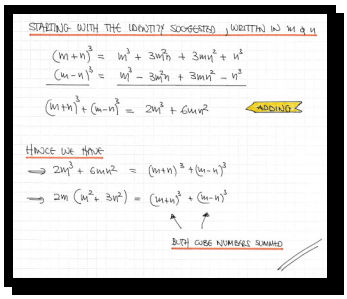
It is given that

$$f(m,n) \equiv 2m\left(m^2+3n^2\right),$$

where m and n are distinct positive integers, with $m>n$.

By using the expansion of $(A \pm B)^3$, prove that $f(m,n)$ can always be written as the sum of two cubes.

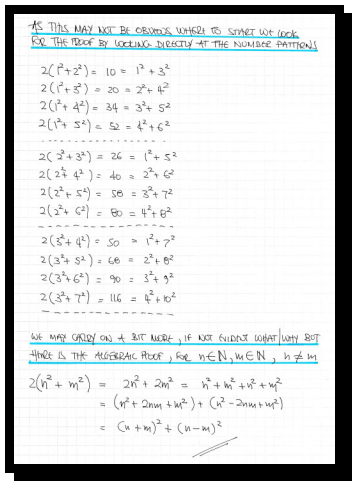
[ ], [proof]



Question 29 (*****)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers

[ ], [proof]



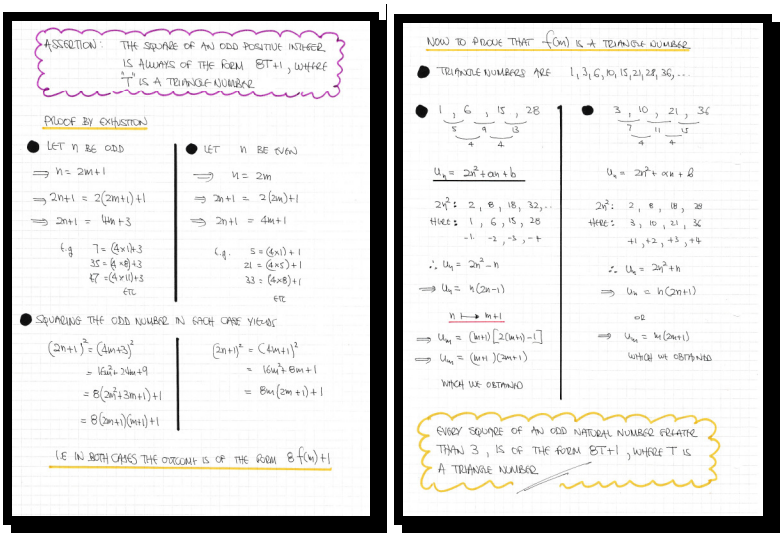
Question 30 (*****)

Show that the square of an odd positive integer greater than 1 is of the form

$$8T+1,$$

where T is a triangular number.

[ ], [proof]



Question 31 (*****)

The product operator $\prod$, is defined as

$$\prod_{r=1}^{k}[u_r] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k .$$

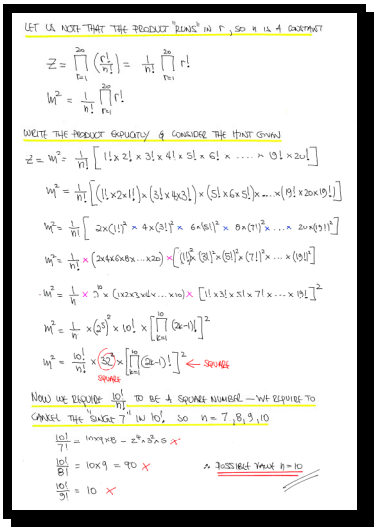
The integer Z is a **square number** and defined as

$$Z = \prod_{r=1}^{20}\left(\frac{r!}{n!}\right), \{n \in \mathbb{N} : 1 \leq n \leq 20\}.$$

By considering the terms inside the product operator in pairs, or otherwise, determine a possible value of n.

You must show a detailed method in this question.

$\boxed{}$, $\boxed{n = 10}$



Question 32 (*****)

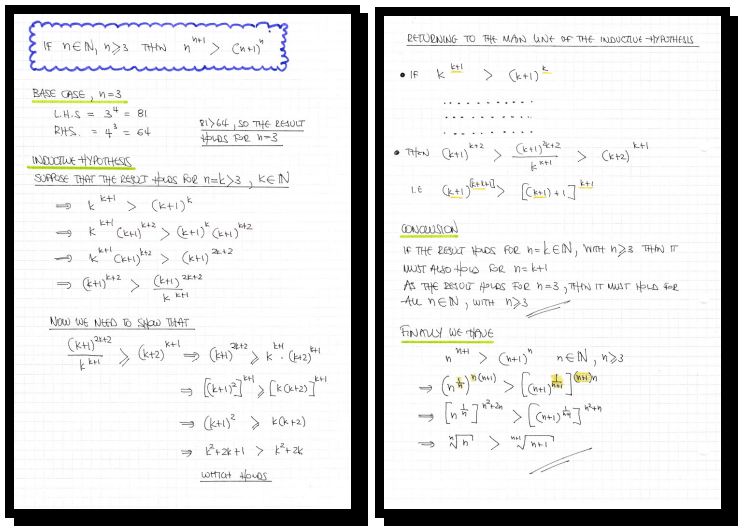
Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n ,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

, proof



Question 33 (*****)

It is given that $11a+13b$ is a multiple of $13-a$, where $a\in\mathbb{N}$, $b\in\mathbb{N}$.

It is then asserted that $(13+a)(11+b)$ is also a multiple of $13-a$.

Prove the validity of this assertion.

[ ], [proof]

