MATRICES PRACTICE

MATRIX INTRODUCTION

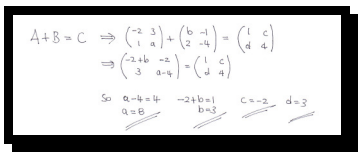
Question 1

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b, c and d, by

$$\mathbf{A}=\begin{pmatrix}-2 & 3\\ 1 & a\end{pmatrix},\qquad \mathbf{B}=\begin{pmatrix}b & -1\\ 2 & -4\end{pmatrix},\qquad \mathbf{C}=\begin{pmatrix}1 & c\\ d & 4\end{pmatrix}.$$

Given that $\mathbf{A}+\mathbf{B}=\mathbf{C}$, find the value of a, b, c and d.

$$\boxed{a=8,\ b=3,\ c=2,\ d=3}$$



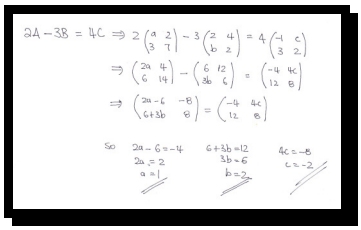
Question 2

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b and c, by

$$\mathbf{A}=\begin{pmatrix}a & 2\\ 3 & 7\end{pmatrix},\qquad \mathbf{B}=\begin{pmatrix}2 & 4\\ b & 2\end{pmatrix},\qquad \mathbf{C}=\begin{pmatrix}-1 & c\\ 3 & 2\end{pmatrix}.$$

Given that $2\mathbf{A}-3\mathbf{B}=4\mathbf{C}$, find the value of a, b and c.

$$\boxed{a=1,\ b=-2,\ c=-2}$$



Question 3

Multiply each of the following matrices.

a) $\begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

b) $\begin{pmatrix} -4 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -2 & 7 \end{pmatrix}$

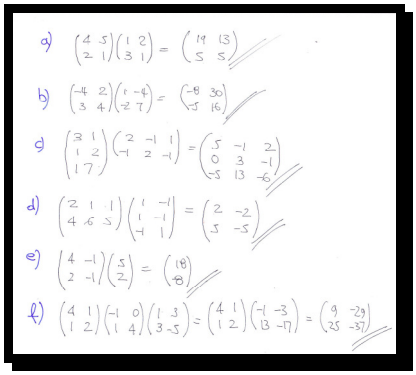
c) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 4 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

f) $\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & -5 \end{pmatrix}$

$\boxed{\begin{pmatrix} 19 & 13 \\ 5 & 5 \end{pmatrix}}$, $\boxed{\begin{pmatrix} -8 & 30 \\ -5 & 16 \end{pmatrix}}$, $\boxed{\begin{pmatrix} 5 & -1 & 2 \\ 0 & 3 & -1 \\ -5 & 13 & -6 \end{pmatrix}}$, $\boxed{\begin{pmatrix} 2 & -2 \\ 5 & -5 \end{pmatrix}}$, $\boxed{\begin{pmatrix} 18 \\ 8 \end{pmatrix}}$, $\boxed{\begin{pmatrix} 9 & -29 \\ 25 & -37 \end{pmatrix}}$



Question 4

Multiply each of the following matrices.

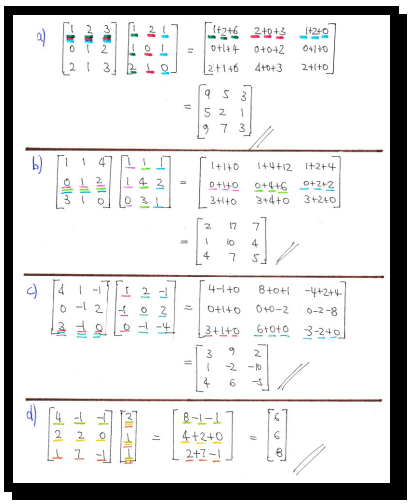
a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 4 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 2 \\ 0 & -1 & -4 \end{pmatrix}$

d) $\begin{pmatrix} 4 & -1 & -1 \\ 2 & 2 & 0 \\ 1 & 7 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\begin{bmatrix} \begin{pmatrix} 9 & 5 & 3 \\ 5 & 2 & 1 \\ 9 & 7 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 17 & 7 \\ 1 & 10 & 4 \\ 4 & 7 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 9 & 2 \\ 1 & -2 & -10 \\ 4 & 6 & -5 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 8 \end{pmatrix} \end{bmatrix}$



Question 5

A matrix $\mathbf{T}$ represents the linear transformation

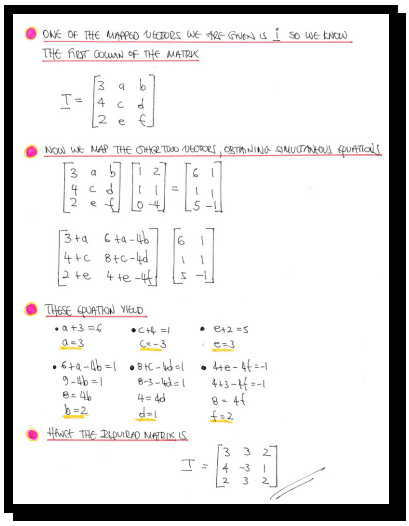
$$\mathbf{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} :\mapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

so that

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} :\mapsto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find the elements of $\mathbf{T}$.

$$\mathbf{T} = \begin{pmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$



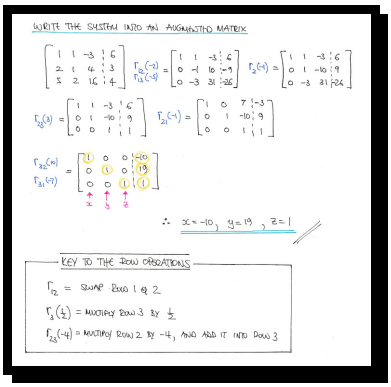
MATRIX ROW REDUCTION

Question 1

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

V, $\boxed{x=-10, \quad y=19, \quad z=1}$

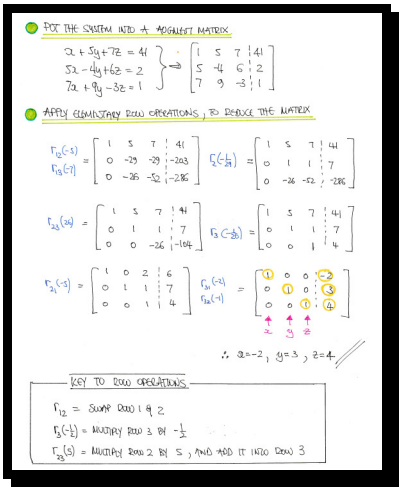


Question 2

$$\begin{aligned} x+5y+7z &= 41 \\ 5x-4y+6z &= 2 \\ 7x+9y-3z &= 1 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=-2,\quad y=3,\quad z=4}$$

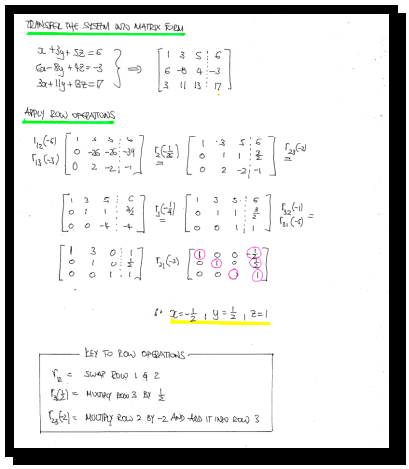


Question 3

$$\begin{aligned} x\ +3y\ +5z &= \ \ 6 \\ 6x\ -8y\ +4z &= -3 \\ 3x+11y+13z &= \ 17 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

V, [ ], $\boxed{x=-\frac{1}{2},\ y=\frac{1}{2},\ z=1}$

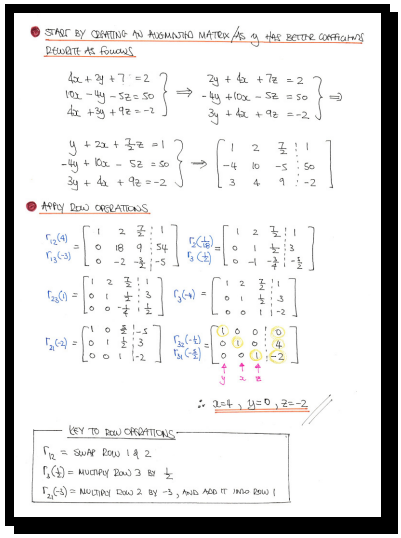


Question 4

$$\begin{aligned} 4x &+2y+7z= 2 \\ 10x &-4y-5z= 50 \\ 4x &+3y+9z=-2 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=4,\ y=0,\ z=-2}$$

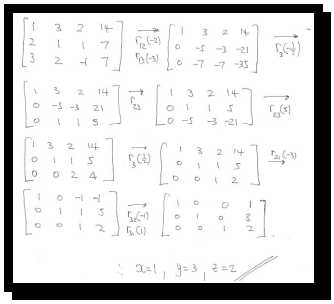


Question 5

$$\begin{aligned} x+3y+2z&=14\\ 2x+y+z&=7\\ 3x+2y-z&=7 \end{aligned}$$

Solve the system simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=1,\ y=3,\ z=2}$$

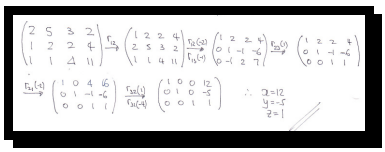


Question 6

$$\begin{aligned} 2x+5y+3z&=2\\ x+2y+2z&=4\\ x+y+4z&=11 \end{aligned}$$

Solve the above simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=12,\ y=-5,\ z=1}$$

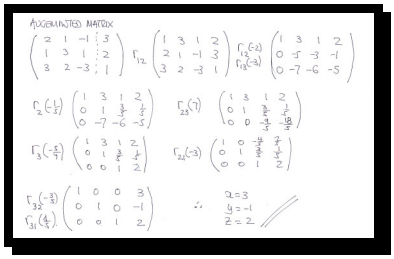


Question 7

$$\begin{aligned} 2x + y - z &= 3 \\ x + 3y + z &= 2 \\ 3x + 2y - 3z &= 1 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x = 3,\ y = -1,\ z = 2}$$

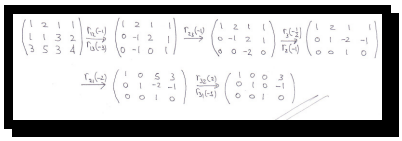


Question 8

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\boxed{x = 3,\ y = -1,\ z = 0}$$

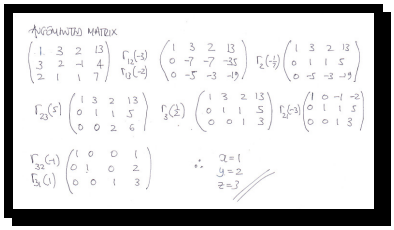


Question 9

$$\begin{aligned} x+3y+2z&=13\\ 3x+2y-z&=4\\ 2x+y+z&=7 \end{aligned}$$

Solve the system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\boxed{x=1,\ y=2,\ z=3}$$

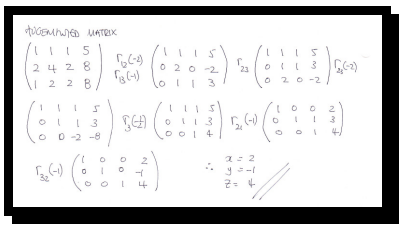


Question 10

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & 1\\ 2 & 4 & 2\\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 5\\ 8\\ 8 \end{pmatrix}$$

$$\boxed{x=2,\ y=-1,\ z=4}$$



Question 11

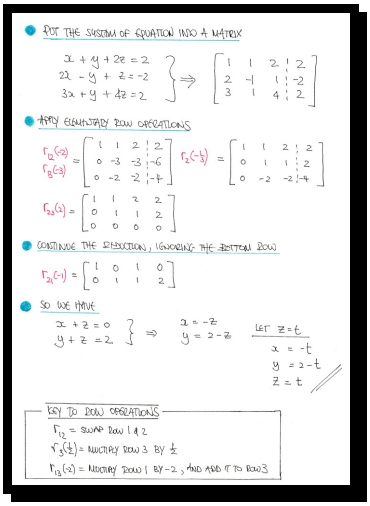
$$\begin{aligned} x+y+2z &= 2 \\ 2x-y+z &= -2 \\ 3x+y+4z &= 2 \end{aligned}$$

Show, by reducing the augmented matrix of the above system of equations into row echelon form, that the solution can be written as

$$x=-t,\ y=2-t,\ z=t$$

where t is a scalar parameter.

proof



Question 12

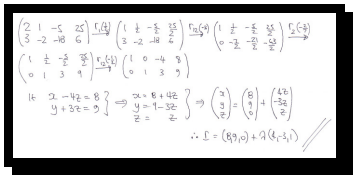
$$\begin{aligned} 3x-2y-18z&=6\\ 2x+y-5z&=25 \end{aligned}$$

Show, by reducing the above system of equations into row echelon form, that the solution can be written as

$$\mathbf{r}=8\mathbf{i}+9\mathbf{j}+\lambda\left(4\mathbf{i}-3\mathbf{j}+\mathbf{k}\right),$$

where λ is a scalar parameter.

proof



Question 13

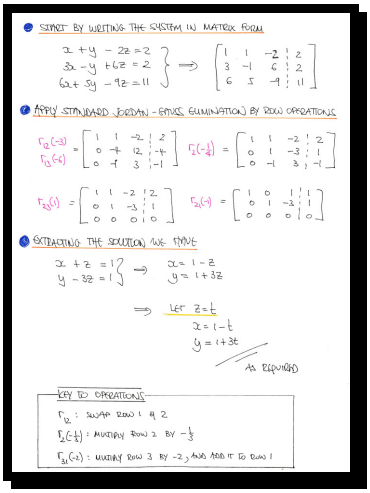
$$\begin{aligned} x+y-2z&=2\\ 3x-y+6z&=2\\ 6x+5y-9z&=11 \end{aligned}$$

Show, by reducing the above equation system into row echelon form, that the consistent solution of the system can be written as

$$x=1-t\,,\qquad y=3t+1\,,\qquad z=t$$

where t is a scalar parameter.

proof



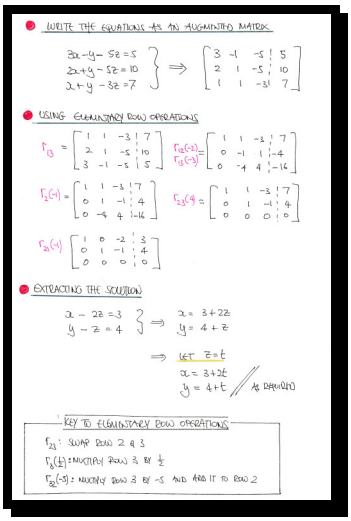
Question 14

$$\begin{aligned} 3x - y - 5z &= 5 \\ 2x + y - 5z &= 10 \\ x + y - 3z &= 7 \end{aligned}$$

Show, by reducing the above system into row echelon form, that the consistent solution of the system can be written as

$$x = 2t + 3, \quad y = t + 4, \quad z = t.$$

V, proof



DETERMINANTS

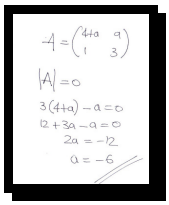
Question 1

The 2×2 matrix $\mathbf{A}$ is defined, in terms of a scalar constant a, by

$$\mathbf{A}=\begin{pmatrix} 4+a & a \\ 1 & 3 \end{pmatrix}.$$

Given that $\mathbf{A}$ is singular, find the value of a.

$$\boxed{a=-6}$$



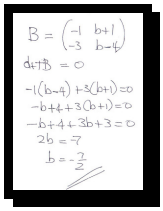
Question 2

The 2×2 matrix $\mathbf{B}$ is defined, in terms of a scalar constant b, by

$$\mathbf{B}=\begin{pmatrix} -1 & b+1 \\ -3 & b-4 \end{pmatrix}.$$

Given that $\mathbf{B}$ is singular, determine the value of b.

$$\boxed{b=-\frac{7}{2}}$$



Question 3

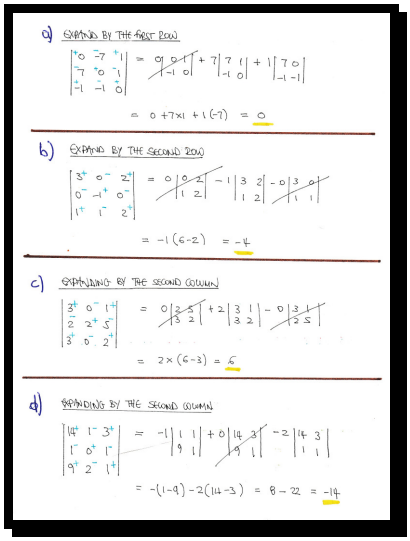
Evaluate each of the following determinants to the answer given.

a) $\begin{vmatrix} 0 & -7 & 1 \\ 7 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix} = 0$

b) $\begin{vmatrix} 3 & 0 & 2 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -4$

c) $\begin{vmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ 3 & 0 & 2 \end{vmatrix} = 6$

d) $\begin{vmatrix} 14 & 1 & 3 \\ 1 & 0 & 1 \\ 9 & 2 & 1 \end{vmatrix} = -14$



Question 4

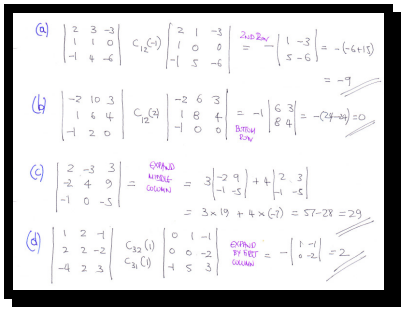
Evaluate each of the following determinants to the answer given.

a) $\begin{vmatrix} 2 & 3 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{vmatrix} = -9$

b) $\begin{vmatrix} -2 & 10 & 3 \\ 1 & 6 & 4 \\ -1 & 2 & 0 \end{vmatrix} = 0$

c) $\begin{vmatrix} 2 & -3 & 3 \\ -2 & 4 & 9 \\ -1 & 0 & -5 \end{vmatrix} = 29$

d) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 2 & -2 \\ -4 & 2 & 3 \end{vmatrix} = 2$



Question 5

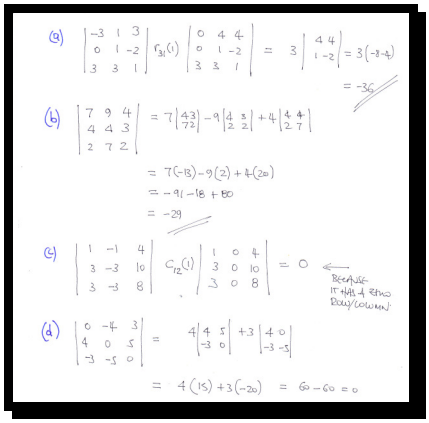
Evaluate the each of the following determinants to the answer given.

a) $\begin{vmatrix} -3 & 1 & 3 \\ 0 & 1 & -2 \\ 3 & 3 & 1 \end{vmatrix} = -36$

b) $\begin{vmatrix} 7 & 9 & 4 \\ 4 & 4 & 3 \\ 2 & 7 & 2 \end{vmatrix} = -29$

c) $\begin{vmatrix} 1 & -1 & 4 \\ 3 & -3 & 10 \\ 3 & -3 & 8 \end{vmatrix} = 0$

d) $\begin{vmatrix} 0 & -4 & 3 \\ 4 & 0 & 5 \\ -3 & -5 & 0 \end{vmatrix} = 0$



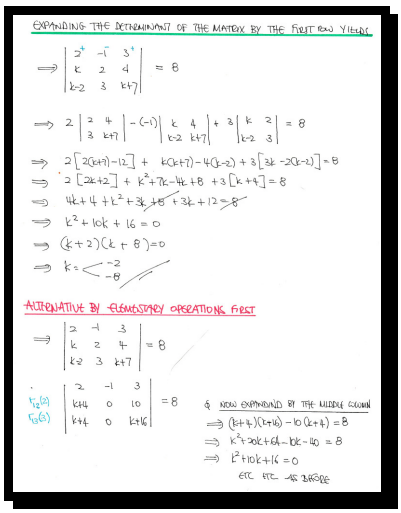
Question 6

The 3×3 matrix $\mathbf{A}$ is defined in terms of the scalar constant k by

$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{pmatrix}.$$

Given that $|\mathbf{A}|=8$, find the possible values of k.

$$\boxed{k=-2,\quad k=-8}$$



Question 7

A transformation in three dimensional space is defined by the following 3×3 matrix.

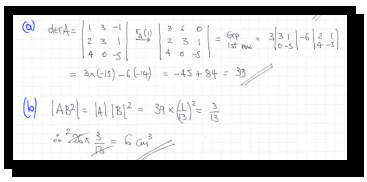
$$\mathbf{A}=\begin{pmatrix}1 & 3 & -1\\ 2 & 3 & 1\\ 4 & 0 & -5\end{pmatrix}.$$

a) Find the value of $\det\mathbf{A}$.

A cone with a volume of 26 cm^3 is transformed by the matrix composition $\mathbf{AB}^2$.

b) Given that $\det\mathbf{B}=\dfrac{1}{13}$, calculate the volume of the transformed cone.

$\boxed{\det\mathbf{A}=39}$, $\boxed{\text{volume}=6}$



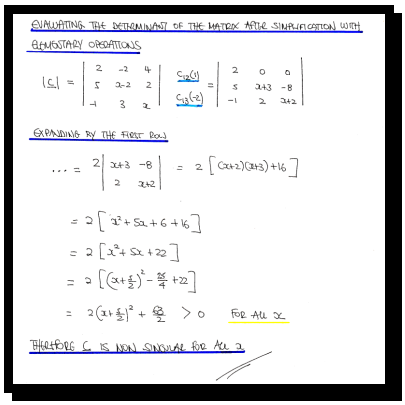
Question 8

A transformation in three dimensional space is defined by the following 3×3 matrix, where x is a scalar constant.

$$\mathbf{C}=\begin{pmatrix}2 & -2 & 4\\ 5 & x-2 & 2\\ -1 & 3 & x\end{pmatrix}.$$

Show that $\mathbf{C}$ is non singular for all values of x.

proof



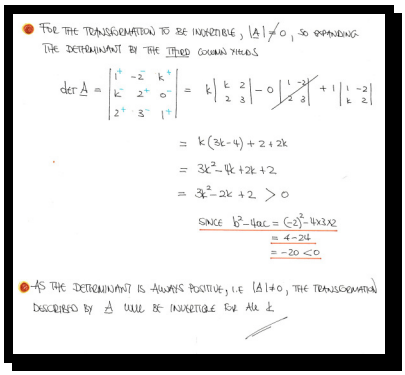
Question 9

A transformation in three dimensional space is defined by the following 3×3 matrix, where k is a scalar constant.

$$\mathbf{A}=\begin{pmatrix}1 & -2 & k\\ k & 2 & 0\\ 2 & 3 & 1\end{pmatrix}.$$

Show that the transformation defined by $\mathbf{A}$ can be inverted for all values of k.

proof



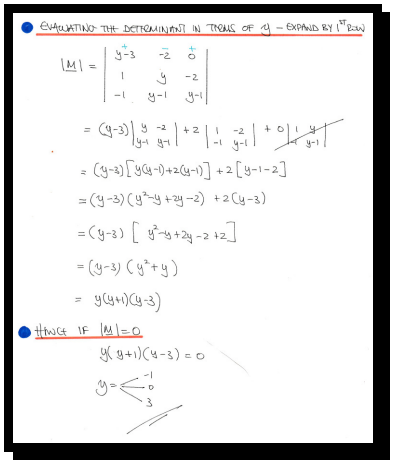
Question 10

A transformation in three dimensional space is defined by the following 3×3 matrix, where y is a scalar constant.

$$\mathbf{M}=\begin{pmatrix} y-3 & -2 & 0 \\ 1 & y & -2 \\ -1 & y-1 & y-1 \end{pmatrix}$$

If $|\mathbf{M}|=0$, find the possible values of y.

$$\boxed{y=-1,\quad y=0,\quad y=3}$$



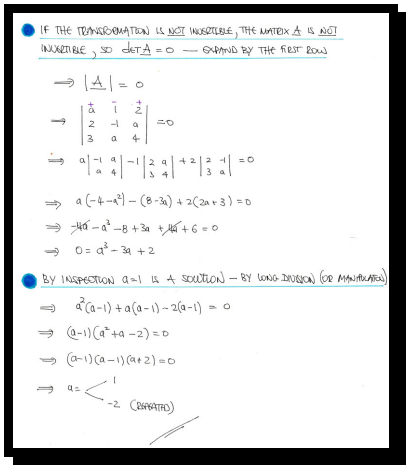
Question 11

A non invertible transformation in three dimensional space is defined by the following 3×3 matrix, where a is a scalar constant.

$$\mathbf{A}=\begin{pmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{pmatrix}$$

Determine the possible values of a.

$$\boxed{a=1,\ \ a=-2}$$



Question 12

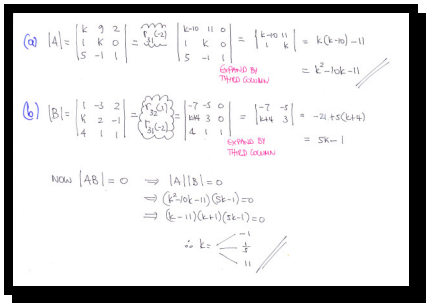
The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{pmatrix}.$$

a) Find an expression for $\det \mathbf{A}$, in terms of k.

b) Find the possible values of k given that $\mathbf{AB}$ is singular.

$$\boxed{\det \mathbf{A}=k^2-10k-11}\,, \qquad \boxed{k=-1,\ 11,\ \frac{1}{5}}$$

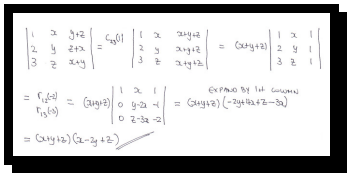


Question 13

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix}.$$

$$\boxed{(x+y+z)(x-2y+z)}$$

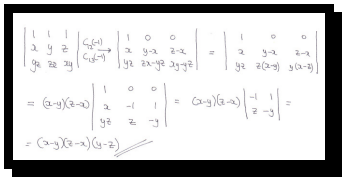


Question 14

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)}$$

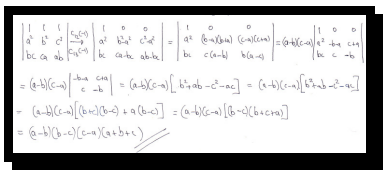


Question 15

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}.$$

$$\boxed{(a-b)(b-c)(c-a)(a+b+c)}$$

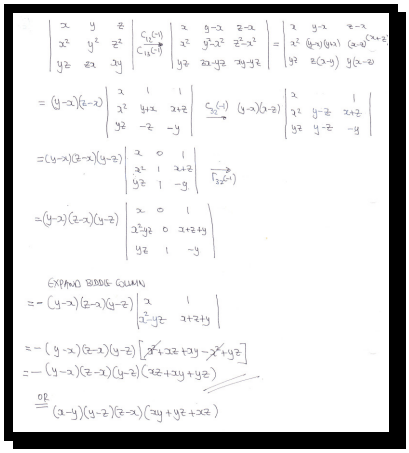


Question 16

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)(xy+yz+zx)}$$



MATRIX INVERSE

Question 1

Find the inverse for each of the following 2×2 matrices.

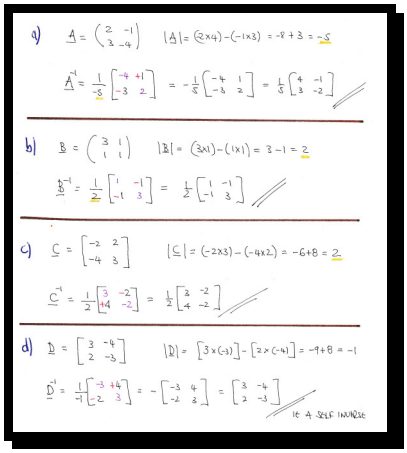
a) $\mathbf{A}=\begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix} -2 & 2 \\ -4 & 3 \end{pmatrix}$

d) $\mathbf{D}=\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{5}\begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}},\quad \boxed{\mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}},\quad \boxed{\mathbf{C}^{-1}=\frac{1}{2}\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}},\quad \boxed{\mathbf{D}^{-1}=\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}}$$



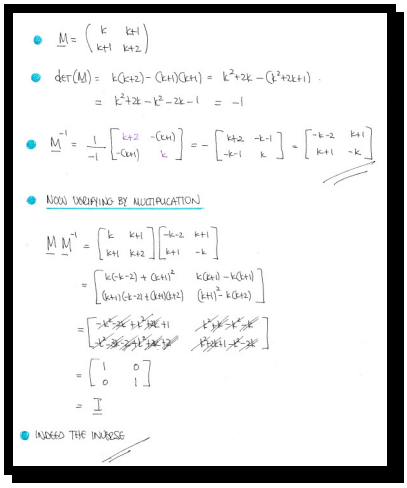
Question 2

Find, in terms of k, the inverse of the following 2×2 matrix.

$$\mathbf{M}=\begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}.$$

Verify your answer by multiplication.

$$\boxed{\mathbf{M}^{-1}=\begin{pmatrix} -k-2 & k+1 \\ k+1 & -k \end{pmatrix}}$$



Question 3

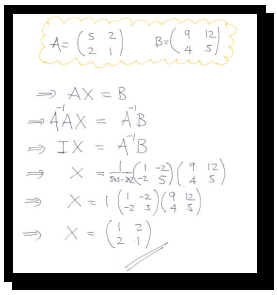
The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A}=\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}9 & 12\\ 4 & 5\end{pmatrix}.$$

Find the 2×2 matrix $\mathbf{X}$ that satisfy the equation

$$\mathbf{AX}=\mathbf{B}.$$

$$\boxed{\mathbf{X}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}}$$



Question 4

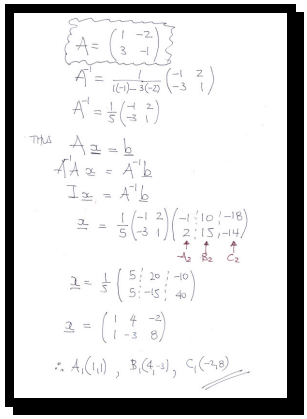
The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{A}=\begin{pmatrix}1 & -2\\ 3 & -1\end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(-1,2)$, $B_2(10,15)$ and $C_2(-18,-14)$.

Find the coordinates of the vertices of T_1.

$$\boxed{A_1(1,1)},\ \boxed{B_1(4,-3)},\ \boxed{C_1(-2,8)}$$



Question 5

The triangle T_1 is mapped by the 2×2 matrix

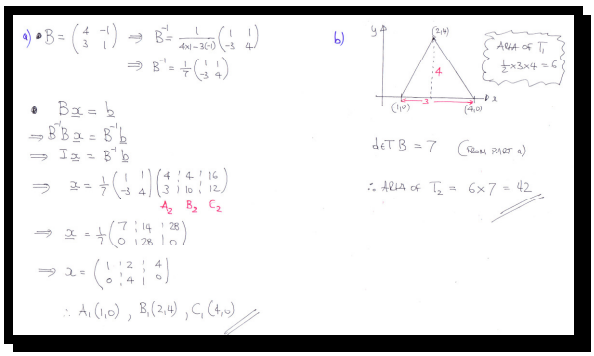
$$\mathbf{B}=\begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(4,3)$, $B_2(4,10)$ and $C_2(16,12)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{A_1(1,0)}$, $\boxed{B_1(2,4)}$, $\boxed{C_1(4,0)}$, $\boxed{\text{area}=42}$



Question 6

The triangle T_1 is mapped by the 2×2 matrix

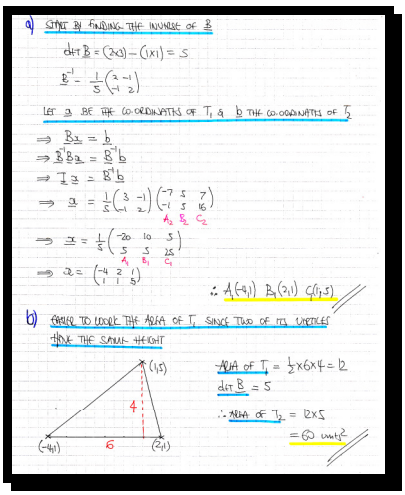
$$\mathbf{B}=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(-7,-1)$, $B_2(5,5)$ and $C_2(7,16)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{A_1(-4,1)}$, $\boxed{B_1(2,1)}$, $\boxed{C_1(1,5)}$, $\boxed{\text{area}=20}$



Question 7

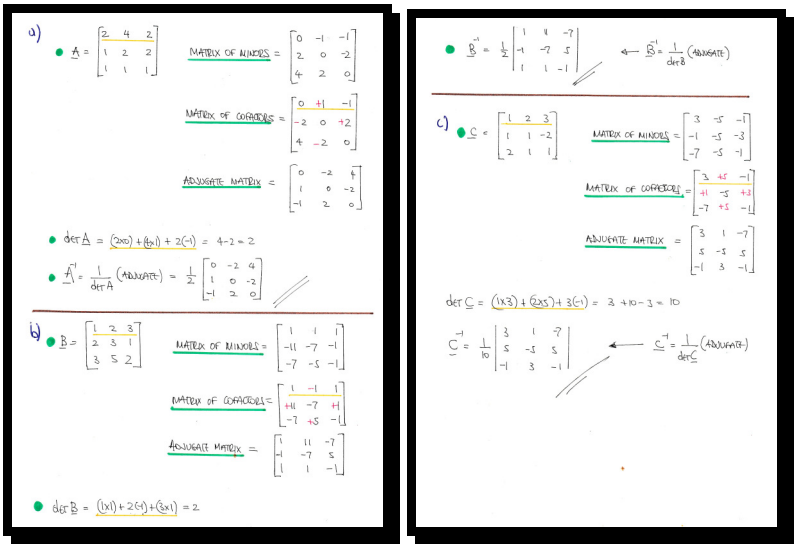
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix}2&4&2\\1&2&2\\1&1&1\end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix}1&2&3\\2&3&1\\3&5&2\end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix}1&2&3\\1&1&-2\\2&1&1\end{pmatrix}$

$$\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}0&-2&4\\1&0&-2\\-1&2&0\end{pmatrix},\quad \mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix}1&11&-7\\-1&-7&5\\1&1&-1\end{pmatrix},\quad \mathbf{C}^{-1}=\frac{1}{10}\begin{pmatrix}-3&-1&7\\5&5&-5\\1&-3&1\end{pmatrix}$$



Question 8

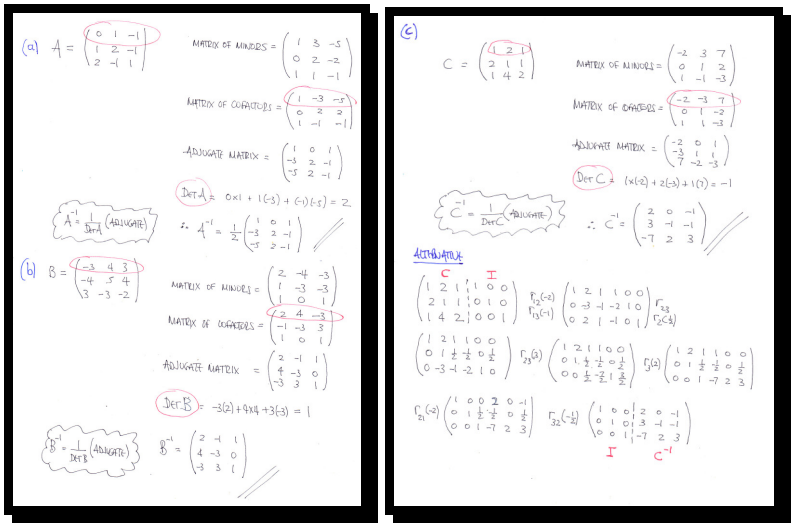
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 4 & 2 \end{pmatrix}$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ -3 & 2 & -1 \\ -5 & 2 & -1 \end{pmatrix}}\,,\quad \boxed{\mathbf{B}^{-1}=\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}}\,,\quad \boxed{\mathbf{C}^{-1}=\begin{pmatrix} 2 & 0 & -1 \\ 3 & -1 & -1 \\ -7 & 2 & 3 \end{pmatrix}}$$



Question 9

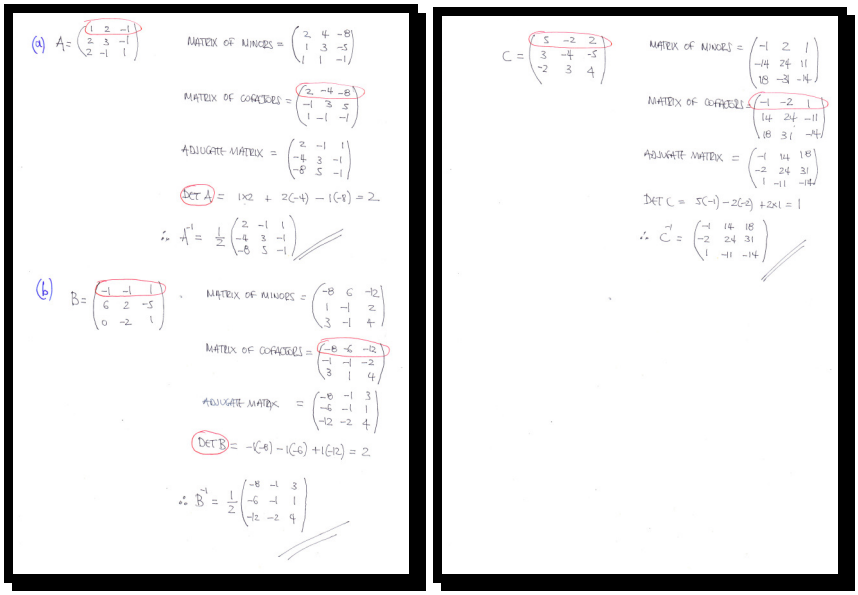
Find the inverse of each the following 3×3 matrices.

a) $\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}$

b) $\mathbf{B}=\begin{pmatrix}-1 & -1 & 1\\ 6 & 2 & -5\\ 0 & -2 & 1\end{pmatrix}$

c) $\mathbf{C}=\begin{pmatrix}5 & -2 & 2\\ 3 & -4 & -5\\ -2 & 3 & 4\end{pmatrix}$

$$\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}2 & -1 & 1\\ -4 & 3 & -1\\ -8 & 5 & -1\end{pmatrix},\quad \mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix}-8 & -1 & 3\\ -6 & -1 & 1\\ -12 & -2 & 4\end{pmatrix},\quad \mathbf{C}^{-1}=\begin{pmatrix}-1 & 14 & 18\\ -2 & 24 & 31\\ 1 & -11 & -14\end{pmatrix}$$



Question 10

The 3×3 matrix $\mathbf{A}$ is given below.

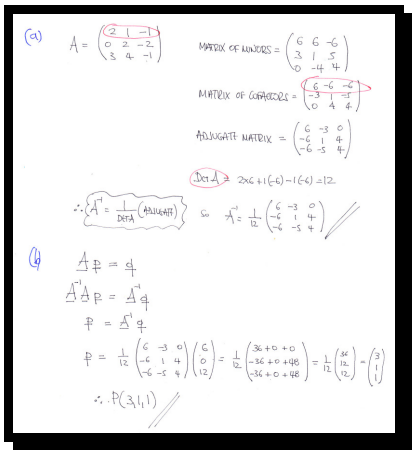
$$\mathbf{A}=\begin{pmatrix}2&1&-1\\0&2&-2\\3&4&-1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

The point P has been mapped by $\mathbf{A}$ onto the point $Q(6,0,12)$.

b) Determine the coordinates of P.

$$\mathbf{A}^{-1}=\frac{1}{12}\begin{pmatrix}6&-3&0\\-6&1&4\\-6&-5&4\end{pmatrix},\qquad \boxed{P(3,1,1)}$$



Question 11

The 3×3 matrix $\mathbf{M}$ is given below.

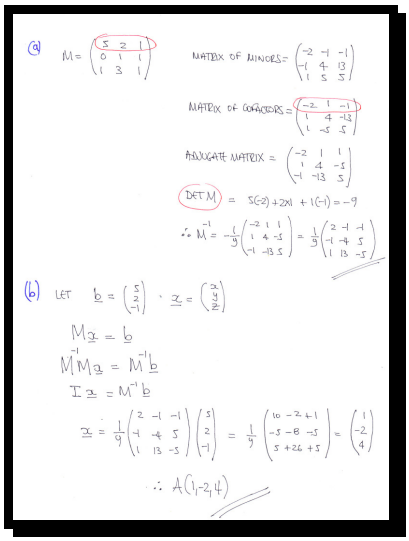
$$\mathbf{M}=\begin{pmatrix}5&2&1\\0&1&1\\1&3&1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

The point A has been transformed by $\mathbf{M}$ into the point $B(5,2,-1)$.

b) Determine the coordinates of A.

$$\boxed{\mathbf{M}^{-1}=\frac{1}{9}\begin{pmatrix}2&-1&-1\\-1&-4&5\\1&13&-5\end{pmatrix}},\quad \boxed{A(1,-2,4)}$$



Question 12

The 3×3 matrix $\mathbf{A}$ is given below.

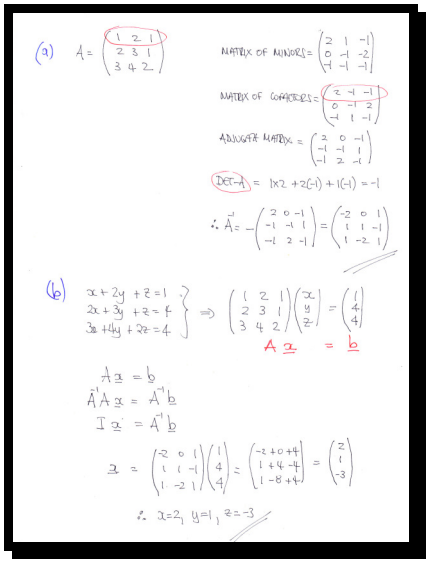
$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&3&1\\3&4&2\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

b) Hence, or otherwise, solve the system of equations

$$\begin{aligned}x+2y+\ z&=1\\2x+3y+\ z&=4\\3x+4y+2z&=4\end{aligned}$$

$$\boxed{\mathbf{A}^{-1}=\begin{pmatrix}-2&0&1\\1&1&-1\\1&-2&1\end{pmatrix}},\quad \boxed{x=2,\ \ y=1,\ \ z=-3}$$



Question 13

The 3×3 matrix $\mathbf{M}$ is given below.

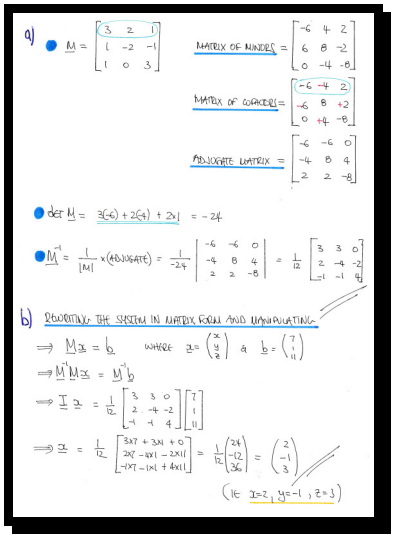
$$\mathbf{M}=\begin{pmatrix}3 & 2 & 1\\ 1 & -2 & -1\\ 1 & 0 & 3\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

b) Hence, or otherwise, solve the following system of equations.

$$\begin{aligned}3x+2y+z&=7\\ x-2y-z&=1\\ x\quad\ +3z&=11\end{aligned}$$

$$\boxed{\mathbf{M}^{-1}=\frac{1}{12}\begin{pmatrix}3 & 3 & 0\\ 2 & -4 & -2\\ -1 & -1 & 4\end{pmatrix}},\quad \boxed{x=2,\ \ y=-1,\ \ z=3}$$



Question 14

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}$$

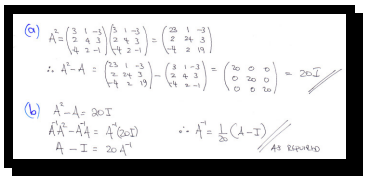
The matrix $\mathbf{A}$ is non singular.

a) Evaluate $\mathbf{A}^2-\mathbf{A}$.

b) Show clearly that

$$\mathbf{A}^{-1}=\frac{1}{20}[\mathbf{A}-\mathbf{I}].$$

$20\mathbf{I}$



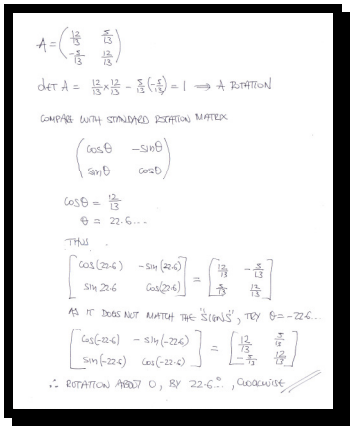
MATRIX TRANSFORMATIONS

Question 1

Describe fully the transformation given by the following 2×2 matrix.

$$\begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix}.$$

rotation, anticlockwise, by $\arcsin\frac{5}{13}$

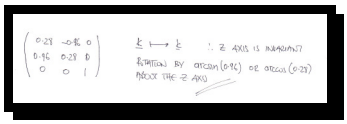


Question 2

Describe fully the transformation given by the following 3×3 matrix.

$$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

rotation in the z axis, anticlockwise, by $\arcsin\frac{24}{25}$



Question 3

A plane transformation maps the general point (x, y) to the general point (X, Y) by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

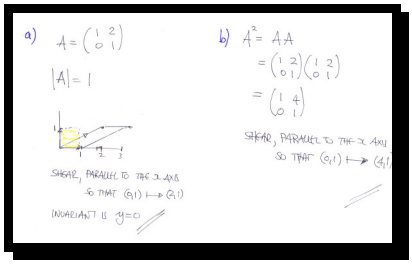
where $\mathbf{A}$ is the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

a) Give a full geometrical description for the transformation represented by $\mathbf{A}$, stating the equation of the line of invariant points under this transformation

b) Calculate $\mathbf{A}^2$ and describe geometrically the transformation it represents.

shear parallel to $y = 0$, $(0,1) \mapsto (2,1)$ | line of invariant points $y = 0$,

shear parallel to $y = 0$, $(0,1) \mapsto (4,1)$



Question 4

A plane transformation maps the points (x, y) to the points (X, Y) such that

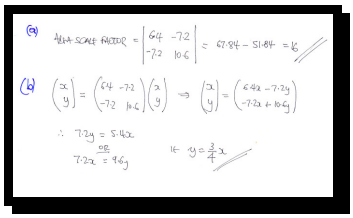
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the **area** scale factor of the transformation.

The points which lie on a straight line through the origin remain invariant under this transformation.

b) Determine the equation of this straight line.

$\boxed{\text{SF} = 16}$, $\boxed{y = \frac{3}{4}x}$

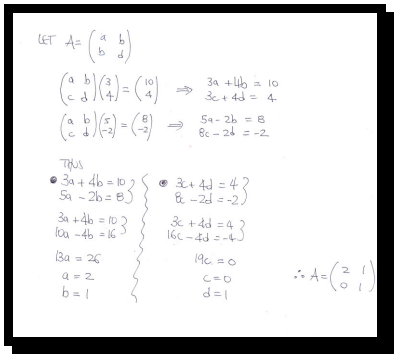


Question 5

The transformation represented by the 2×2 matrix $\mathbf{A}$ maps the point $(3,4)$ onto the point $(10,4)$, and the point $(5,-2)$ onto the point $(8,-2)$.

Determine the elements of $\mathbf{A}$.

$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$$



Question 6

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below

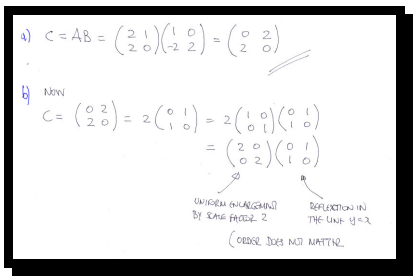
$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 2 & 0\end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix}1 & 0\\ -2 & 2\end{pmatrix}.$$

The matrix $\mathbf{C}$ represents the combined effect of the transformation represented by the $\mathbf{B}$, followed by the transformation represented by $\mathbf{A}$.

a) Determine the elements of $\mathbf{C}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\mathbf{C}=\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$, enlargement by scale factor 2, reflection in the line $y = x$, in any order



Question 7

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\quad\text{and}\quad\mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

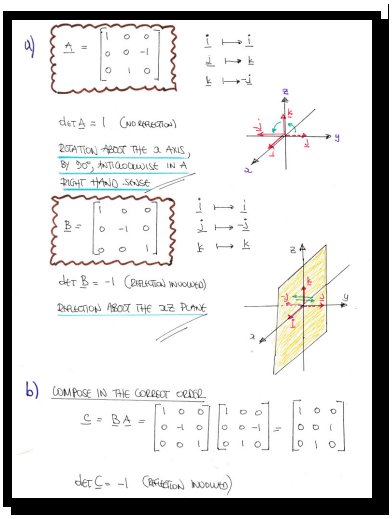
a) Describe geometrically the transformations given by each of the two matrices.

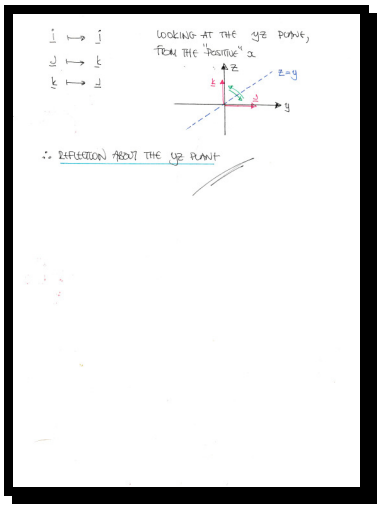
The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{rotation about } x \text{ axis, } 90^\circ \text{ anticlockwise}}$, $\boxed{\mathbf{B}:\text{reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}:\text{reflection in the plane } y=z}$





Question 8

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

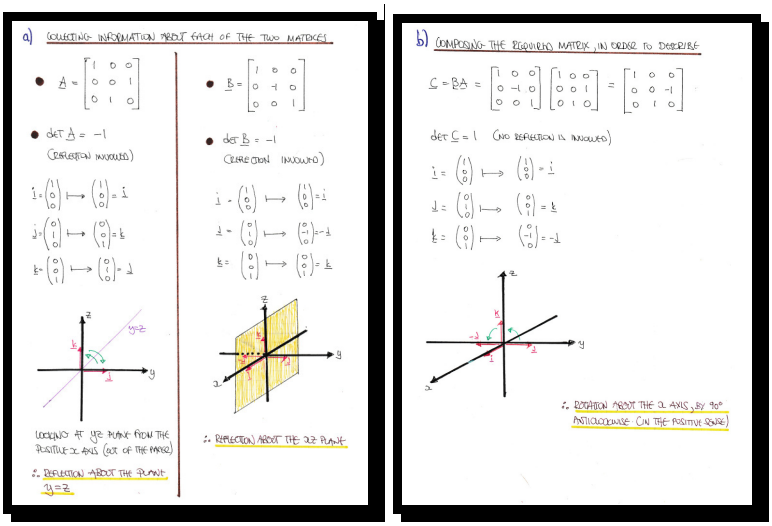
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{reflection in the plane } y=z}$, $\boxed{\mathbf{B}:\text{reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}:\text{rotation in the } x \text{ axis, } 90°, \text{ clockwise}}$



Question 9

The matrix $\mathbf{A}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ and the matrix $\mathbf{A}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ are defined as

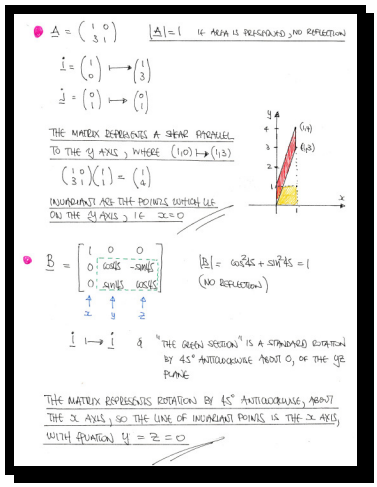
$$\mathbf{A}=\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45^\circ \\ 0 & \sin 45^\circ & \cos 45^\circ \end{pmatrix}.$$

Describe geometrically the transformations given by each of these matrices.

State in each case the equation of the line of invariant points.

$\boxed{\mathbf{A}:\text{shear parallel to } y \text{ axis}, (1,0)\mapsto(3,1)}$,

$\boxed{\mathbf{B}:\text{rotation in the } x \text{ axis, } 45^\circ, \text{ anticlockwise}}$, $\boxed{\mathbf{A}: x=0}$, $\boxed{\mathbf{B}: y=z=0, \text{ i.e. } x \text{ axis}}$



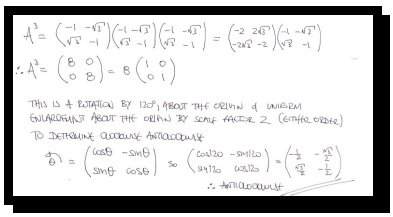
Question 10

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

Determine the elements of $\mathbf{A}^3$ and hence describe geometrically the transformation represented by $\mathbf{A}$.

$\boxed{\mathbf{A}^3=\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}}$, rotation of 120°, anticlockwise & enlargement of S.F. 2, both about the origin and in any order.



Question 11

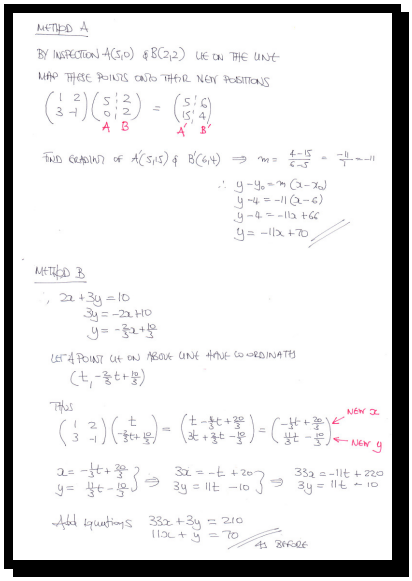
Find the image of the straight line with equation

$$2x+3y=10,$$

under the transformation represented by the 2×2 matrix

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}.$$

$$\boxed{11x+y=70}$$



Question 12

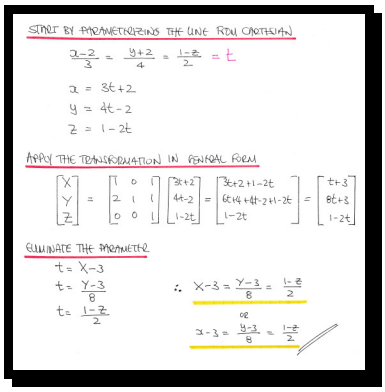
Find in Cartesian form the image of the straight line with equation

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2},$$

under the transformation represented by the 3×3 matrix $\mathbf{A}$, shown below.

$$\mathbf{A}=\begin{pmatrix}1&0&1\\2&1&1\\0&0&1\end{pmatrix}.$$

$$\boxed{x-3=\frac{y-3}{8}=\frac{1-z}{2}}$$



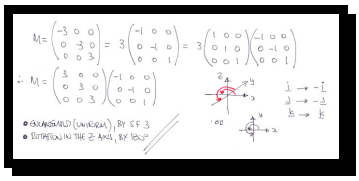
Question 13

$$\mathbf{M}=\begin{pmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 3\end{pmatrix}$$

The 3×3 matrix $\mathbf{M}$ above, describes two consecutive geometrical transformations of 3 dimensional space, which can be carried out in any order.

Describe geometrically each these two transformations.

$\boxed{\text{rotation about } z \text{ axis}, 180^\circ}$, $\boxed{\text{uniform enlargement, S.F.} = 3}$



Question 14

A plane transformation maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the straight line of invariant points under this transformation.

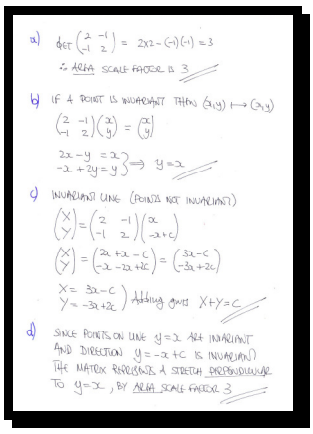
c) Show that all the straight lines with equation of the form

$$x + y = c,$$

where c is a constant, are invariant lines under this transformation.

d) Hence describe the transformation geometrically.

$\boxed{\text{SF} = 3}$, $\boxed{y = x}$, $\boxed{\text{stretch perpendicular to the line } y = x, \text{ by area scale factor } 3}$



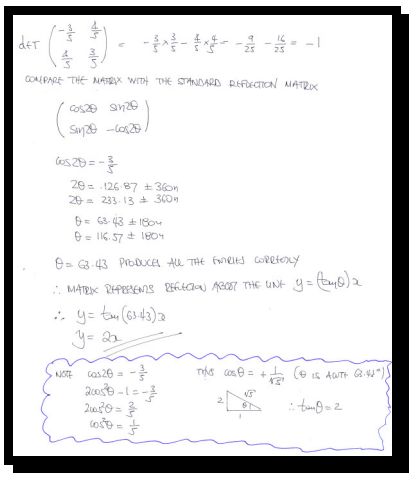
Question 15

Describe fully the transformation given by the following 2×2 matrix

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

The description must be supported by mathematical calculations.

reflection in $y = 2x$



Question 16

The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix},$$

where k is a scalar constant.

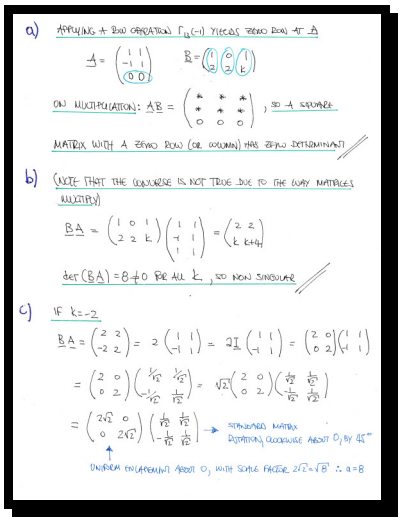
a) Without calculating $\mathbf{AB}$, show that $\mathbf{AB}$ is singular for all values of k.

b) Show that $\mathbf{BA}$ is non singular for all values of k.

When $k=-2$ the matrix $\mathbf{BA}$ represents a combination of a uniform enlargement with linear scale factor $\sqrt{a}$ and another transformation T.

c) Find the value of a and describe T geometrically.

$\boxed{a=8}$, $\boxed{\text{rotation about } O\text{, clockwise, by } 45^\circ}$



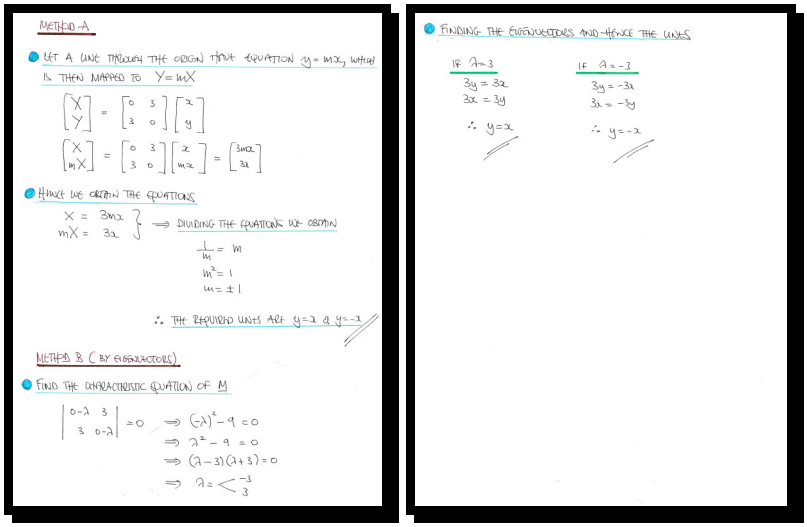
Question 17

The 2×2 matrix $\mathbf{M}$ is defined by

$$\mathbf{M}=\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}.$$

Find, by calculation, the equations of the two lines which pass through the origin, that remain invariant under the transformation represented by $\mathbf{M}$.

$$\boxed{y=\pm x}$$



Question 18

The curve C has equation

$$5x^2 - 16xy + 13y^2 = 25 .$$

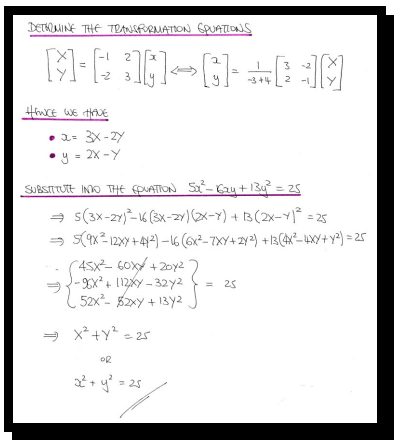
This curve is to be transformed by the 2×2 matrix $\mathbf{A}$, given below.

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Show that, under this transformation matrix, the image of C is the circle with equation

$$x^2 + y^2 = 25 .$$

proof



Question 19

The 3×3 matrix $\mathbf{A}$ is defined by

$$\mathbf{A}=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}.$$

a) Describe geometrically the transformation given by $\mathbf{A}$.

The 3×3 matrix $\mathbf{B}$ represents a rotation of $180°$ about the line $x=z$, $y=0$.

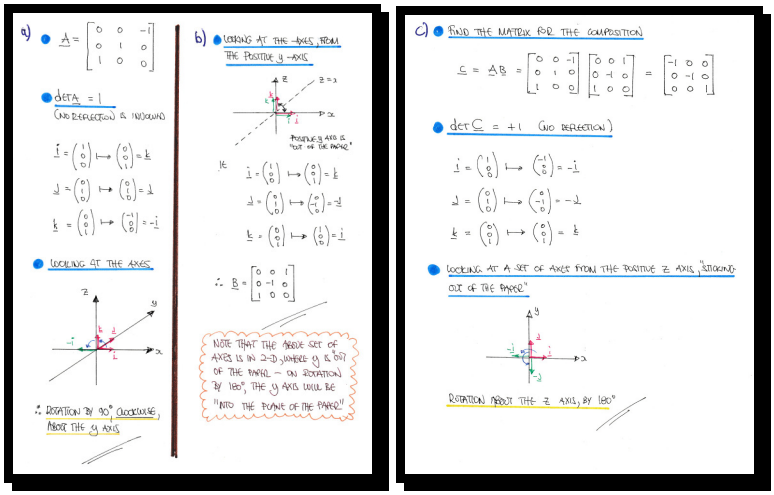
b) Determine the elements of $\mathbf{B}$.

The 3×3 matrix $\mathbf{C}$ is represents the transformation defined by $\mathbf{B}$, followed by the transformation defined by $\mathbf{A}$.

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}:\text{rotation about } y \text{ axis, } 90° \text{ clockwise}}$, $\boxed{\mathbf{B}=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}}$,

$\boxed{\mathbf{C}:\text{rotation about } z \text{ axis, } 180°}$



Question 20

The 3×3 matrix $\mathbf{R}$ is defined by

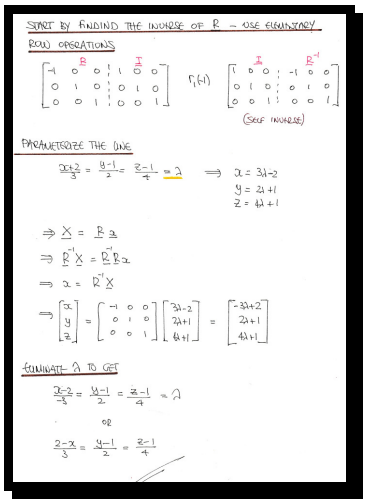
$$\mathbf{R}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The image of the straight line L, when transformed by $\mathbf{R}$, is the straight line with Cartesian equation

$$\frac{x+2}{3}=\frac{y-1}{2}=\frac{z-1}{4}.$$

Find a Cartesian equation for L.

$$\boxed{\frac{x-2}{-3}=\frac{y-1}{2}=\frac{z-1}{4}}$$



Question 21

The 3×3 matrix $\mathbf{C}$ is defined by

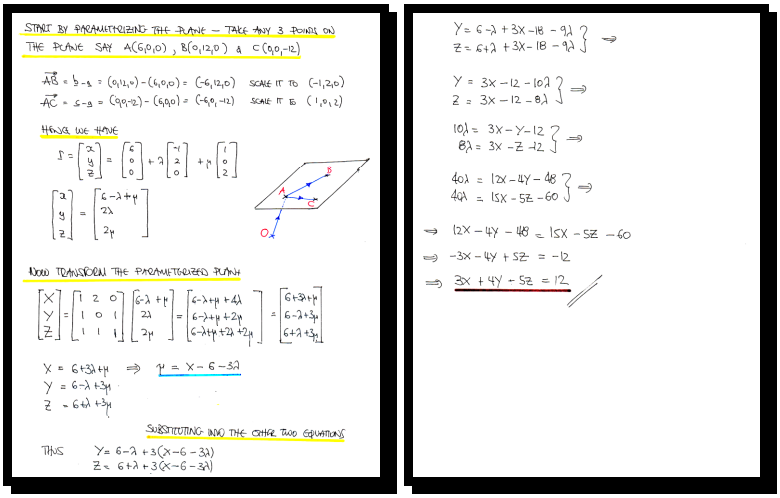
$$\mathbf{C}=\begin{pmatrix}1&2&0\\1&0&1\\1&1&1\end{pmatrix}.$$

Find, in Cartesian form, the image of the plane with Cartesian equation

$$2x+y-z=12$$

under the transformation defined by $\mathbf{C}$.

$$\boxed{3x+4y-5z=12}$$



Question 22

A transformation $T:\mathbb{R}^2 \mapsto \mathbb{R}^2$ is represented by the 2×2 matrix $\mathbf{A}$ below.

$$\mathbf{A}=\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}.$$

a) Find the determinant of $\mathbf{A}$ and explain its significance in sign and size.

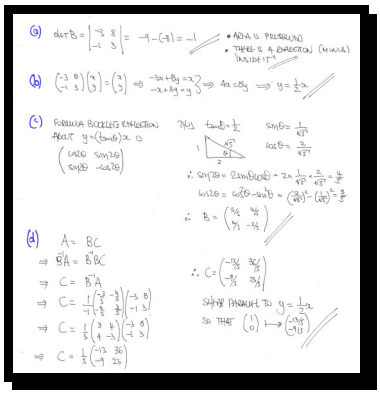
b) Find the equation of the line of the invariant points of $\mathbf{A}$.

c) Determine the entries of the 2×2 matrix $\mathbf{B}$ which represents a reflection about the line found in part **(b)**, giving all its entries as simple fractions.

The 2×2 matrix $\mathbf{A}$, consists of a shear represented by the matrix $\mathbf{C}$, followed by a reflection represented by the matrix $\mathbf{B}$.

d) Determine the elements of $\mathbf{C}$ and describe the shear.

$$\boxed{\det \mathbf{A}=-1}, \quad \boxed{y=\frac{1}{2}x}, \quad \boxed{\mathbf{B}=\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}}, \quad \boxed{\mathbf{C}=\begin{pmatrix} -\frac{13}{5} & \frac{36}{5} \\ -\frac{9}{5} & \frac{23}{5} \end{pmatrix}}$$



Question 23

A transformation T, maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the line of invariant points under this transformation.

c) Show that all the straight lines of the form

$$y = x + c,$$

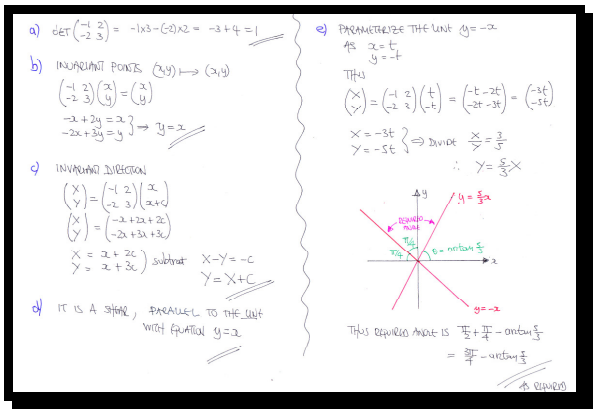
where c is a constant, are invariant lines under T.

d) Hence state the name of T.

e) Show that the acute angle formed by the straight line with equation $y = -x$ and its the image under T is

$$\frac{3\pi}{4} - \arctan\left(\frac{5}{3}\right).$$

$\boxed{\text{SF} = 1}$, $\boxed{y = x}$, $\boxed{\text{shear}}$



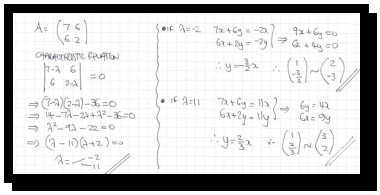
EIGENVALUES
&
EIGENVECTORS

Question 1

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

$$\boxed{\lambda=-2,\quad \mathbf{u}=\alpha\begin{pmatrix}2\\ -3\end{pmatrix}},\quad \boxed{\lambda=11,\quad \mathbf{u}=\beta\begin{pmatrix}3\\ 2\end{pmatrix}}$$

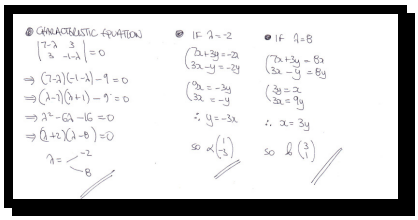


Question 2

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{C}=\begin{pmatrix}7 & 3\\ 3 & -1\end{pmatrix}.$$

$$\boxed{\lambda=-2,\quad \mathbf{u}=\alpha\begin{pmatrix}1\\ -3\end{pmatrix}},\quad \boxed{\lambda=8,\quad \mathbf{u}=\beta\begin{pmatrix}3\\ 1\end{pmatrix}}$$

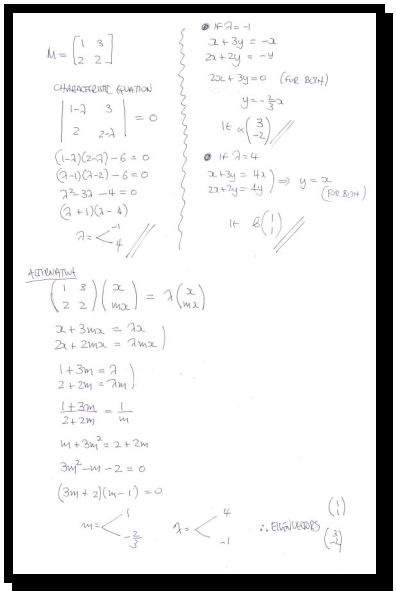


Question 3

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{M}=\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}.$$

$$\boxed{\lambda=-1,\quad \mathbf{u}=\alpha\begin{pmatrix}3\\ -2\end{pmatrix}},\qquad \boxed{\lambda=4,\quad \mathbf{u}=\beta\begin{pmatrix}1\\ 1\end{pmatrix}}$$

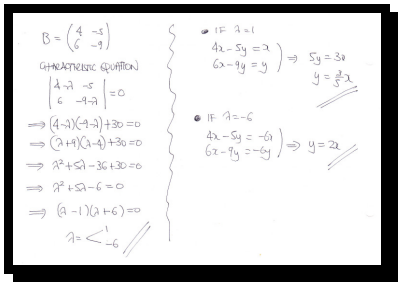


Question 4

Determine the eigenvalues and the corresponding equations of invariant lines of the following 2×2 matrix.

$$\mathbf{B}=\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}.$$

$$\boxed{\lambda=1,\ y=\frac{3}{5}x}\ ,\ \boxed{\lambda=-6,\ y=2x}$$

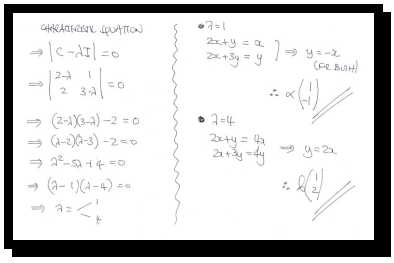


Question 5

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{C}=\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

$$\boxed{\lambda=1,\quad \mathbf{u}=\alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}}\ ,\ \boxed{\lambda=4,\quad \mathbf{u}=\beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$



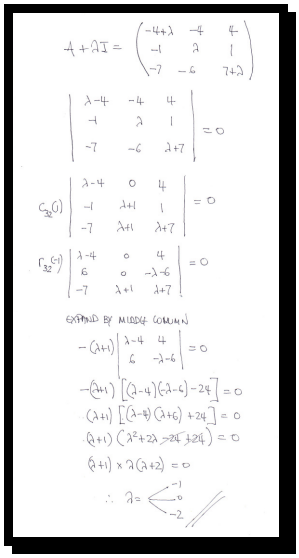
Question 6

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}-4 & -4 & 4\\ -1 & 0 & 1\\ -7 & -6 & 7\end{pmatrix}. \qquad .$$

Given that $\mathbf{I}$ is the 3×3 identity matrix, determine the values of the constant λ, so that $\mathbf{A}+\lambda\mathbf{I}$ is singular.

$$\boxed{\lambda=0,-1,-2}$$

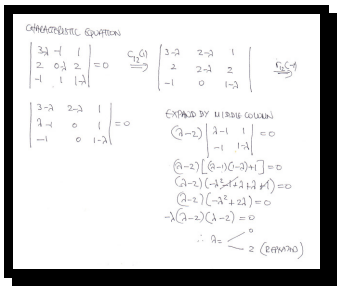


Question 7

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$\boxed{\lambda=0,\quad \lambda=2}$$

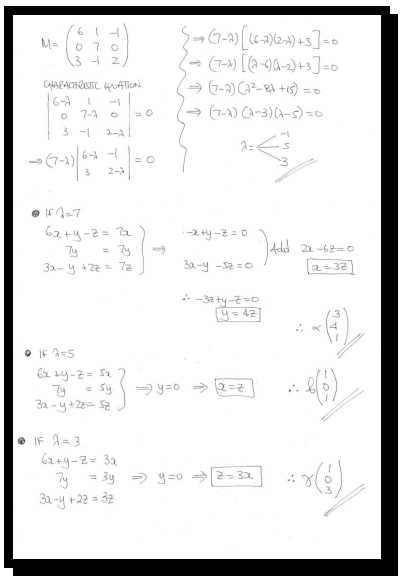


Question 8

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{M}=\begin{pmatrix}6 & 1 & -1\\ 0 & 7 & 0\\ 3 & -1 & 2\end{pmatrix}.$$

$$\boxed{\lambda_1=3,\quad \mathbf{u}=\begin{pmatrix}1\\0\\3\end{pmatrix}},\quad \boxed{\lambda_2=5,\quad \mathbf{v}=\begin{pmatrix}1\\0\\1\end{pmatrix}},\quad \boxed{\lambda_3=7,\quad \mathbf{w}=\begin{pmatrix}3\\4\\1\end{pmatrix}}$$

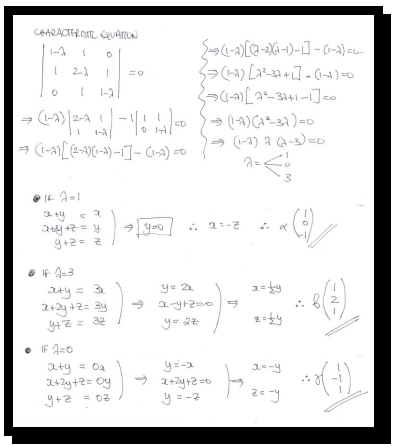


Question 9

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1&1&0\\1&2&1\\0&1&1\end{pmatrix}.$$

$$\boxed{\lambda_1=0,\quad \mathbf{u}=\begin{pmatrix}1\\-1\\1\end{pmatrix}},\quad \boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix}1\\0\\-1\end{pmatrix}},\quad \boxed{\lambda_3=3,\quad \mathbf{w}=\begin{pmatrix}1\\2\\1\end{pmatrix}}$$

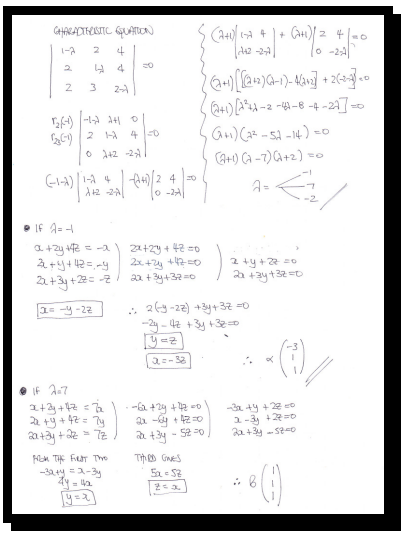


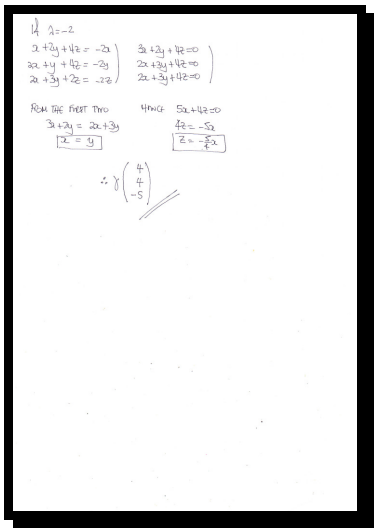
Question 10

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1&2&4\\2&1&4\\2&3&2\end{pmatrix}.$$

$$\boxed{\lambda_1=-1,\quad \mathbf{u}=\begin{pmatrix}-3\\1\\1\end{pmatrix}},\quad \boxed{\lambda_2=-2,\quad \mathbf{v}=\begin{pmatrix}4\\4\\-5\end{pmatrix}},\quad \boxed{\lambda_3=7,\quad \mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$$





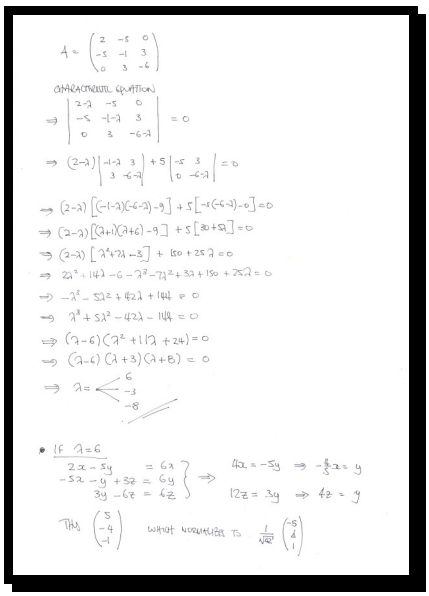
Question 11

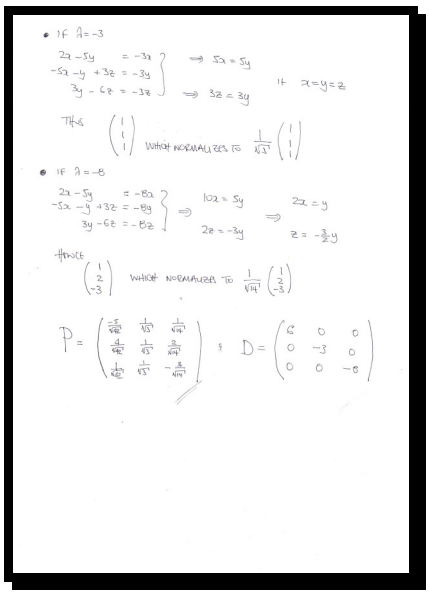
The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix} 2 & -5 & 0 \\ -5 & -1 & 3 \\ 0 & 3 & -6 \end{pmatrix}.$$

Since $\mathbf{A}$ is symmetric, determine an orthogonal 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$.

$$\boxed{\mathbf{P}=\begin{pmatrix} -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \end{pmatrix}}, \quad \boxed{\mathbf{D}=\begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -8 \end{pmatrix}}$$





Question 12

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}.$$

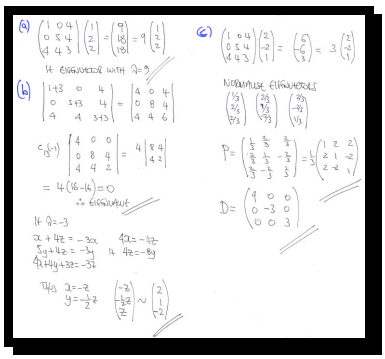
a) Verify that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue.

b) Show that -3 is an eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

c) Given further that $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is another eigenvector of $\mathbf{A}$, find the 3×3 matrices $\mathbf{P}$ and $\mathbf{D}$ such that

$$\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$$

$$\boxed{\lambda=9},\ \boxed{\begin{pmatrix}2\\1\\-2\end{pmatrix}},\ \boxed{\mathbf{D}=\begin{pmatrix}9&0&0\\0&-3&0\\0&0&3\end{pmatrix}},\ \boxed{\mathbf{P}=\frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}}$$



Question 13

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}4&-1&1\\-1&6&-1\\1&-1&4\end{pmatrix}.$$

a) Show that $\lambda=7$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Find the eigenvector associated with the eigenvalue $\lambda=7$.

The other two eigenvectors of $\mathbf{A}$ are

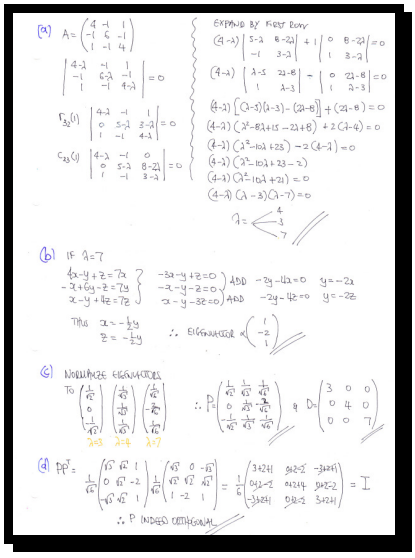
$$\mathbf{u}=\mathbf{i}-\mathbf{k} \qquad \text{and} \qquad \mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k},$$

where the eigenvalue of $\mathbf{v}$ is greater than the eigenvalue of $\mathbf{u}$.

c) Find a 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$.

d) Show that $\mathbf{P}$ is an orthogonal matrix.

$$\boxed{\lambda=4,3},\quad \boxed{\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}},\quad \boxed{\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}},\quad \boxed{\mathbf{D}=\begin{pmatrix}3&0&0\\0&4&0\\0&0&7\end{pmatrix}}$$



Question 14

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

A straight line with equation $y=mx$, where m is a constant, remains invariant under the transformation represented by $\mathbf{A}$.

a) Show that

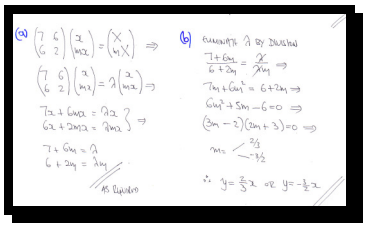
$$7+6m=\lambda$$

$$6+2m=\lambda m$$

where λ is a constant.

b) Hence find the two possible equations of this straight line.

$$\boxed{y=\frac{2}{3}x}\,,\quad \boxed{y=-\frac{3}{2}x}$$



Question 15

The 3×3 matrix $\mathbf{C}$ represents a geometric transformation $T:\mathbb{R}^3\mapsto\mathbb{R}^3$.

$$\mathbf{C}=\begin{pmatrix}3&-1&1\\-1&3&1\\1&1&3\end{pmatrix}$$

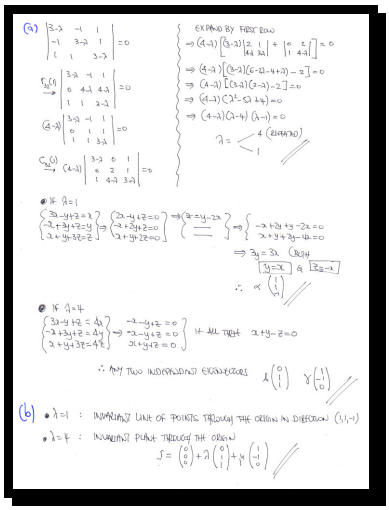
a) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{C}$.

b) Describe the geometrical significance of the eigenvectors of $\mathbf{C}$ in relation to T.

$$\boxed{\lambda=1,\quad \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}},\ \boxed{\lambda=4,\quad \beta\begin{pmatrix}0\\1\\1\end{pmatrix},\gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}},$$

$$\boxed{\lambda=1\Leftrightarrow\text{invariant line of points through the origin}},$$

$$\boxed{\lambda=4\Leftrightarrow\text{invariant plane through the origin}}$$



Question 16

The 2×2 matrix $\mathbf{A}$ is defined in terms of a constant k.

$$\mathbf{A}=\begin{pmatrix}2 & 7\\ 4 & k\end{pmatrix}$$

a) Given that $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find …

i. … the corresponding eigenvalue to the eigenvector.

ii. … the value of k

b) Find another eigenvector and the corresponding eigenvalue of $\mathbf{A}$.

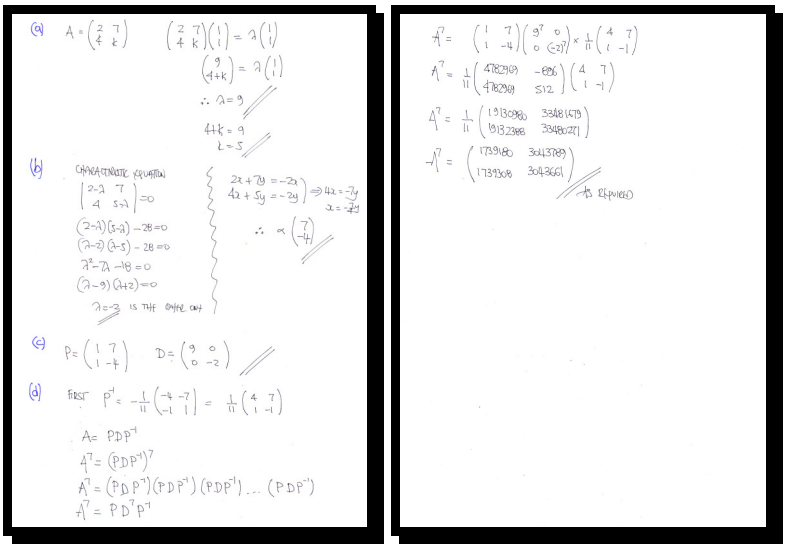
It is further given that $\mathbf{A}$ can be written as $\mathbf{A}=\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a 2×2 diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

c) Write down possible forms for the matrices $\mathbf{D}$ and $\mathbf{P}$.

d) Hence show clearly that

$$\mathbf{A}^7=\begin{pmatrix}1739180 & 3043789\\ 1739308 & 3043661\end{pmatrix}.$$

$\boxed{\lambda=9}$, $\boxed{k=5}$, $\boxed{\lambda=-2,\mathbf{u}=\begin{pmatrix}7\\-4\end{pmatrix}}$, $\boxed{\mathbf{D}=\begin{pmatrix}9 & 0\\ 0 & -2\end{pmatrix}}$, $\boxed{\mathbf{P}=\begin{pmatrix}1 & 7\\ 1 & -4\end{pmatrix}}$



Question 17

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1 & -1 & 1\\ 3 & -3 & 1\\ 3 & -5 & 3\end{pmatrix}.$$

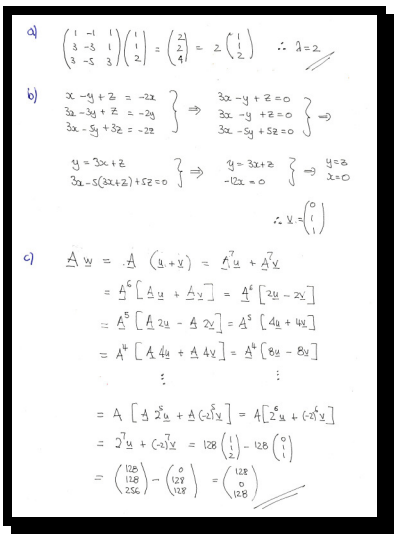
a) Given that $\mathbf{u}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue.

b) Given that $\lambda=-2$ is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector $\mathbf{v}$.

The vector $\mathbf{w}$ is defined as $\mathbf{w}=\mathbf{u}+\mathbf{v}$.

c) Determine the vector $\mathbf{A}^7\mathbf{w}$.

$$\boxed{\lambda=2},\quad \boxed{\mathbf{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}},\quad \boxed{\mathbf{A}^7\mathbf{w}=\begin{pmatrix}128\\0\\128\end{pmatrix}}$$



Question 18

The 2×2 matrix $\mathbf{C}$ is defined as

$$\mathbf{C}=\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix},$$

where a and b are constants.

a) Determine the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors, giving the answers in terms of a and b where appropriate.

It is further given that $\mathbf{C}=\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

b) Write down the possible form of $\mathbf{D}$ and the possible form of $\mathbf{P}$ and hence show that

$$\mathbf{C}^9=b^8\mathbf{C}.$$

$$\boxed{\lambda_1=b,\quad \mathbf{u}=\begin{pmatrix} 1 \\ \dfrac{b-a}{b+a} \end{pmatrix} \text{ or } \mathbf{u}=\begin{pmatrix} b+a \\ b-a \end{pmatrix}},\quad \boxed{\lambda=-b,\quad \mathbf{v}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}},\quad \boxed{\mathbf{D}=\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}},$$

$$\boxed{\mathbf{P}=\begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}}$$

