MATRICES

EXAM QUESTIONS

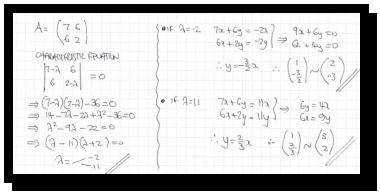
(Part Two)

Question 1 (**)

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

$$\boxed{\lambda=-2,\mathbf{u}=\alpha\begin{pmatrix}2\\ -3\end{pmatrix}},\ \boxed{\lambda=11,\mathbf{u}=\beta\begin{pmatrix}3\\ 2\end{pmatrix}}$$



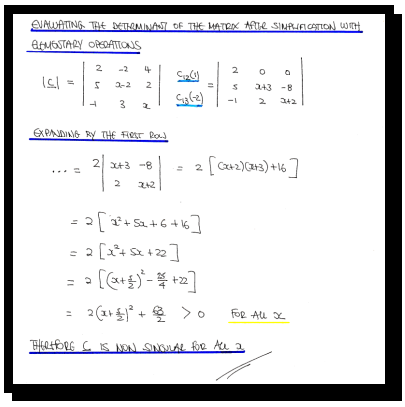
Question 2 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where x is a scalar constant.

$$\mathbf{C}=\begin{pmatrix}2 & -2 & 4\\ 5 & x-2 & 2\\ -1 & 3 & x\end{pmatrix}.$$

Show that $\mathbf{C}$ is non singular for all values of x.

$$\boxed{},\ \boxed{\text{proof}}$$



Question 3 (**)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1 & 8\\ 8 & -11\end{pmatrix}.$$

a) Find the eigenvalues of $\mathbf{A}$.

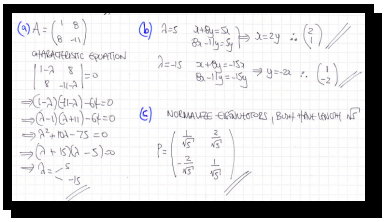
b) Determine an eigenvector for each of the corresponding eigenvalues of $\mathbf{A}$.

c) Find a 2×2 matrix $\mathbf{P}$, so that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{A}$, with $\lambda_1<\lambda_2$.

$$\boxed{\lambda_1=-15,\ \lambda_2=5},\quad \boxed{\mathbf{u}=\begin{pmatrix}1\\ -2\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}2\\ 1\end{pmatrix}},\quad \boxed{\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}}$$

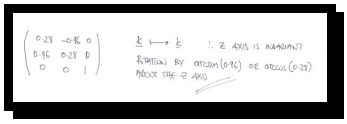


Question 4 (**)

Describe fully the transformation given by the following 3×3 matrix.

$$\begin{pmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

rotation in the z axis, anticlockwise, by $\arcsin(0.96)$



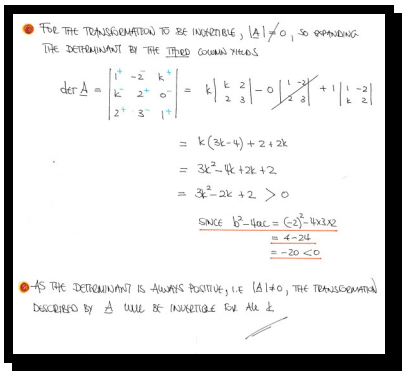
Question 5 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where k is a scalar constant.

$$\mathbf{A}=\begin{pmatrix} 1 & -2 & k \\ k & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

Show that the transformation defined by $\mathbf{A}$ can be inverted for all values of k.

$\square$, proof



Question 6 **(**)**

The 3×3 matrix $\mathbf{A}$ is given below.

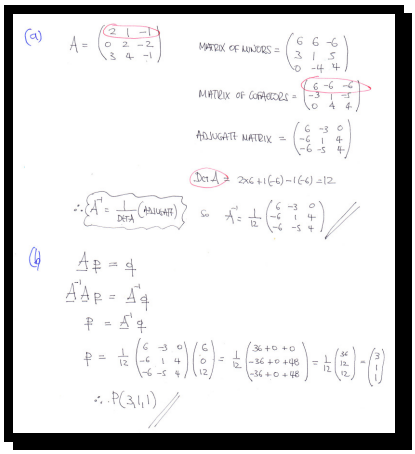
$$\mathbf{A}=\begin{pmatrix}2&1&-1\\0&2&-2\\3&4&-1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

The point P has been mapped by $\mathbf{A}$ onto the point $Q(6,0,12)$.

b) Determine the coordinates of P.

$$\mathbf{A}^{-1}=\frac{1}{12}\begin{pmatrix}6&-3&0\\-6&1&4\\-6&-5&4\end{pmatrix},\qquad \boxed{P(3,1,1)}$$



Question 7 (**)

The 3×3 matrix $\mathbf{A}$ is given below.

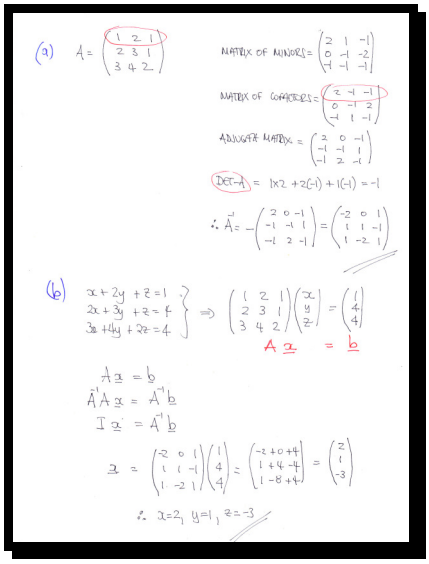
$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&3&1\\3&4&2\end{pmatrix}.$$

a) Find the inverse of $\mathbf{A}$.

b) Hence, or otherwise, solve the system of equations

$$\begin{aligned}x+2y+\ z&=1\\2x+3y+\ z&=4\\3x+4y+2z&=4\end{aligned}$$

$$\boxed{\mathbf{A}^{-1}=\begin{pmatrix}-2&0&1\\1&1&-1\\1&-2&1\end{pmatrix}},\quad \boxed{x=2,\ \ y=1,\ \ z=-3}$$



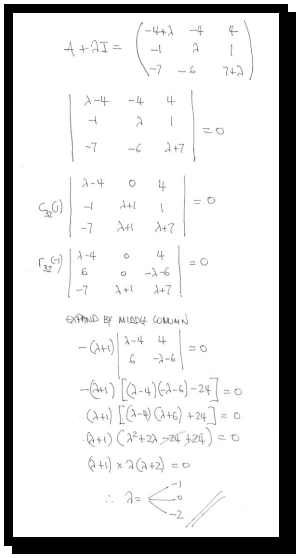
Question 8 (**)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}-4 & -4 & 4\\ -1 & 0 & 1\\ -7 & -6 & 7\end{pmatrix}.$$

Given that $\mathbf{I}$ is the 3×3 identity matrix, determine the values of the constant λ, so that $\mathbf{A}+\lambda\mathbf{I}$ is singular.

$$\boxed{\lambda=0,-1,-2}$$



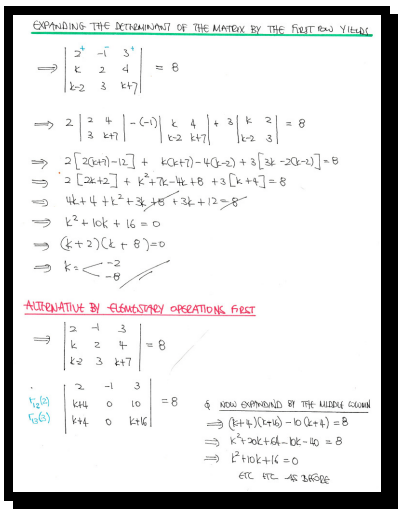
Question 9 **(**)**

The 3×3 matrix $\mathbf{A}$ is defined in terms of the scalar constant k by

$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ k-2 & 3 & k+7 \end{pmatrix}.$$

Given that $|\mathbf{A}|=8$, find the possible values of k.

$\boxed{}$, $\boxed{k=-2,\quad k=-8}$

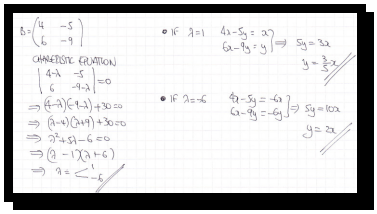


Question 10 (**)

Find the eigenvalues and the corresponding equations of invariant lines of the following 2×2 matrix

$$\mathbf{B}=\begin{pmatrix}4 & -5\\ 6 & -9\end{pmatrix}.$$

$$\boxed{\lambda=1,\ y=\frac{3}{5}x},\ \boxed{\lambda=-6,\ y=2x}$$



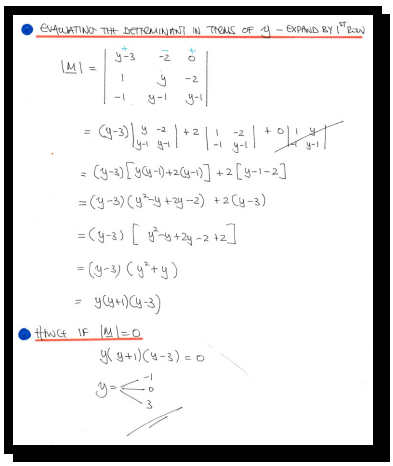
Question 11 (**)

A transformation in three dimensional space is defined by the following 3×3 matrix, where y is a scalar constant.

$$\mathbf{M}=\begin{pmatrix}y-3 & -2 & 0\\ 1 & y & -2\\ -1 & y-1 & y-1\end{pmatrix}$$

If $|\mathbf{M}|=0$, find the possible values of y.

$$\boxed{\ \ },\ \boxed{y=-1,\quad y=0,\quad y=3}$$



Question 12 (**)

The 3×3 matrix $\mathbf{M}$ is given below.

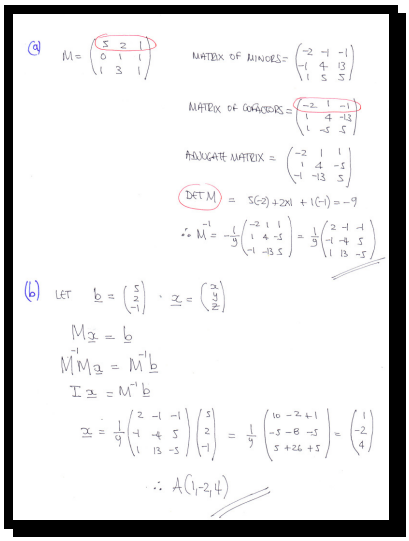
$$\mathbf{M}=\begin{pmatrix}5&2&1\\0&1&1\\1&3&1\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

The point A has been transformed by $\mathbf{M}$ into the point $B(5,2,-1)$.

b) Determine the coordinates of A.

$$\boxed{\mathbf{M}^{-1}=\frac{1}{9}\begin{pmatrix}2&-1&-1\\-1&-4&5\\1&13&-5\end{pmatrix}},\quad \boxed{A(1,-2,4)}$$



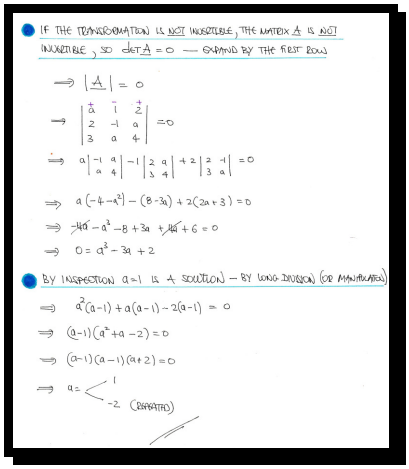
Question 13 **(**)**

A non invertible transformation in three dimensional space is defined by the following 3×3 matrix, where a is a scalar constant.

$$\mathbf{A}=\begin{pmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{pmatrix}$$

Determine the possible values of a.

$\boxed{}$, $\boxed{a=1,\ a=-2}$



Question 14 (**)

The 3×3 matrix $\mathbf{M}$ is given below.

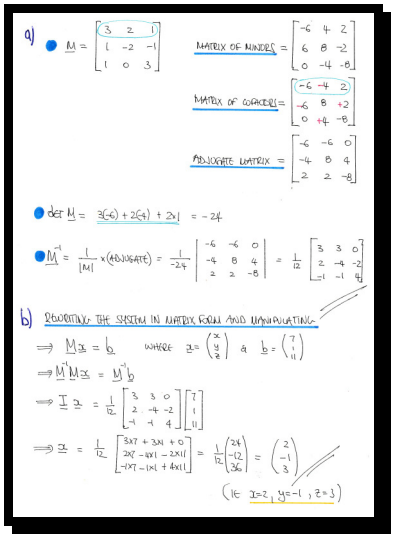
$$\mathbf{M}=\begin{pmatrix}3&2&1\\1&-2&-1\\1&0&3\end{pmatrix}.$$

a) Find the inverse of $\mathbf{M}$.

b) Hence, or otherwise, solve the following system of equations.

$$\begin{aligned}3x+2y+z&=7\\x-2y-z&=1\\x\quad\quad+3z&=11\end{aligned}$$

$\square$, $\boxed{\mathbf{M}^{-1}=\frac{1}{12}\begin{pmatrix}3&3&0\\2&-4&-2\\-1&-1&4\end{pmatrix}}$, $\boxed{x=2,\ \ y=-1,\ \ z=3}$



Question 15 **(**)**

A 3×3 matrix $\mathbf{T}$ represents the linear transformation

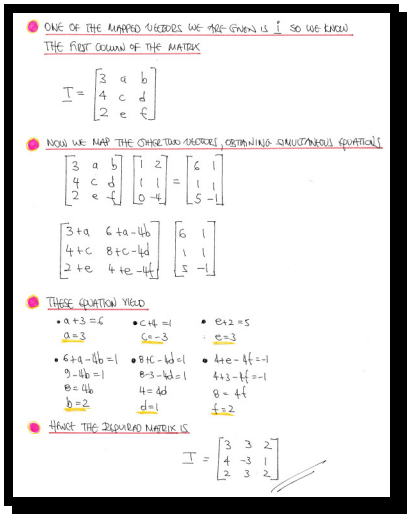
$$\mathbf{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} :\mapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

so that

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} :\mapsto \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} :\mapsto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Find the elements of $\mathbf{T}$.

$$\boxed{}, \qquad \boxed{\mathbf{T}=\begin{pmatrix} 3 & 3 & 2 \\ 4 & -3 & 1 \\ 2 & 3 & 2 \end{pmatrix}}$$

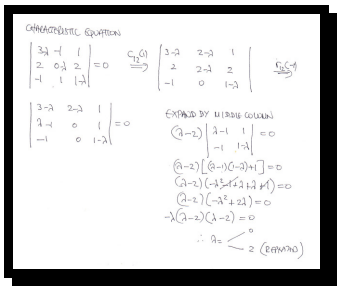


Question 16 (+)**

Find the eigenvalues of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$\boxed{\lambda=0,\ 2}$$



Question 17 (+)**

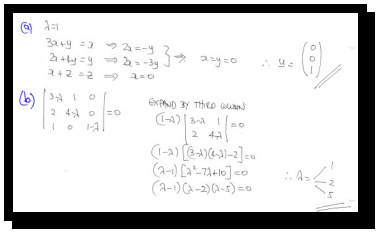
The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

a) Given that $\lambda=1$ is an eigenvalue of $\mathbf{A}$ find the corresponding eigenvector.

b) Find the other two eigenvalues of $\mathbf{A}$.

$$\boxed{\mathbf{u}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}},\quad \boxed{\lambda=2,\ 5}$$



Question 18 **(**+)**

A transformation in three dimensional space is defined by the following 3×3 matrix.

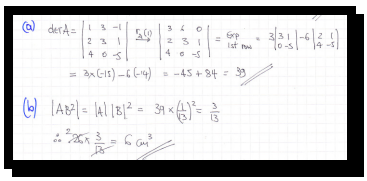
$$\mathbf{A}=\begin{pmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{pmatrix}.$$

a) Find the value of $\det \mathbf{A}$.

A cone with a volume of 26 cm^3 is transformed by the matrix composition $\mathbf{AB}^2$.

b) Given that $\det \mathbf{B}=\dfrac{1}{13}$, calculate the volume of the transformed cone.

$\boxed{\det\mathbf{A}=39}$, $\boxed{\text{volume}=6}$



Question 19 (+)**

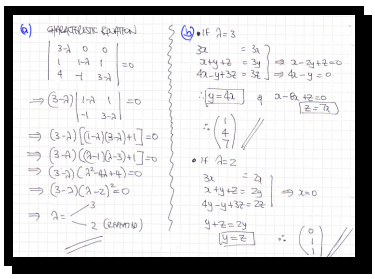
The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}3&0&0\\1&1&1\\4&-1&3\end{pmatrix}.$$

a) Show that $\mathbf{A}$ only has two eigenvalues.

b) Find the eigenvectors associated with each of these eigenvalues.

$$\boxed{\lambda=2,\ \lambda=2,\ \lambda=3},\ \boxed{\mathbf{u}=\begin{pmatrix}0\\1\\1\end{pmatrix}},\ \boxed{\mathbf{v}=\begin{pmatrix}1\\4\\7\end{pmatrix}}$$



Question 20 (+)**

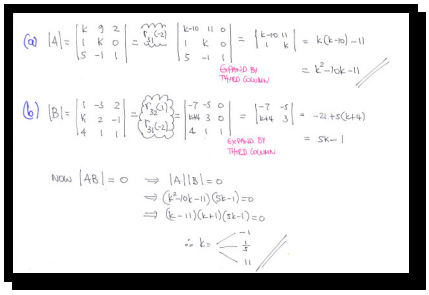
The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix} k & 9 & 2 \\ 1 & k & 0 \\ 5 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix} 1 & -3 & 2 \\ k & 2 & -1 \\ 4 & 1 & 1 \end{pmatrix}.$$

a) Find an expression for $\det\mathbf{A}$, in terms of k.

b) Find the possible values of k given that $\mathbf{AB}$ is singular.

$\det\mathbf{A}=k^2-10k-11$, $k=-1,\ 11,\ \frac{1}{5}$



Question 21 (+)**

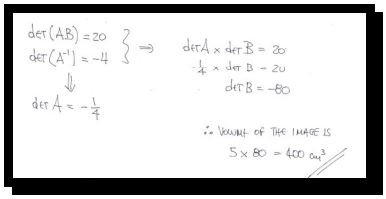
It is given that $\mathbf{A}$ and $\mathbf{B}$ are 3×3 matrices that satisfy

$$\det(\mathbf{AB})=20 \qquad \text{and} \qquad \det\left(\mathbf{A}^{-1}\right)=-4.$$

A solid S, of volume 5 cm^3, is transformed by $\mathbf{B}$ to produce an image S'.

Find the volume of S'.

400 cm^3



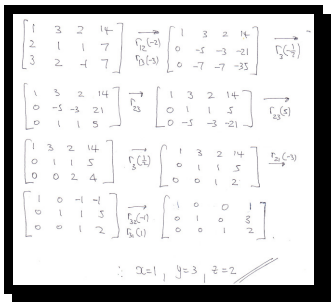
Question 22 (***)

$$\begin{aligned} x+3y+2z&=14\\ 2x+y+z&=7\\ 3x+2y-z&=7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$$\boxed{x=1,\quad y=3,\quad z=2}$$



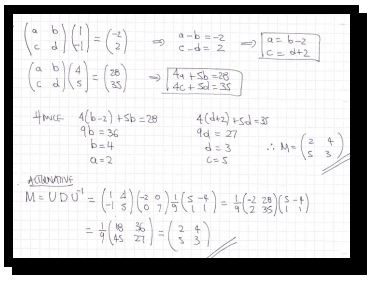
Question 23 (***)

A 2×2 matrix $\mathbf{M}$ has eigenvalues $\lambda=-2$ and $\lambda=7$, with respective eigenvectors

$$\mathbf{i}-\mathbf{j} \qquad \text{and} \qquad 4\mathbf{i}+5\mathbf{j}.$$

Find the elements of $\mathbf{M}$.

$$\boxed{\mathbf{M}=\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}}$$



Question 24 (***)

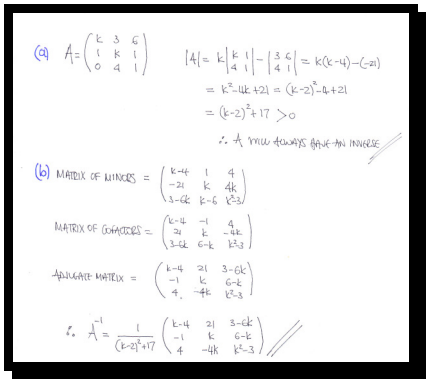
The 3×3 matrix $\mathbf{A}$ is given in terms of a constant k below.

$$\mathbf{A}=\begin{pmatrix} k & 3 & 6 \\ 1 & k & 1 \\ 0 & 4 & 1 \end{pmatrix}.$$

a) Show that $\mathbf{A}$ has an inverse for all values of k.

b) Find $\mathbf{A}^{-1}$ in terms of k.

$$\mathbf{A}=\frac{1}{(k-2)^2+17}\begin{pmatrix} k-4 & 21 & 3-6k \\ -1 & k & 6-k \\ 4 & -4k & k^2-3 \end{pmatrix}$$



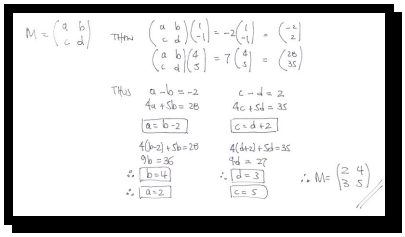
Question 25 (***)

The 2×2 matrix $\mathbf{M}$ has eigenvalues -2 and 7.

The respective eigenvectors of $\mathbf{M}$ are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Find the entries of $\mathbf{M}$.

$$\boxed{\mathbf{M}=\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}}$$



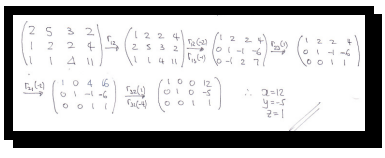
Question 26 (***)

$$\begin{aligned} 2x+5y+3z&=2 \\ x+2y+2z&=4 \\ x+y+4z&=11 \end{aligned}$$

Solve the above simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$$\boxed{x=12,\quad y=-5,\quad z=1}$$



Question 27 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

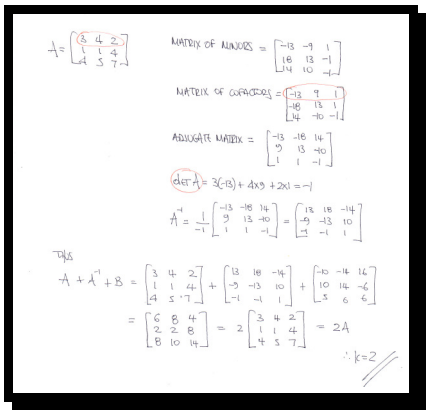
$$\mathbf{A}=\begin{pmatrix}3&4&2\\1&1&4\\4&5&7\end{pmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{pmatrix}-10&-14&16\\10&14&-6\\5&6&6\end{pmatrix}.$$

Show clearly that

$$\mathbf{A}+\mathbf{A}^{-1}+\mathbf{B}=k\mathbf{A}\,,$$

stating the value of the scalar constant k

$$\boxed{k=2}$$



Question 28 (***)

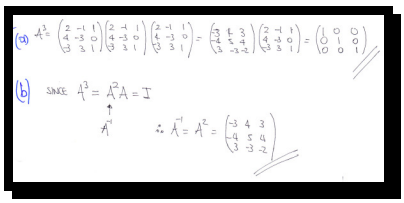
The 3×3 matrix $\mathbf{A}$ below, represents a transformation such that $\mathbb{R}^3 \mapsto \mathbb{R}^3$.

$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}.$$

a) Find the entries of $\mathbf{A}^3$.

b) Determine the entries of $\mathbf{A}^{-1}$.

$$\boxed{\mathbf{A}^3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \qquad \boxed{\mathbf{A}^{-1}=\begin{pmatrix} -3 & 4 & 3 \\ -4 & 5 & 4 \\ 3 & -3 & -2 \end{pmatrix}}$$

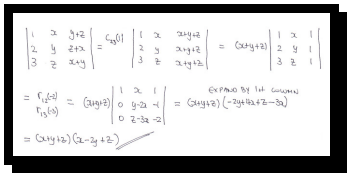


Question 29 (***)

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & x & y+z \\ 2 & y & z+x \\ 3 & z & x+y \end{vmatrix}.$$

$$\boxed{(x+y+z)(x-2y+z)}$$



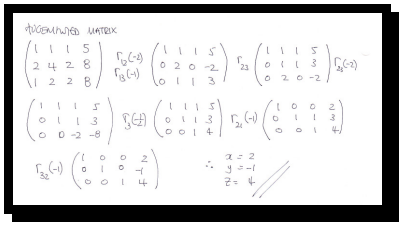
Question 30 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}$$

No credit will be given for alternative solution methods.

$$\boxed{x = 2, \quad y = -1, \quad z = 4}$$



Question 31 (***)

The 3×3 matrix $\mathbf{D}$ is given below in terms of the constants a, b, c and d.

$$\mathbf{D}=\begin{pmatrix} a & 1 & b \\ c & 7 & 0 \\ 3 & d & 2 \end{pmatrix}.$$

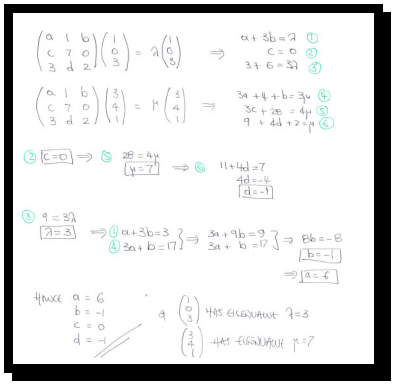
It is further given that

$$\mathbf{u}=\mathbf{i}+3\mathbf{k} \qquad \text{and} \qquad \mathbf{u}=3\mathbf{i}+4\mathbf{j}+\mathbf{k}$$

are eigenvectors of $\mathbf{D}$ with corresponding eigenvalues λ and μ.

Determine in any order the value of a, b, c, d, λ and μ.

$$\boxed{a=6},\ \boxed{b=-1},\ \boxed{c=0},\ \boxed{d=-1},\ \boxed{\lambda=3},\ \boxed{\mu=7},$$



Question 32 (***)

The 2×2 matrix $\mathbf{A}$ and the 3×3 matrix $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 2 & -4\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}2 & 1 & 1\\ 1 & 1 & 0\\ 0 & 2 & 1\end{pmatrix}.$$

The straight line L_1 with equation

$$y=x+k\,,$$

where k is a constant, is transformed by $\mathbf{A}$.

a) Find an equation for the image of L_1 under $\mathbf{A}$.

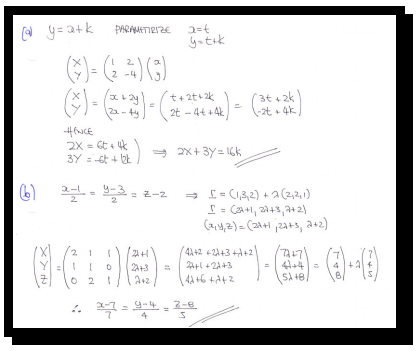
The straight line L_2 with Cartesian equation

$$\frac{x-1}{2}=\frac{y-3}{2}=z-2\,,$$

is transformed by $\mathbf{B}$.

b) Find a Cartesian equation for the image of L_2 under $\mathbf{B}$.

$\boxed{2x+3y=16k}$, $\boxed{\dfrac{x-7}{7}=\dfrac{y-4}{4}=\dfrac{z-8}{5}}$



Question 33 (***)

The 3×3 matrix $\mathbf{A}$ is given in terms of the scalar constant k by

$$\mathbf{A}=\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & k \\ 0 & 1 & 1 \end{pmatrix}.$$

a) Find, in terms of k, the inverse of $\mathbf{A}$.

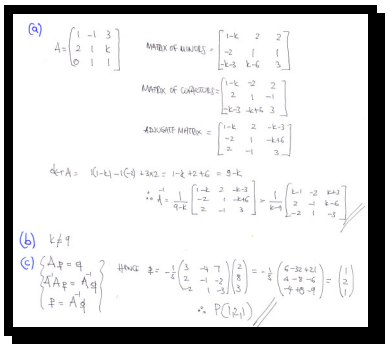
b) State the condition that k must satisfy, so that the inverse matrix exists.

Now suppose that $k=4$.

The point P has been transformed by the matrix $\mathbf{A}$ into the point $Q(2,8,3)$.

c) Determine the coordinates of P.

$$\boxed{\mathbf{A}^{-1}=\frac{1}{k-9}\begin{pmatrix} k-1 & -4 & k+3 \\ 2 & -1 & k-6 \\ -2 & 1 & -3 \end{pmatrix}}\,,\quad \boxed{k\neq 9}\,,\quad \boxed{P(1,2,1)}$$



Question 34 (***)

The 3×3 matrix $\mathbf{M}$ is given below, in terms of a scalar constant k.

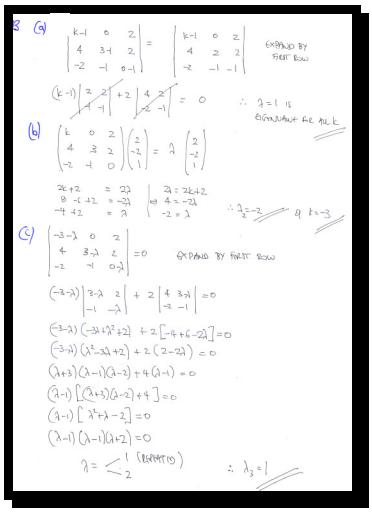
$$\mathbf{M}=\begin{pmatrix} k & 0 & 2 \\ 4 & 3 & 2 \\ -2 & -1 & 0 \end{pmatrix}$$

a) Show that $\lambda_1=1$ is an eigenvalue of $\mathbf{M}$ for all values of k.

b) Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$ with corresponding eigenvalue $\lambda_2\neq1$, find the values of λ_2 and the value of k.

c) Find the value of the third eigenvalue of $\mathbf{M}$.

$\boxed{\lambda_2=-2}$, $\boxed{k=-3}$, $\boxed{\lambda_3=1}$



Question 35 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

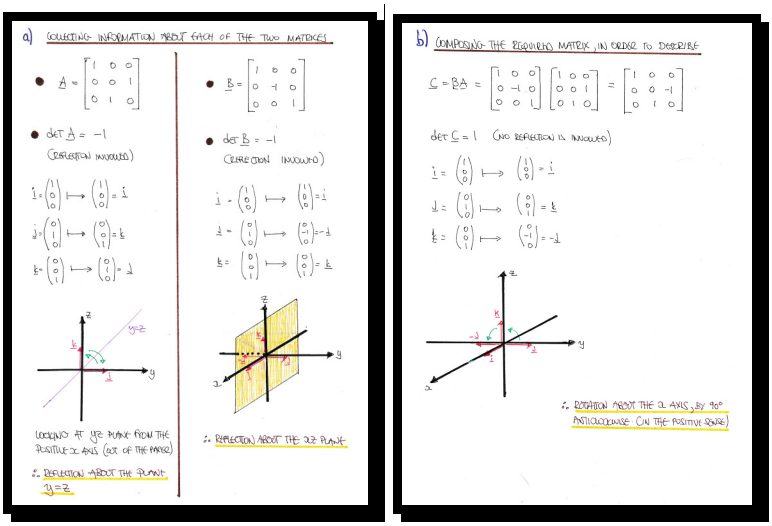
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{}$, $\boxed{\mathbf{A}:\text{reflection in the plane } y=z}$, $\boxed{\mathbf{B}:\text{reflection in the } xz \text{ plane}}$,

$\boxed{\mathbf{C}:\text{rotation in the } x \text{ axis, } 90^\circ\text{, anticlockwise}}$

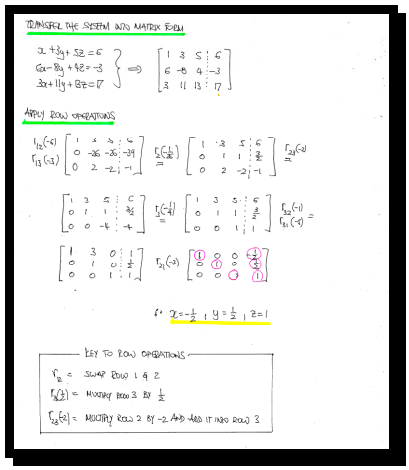


Question 36 (***)

$$\begin{aligned} x\ +3y\ +5z &= 6 \\ 6x\ -8y\ +4z &= -3 \\ 3x+11y+13z &= 17 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

V, [ ], $\boxed{x=-\frac{1}{2},\ y=\frac{1}{2},\ z=1}$



Question 37 (***)

The matrix $\mathbf{A}:\mathbb{R}^2\mapsto\mathbb{R}^2$ and the matrix $\mathbf{B}:\mathbb{R}^3\mapsto\mathbb{R}^3$ are defined as

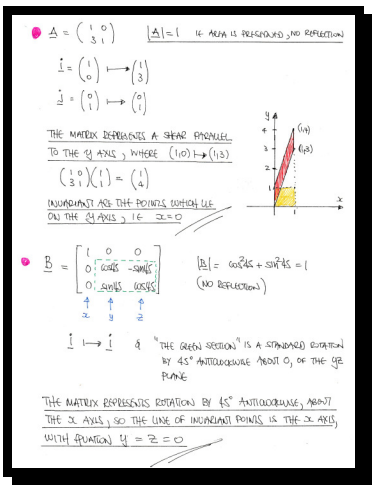
$$\mathbf{A}=\begin{pmatrix}1&0\\3&1\end{pmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{pmatrix}1&0&0\\0&\cos 45^\circ&-\sin 45^\circ\\0&\sin 45^\circ&\cos 45^\circ\end{pmatrix}.$$

Describe geometrically the transformations given by each of these matrices.

State in each case the equation of the line of invariant points.

$\boxed{}$, $\boxed{\mathbf{A}:\text{shear parallel to } y \text{ axis}, (1,0)\mapsto(3,1)}$,

$\boxed{\mathbf{B}:\text{rotation in the } x \text{ axis, } 45^\circ, \text{ anticlockwise}}$, $\boxed{\mathbf{A}: x=0}$, $\boxed{\mathbf{B}: y=z=0, \text{ i.e. } x \text{ axis}}$

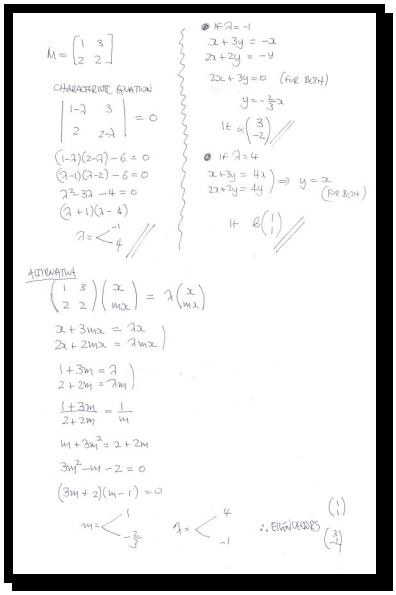


Question 38 (***)

Find the eigenvalues and the corresponding eigenvectors of the following 2×2 matrix.

$$\mathbf{M}=\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}.$$

$$\boxed{\lambda=-1,\ \ \mathbf{u}=\alpha\begin{pmatrix}3\\ -2\end{pmatrix}},\ \boxed{\lambda=4,\ \ \mathbf{u}=\beta\begin{pmatrix}1\\ 1\end{pmatrix}}$$



Question 39 (***)

A transformation of the x-y plane is represented by the following 2×2 matrix.

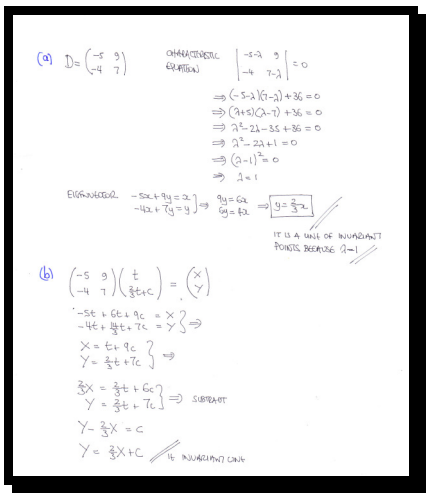
$$\mathbf{D}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}.$$

The straight line with equation of the form $y=ax$, where a is the gradient, is in the direction of the eigenvector of $\mathbf{D}$.

a) Find the equation of this straight line, stating whether this line is an invariant line or a line of invariant points.

b) Show that all the straight lines of the form $y=ax+c$, where c is a constant, remain invariant under the transformation represented by $\mathbf{D}$.

$$\boxed{y=\frac{2}{3}x}$$

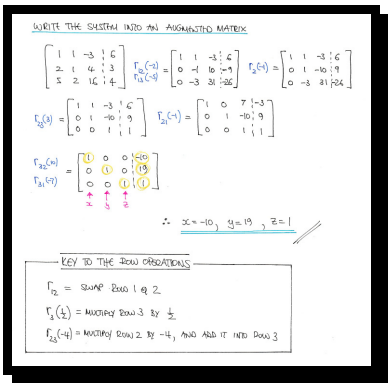


Question 40 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

V , $\boxed{}$, $\boxed{x=-10, \quad y=19, \quad z=1}$



Question 41 (***)

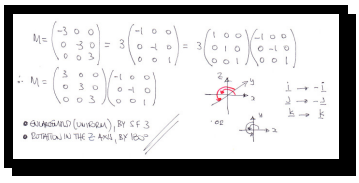
The 3×3 matrix $\mathbf{M}$ is given below.

$$\mathbf{M}=\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$\mathbf{M}$ describes two consecutive linear transformations of 3 dimensional space, which can be carried out in any order.

Describe geometrically each these two transformations.

$\boxed{\text{rotation about } z \text{ axis}, 180^\circ}$, $\boxed{\text{uniform enlargement, S.F.} = 3}$



Question 42 (***)

The system of simultaneous equations

$$\begin{aligned} x+\ \ y+2z&=2\\ x+2y+\ \ z&=2\\ 2x+ay+5z&=b \end{aligned}$$

where a and b are constants, does **not** have a unique solution, but it is **consistent**.

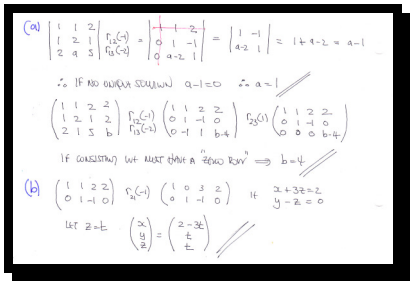
a) Determine the value of a and the value of b.

b) Show that the general solution of the system can be written as

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 2-3t\\ t\\ t \end{pmatrix},$$

where t is a parameter.

$\boxed{a=1}$, $\boxed{b=4}$



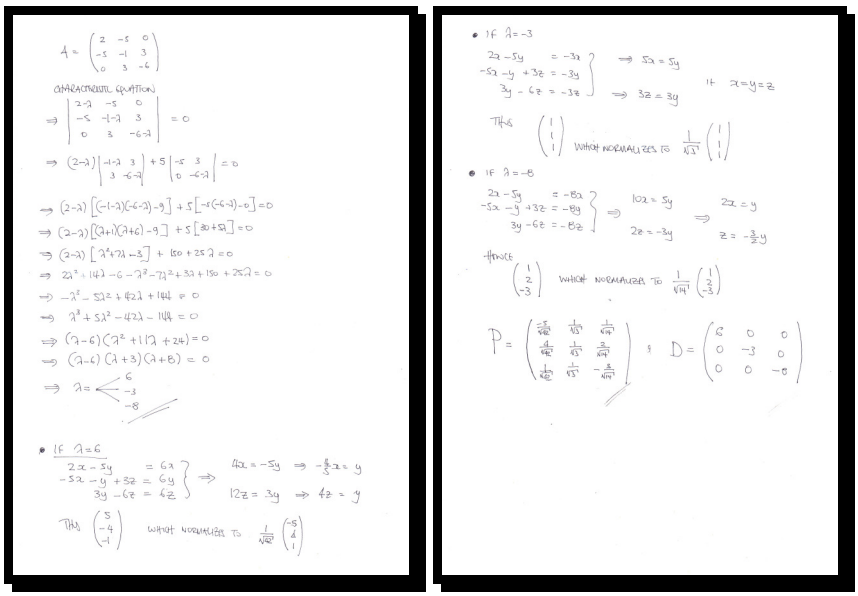
Question 43 (***)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}2&-5&0\\-5&-1&3\\0&3&-6\end{pmatrix}.$$

As $\mathbf{A}$ is a symmetric matrix, find the orthogonal 3×3 matrix $\mathbf{P}$ and a diagonal 3×3 matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}=\mathbf{D}$.

$$\mathbf{P}=\begin{pmatrix}-\frac{5}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{14}}\\\frac{4}{\sqrt{42}}&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{14}}\\\frac{1}{\sqrt{42}}&\frac{1}{\sqrt{3}}&-\frac{3}{\sqrt{14}}\end{pmatrix},\qquad \mathbf{D}=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-8\end{pmatrix}$$

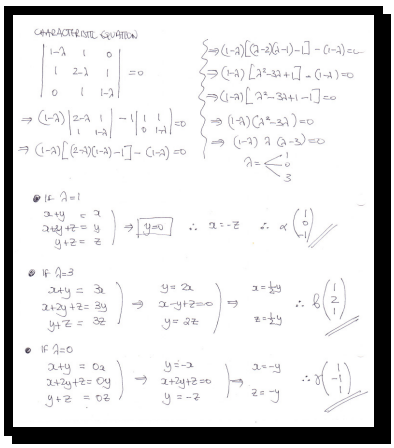


Question 44 (***)

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

$$\boxed{\lambda_1=0,\quad \mathbf{u}=\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}},\quad \boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}},\quad \boxed{\lambda_3=3,\quad \mathbf{w}=\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}}$$



Question 45 (***)

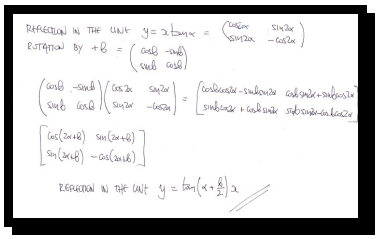
A linear transformation T in the $x-y$ plane consists of a reflection about the straight line with equation

$$y = x\tan\alpha^\circ,$$

followed by an anticlockwise rotation about the origin O, by an angle of β°.

By considering matrix compositions, or otherwise, describe T geometrically.

$$\boxed{\text{reflection in the line } y = \tan\left(\alpha^\circ + \frac{\beta^\circ}{2}\right)}$$



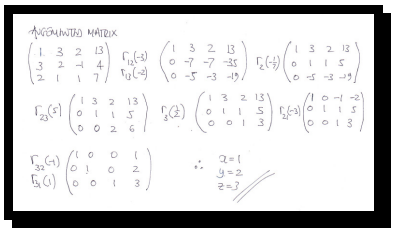
Question 46 (***)

$$\begin{aligned} x+3y+2z &= 13 \\ 3x+2y-z &= 4 \\ 2x+y+z &= 7 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

No credit will be given for alternative solution methods.

$$\boxed{x=1,\quad y=2,\quad z=3}$$



Question 47 (***)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{pmatrix}$$

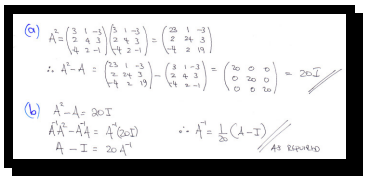
The matrix $\mathbf{A}$ is non singular.

a) Evaluate $\mathbf{A}^2-\mathbf{A}$.

b) Show clearly that

$$\mathbf{A}^{-1}=\frac{1}{20}\left[\mathbf{A}-\mathbf{I}\right].$$

$\boxed{20\mathbf{I}}$



Question 48 (***)

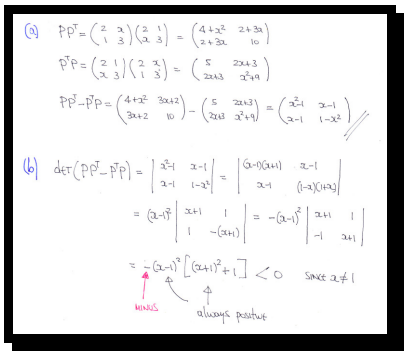
The 2×2 matrix $\mathbf{P}$ is defined in terms of x, where $x\neq 1$.

$$\mathbf{P}=\begin{pmatrix}2 & x\\ 1 & 3\end{pmatrix}.$$

a) Find in its simplest form the matrix $\mathbf{PP}^{\mathrm{T}}-\mathbf{P}^{\mathrm{T}}\mathbf{P}$.

b) Show clearly that $\det\left(\mathbf{PP}^{\mathrm{T}}-\mathbf{P}^{\mathrm{T}}\mathbf{P}\right)<0$.

$$\mathbf{PP}^{\mathrm{T}}-\mathbf{P}^{\mathrm{T}}\mathbf{P}=\begin{pmatrix}x^2-1 & x-1\\ x-1 & 1-x^2\end{pmatrix}$$



Question 49 **(***)**

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}.$$

Show that the above system of simultaneous equations …

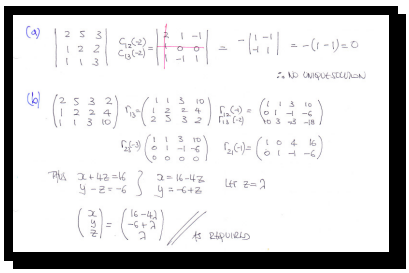
a) … does **not** have a unique solution.

b) … is **consistent** and the general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 16-4\lambda \\ \lambda-6 \\ \lambda \end{pmatrix},$$

where λ is a scalar parameter.

proof



Question 50 (***)

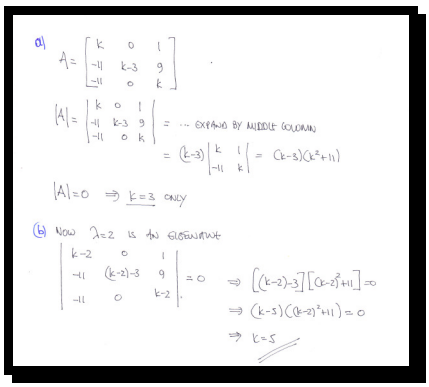
The 3×3 matrix $\mathbf{A}$ is given below, in terms of a scalar constant k.

$$\mathbf{A}=\begin{pmatrix} k & 0 & 1 \\ -11 & k-3 & 9 \\ -11 & 0 & k \end{pmatrix}.$$

a) Given that $\mathbf{A}$ is singular, find the value of k.

b) Given instead that $\lambda=2$ is an eigenvalue of $\mathbf{A}$, determine the value of k on this occasion.

$\boxed{k=3}$, $\boxed{k=5}$



Question 51 (***)

Three planes, the equations of which are given below, intersect along a straight line L.

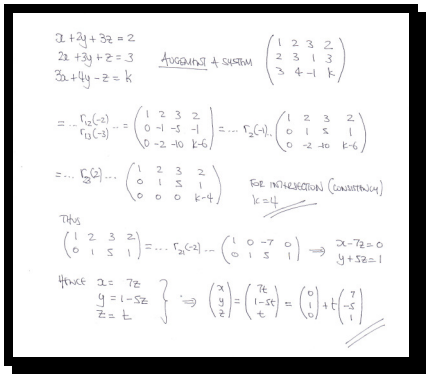
$$\begin{aligned} x+2y+3z&=2\\ 2x+3y+z&=3\\ 3x+4y-z&=k \end{aligned}$$

Show, by reducing an augmented matrix into row echelon form, that the equation of L can be written in the form

$$\mathbf{r}=\mathbf{j}+t\left(7\mathbf{i}-5\mathbf{j}+\mathbf{k}\right),$$

where t is a scalar parameter.

proof



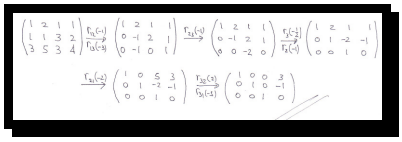
Question 52 (***)

Solve the following simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

No credit will be given for alternative solution methods.

$$\boxed{x=3,\quad y=-1,\quad z=0}$$



Question 53 (***)

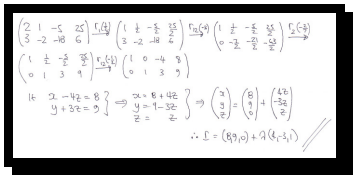
$$\begin{aligned} 3x-2y-18z&=6 \\ 2x+y-5z&=25 \end{aligned}$$

Show, by reducing the system into row echelon form, that the solution can be written in the form

$$\mathbf{r}=8\mathbf{i}+9\mathbf{j}+\lambda\left(4\mathbf{i}-3\mathbf{j}+\mathbf{k}\right),$$

where λ is a scalar parameter.

proof



Question 54 (***)

The 3×3 matrix $\mathbf{R}$ is defined by

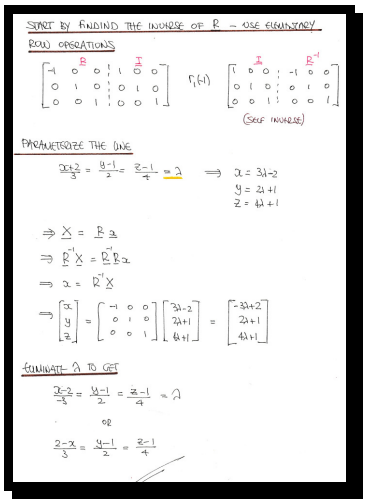
$$\mathbf{R}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

The image of the straight line L, when transformed by $\mathbf{R}$, is the straight line with Cartesian equation

$$\frac{x+2}{3}=\frac{y-1}{2}=\frac{z-1}{4}.$$

Find a Cartesian equation for L.

$\boxed{}$, $\boxed{\dfrac{x-2}{-3}=\dfrac{y-1}{2}=\dfrac{z-1}{4}}$



Question 55 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

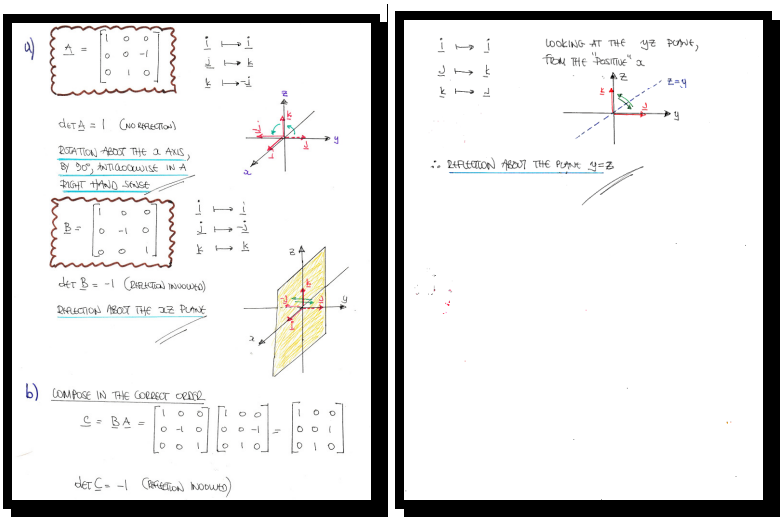
a) Describe geometrically the transformations given by each of the two matrices.

The matrix $\mathbf{C}$ is defined as the transformation defined by the matrix $\mathbf{A}$, followed by the transformation defined by the matrix $\mathbf{B}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

, $\mathbf{A}$: rotation about x axis, 90° anticlockwise, $\mathbf{B}$: reflection in the xz plane,

$\mathbf{C}$: reflection in the plane $y=z$



Question 56 (***)

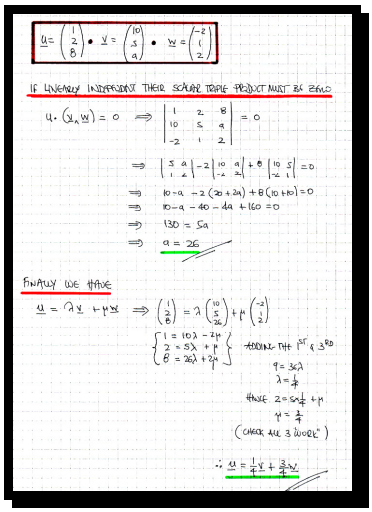
The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are defined as

$$\mathbf{u}=\begin{bmatrix}1\\2\\8\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}10\\5\\a\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}-2\\1\\2\end{bmatrix},$$

where a is a scalar constant.

Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, determine the value of a and hence express $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

$\boxed{}$, $\boxed{a=26}$, $\boxed{\mathbf{u}=\frac{1}{4}\mathbf{v}+\frac{3}{4}\mathbf{w}}$



Question 57 (***)

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, are given in terms of the constants k and h below.

$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&k&4\\3&2&-1\end{pmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{pmatrix}15&-4&-1\\h&4&2\\17&-4&-3\end{pmatrix}.$$

a) Find the matrix composition $\mathbf{AB}$, in terms of k and h.

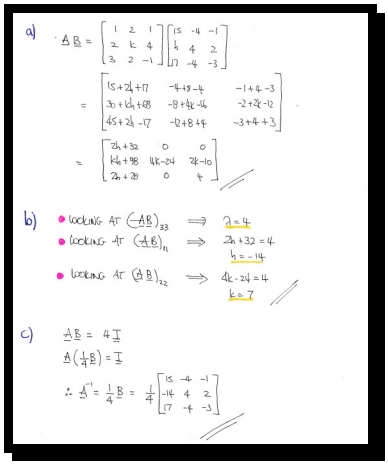
It is further given that $\mathbf{AB}=\lambda\mathbf{I}$ for some values of k and h.

b) Find the value of each of the constants λ, k and h.

c) Deduce $\mathbf{A}^{-1}$, for the values of λ, k and h, found in part **(b)**.

$\square$, $\boxed{\mathbf{AB}=\begin{pmatrix}2h+32&0&0\\hk+98&4k-24&2k-14\\2h+28&0&4\end{pmatrix}}$, $\boxed{\lambda=4,\ h=14,\ k=7}$,

$$\boxed{\mathbf{A}^{-1}=\frac{1}{4}\begin{pmatrix}15&-4&-1\\-14&4&2\\17&-4&-3\end{pmatrix}}$$



Question 58 (***)

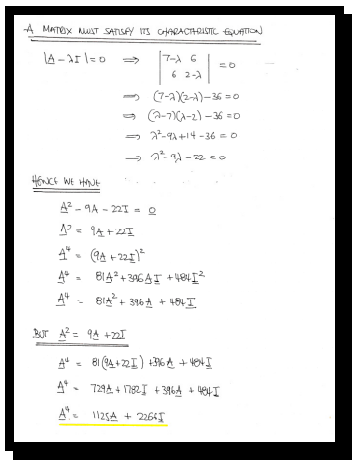
The 2×2 matrix $\mathbf{A}=\begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}$ is given.

Use the Caley- Hamilton theorem to show that

$$\mathbf{A}^4=\lambda\mathbf{A}+\mu\mathbf{I},$$

where $\mathbf{I}$ is 2×2 identity matrix.

$\boxed{}$, $\boxed{\mathbf{A}^4=1125\mathbf{A}+2266\mathbf{I}}$



Question 59 (*+)**

A system of equation is given in matrix form below

$$\begin{pmatrix} t & 2 & 3 \\ 2 & 3 & -t \\ 3 & 5 & t+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

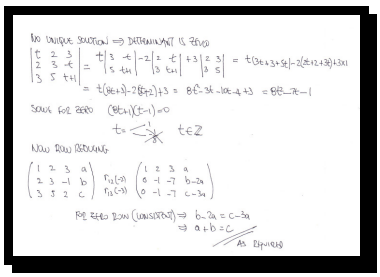
where t is an integer constant, and a, b and c are real constants.

The system of equations does not have a unique solution, but it is consistent.

Show clearly that

$$a+b=c\,.$$

proof

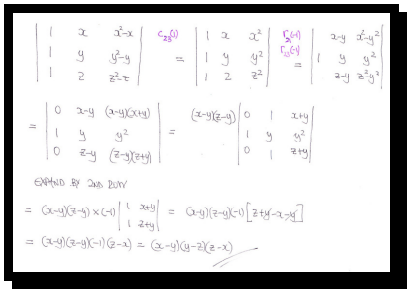


Question 60 (*+)**

Express the following 3×3 determinant as the product of three linear factors.

$$\begin{vmatrix} 1 & x & x^2-x \\ 1 & y & y^2-y \\ 1 & z & z^2-z \end{vmatrix}$$

$$\boxed{(x-y)(y-z)(z-x)}$$

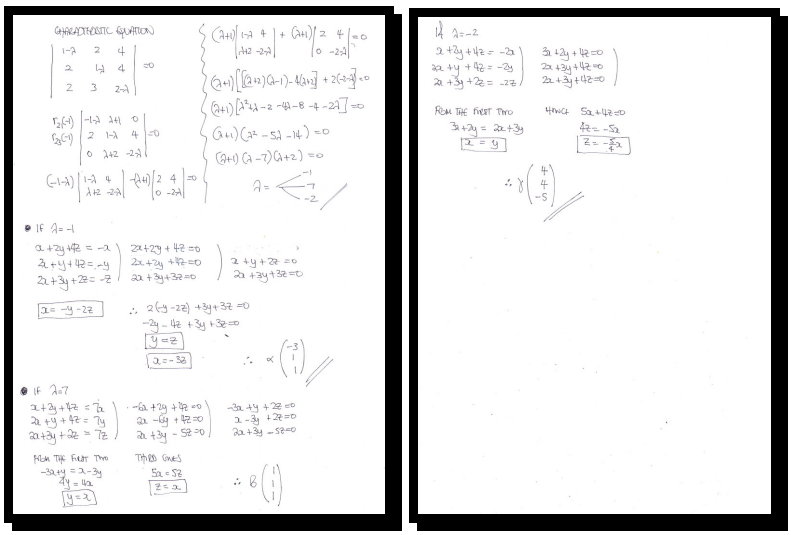


Question 61 (*+)**

Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrix.

$$\mathbf{A}=\begin{pmatrix}1&2&4\\2&1&4\\2&3&2\end{pmatrix}.$$

$$\boxed{\lambda_1=-1,\quad \mathbf{u}=\begin{pmatrix}-3\\1\\1\end{pmatrix}},\quad \boxed{\lambda_2=-2,\quad \mathbf{v}=\begin{pmatrix}4\\4\\-5\end{pmatrix}},\quad \boxed{\lambda_3=7,\quad \mathbf{w}=\begin{pmatrix}1\\1\\1\end{pmatrix}}$$



Question 62 **(***+)**

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k as

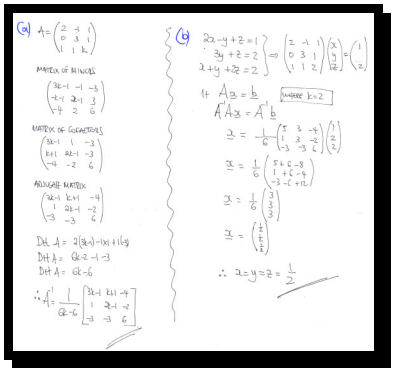
$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & k \end{pmatrix}$$

a) Find $\mathbf{A}^{-1}$, in terms of k.

b) Hence solve the following simultaneous equations

$$\begin{aligned} 2x-y+z&=1 \\ 3y+z&=2 \\ x+y+2z&=2 \end{aligned}$$

$$\boxed{\mathbf{A}^{-1}=\frac{1}{6(k-1)}\begin{pmatrix} 3k-1 & k+1 & -4 \\ 1 & 2k-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}}\,,\qquad \boxed{x=\frac{1}{2},\ y=\frac{1}{2},\ z=\frac{1}{2}}$$



Question 63 (*+)**

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix} 1 & -2 & 2 \\ k & 1 & k-1 \\ 2 & 2k-1 & 2-k \end{pmatrix}.$$

a) Show that $\det \mathbf{A}$ is independent of k.

b) Determine, with full justification, whether the vectors

$$\mathbf{i}+2\mathbf{j}+2\mathbf{k}\,, \quad -2\mathbf{i}+\mathbf{j}+3\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}$$

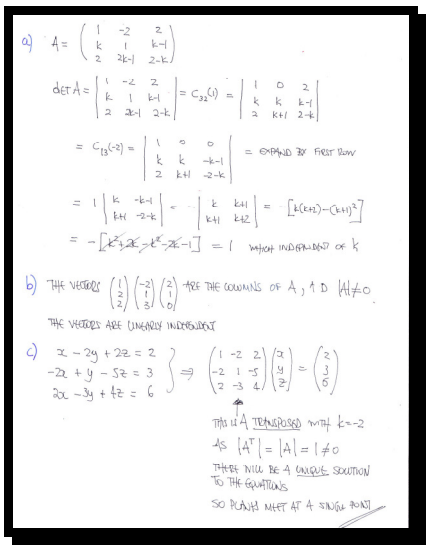
are linearly dependent or linearly independent.

The equations of three planes are given below.

$$\begin{aligned} x-2y+2z&=2 \\ -2x+y-5z&=3 \\ 2x-3y+4z&=6 \end{aligned}$$

c) Determine, with full justification, the geometrical configuration of these three planes.

linearly independent , all 3 planes meet at a single point

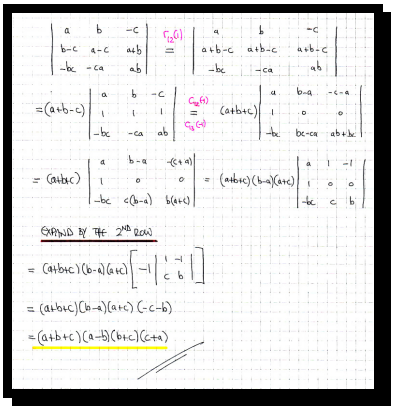


Question 64 (*+)**

By using elementary row and column operations, or otherwise, factorize the following determinant completely.

$$\begin{vmatrix} a & b & -c \\ b-c & a-c & a+b \\ -bc & -ca & ab \end{vmatrix}.$$

$\boxed{}$, $\boxed{(a+b+c)(a-b)(b+c)(c+a)}$



Question 65 (*+)**

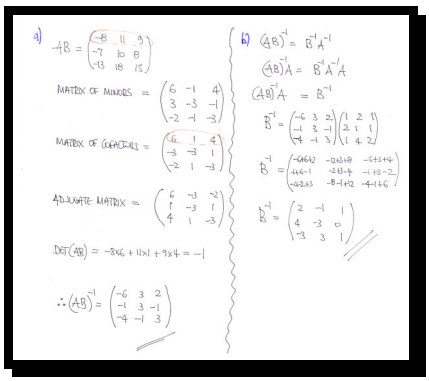
The 3×3 matrices $\mathbf{A}$ and $\mathbf{AB}$ are given below.

$$\mathbf{A}=\begin{pmatrix}1&2&1\\2&1&1\\1&4&2\end{pmatrix}\qquad\text{and}\qquad\mathbf{AB}=\begin{pmatrix}-8&11&9\\-7&10&8\\-13&18&15\end{pmatrix}.$$

a) Find the inverse of $\mathbf{AB}$.

b) Hence determine the inverse of $\mathbf{B}$.

$$\boxed{(\mathbf{AB})^{-1}=\begin{pmatrix}-6&3&2\\-1&3&-1\\-4&-1&3\end{pmatrix}},\qquad\boxed{\mathbf{B}^{-1}=\begin{pmatrix}2&-1&1\\4&-3&0\\-3&3&1\end{pmatrix}}$$



Question 66 **(***+)**

The 3×3 matrix $\mathbf{A}$ is defined by

$$\mathbf{A}=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}.$$

a) Describe geometrically the transformation given by $\mathbf{A}$.

The 3×3 matrix $\mathbf{B}$ represents a rotation of $180°$ about the line $x=z$, $y=0$.

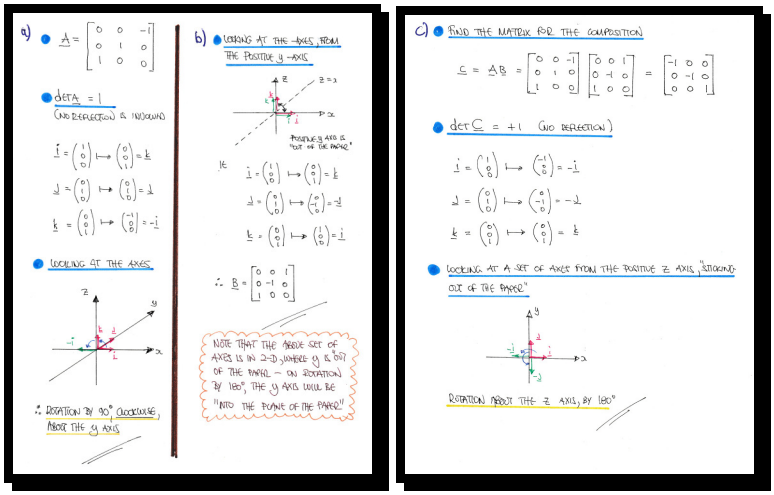
b) Determine the elements of $\mathbf{B}$.

The 3×3 matrix $\mathbf{C}$ is represents the transformation defined by $\mathbf{B}$, followed by the transformation defined by $\mathbf{A}$.

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{}$, $\boxed{\mathbf{A}:\text{rotation about } y \text{ axis, } 90° \text{ clockwise}}$, $\boxed{\mathbf{B}=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}}$,

$\boxed{\mathbf{C}:\text{rotation about } z \text{ axis, } 180°}$



Question 67 (*+)**

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&1&0\\3&-3&1\\0&3&2\end{pmatrix}.$$

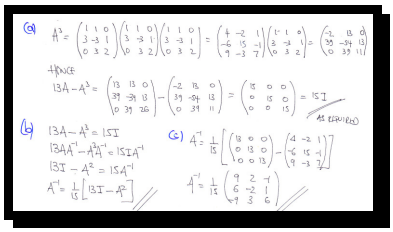
a) Show that

$$13\mathbf{A}-\mathbf{A}^3=15\mathbf{I}.$$

b) Hence find an expression for $\mathbf{A}^{-1}$ in terms of other matrices.

c) Use this expression to find $\mathbf{A}^{-1}$.

$$\boxed{\mathbf{A}^{-1}=\frac{1}{15}\left(13\mathbf{I}-\mathbf{A}^2\right)},\quad \boxed{\mathbf{A}^{-1}=\frac{1}{15}\begin{pmatrix}9&2&-1\\6&-2&1\\-9&3&6\end{pmatrix}}$$



Question 68 (*+)**

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}.$$

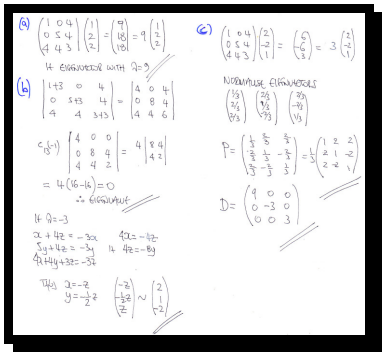
a) Verify that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue.

b) Show that -3 is an eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

c) Given further that $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is another eigenvector of $\mathbf{A}$, find 3×3 matrices $\mathbf{P}$ and $\mathbf{D}$ such that

$$\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}.$$

$\boxed{\lambda=9}$, $\boxed{\begin{pmatrix}2\\1\\-2\end{pmatrix}}$, $\boxed{\mathbf{D}=\begin{pmatrix}9&0&0\\0&-3&0\\0&0&3\end{pmatrix}}$, $\boxed{\mathbf{P}=\frac{1}{3}\begin{pmatrix}1&2&2\\2&1&-2\\2&-2&1\end{pmatrix}}$



Question 69 (*+)**

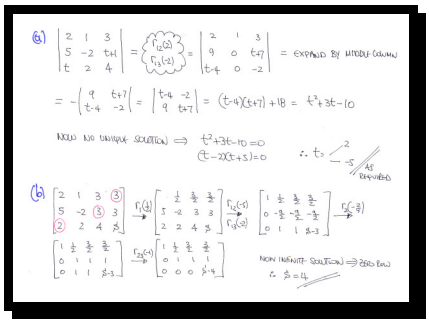
A system of equations is given below in terms of the scalar parameters t and s.

$$\begin{aligned} 2x\ \ +y\ \ \ \ \ \ +3z&=t+1\\ 5x\ -2y\ +(t+1)z&=3\\ tx\ \ +2y\ \ \ \ \ \ +4z&=s \end{aligned}$$

a) Show that if $t=-5$ or $t=2$, the system does not have a unique solution.

b) Determine the value of s is the system is to have infinite solutions with $t=2$.

$$\boxed{s=4}$$



Question 70 **(***+)**

The 3×3 matrix $\mathbf{A}$ is defined in terms of a scalar constant k by

$$\mathbf{A}=\begin{pmatrix} k & 8 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}.$$

The straight line L is a line of invariant points under $\mathbf{A}$.

Determine, in any order, …

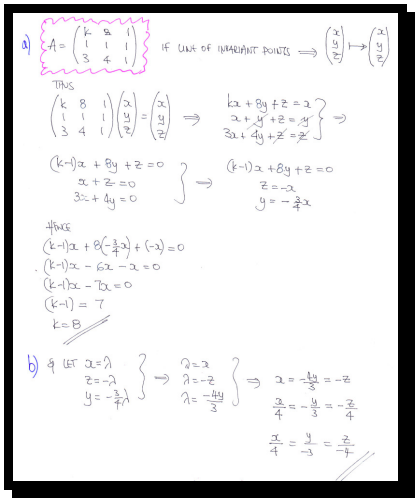
a) … the value of k.

b) … the Cartesian equation of L, giving the answer in the form

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

where l, m and n are integers to be found.

$$\boxed{k=8}\,,\quad \boxed{\frac{x}{4}=\frac{y}{-3}=\frac{z}{-4}}$$



Question 71 (*+)**

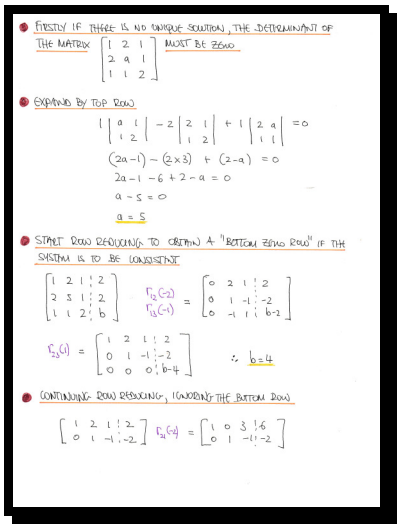
The three planes defined by the equations

$$\begin{aligned} x+2y+\ z&=2\\ 2x+a\,y+\ z&=1\\ x+\ y+2z&=b \end{aligned}$$

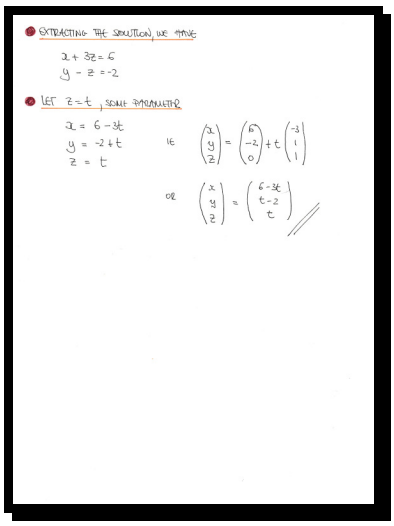
where a and k are constants, intersect along a straight line L.

Determine an equation of L.

$\boxed{}$, $\boxed{\mathbf{r}=(6-3t)\mathbf{i}+(t-2)\mathbf{j}+t\mathbf{k}}$



- FIRSTLY IF THERE IS NO UNIQUE SOLUTION, THE DETERMINANT OF THE MATRIX $\begin{bmatrix}1&2&1\\2&a&1\\1&1&2\end{bmatrix}$ MUST BE ZERO
- EXPANDING BY TOP ROW
$$1\begin{vmatrix}a&1\\1&2\end{vmatrix}-2\begin{vmatrix}2&1\\1&2\end{vmatrix}+1\begin{vmatrix}2&a\\1&1\end{vmatrix}=0$$
$$(2a-1)-(2\times3)+(2-a)=0$$
$$2a-1-6+2-a=0$$
$$a-5=0$$
$$a=5$$
- START ROW REDUCING TO OBTAIN A "BOTTOM ZERO ROW" IF THE SYSTEM IS TO BE CONSISTENT
$$\begin{bmatrix}1&2&1&2\\2&5&1&2\\1&1&2&b\end{bmatrix}\ \begin{matrix}r_2-2r_1\\r_3-r_1\end{matrix}\ =\ \begin{bmatrix}1&2&1&2\\0&1&-1&-2\\0&-1&1&b-2\end{bmatrix}$$
$$r_{23}(1)=\begin{bmatrix}1&2&1&2\\0&1&-1&-2\\0&0&0&b-4\end{bmatrix}\qquad\therefore\ b=4$$
- CONTINUING ROW REDUCING, IGNORING THE BOTTOM ROW
$$\begin{bmatrix}1&2&1&2\\0&1&-1&-2\end{bmatrix}\ r_{21}(-2)\ =\begin{bmatrix}1&0&3&6\\0&1&-1&-2\end{bmatrix}$$



- EXTRACTING THE SOLUTION, WE HAVE
$$x+3z=6$$
$$y-z=-2$$
- LET $z=t$, SOME PARAMETER
$$\begin{aligned}x&=6-3t\\y&=-2+t\\z&=t\end{aligned}\qquad\text{IE}\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-2\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\1\end{pmatrix}$$
$$\text{OR}\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6-3t\\t-2\\t\end{pmatrix}$$

Question 72 (*+)**

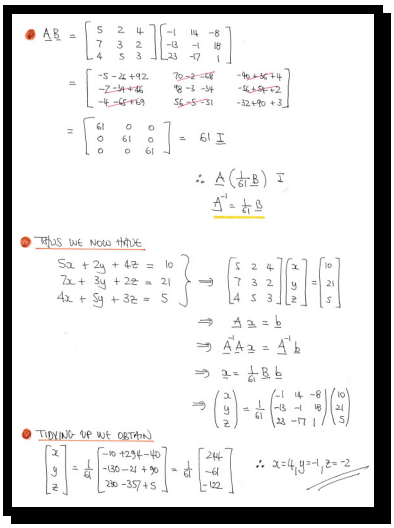
The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

$$\mathbf{A}=\begin{pmatrix}5&2&4\\7&3&2\\4&5&3\end{pmatrix} \qquad \text{and} \qquad \mathbf{B}=\begin{pmatrix}-1&14&-8\\-13&-1&18\\23&-17&1\end{pmatrix}$$

Find an expression for $\mathbf{AB}$ and use it to solve the following system of equations.

$$\begin{aligned}5x+2y+4z&=10\\7x+3y+2z&=21\\4x+5y+3z&=5\end{aligned}$$

$\boxed{}$, $\boxed{x=4,\ \ y=-1,\ \ z=-2}$



Question 73 (*+)**

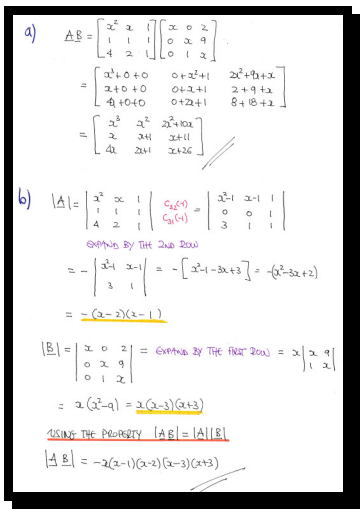
The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, are defined in terms of the scalar constants x as follows.

$$\mathbf{A}=\begin{pmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix} x & 0 & 2 \\ 0 & x & 9 \\ 0 & 1 & x \end{pmatrix}.$$

a) Find an expression for $\mathbf{AB}$, in terms of x.

b) By considering the properties of the determinants, or otherwise, find $\det(\mathbf{AB})$ in fully factorized form.

$$\boxed{}, \quad \boxed{\mathbf{AB}=\begin{pmatrix} x^3 & x^2+1 & 2x^2+10x \\ x & x+1 & x+11 \\ 4x & 2x+1 & x+26 \end{pmatrix}}, \quad \boxed{-x(x-1)(x-2)(x-3)(x+3)}$$

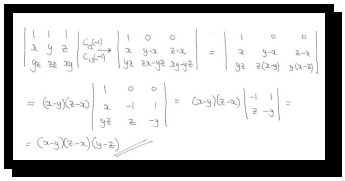


Question 74 (*+)**

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)}$$



Question 75 (*+)**

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}4&-1&1\\-1&6&-1\\1&-1&4\end{pmatrix}.$$

a) Show that $\lambda=7$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Find the eigenvector associated with the eigenvalue $\lambda=7$.

The other two eigenvectors of $\mathbf{A}$ are

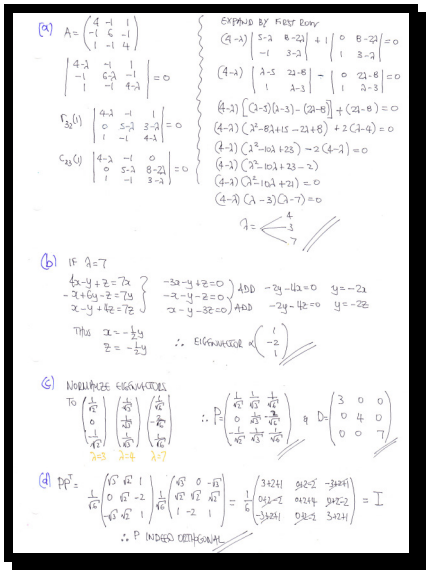
$$\mathbf{u}=\mathbf{i}-\mathbf{k} \qquad \text{and} \qquad \mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k},$$

where the eigenvalue of $\mathbf{v}$ is greater than the eigenvalue of $\mathbf{u}$.

c) Find a 3×3 matrix $\mathbf{P}$ and a 3×3 diagonal matrix $\mathbf{D}$ such that $\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$.

d) Show that $\mathbf{P}$ is an orthogonal matrix.

$$\boxed{\lambda=4,3},\quad \boxed{\alpha\begin{pmatrix}1\\-2\\1\end{pmatrix}},\quad \boxed{\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}},\quad \boxed{\mathbf{D}=\begin{pmatrix}3&0&0\\0&4&0\\0&0&7\end{pmatrix}}$$



Question 76 (*+)**

$$\begin{aligned} x+y-2z&=2\\ 3x-y+6z&=2\\ 6x+5y-9z&=k \end{aligned}$$

a) Show that the system of equations does not have a unique solution.

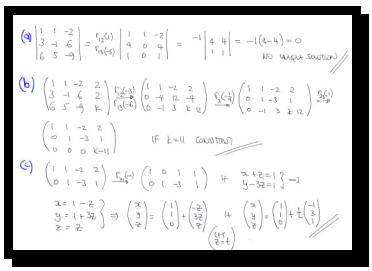
b) Show that there exists a value of k for which the system is consistent.

c) Show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x=1-t\,,\ y=3t+1\,,\ z=t$$

where t is a scalar parameter.

$\boxed{k=11}$

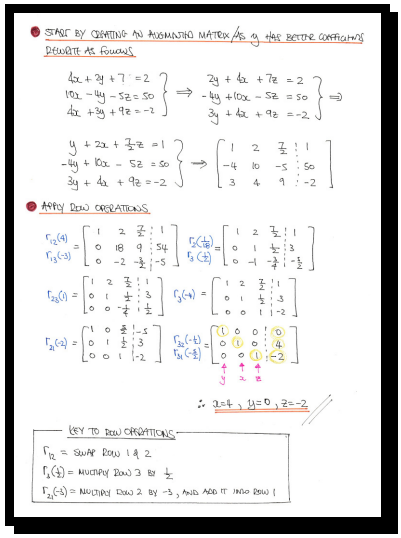


Question 77 (*+)**

$$\begin{aligned} 4x\ \ +2y+7z&=\ \ 2\\ 10x-4y-5z&=\ 50\\ 4x\ \ +3y+9z&=-2 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

$\boxed{}$, $\boxed{x=4,\ \ y=0,\ \ z=-2}$

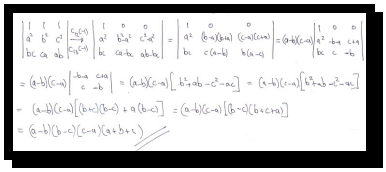


Question 78 (*+)**

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}.$$

$$\boxed{(a-b)(b-c)(c-a)(a+b+c)}$$



Question 79 (*+)**

The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below.

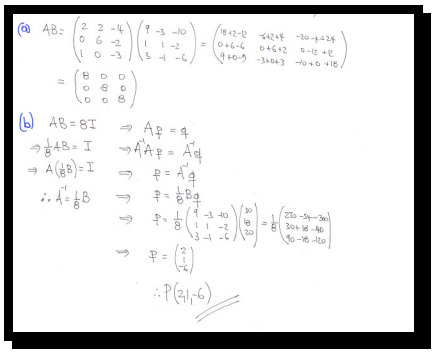
$$\mathbf{A}=\begin{pmatrix}2&2&-4\\0&6&-2\\1&0&-3\end{pmatrix}\quad\text{and}\quad\mathbf{B}=\begin{pmatrix}9&-3&-10\\1&1&-2\\3&-1&-6\end{pmatrix}.$$

a) Find the matrix composition $\mathbf{AB}$.

The point P has been transformed by $\mathbf{A}$ into the point $Q(30,18,20)$.

b) Determine the coordinates of P.

$$\boxed{\mathbf{AB}=\begin{pmatrix}8&0&0\\0&8&0\\0&0&8\end{pmatrix}},\quad\boxed{P(2,1,-6)}$$



Question 80 (*+)**

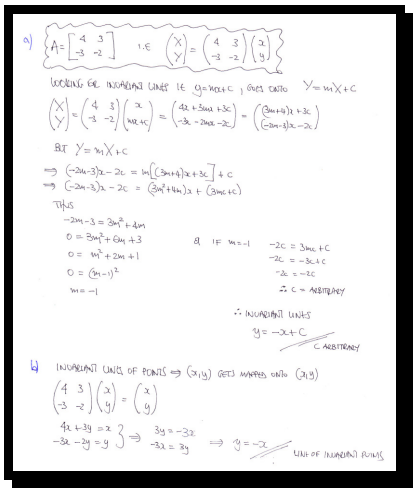
The 2×2 matrix $\mathbf{A}$ maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$ and is given by

$$\mathbf{A}=\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}.$$

a) Determine an equation of an invariant straight line under $\mathbf{A}$.

b) Find an equation of a straight line of invariant points under $\mathbf{A}$.

$\boxed{y=-x+c}$, $\boxed{y=-x}$



Question 81 (*+)**

The 2×2 matrix $\mathbf{A}$ is defined below in terms of the scalar constants p, q and r.

$$\mathbf{A}=\begin{pmatrix} 4 & p \\ q & r \end{pmatrix}.$$

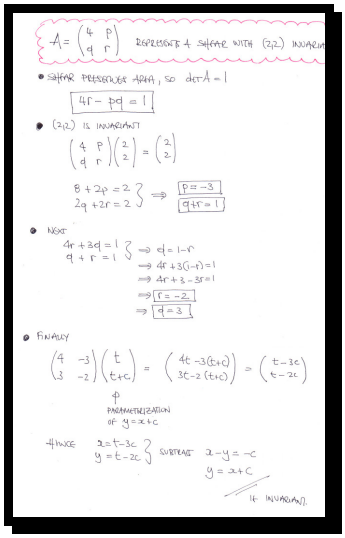
It is further given that $\mathbf{A}$ represents a shear under which the point $(2,2)$ is invariant.

Show that all straight lines of the form

$$y=x+c,$$

where c is a constant, are invariant under the shear represented by $\mathbf{A}$.

proof



Question 82 (*+)**

The 3×3 matrix $\mathbf{A}$ is given below.

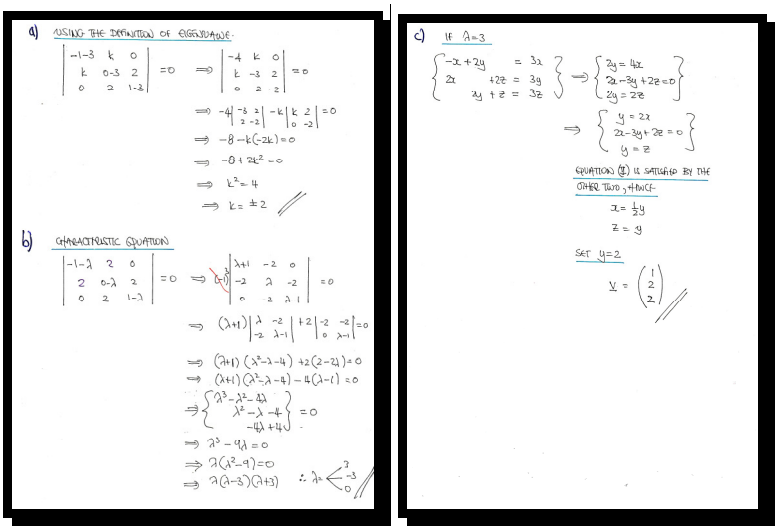
$$\mathbf{A}=\begin{pmatrix}-1 & k & 0\\ k & 0 & 2\\ 0 & 2 & 1\end{pmatrix}.$$

a) If one of the eigenvalues of $\mathbf{A}$ is 3, find the possible values of k.

b) Determine the other two eigenvalues of $\mathbf{A}$, given that $k>0$.

c) Find an eigenvector corresponding to the eigenvalue 3.

$$\boxed{},\ \boxed{k=\pm2},\ \boxed{\lambda=0,\quad \lambda=-3},\ \boxed{\mathbf{v}=\alpha\left(\mathbf{i}+2\mathbf{j}+2\mathbf{k}\right)}$$



Question 83 **(***+)**

Consider the following matrix equation

$$\begin{pmatrix} k & 1 & 0 \\ 3 & -2 & k-3 \\ 10k & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} a \\ b \\ 15 \end{pmatrix},$$

where a, b and k are scalar constants.

a) Find the values of k for which the equation has a unique solution.

It is further asserted that $k=2$.

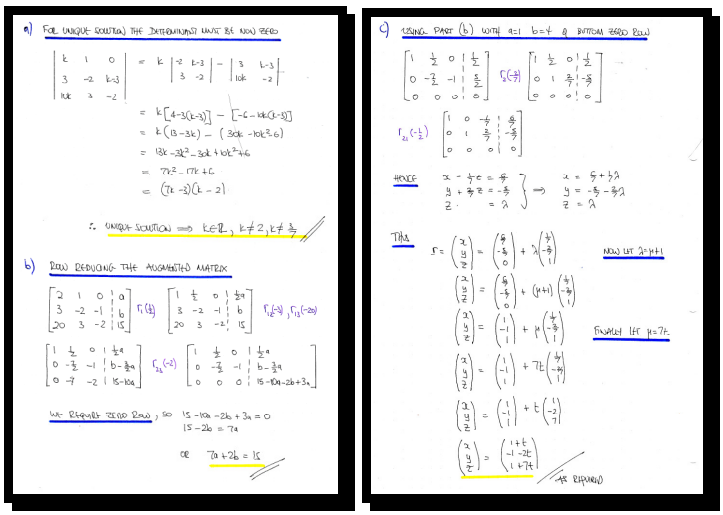
b) Express a in terms of b if the matrix equation is to be consistent.

c) Show that if $a=1$ and $b=4$, the solution of the matrix equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} t+1 \\ -2t-1 \\ 7t+1 \end{pmatrix},$$

where t is a scalar parameter.

$\boxed{}$, $\boxed{k\neq 2 \ \cup \ k\neq \frac{3}{7}}$, $\boxed{2a+7b=15}$



Question 84 (*+)**

The 3×3 matrix $\mathbf{A}$ is defined as

$$\mathbf{A}=\begin{pmatrix}3 & a & 0\\ 2 & b & 0\\ c & 0 & 1\end{pmatrix},$$

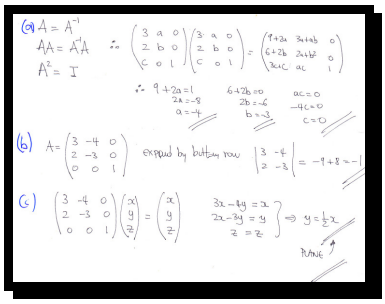
where a, b and c are scalar constants.

a) If $\mathbf{A}=\mathbf{A}^{-1}$, find the value of a, b and c,

b) Evaluate the determinant of $\mathbf{A}$.

c) Determine an equation of a plane of invariant points under the transformation described by $\mathbf{A}$.

$\boxed{a=-4}$, $\boxed{b=-3}$, $\boxed{c=0}$, $\boxed{\det\mathbf{A}=-1}$, $\boxed{\text{plane: } x=2y}$

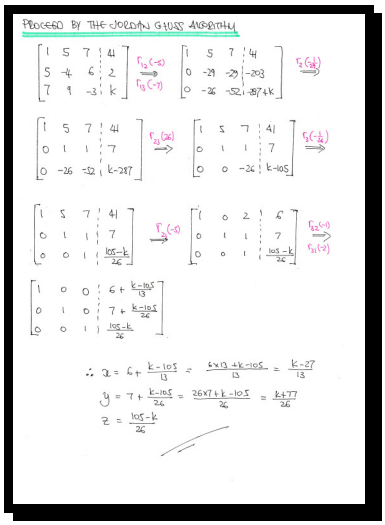


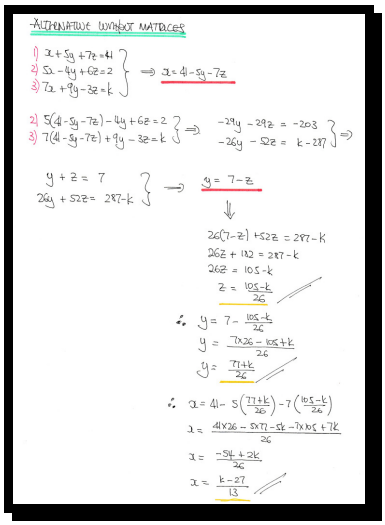
Question 85 (*+)**

$$\begin{aligned} x+5y+7z&=41\\ 5x-4y+6z&=2\\ 7x+9y-3z&=k \end{aligned}$$

Find the solution of the system of simultaneous equations above, giving the answers in terms of the constant k.

$\square$, $\boxed{x=\dfrac{k-27}{13},\quad y=\dfrac{k+77}{26},\quad z=\dfrac{105-k}{26}}$





Question 86 (*+)**

The matrices $\mathbf{A}$ and $\mathbf{B}$, where k is a scalar constant, are given below.

$$\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 4 & k & -2\\ 0 & 0 & -1\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}-k & 2 & k-4\\ 4 & -1 & -2\\ 0 & 0 & k-8\end{pmatrix}.$$

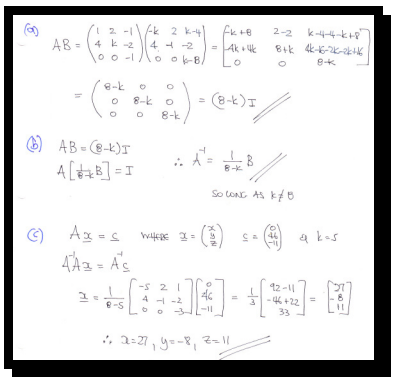
a) Find $\mathbf{AB}$ in its simplest form.

b) Hence, or otherwise, find the inverse of $\mathbf{A}$ in terms of k, stating the condition for its existence.

c) Use the inverse of $\mathbf{A}$ to solve the equation $\mathbf{Ax}=\mathbf{c}$ where

$$\mathbf{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad \text{and} \quad \mathbf{c}=\begin{pmatrix}0\\ 46\\ -11\end{pmatrix}.$$

$$\boxed{\mathbf{AB}=(8-k)\mathbf{I}}, \quad \boxed{\mathbf{A}^{-1}=\frac{1}{8-k}\mathbf{B}}, \quad \boxed{\mathbf{x}=\begin{pmatrix}27\\ -8\\ 11\end{pmatrix}}$$



Question 87 (****)

A 3×3 matrix $\mathbf{A}$ has characteristic equation

$$2\lambda^3-7\lambda^2+\lambda+10=0\,.$$

a) Show that $\lambda=2$ is an eigenvalue of $\mathbf{A}$ and find the other two eigenvalues.

b) Show further that

$$2\mathbf{A}^4+71\mathbf{A}^2=27\mathbf{A}^3+100\mathbf{I}\,.$$

An eigenvector corresponding to $\lambda=2$ is $\mathbf{u}$.

It is further given that $\mathbf{u}=\begin{pmatrix}2\\-4\\-5\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0.4\\-0.8\\-1\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

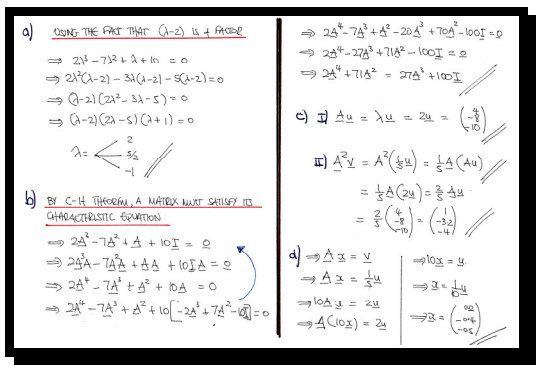
c) Evaluate each of the following expressions.

i. $\mathbf{Au}$.

ii. $\mathbf{A}^2\mathbf{v}$.

d) Solve the equation $\mathbf{Ax}=\mathbf{v}$.

$\square$, $\boxed{\lambda=-1,\quad \lambda=\frac{5}{2}}$, $\boxed{\mathbf{Au}=\begin{pmatrix}4\\-8\\-10\end{pmatrix}}$, $\boxed{\mathbf{A}^2\mathbf{v}=\begin{pmatrix}1.6\\-3.2\\-4\end{pmatrix}}$, $\boxed{\mathbf{x}=\begin{pmatrix}0.2\\-0.4\\-0.5\end{pmatrix}}$



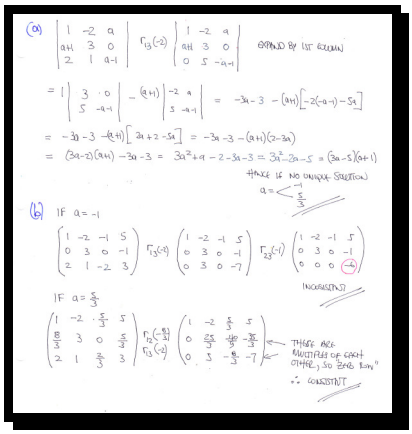
Question 88 (****)

$$\begin{aligned} x\ -2y\ + \quad az &= 5 \\ (a+1)x\ +3y\ \quad\quad &= a \\ 2x\ +\ y\ +(a-1)z &= 3 \end{aligned}$$

a) Determine the two values of the constant a for which the above system of equations does **not** have a unique solution.

b) Show clearly that the system is consistent for one of these values and inconsistent for the other.

$$\boxed{a=-1,\ \frac{5}{3}}$$



Question 89 (****)

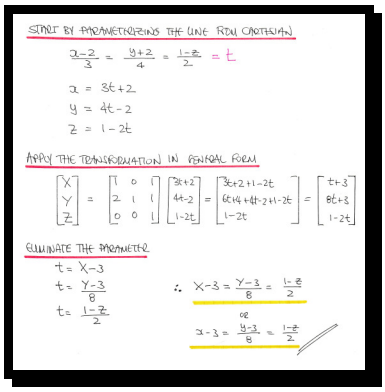
Find in Cartesian form the image of the straight line with equation

$$\frac{x-2}{3}=\frac{y+2}{4}=\frac{1-z}{2},$$

under the transformation represented by the 3×3 matrix $\mathbf{A}$, shown below.

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

$\square$, $\boxed{x-3=\frac{y-3}{8}=\frac{1-z}{2}}$



Question 90 (****)

The 3×3 matrix $\mathbf{C}$ represents a geometric transformation $T:\mathbb{R}^3\mapsto\mathbb{R}^3$.

$$\mathbf{C}=\begin{pmatrix}3&-1&1\\-1&3&1\\1&1&3\end{pmatrix}.$$

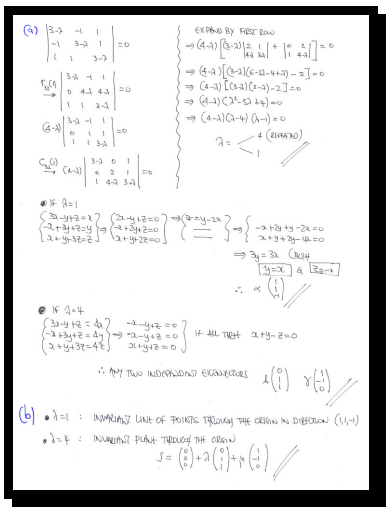
a) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{C}$.

b) Describe the geometrical significance of the eigenvectors of $\mathbf{C}$ in relation to T.

$$\boxed{\lambda=1,\quad \alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}},\qquad \boxed{\lambda=4,\quad \beta\begin{pmatrix}0\\1\\1\end{pmatrix},\quad \gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}},$$

$$\boxed{\lambda=1\Leftrightarrow\text{invariant line of points through the origin}},$$

$$\boxed{\lambda=4\Leftrightarrow\text{invariant plane through the origin}}$$



Question 91 (****)

A 3×3 determinant, Δ, is given below.

$$\Delta=\begin{vmatrix} n(n+1) & n+1 & -1 \\ 0 & 1 & n \\ 1 & -n-1 & 1 \end{vmatrix}.$$

a) Show that

$$\Delta=\left(an^2+bn+c\right)^2,$$

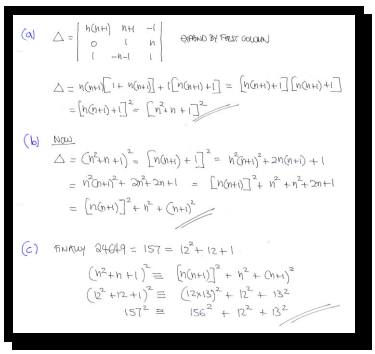
where a, b and c are constants.

b) Show further that

$$\Delta=\left[n(n+1)\right]^2+n^2+(n+1)^2.$$

c) Hence or otherwise express 24649 as the sum of three square numbers.

$$\boxed{\Delta=\left(n^2+n+1\right)^2}, \quad \boxed{24649=156^2+13^2+12^2}$$



Question 92 (****)

$$\begin{aligned} 3x-y+5z&=5\\ 2x+y-5z&=10\\ x+y+kz&=7 \end{aligned}$$

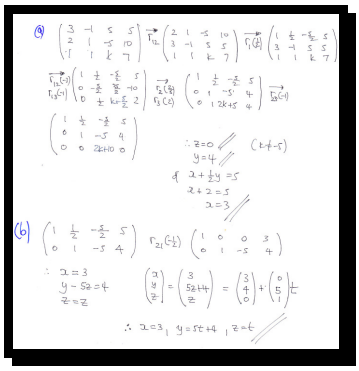
where k is a constant.

a) Given that $k\neq -5$ find the unique solution of the system of equations.

b) Given instead that $k=-5$ show, by reducing the system into row echelon form, that the consistent solution of the system can be written as

$$x=3\,,\quad y=5t+4\,,\quad z=t\,.$$

$$\boxed{x=3,\quad y=4,\quad z=0}$$



Question 93 (****)

The 3×3 matrix $\mathbf{A}$, where a is a scalar constant, is given below.

$$\mathbf{A}=\begin{pmatrix} a & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

a) Find the elements of $\mathbf{A}^{-1}$, in terms of a where appropriate.

The straight line L_1 was mapped onto another straight line L_2 by the following 3×3 matrix .

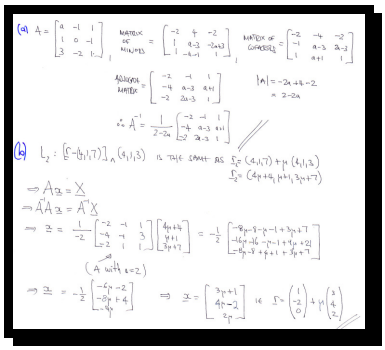
$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

b) Given that L_2 has vector equation

$$\left[\mathbf{r}-\left(4\mathbf{i}+\mathbf{j}+7\mathbf{k}\right)\right]\wedge\left(4\mathbf{i}+\mathbf{j}+3\mathbf{k}\right)=\mathbf{0}$$

find a vector equation for L_1.

$$\boxed{\mathbf{A}^{-1}=\frac{1}{2-2a}\begin{pmatrix} -2 & -1 & 1 \\ -4 & a-3 & a+1 \\ -2 & 2a-3 & 1 \end{pmatrix}}\ ,\quad \boxed{\mathbf{r}=\mathbf{i}-2\mathbf{j}+\lambda\left(3\mathbf{i}+4\mathbf{j}+2\mathbf{k}\right)}$$



Question 94 (****)

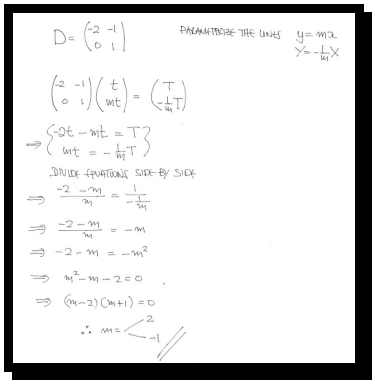
The 2×2 matrix $\mathbf{D}$ shown below, represents a linear transformation in the x-y plane.

$$\mathbf{D}=\begin{pmatrix}-2 & -1\\ 0 & 1\end{pmatrix}.$$

The straight line with equation $y=mx$ is rotated by 90° about the origin under the transformation represented by $\mathbf{D}$.

Determine the possible values of m.

$$\boxed{m=-1,\quad m=2}$$



Question 95 (****)

The 2×2 matrix $\mathbf{C}$ is given below.

$$\mathbf{C}=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}.$$

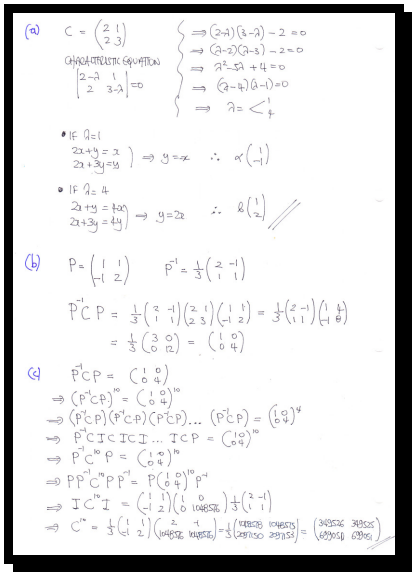
a) Find the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors.

b) Find a 2×2 matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{CP}$ is a diagonal 2×2 matrix and evaluate $\mathbf{P}^{-1}\mathbf{CP}$ explicitly.

c) Hence show that

$$\mathbf{C}^7=\begin{pmatrix}349526 & 349525\\ 699050 & 699051\end{pmatrix}.$$

$$\boxed{\lambda_1=1,\quad \mathbf{u}=\begin{pmatrix}1\\ -1\end{pmatrix}},\quad \boxed{\lambda_2=1,\quad \mathbf{v}=\begin{pmatrix}1\\ 2\end{pmatrix}},\quad \boxed{\mathbf{P}=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}},\quad \boxed{\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}}$$



Question 96 (****)

Consider the system of simultaneous equations

$$\begin{aligned} kx+ky\ \ -z&=-1\\ ky+2z&=\ \ 2k\\ x+2y\ \ +z&=\ \ 1 \end{aligned}$$

where the constant k can **only** take the values 0, 1 and 2.

Determine for each of the possible values of k whether the system …

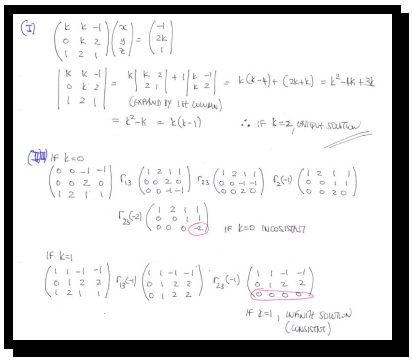
i. … has a unique solution

ii. … has no unique solution, but it is consistent.

iii. … is inconsistent.

$\boxed{a=0\Rightarrow \text{incosistent}}$, $\boxed{a=1\Rightarrow \text{no unique solution/consistent}}$,

$\boxed{a=2\Rightarrow \text{unique solution}}$



Question 97 (****)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}.$$

A straight line with equation $y=mx$, where m is a constant, remains invariant under the transformation represented by $\mathbf{A}$.

a) Show that

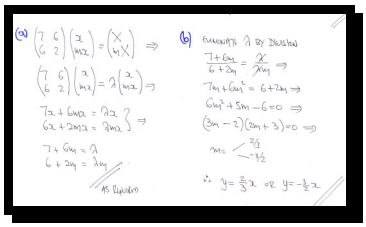
$$7+6m=\lambda$$

$$6+2m=\lambda m$$

where λ is a constant.

b) Hence find the two possible equations of this straight line.

$$\boxed{y=\frac{2}{3}x}\,,\quad \boxed{y=-\frac{3}{2}x}$$



Question 98 **(****)**

A plane Π is defined parametrically by

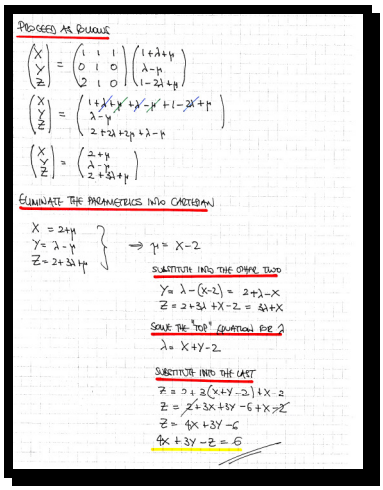
$$\mathbf{r}=\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-2\mathbf{k})+\mu(\mathbf{i}-\mathbf{j}+\mathbf{k}),$$

where λ and μ are a scalar parameters.

Determine a Cartesian equation for the transformation of Π under the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

$\boxed{}$, $\boxed{4x+3y-z=6}$



Question 99 (****)

The 3×3 matrix $\mathbf{C}$ is defined by

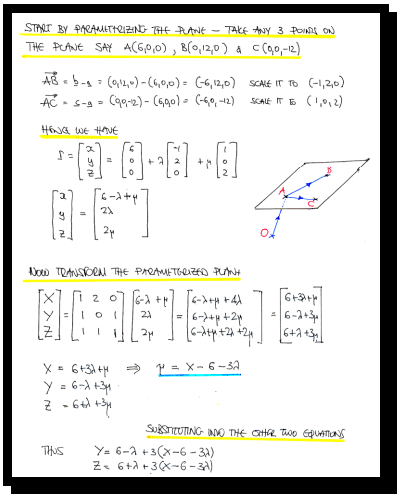
$$\mathbf{C}=\begin{pmatrix}1&2&0\\1&0&1\\1&1&1\end{pmatrix}.$$

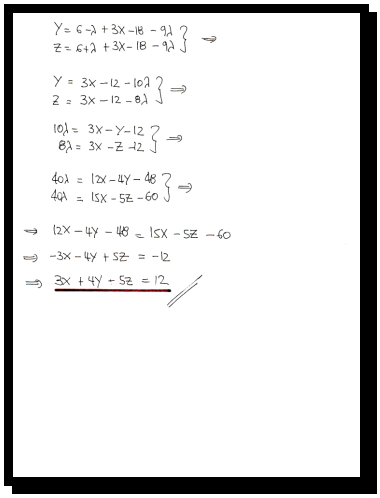
Find, in Cartesian form, the image of the plane with Cartesian equation

$$2x+y-z=12$$

under the transformation defined by $\mathbf{C}$.

$\boxed{}$, $\boxed{3x+4y-5z=12}$





Question 100 (****)

A linear transformation T, acting in the x-y plane, consists of …

- … a reflection about the line $y=-x$,

  followed by ....

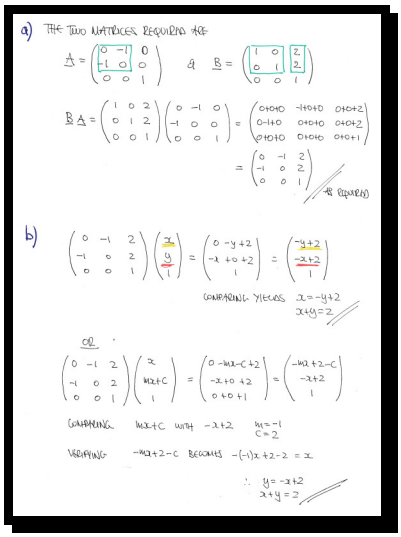
- … a translation such that $(x,y)\mapsto(x+2,y+2)$.

a) Show that the matrix that represents T is given by the matrix

$$\mathbf{T}=\begin{pmatrix}0&-1&2\\-1&0&2\\0&0&1\end{pmatrix}.$$

b) Determine the invariant line under T.

$\boxed{}$, $\boxed{y+x=2}$



Question 101 (****)

The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix},$$

where k is a scalar constant.

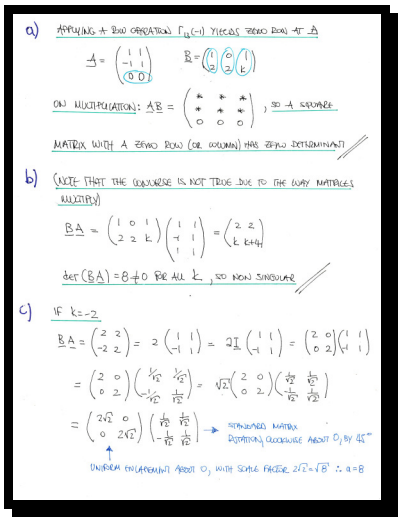
a) Without calculating $\mathbf{AB}$, show that $\mathbf{AB}$ is singular for all values of k.

b) Show that $\mathbf{BA}$ is non singular for all values of k.

When $k=-2$ the matrix $\mathbf{BA}$ represents a combination of a uniform enlargement with linear scale factor $\sqrt{a}$ and another transformation T.

c) Find the value of a and describe T geometrically.

$\boxed{}$, $\boxed{a=8}$, $\boxed{\text{rotation about } O\text{, clockwise, by } 45°}$



Question 102 **(****)**

The equation of a plane Π is given by

$$\mathbf{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1\\0\\2\end{pmatrix},$$

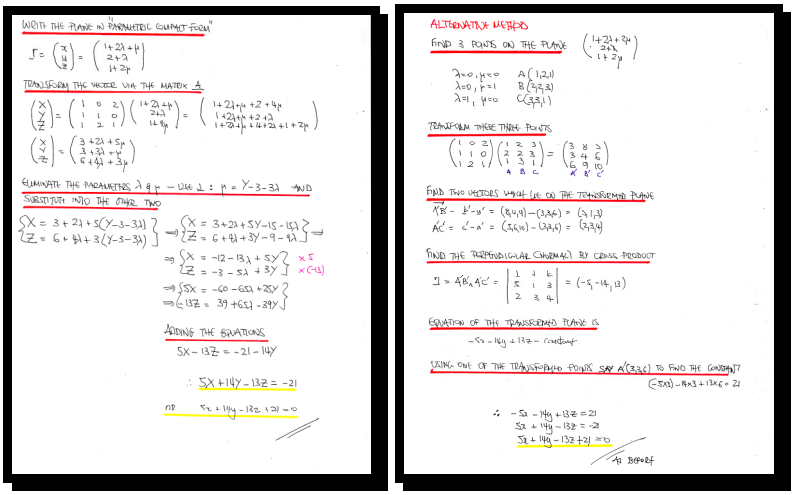
where λ and μ are parameters.

The plane Π is transformed to the plane Π' by the matrix

$$\mathbf{A}=\begin{pmatrix}1&0&2\\1&1&0\\1&2&1\end{pmatrix}.$$

Find a Cartesian equation of Π'.

$\boxed{}$, $\boxed{5x+14y-13z+21=0}$



Question 103 (****)

The Cartesian equation of a plane Π is given by

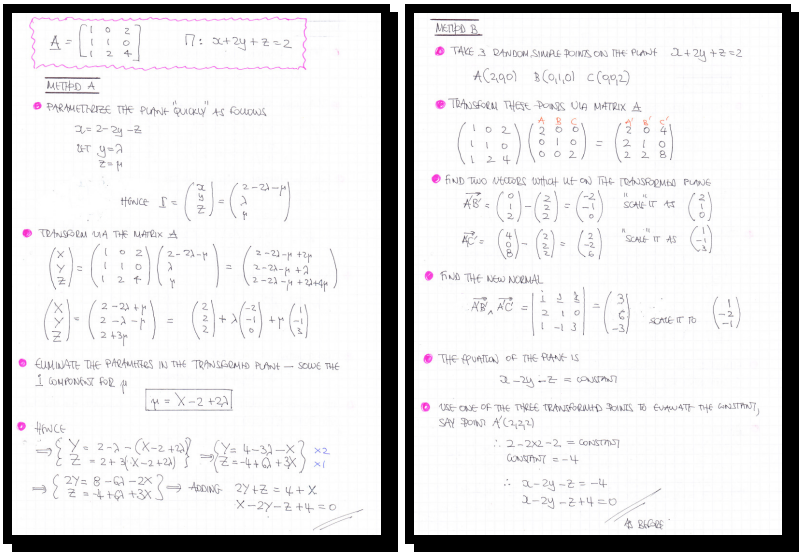
$$x+2y+z=2\,.$$

The plane Π is transformed to the plane Π' by the matrix

$$\mathbf{A}=\begin{pmatrix}1&0&2\\1&1&0\\1&2&4\end{pmatrix}.$$

Find a Cartesian equation of Π'.

$\boxed{}$, $\boxed{x-2y-z+4=0}$



Question 104 **(****)**

A linear transformation T, acting in the ***x*-*y*** plane, consists of …

- … a reflection about the line $y=-x$,

  followed by ...

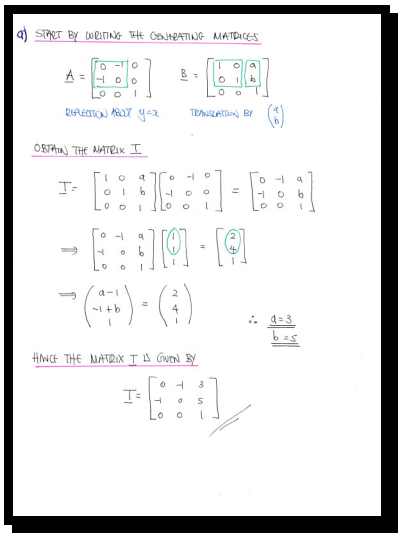
- … a translation such that $(x,y)\mapsto(x+a,y+b)$.

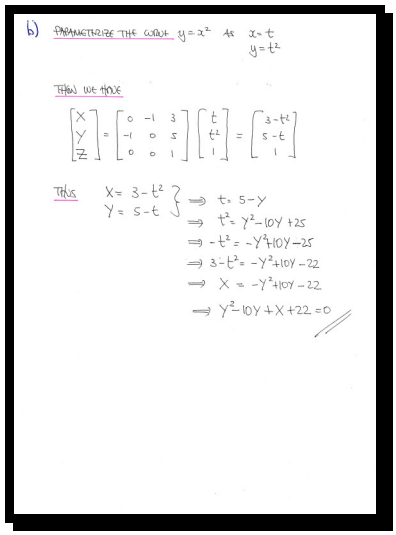
The transformation T is represented by the matrix $\mathbf{T}$.

a) Given the point $(1,1)$ is mapped to $(2,4)$, find the matrix $\mathbf{T}$.

b) Determine the equation of the image of the curve with equation $y=x^2$, under the transformation represented by $\mathbf{T}$.

$$\boxed{},\quad \boxed{\mathbf{T}=\begin{pmatrix}0&-1&3\\-1&0&5\\0&0&1\end{pmatrix}},\quad \boxed{y^2-10y+x+22=0}$$





Question 105 (****)

The 3×3 matrix $\mathbf{A}$, is defined in terms of a scalar constant k, below.

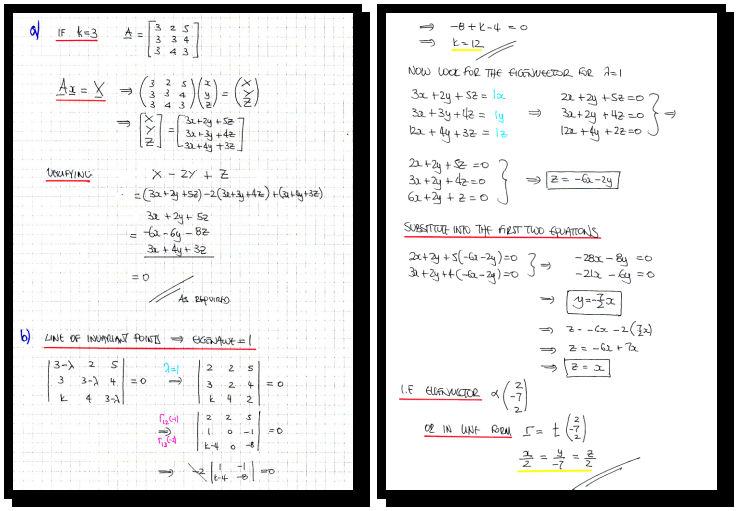
$$\mathbf{A}=\begin{pmatrix} 3 & 2 & 5 \\ 3 & 3 & 4 \\ k & 4 & 3 \end{pmatrix}.$$

a) If $k=3$, verify that $\mathbf{A}$ maps every point of the three dimensional space onto the plane with Cartesian equation

$$x-2y+z=0.$$

b) If $k\neq 3$, determine the value k so that the transformation represented by $\mathbf{A}$ has a line of invariant points, and state the Cartesian equation of this line.

$$\boxed{}\,,\quad \boxed{k=12}\,,\quad \boxed{\frac{1}{2}x=-\frac{1}{7}y=\frac{1}{2}z}$$



Question 106 (****)

The 2×2 matrix $\mathbf{M}$ satisfies $\mathbf{M}=\mathbf{PDP}^{-1}$ where

$$\mathbf{P}=\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}.$$

a) Determine the elements of $\mathbf{M}$.

b) State the eigenvalues, and the corresponding eigenvectors of $\mathbf{M}$.

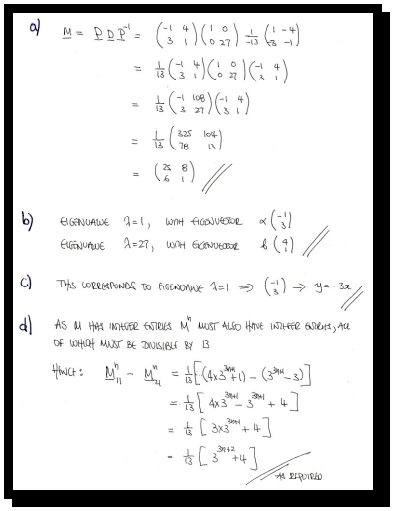
c) Find an equation of the straight line of invariant points under the transformation described by $\mathbf{M}$.

It is further given that

$$\mathbf{M}^n=\frac{1}{13}\begin{pmatrix}4\times 3^{3n+1}+1 & 4\times 3^{3n}-4\\ 3^{3n+1}-3 & 3^{3n}+12\end{pmatrix}.$$

d) Deduce that $3^{3n+2}+4$ is divisible by 13, for all positive integers n.

$\square$, $\boxed{\mathbf{M}=\begin{pmatrix}25 & 8\\ 6 & 3\end{pmatrix}}$, $\boxed{\lambda_1=1,\quad \lambda_2=27,\quad \mathbf{u}_1=4\mathbf{i}+\mathbf{j},\quad \mathbf{u}_2=-\mathbf{i}+3\mathbf{j}}$, $\boxed{y=-3x}$


a) $\underline{M} = \underline{P}\,\underline{D}\,\underline{P}^{-1} = \begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}\frac{1}{-13}\begin{pmatrix}1 & -4\\ -3 & -1\end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 27\end{pmatrix}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix}-1 & 108\\ 3 & 27\end{pmatrix}\begin{pmatrix}-1 & 4\\ 3 & 1\end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix}325 & 104\\ 78 & 39\end{pmatrix}$

$= \begin{pmatrix}25 & 8\\ 6 & 3\end{pmatrix}$

b) EIGENVALUE $\lambda=1$, WITH EIGENVECTOR $\propto \begin{pmatrix}-1\\ 3\end{pmatrix}$

EIGENVALUE $\lambda=27$, WITH EIGENVECTOR $\propto \begin{pmatrix}4\\ 1\end{pmatrix}$

c) THIS CORRESPONDS TO EIGENVALUE $\lambda=1 \Rightarrow \begin{pmatrix}-1\\ 3\end{pmatrix} \Rightarrow y=-3x$

d) AS M HAS INTEGER ENTRIES M^n MUST ALSO HAVE INTEGER ENTRIES, AND OF WHICH MUST BE DIVISIBLE BY 13

HENCE: $M_{11}^n - M_{12}^n = \frac{1}{13}\left[\left(4\times 3^{3n+1}+1\right)-\left(3^{3n+1}-3\right)\right]$

$= \frac{1}{13}\left[4\times 3^{3n+1}-3^{3n+1}+4\right]$

$= \frac{1}{13}\left[3\times 3^{3n+1}+4\right]$

$= \frac{1}{13}\left[3^{3n+2}+4\right]$

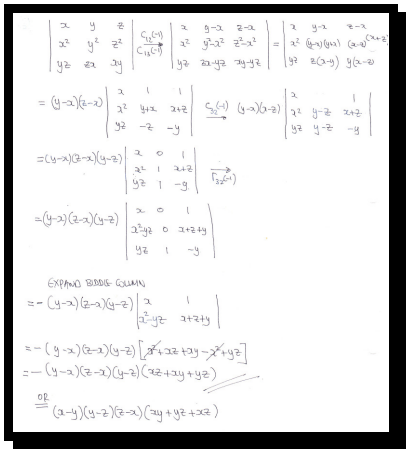
AS REQUIRED

Question 107 (****)

Factorize fully the following 3×3 determinant.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}.$$

$$\boxed{(x-y)(y-z)(z-x)(xy+yz+zx)}$$



Question 108 (****)

A system of equations is given below

$$\begin{aligned} 3x+2y\ \ -z&=10\\ 5x\ \ -y-4z&=17\\ x+5y+pz&=q \end{aligned}$$

where p and q are constants.

a) Find the value of p so that the above system does not have a unique solution.

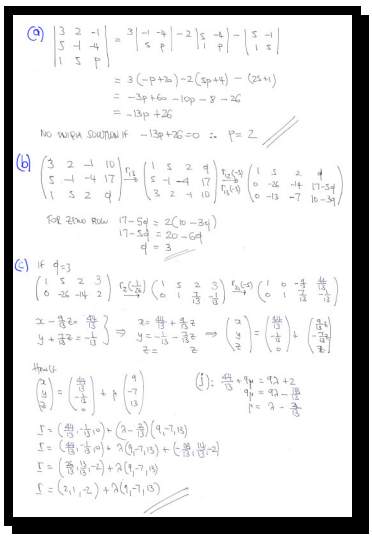
b) Show that for this value of p the system is consistent if $q=3$.

c) Show that the general solution of the system can be written as

$$\mathbf{r}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda\left(9\mathbf{i}-7\mathbf{j}+13\mathbf{k}\right),$$

where λ is a scalar parameter.

$\boxed{p=2}$



Question 109 (****)

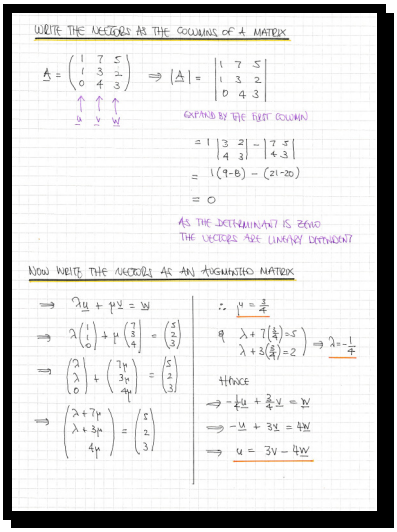
The following three vectors are given.

$$\mathbf{u}=\begin{bmatrix}2\\-1\\1\end{bmatrix},\qquad \mathbf{v}=\begin{bmatrix}-1\\1\\2\end{bmatrix},\qquad \mathbf{w}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

b) Find a linear relationship, with integer coefficients, between $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{u}=3\mathbf{v}-4\mathbf{w}}$$



Question 110 (****)

The 2×2 matrix $\mathbf{A}$ is defined in terms of a constant k.

$$\mathbf{A}=\begin{pmatrix}2 & 7\\ 4 & k\end{pmatrix}$$

a) Given that $\begin{pmatrix}1\\ 1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find …

i. … the corresponding eigenvalue to the eigenvector.

ii. … the value of k

b) Find another eigenvector and the corresponding eigenvalue of $\mathbf{A}$.

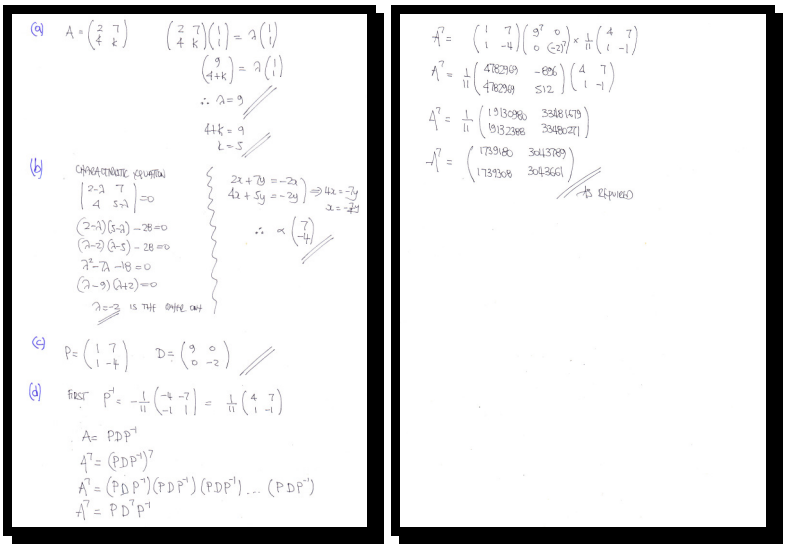
It is further given that $\mathbf{A}=\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another matrix.

c) Write down possible forms for the matrices $\mathbf{D}$ and $\mathbf{P}$.

d) Hence show clearly that

$$\mathbf{A}^7=\begin{pmatrix}1739180 & 3043789\\ 1739308 & 3043661\end{pmatrix}.$$

$\boxed{\lambda=9}$, $\boxed{k=5}$, $\boxed{\lambda=-2,\mathbf{u}=\begin{pmatrix}7\\ -4\end{pmatrix}}$, $\boxed{\mathbf{D}=\begin{pmatrix}9 & 0\\ 0 & -2\end{pmatrix}}$, $\boxed{\mathbf{P}=\begin{pmatrix}1 & 7\\ 1 & -4\end{pmatrix}}$



Question 111 (****)

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1&-1&1\\3&-3&1\\3&-5&3\end{pmatrix}.$$

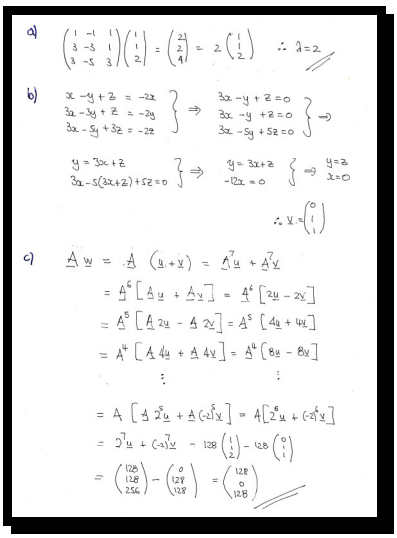
a) Given that $\mathbf{u}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue.

b) Given that $\lambda=-2$ is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector $\mathbf{v}$.

The vector $\mathbf{w}$ is defined as $\mathbf{w}=\mathbf{u}+\mathbf{v}$.

c) Determine the vector $\mathbf{A}^7\mathbf{w}$.

$\boxed{}$, $\boxed{\lambda=2}$, $\boxed{\mathbf{v}=\begin{pmatrix}0\\1\\1\end{pmatrix}}$, $\boxed{\mathbf{A}^7\mathbf{w}=\begin{pmatrix}128\\0\\128\end{pmatrix}}$



Question 112 (****)

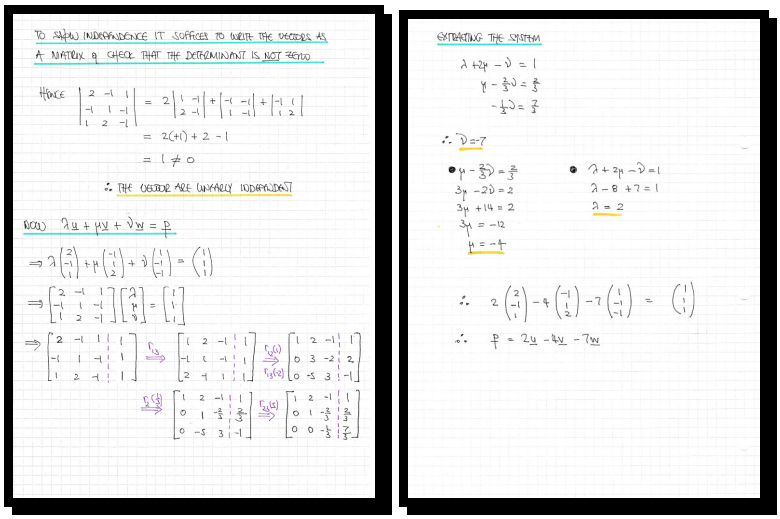
The following four vectors are given.

$$\mathbf{u}=\begin{bmatrix}2\\-1\\1\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}-1\\1\\2\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}1\\-1\\-1\end{bmatrix},\quad \mathbf{p}=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{p}=2\mathbf{u}-4\mathbf{v}-7\mathbf{w}}$$



Question 113 **(****)**

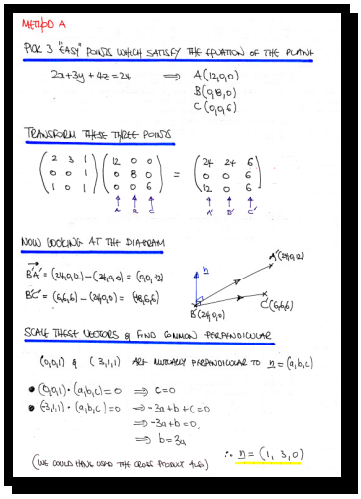
A plane Π has Cartesian equation

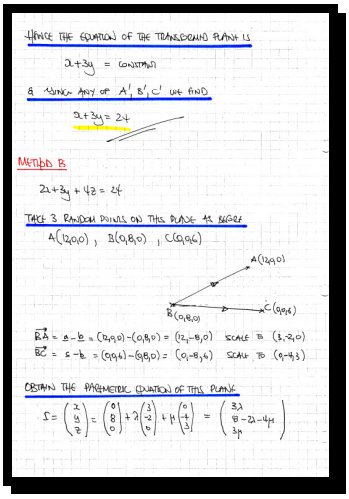
$$2x+3y+4z=24\,.$$

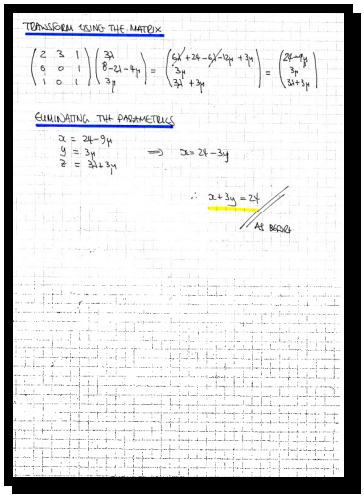
Determine a Cartesian equation for the transformation of Π under the matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$\boxed{}$, $\boxed{x+3y=24}$







Question 114 (**+)**

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

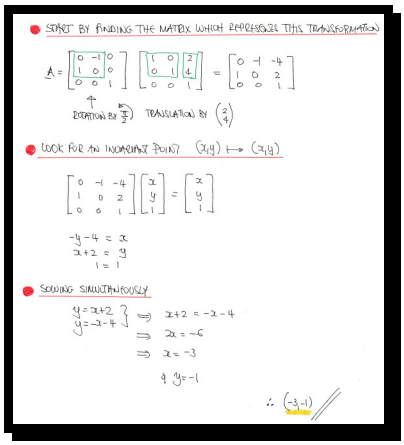
- … a translation such that $(x, y) \mapsto (x+2, y+4)$,

  followed by ...

- … an anticlockwise rotation about the origin by $\dfrac{\pi}{2}$.

Determine the coordinates of the invariant point under T.

$\boxed{}$, $\boxed{(-3,-1)}$



Question 115 (**+)**

A linear transformation T, acting in the x-y plane, consists of …

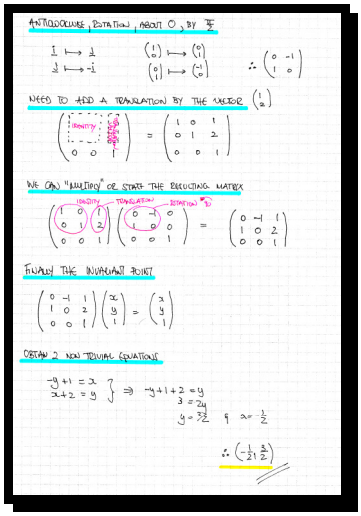
- … an anticlockwise rotation about the origin by $\frac{\pi}{2}$

followed by ...

- … a translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Determine the coordinates of the invariant point under T.

$\square$, $\boxed{\left(-\frac{1}{2}, \frac{3}{2}\right)}$



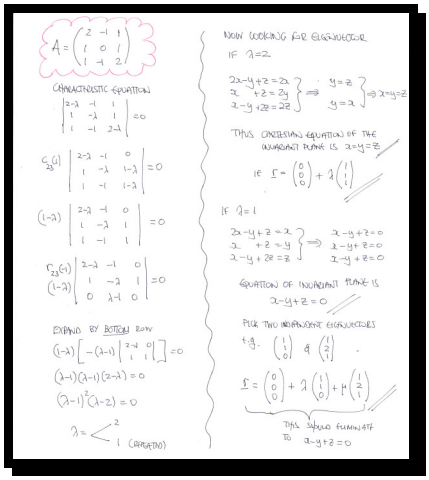
Question 116 (**+)**

The 3×3 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}1 & -1 & 1\\ 3 & 0 & 1\\ 1 & -1 & 2\end{pmatrix}.$$

Find in Cartesian and parametric form the equation of the invariant line and the equation of the invariant plane under the transformation represented by $\mathbf{A}$.

$$\boxed{\mathbf{r}=\lambda\mathbf{i}+\lambda\mathbf{j}+\lambda\mathbf{k},\ \ x=y=z},\ \boxed{\mathbf{r}=(\lambda+\mu)\mathbf{i}+(\lambda+2\mu)\mathbf{j}+\mu\mathbf{k},\ \ x-y+z=0}$$



Question 117 (**+)**

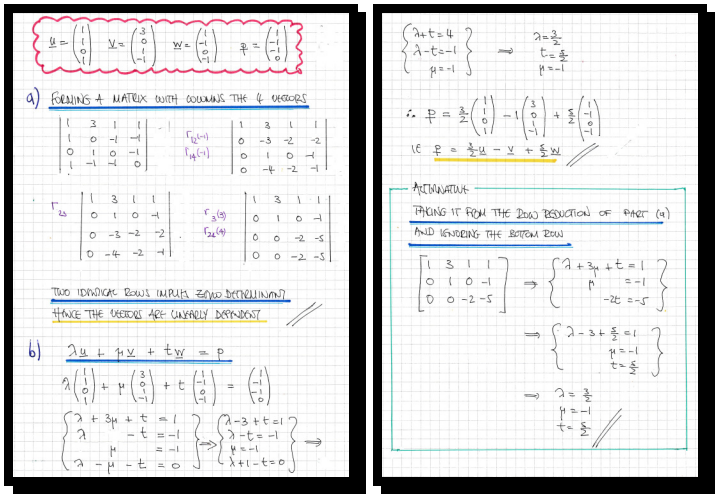
The following four vectors are given.

$$\mathbf{u}=\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}3\\0\\1\\-1\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\quad \mathbf{p}=\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}.$$

a) Show that these four vectors are linearly dependent.

b) Express $\mathbf{p}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{\mathbf{p}=\frac{3}{2}\mathbf{u}-\mathbf{v}+\frac{5}{2}\mathbf{w}}$$



Question 118 (**+)**

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a translation such that $(x, y) \mapsto (x+2, y-3)$,

  followed by ...
- … a rotation about the origin, by $\frac{1}{2}\pi$, anticlockwise.

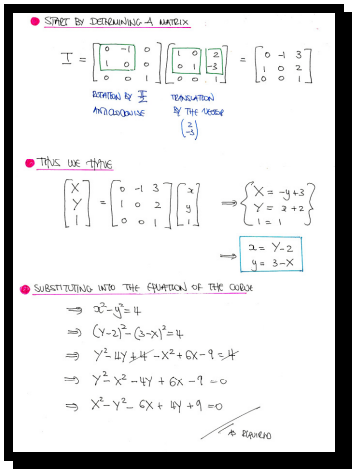
Show that under T, the curve with equation

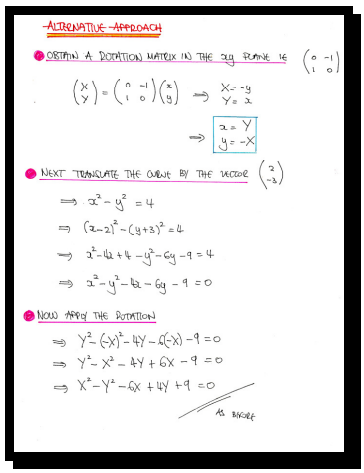
$$x^2 - y^2 = 4,$$

is mapped onto the curve with equation

$$x^2 - y^2 - 6x + 4y + 9 = 0.$$

[ ], proof





Question 119 (**+)**

The 2×2 matrix $\mathbf{C}$ is defined as

$$\mathbf{C}=\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix},$$

where a and b are constants.

a) Determine the eigenvalues of $\mathbf{C}$ and their corresponding eigenvectors, giving the answers in terms of a and b where appropriate.

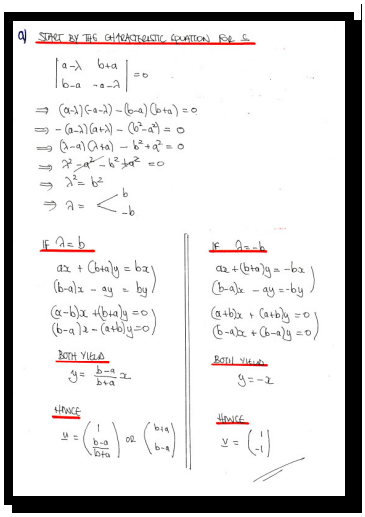
It is further given that $\mathbf{C}=\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is another 2×2 matrix.

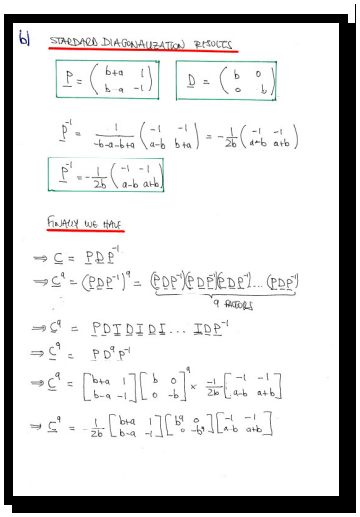
b) Write down the possible form of $\mathbf{D}$ and the possible form of $\mathbf{P}$ and hence show that

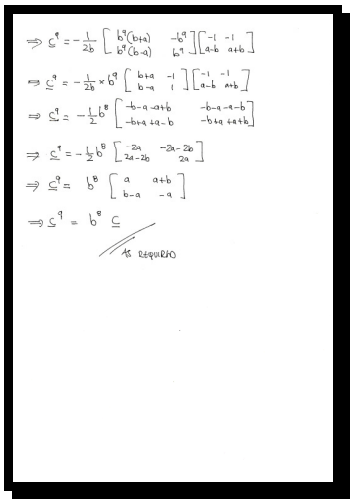
$$\mathbf{C}^9=b^8\mathbf{C}.$$

$\square$, $\boxed{\lambda_1=b,\quad \mathbf{u}=\begin{pmatrix} 1 \\ \dfrac{b-a}{b+a} \end{pmatrix} \text{ or } \mathbf{u}=\begin{pmatrix} b+a \\ b-a \end{pmatrix}}$, $\boxed{\lambda=-b,\quad \mathbf{v}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$,

$\boxed{\mathbf{D}=\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}}$, $\boxed{\mathbf{P}=\begin{pmatrix} b+a & 1 \\ b-a & -1 \end{pmatrix}}$







Question 120 (**+)**

A system of equation is given below

$$\begin{aligned} 3x-2y-18z&=6\\ 2x+y-5z&=25 \end{aligned}$$

a) Show, by reducing the system into row echelon form, that the solution of the system can be written as

$$\mathbf{r}=8\mathbf{i}+9\mathbf{j}+\lambda\left(4\mathbf{i}-3\mathbf{j}+\mathbf{k}\right),$$

where λ is a scalar parameter.

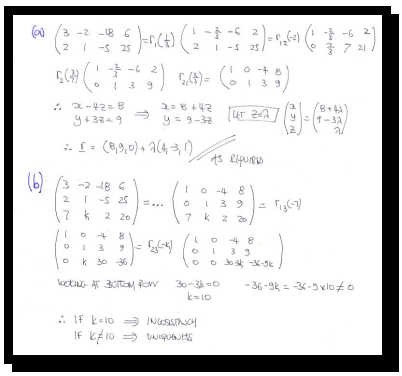
A new system is now given

$$\begin{aligned} 3x-2y-18z&=6\\ 2x+y-5z&=25\\ 7x+ky+2z&=20 \end{aligned}$$

where k is a constant.

b) Determine if the system has solutions for different values of k.

$\boxed{k\neq 10\Rightarrow \text{unique, otherwise incosistent}}$



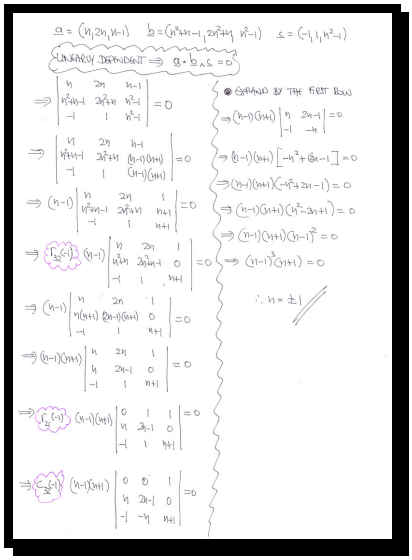
Question 121 (**+)**

The following vectors, given in terms of a scalar constant n, are linearly dependent.

$$\mathbf{a}=n\mathbf{i}+2n\mathbf{j}+(n-1)\mathbf{k},$$
$$\mathbf{b}=\left(n^2+n-1\right)\mathbf{i}+\left(2n^2+n\right)\mathbf{j}+\left(n^2-1\right)\mathbf{k},$$
$$\mathbf{c}=-\mathbf{i}+\mathbf{j}+\left(n^2-1\right)\mathbf{k}.$$

Determine possible values of n.

$\boxed{}$, $\boxed{n=\pm 1}$



Question 122 (**+)**

A linear transformation T, acting in the x-y plane, consists of …

- … a translation such that $(x, y) \mapsto (x+h, y+k)$,

  followed by

- … a reflection about the line $y = x$.

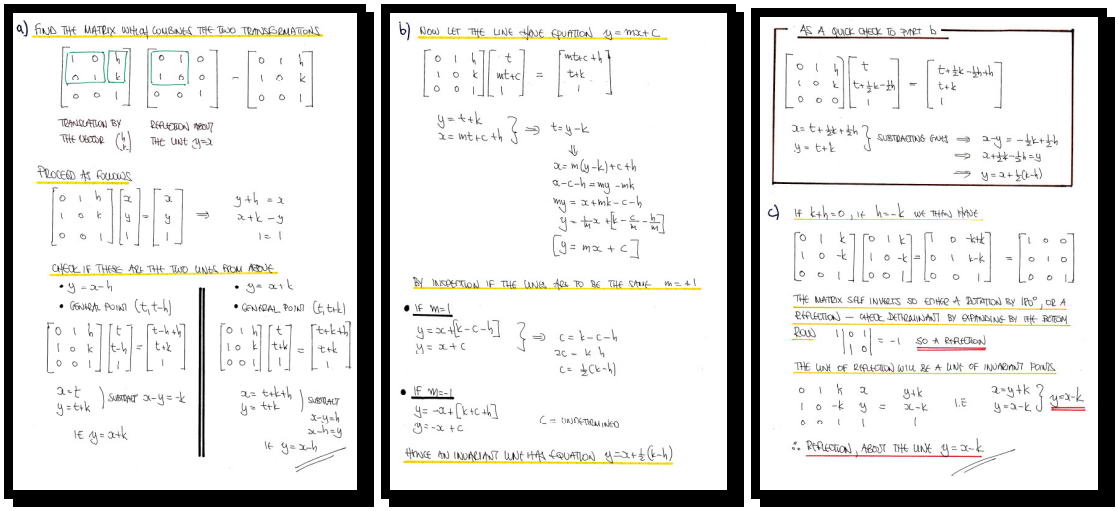
a) Determine, in terms of k and h, the equations of the two straight lines which map onto each other under T.

b) Find, in terms of k and h, the equation of the invariant line under T.

c) Give a full geometrical description for T, in the case where $h+k=0$, by considering the single transformation that is equivalent to T applied twice in succession.

$\boxed{}$, $\boxed{y = x-h, \qquad y = x+k}$, $\boxed{y = x+\frac{1}{2}(k-h)}$,

$\boxed{\text{Reflection about the line } y = x-k}$



Question 123 (**+)**

The 2×2 matrix $\mathbf{A}$ is defined as

$$\mathbf{A}=\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}.$$

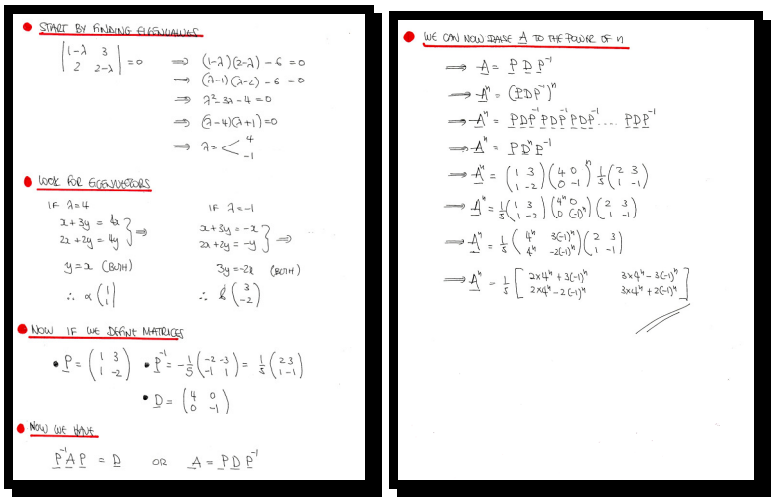
Use linear matrix algebra techniques to show that

$$\mathbf{A}^n=\frac{1}{5}\begin{pmatrix}\alpha\times 4^n+\beta(-1)^n & \beta\times 4^n-\beta(-1)^n\\ \alpha\times 4^n-\alpha(-1)^n & \beta\times 4^n+\alpha(-1)^n\end{pmatrix},$$

where α and β are positive constants.

You may not use proof by induction in this question.

$\boxed{}$, $\boxed{\alpha=2,\quad \beta=3}$



Question 124 (**+)**

A linear transformation T, acting in the x-y plane, consists of …

- … a reflection about the line $y = x$,

  followed by

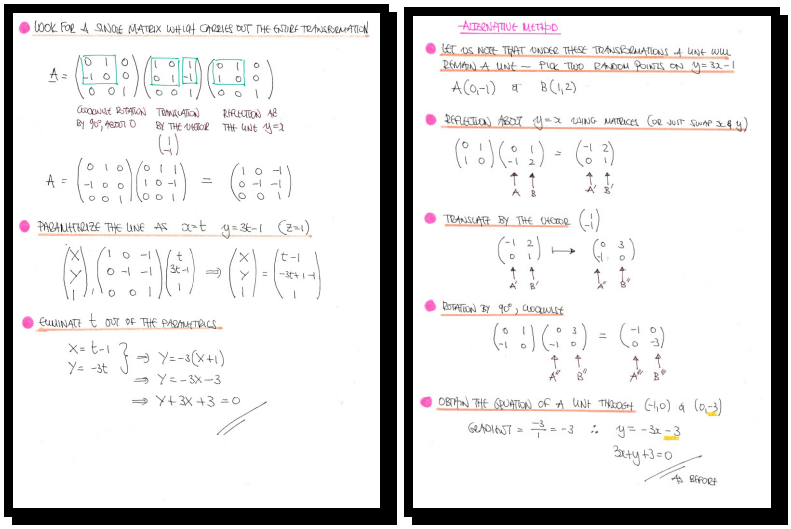
- … a translation such that $(x, y) \mapsto (x+1, y-1)$,

  followed by

- … a clockwise rotation about the origin O by $90°$.

Find, under T, the equation of the image of the straight line with equation $y = 3x - 1$.

$\boxed{}$, $\boxed{3x + y + 3 = 0}$



Question 125 (**+)**

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … a reflection about the line $y = x$,

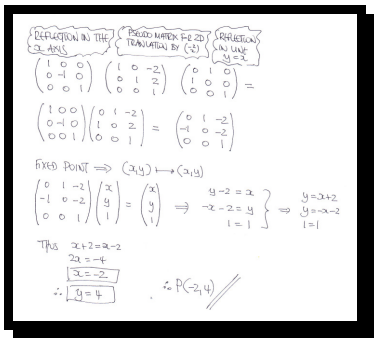
  followed by ...
- … a translation such that $(x, y) \mapsto (x-2, y+2)$,

  followed by ...
- … a reflection about the line $y = 0$,

The point P is invariant under T.

Determine the coordinates of P.

$$\boxed{P(-2,4)}$$



Question 126 **(****+)**

The 3×3 matrix $\mathbf{T}$ is given below.

$$\mathbf{T}=\begin{pmatrix}-1 & 0 & -1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}.$$

The matrix $\mathbf{T}$ describes a composite transformation in the x-y plane.

a) Verify that $\mathbf{T}$ consists of …

- … a reflection in the line $y=-x$,

  followed by ...
- … a translation by the vector $2\mathbf{i}-\mathbf{j}$,

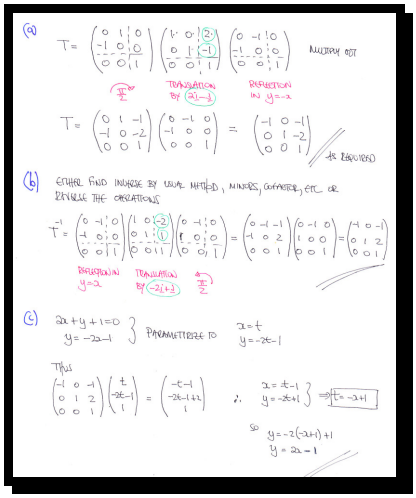
  followed by ...
- … a clockwise rotation by $\frac{1}{2}\pi$, about the origin O.

b) Determine the inverse of the matrix $\mathbf{T}$.

The straight line with equation $2x+y+1=0$ is transformed by $\mathbf{T}$.

c) Find a Cartesian equation of the image of the line after the transformation.

$$\mathbf{T}^{-1}=\begin{pmatrix}-1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix},\qquad \boxed{y=2x-1}$$



Question 127 (**+)**

A linear transformation T, acting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, consists of …

- … an anticlockwise rotation about the origin O by a non zero angle θ,

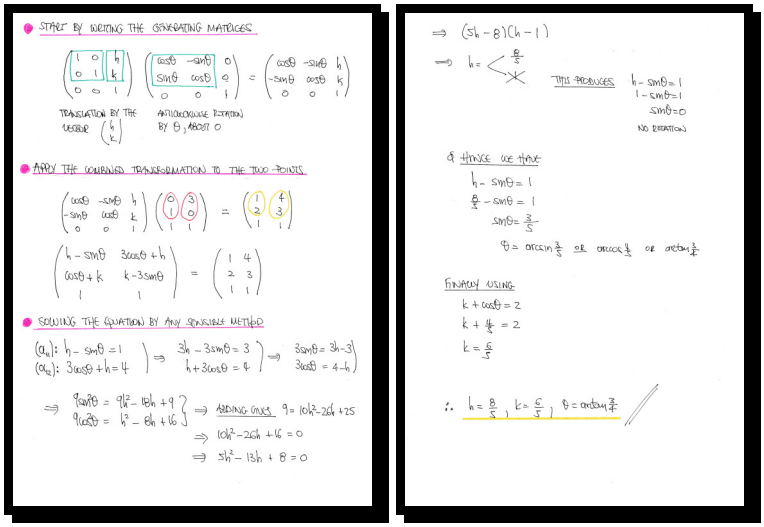
... followed by ...

- … a translation such that $(x,y)\mapsto(x+h,y+k)$

Under this transformation $(0,1)\mapsto(1,2)$ and $(3,0)\mapsto(4,3)$.

Find the value of each of the constants θ, k and h.

$\boxed{}$, $\boxed{\theta=\arctan\frac{3}{4},\ h=\frac{8}{5},\ k=\frac{6}{5}}$



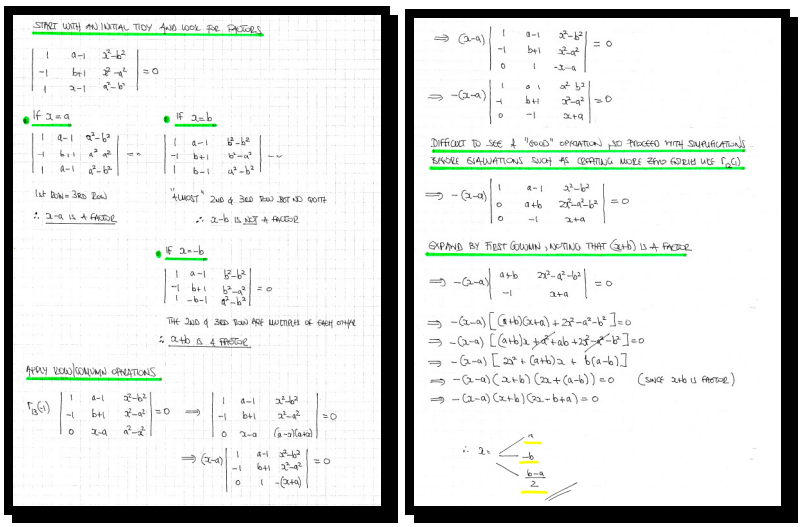
Question 128 (*****)

An equation in x is summarized by the following determinant.

$$\begin{vmatrix} 1 & a-1 & (x-b)(x+b) \\ -1 & b+1 & (x-a)(x+a) \\ 1 & x-1 & (a-b)(a+b) \end{vmatrix} = 0$$

Give the solutions in terms of a and/or b where appropriate.

$\square$, $\boxed{x=a \ \cup \ x=-b \ \cup \ x=\frac{1}{2}(b-a)}$



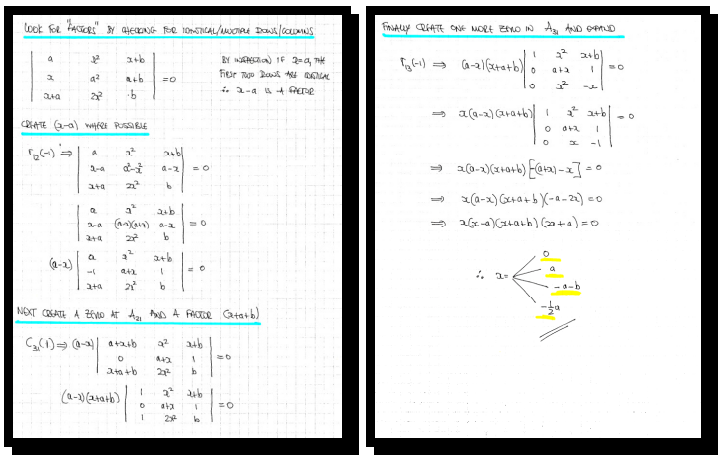
Question 129 (*****)

An equation in x is summarized by the following determinant.

$$\begin{vmatrix} a & x^2 & x+b \\ x & a^2 & a+b \\ x+a & 2x^2 & b \end{vmatrix} = 0$$

Give the solutions in terms of a and/or b where appropriate.

$\square$, $\boxed{x=0 \ \cup \ x=a \ \cup \ x=-a-b \ \cup \ x=\frac{1}{2}a}$



Question 130 (*****)

A transformation is defined by the 2×2 matrix

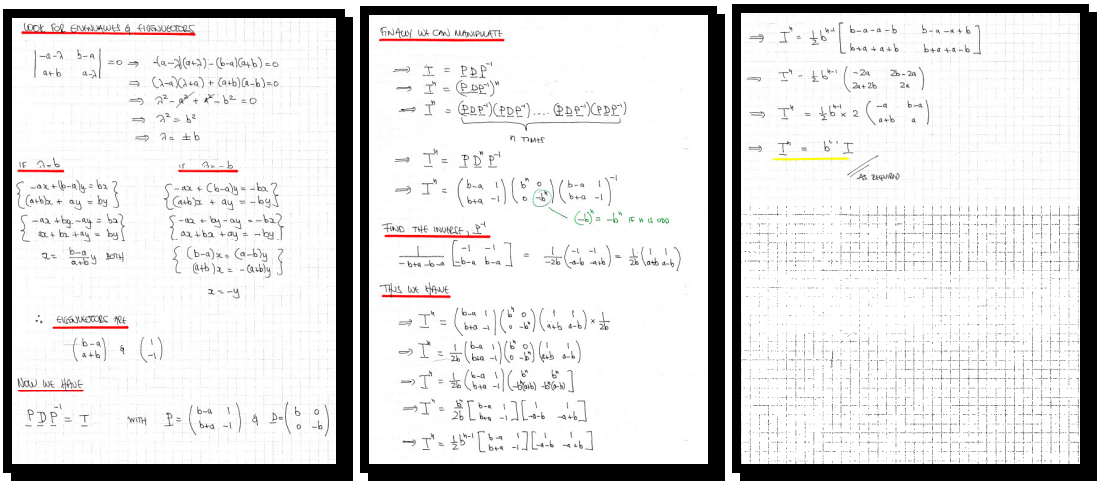
$$\mathbf{T}=\begin{pmatrix} -a & b-a \\ a+b & a \end{pmatrix},$$

where a and b are scalar constants.

If n is an **odd** integer prove that

$$\mathbf{T}^n=b^{n-1}\mathbf{T}\,.$$

, proof



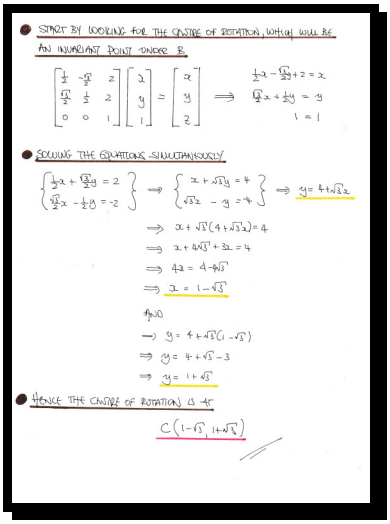
Question 131 (*****)

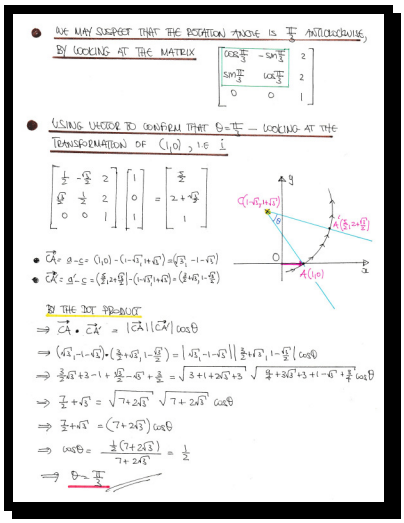
A rotation R, acting in the x - y plane is given by the following 3×3 matrix.

$$\mathbf{R}=\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the centre and angle of this rotation.

$\boxed{}$, $\boxed{\text{centre}\left(1-\sqrt{3},1+\sqrt{3}\right),\quad \theta=\frac{1}{3}\pi}$





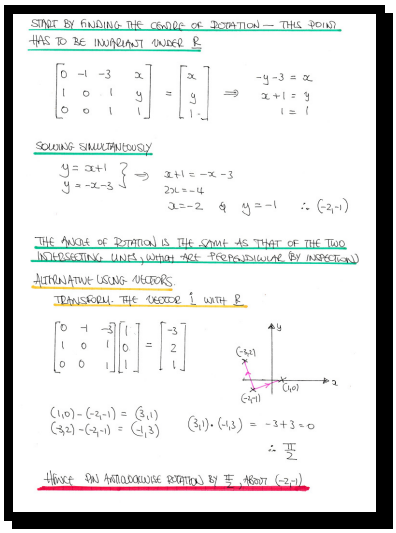
Question 132 (*****)

A rotation R, acting in the x - y plane is given by the following 3×3 matrix.

$$\mathbf{R}=\begin{pmatrix}0&-1&-3\\1&0&1\\0&0&1\end{pmatrix}.$$

Find centre and angle of this rotation.

$\boxed{}$, $\boxed{\text{centre}(-2,-1),\ \theta=\dfrac{\pi}{2}}$



Question 133 (*****)

The point $P(x, y)$ is mapped onto the point $Q(X,Y)$ by the rotation described by the matrix transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The above transformation is used to rotate the hyperbola with equation

$$4x^2 - 44xy - 29y^2 = 120$$

onto the hyperbola with equation

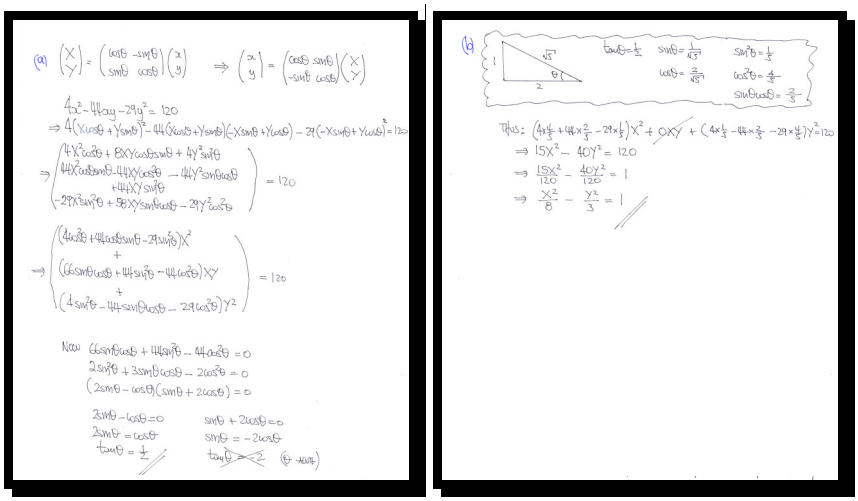
$$\frac{X^2}{a} - \frac{Y^2}{b} = 1,$$

where a and b are positive constants.

a) Given that the rotation is by angle θ, such that θ is acute, find the exact value of $\tan\theta$.

b) Determine the value of a and the value of b .

$\boxed{}$, $\boxed{\tan\theta = \frac{1}{2}}$, $\boxed{a = 8}$, $\boxed{b = 3}$



Question 134 (*****)

A parabola has the following equation

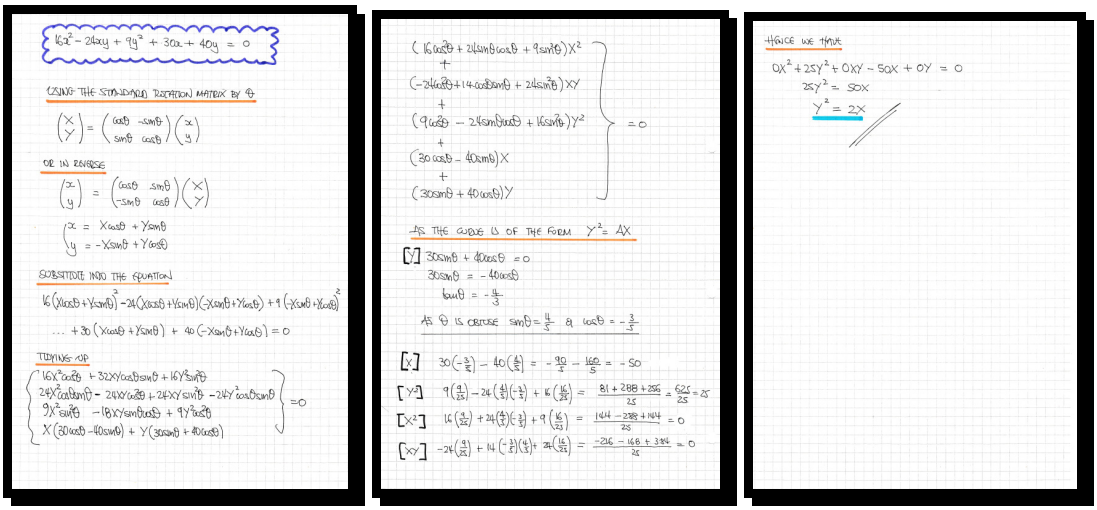
$$y^2 = Ax\,,\ x \geq 0\,,\ A > 0\,.$$

The parabola is rotated about O onto a new parabola with equation

$$16x^2 - 24xy + 9y^2 + 30x + 40y = 0\,.$$

Use algebra to determine the value of A.

$\boxed{}$, $\boxed{A=2}$



Question 135 (*****)

Use the properties of determinants to express the following determinant in fully factorized form.

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

$\square$, $\boxed{(ab+bc+ca)^3}$

