INTEGRATION STRUCTURED EXAM QUESTIONS PART II

Question 1 **(**)**

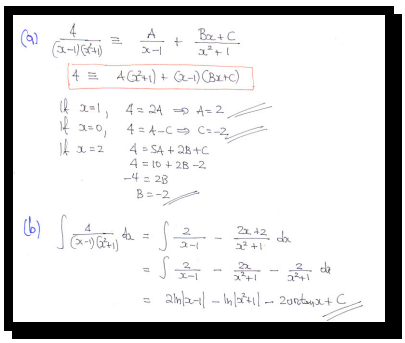
$$\frac{4}{(x-1)\left(x^2+1\right)} \equiv \frac{A}{x-1}+\frac{Bx+C}{x^2+1}.$$

a) Find the values of A, B and C in the above identity.

b) Hence find

$$\int \frac{4}{(x-1)\left(x^2+1\right)}\,dx.$$

$\boxed{A=2}$, $\boxed{B=-2}$, $\boxed{C=-2}$, $\boxed{2\ln|x-1|-\ln\left(x^2+1\right)-2\arctan x+C}$

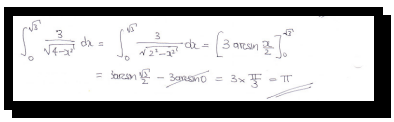


Question 2 **(**)**

Find an exact value for

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}}\,dx.$$

$\boxed{\pi}$

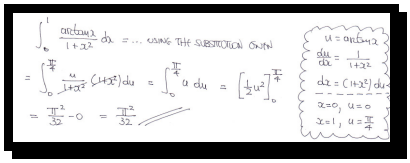


Question 3 (**)

By using the substitution $u = \arctan x$, or otherwise, find an exact value for

$$\int_0^1 \frac{\arctan x}{1+x^2}\, dx .$$

$$\boxed{\frac{\pi^2}{32}}$$

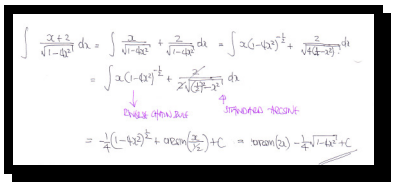


Question 4 (**)

Find an expression for

$$\int \frac{x+2}{\sqrt{1-4x^2}}\, dx .$$

$$\boxed{\arcsin 2x - \frac{1}{4}\sqrt{1-4x^2} + C}$$



Question 5 (**)

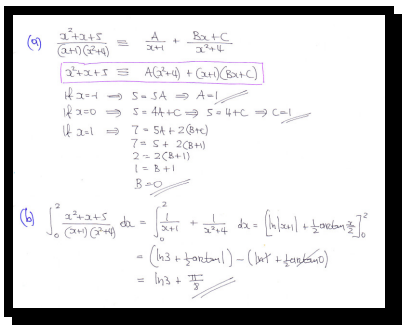
$$\frac{x^2+x+5}{(x+1)\left(x^2+4\right)} \equiv \frac{A}{x+1}+\frac{Bx+C}{x^2+4}.$$

a) Find the values of A, B and C in the above identity.

b) Hence find the exact value of

$$\int_0^2 \frac{x^2+x+5}{(x+1)\left(x^2+4\right)}\,dx.$$

$\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=1}$, $\boxed{\frac{\pi}{8}+\ln 3}$

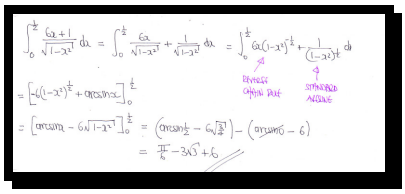


Question 6 (**)

Find an exact value for

$$\int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}}\,dx\,.$$

$$\boxed{\frac{\pi}{6}-3\sqrt{3}+6}$$

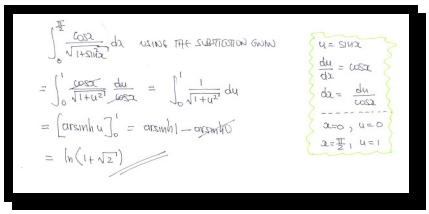


Question 7 (**)

Use the substitution $u=\sin x$ to find an exact value in terms of natural logarithms for

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx\,.$$

$$\boxed{\ln\left(1+\sqrt{2}\right)}$$



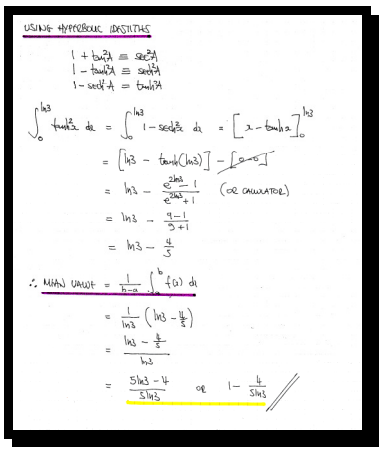
Question 8 **(**)**

The function f is defined as

$$f(x)\equiv \tanh^2 x\,,\ x\in\mathbb{R}\,,\ 0\le x\le \ln 3\,.$$

Determine the mean value of f, in its entire domain.

$\square$, $\boxed{1-\dfrac{4}{5\ln 3}\approx 0.272}$

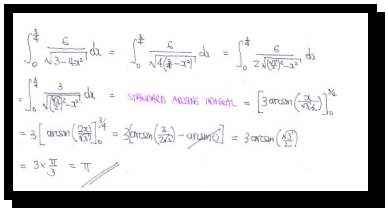


Question 9 (+)**

Find an exact value for

$$\int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}}\,dx .$$

$\boxed{\pi}$

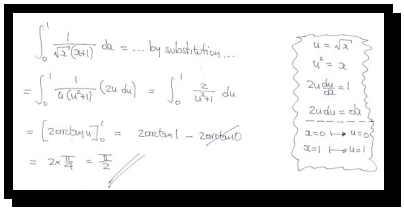


Question 10 (+)**

By using a suitable substitution, find in terms of π, the value of

$$\int_0^1 \frac{1}{(x+1)\sqrt{x}}\,dx .$$

$\boxed{\dfrac{\pi}{2}}$



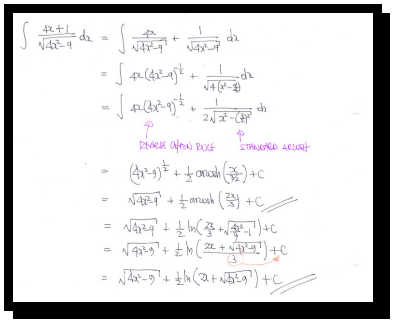
Question 11 **(**+)**

Show clearly that

$$\int \frac{4x+1}{\sqrt{4x^2-9}}\,dx = f(x)+\frac{1}{2}\ln\left[2x+f(x)\right]+C\,,$$

where $f(x)$ is a function to be found.

$$\boxed{f(x)=\sqrt{4x^2-9}}$$

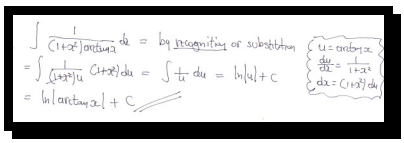


Question 12 **(**+)**

By using a suitable substitution, or otherwise, find

$$\int \frac{1}{\left(1+x^2\right)\arctan x}\,dx\,.$$

$$\boxed{\ln\left|\arctan x\right|+C}$$



Question 13 **(**+)**

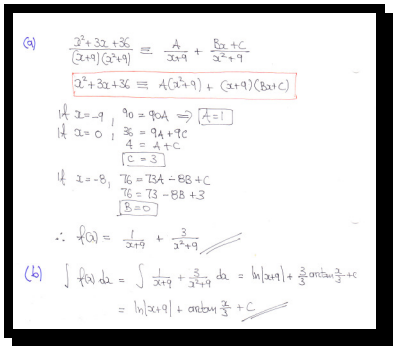
$$f(x)\equiv\frac{x^2+3x+36}{(x+9)\left(x^2+9\right)}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find

$$\int f(x)\,dx\,.$$

$$\boxed{\frac{1}{x+9}+\frac{3}{x^2+9}},\quad \boxed{\ln|x+9|+\arctan\frac{x}{3}+C}$$



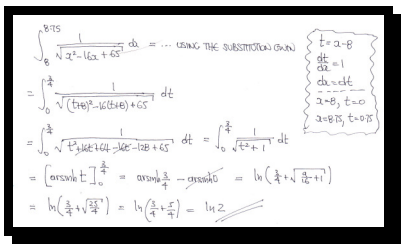
Question 14 (***)

Use the substitution $t = x - 8$ to find the exact value of

$$\int_{8}^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}}\, dx ,$$

giving the answer as a single natural logarithm.

$\ln 2$



Question 15 (***)

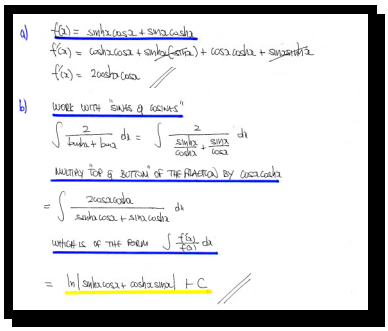
$$f(x) = \sinh x \cos x + \sin x \cosh x , \quad x \in \mathbb{R} .$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part (a) to find

$$\int \frac{2}{\tanh x + \tan x}\, dx .$$

$\square$, $f'(x) = 2\cosh x \cos x$, $\ln\left|\sinh x \cos x + \sin x \cosh x\right| + C$

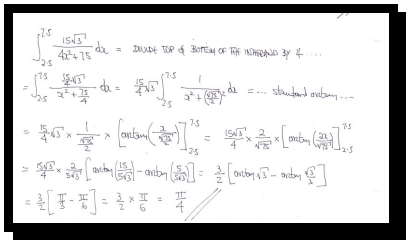


Question 16 (***)

Find the exact value of

$$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2+75}\,dx\,.$$

$$\boxed{\frac{\pi}{4}}$$

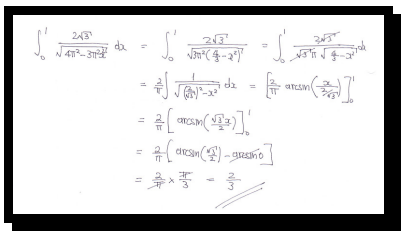


Question 17 (***)

Find the exact value of

$$\int_{0}^{1} \frac{2\sqrt{3}}{\sqrt{4\pi^2-3\pi^2x^2}}\,dx\,.$$

$$\boxed{\frac{2}{3}}$$



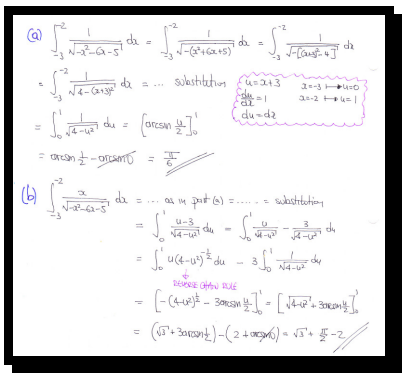
Question 18 (***)

Find the exact value of each of the following integrals.

a) $\displaystyle\int_{-3}^{-2}\frac{1}{\sqrt{-x^2-6x-5}}\,dx$.

b) $\displaystyle\int_{-3}^{-2}\frac{x}{\sqrt{-x^2-6x-5}}\,dx$.

$$\boxed{\frac{\pi}{6}}\,,\qquad \boxed{\sqrt{3}+\frac{\pi}{2}-2}$$

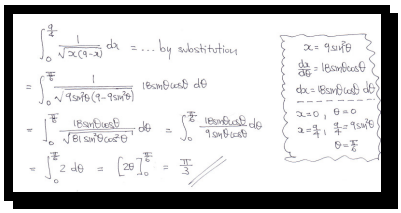


Question 19 (***)

By using the substitution $x=9\sin^2\theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}}\frac{1}{\sqrt{x(9-x)}}\,dx\,.$$

$$\boxed{\frac{\pi}{3}}$$

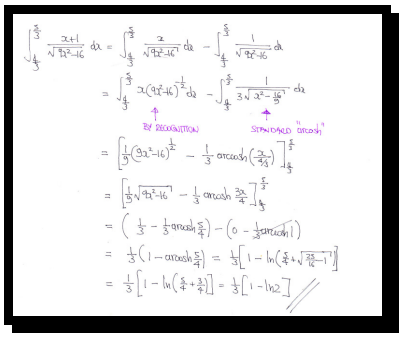


Question 20 (***)

Find the exact value of

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}}\,dx\,.$$

$$\boxed{\frac{1}{3}(1-\ln 2)}$$



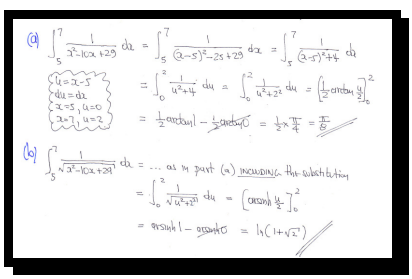
Question 21 (***)

Find the exact value of each of the following integrals.

a) $$\int_{5}^{7} \frac{1}{x^2-10x+29}\,dx\,.$$

b) $$\int_{5}^{7} \frac{x}{\sqrt{x^2-10x+29}}\,dx\,.$$

$$\boxed{\frac{\pi}{8}}\,,\quad \boxed{\ln\left(1+\sqrt{2}\right)}$$

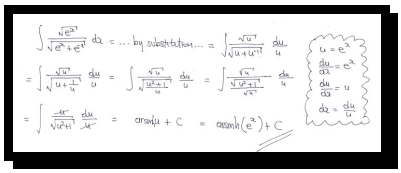


Question 22 (***)

Use the substitution $u = \mathrm{e}^x$ to find

$$\int \frac{\sqrt{\mathrm{e}^x}}{\sqrt{\mathrm{e}^x + \mathrm{e}^{-x}}}\, dx .$$

$$\boxed{\operatorname{arsinh}\left(\mathrm{e}^x\right) + C}$$

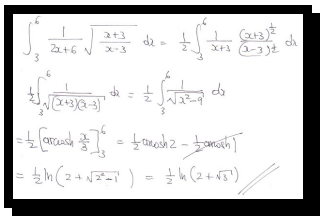


Question 23 (***)

Find in exact simplified form in terms of natural logarithms

$$\int_3^6 \frac{1}{2x+6}\sqrt{\frac{x+3}{x-2}}\, dx .$$

$$\boxed{\frac{1}{2}\ln\left(2+\sqrt{3}\right)}$$



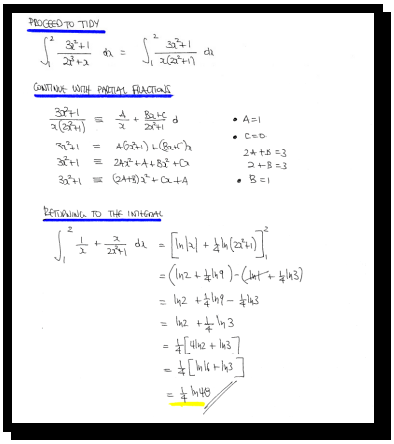
Question 24 **(***)**

Show that the exact value of the following integral

$$\int_1^2 \frac{3x^2+1}{2x^3+x}\,dx$$

is $\frac{1}{4}\ln 48$.

[ ], [proof]

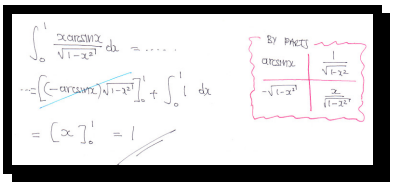


Question 25 (***)

$$\int_0^1 \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx\,.$$

Show that value of the above definite integral is 1.

proof



Question 26 (*+)**

$$f(x)\equiv x\arctan x\,,\ x\in\mathbb{R}\,.$$

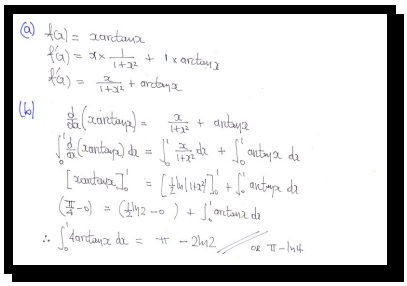
a) Find an expression for $f'(x)$.

b) Use the answer to part (a) to find the exact value of

$$\int_0^1 4\arctan x\ \,dx\,.$$

You may **not** use standard integration by parts to obtain the answer to part (b).

$$f'(x)=\frac{x}{1+x^2}+\arctan x\,,\quad \pi-\ln 4$$

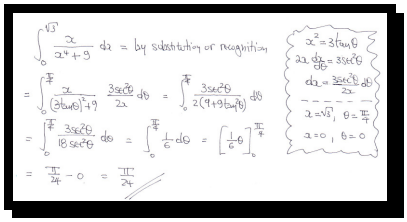


Question 27 (*+)**

By using the substitution $x^2 = 3\tan\theta$, or otherwise, find the exact value of

$$\int_0^{\sqrt{3}} \frac{x}{x^4+9}\,dx\,.$$

$$\boxed{\frac{\pi}{24}}$$



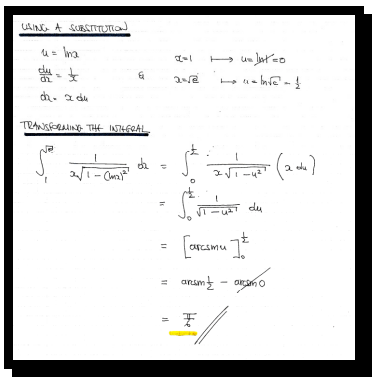
Question 28 (*+)**

Use an appropriate substitution to find an exact value for the following integral.

$$\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx\,.$$

You may assume that the integral converges.

$$\boxed{}\,,\quad \boxed{\frac{1}{6}\pi}$$



Question 29 (*+)**

The curves C_1 and C_2 have respective equations

$$y = 18\cosh x,\ x \in \mathbb{R} \qquad \text{and} \qquad y = 12 + 14\sinh x,\ x \in \mathbb{R}.$$

a) Find the exact coordinates of the points of intersection between C_1 and C_2.

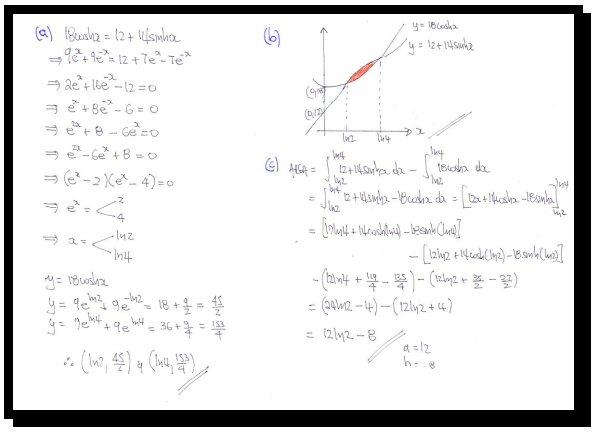
b) Sketch in the same diagram the graph of C_1 and the graph of C_2.

c) Show that the finite region bounded by the graphs of C_1 and C_2 has an area of

$$a\ln 2 + b,$$

where a and b are integers to be found.

$$\boxed{\left(\ln 2, \frac{45}{2}\right) \ \& \ \left(\ln 4, \frac{153}{4}\right)}, \quad \boxed{12\ln 2 - 8}$$



Question 30 (*+)**

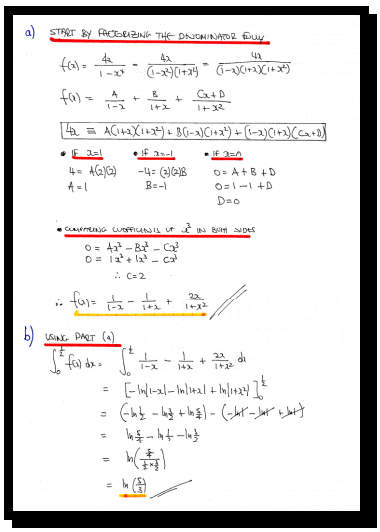
$$f(x)\equiv\frac{4x}{1-x^4}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find, as a single natural logarithm, the value of

$$\int_0^{\frac{1}{2}} f(x)\ dx.$$

$\square$, $\boxed{f(x)=\dfrac{1}{1-x}-\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}}$, $\boxed{\ln\dfrac{5}{3}}$



Question 31 (*+)**

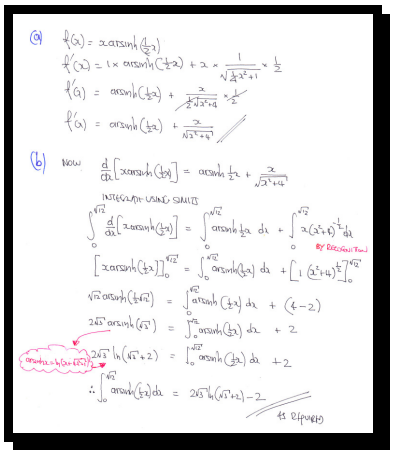
$$f(x)=x\operatorname{arsinh}\left(\frac{1}{2}x\right),\ x\in\mathbb{R}.$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part **(a)** to show that

$$\int_0^{\sqrt{12}}\operatorname{arsinh}\left(\frac{1}{2}x\right)\,dx=2\sqrt{3}\ln\left(2+\sqrt{3}\right)-2.$$

$$\boxed{f'(x)=\operatorname{arsinh}\left(\frac{1}{2}x\right)+\frac{x}{\sqrt{x^2+4}}}$$



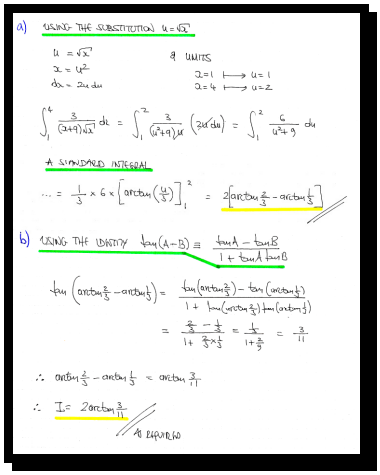
Question 32 **(***+)**

$$I=\int_1^4 \frac{3}{(x+9)\sqrt{x}}\,dx\,.$$

a) By using a suitable substitution find an exact value for I.

b) Show clearly that the answer to part **(a)** can be written as $2\arctan\frac{3}{11}$.

$\square$, $\boxed{I=2\left(\arctan\frac{2}{3}-\arctan\frac{1}{3}\right)}$



Question 33 (*+)**

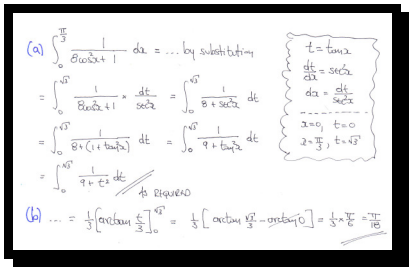
$$I=\int_0^{\frac{\pi}{3}}\frac{1}{1+8\cos^2 x}\,dx\,.$$

a) By using the substitution $t=\tan x$, or otherwise, show clearly that

$$I=\int_0^{\sqrt{3}}\frac{1}{9+t^2}\,dt\,.$$

b) Hence find the exact value of I.

$$\boxed{\frac{\pi}{18}}$$

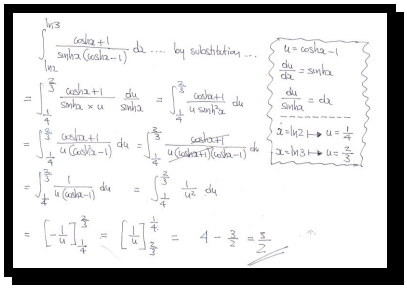


Question 34 (*+)**

By using the substitution $u=\cosh x-1$, or otherwise, find the value of

$$\int_{\ln 2}^{\ln 3}\frac{\cosh x+1}{\sinh x\left(\cosh x-1\right)}\,dx\,.$$

$$\boxed{\frac{5}{2}}$$



Question 35 **(***+)**

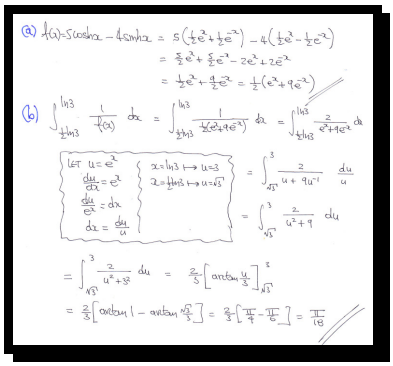
$$f(x)=5\cosh x-4\sinh x,\ x\in\mathbb{R}.$$

a) Find a simplified expression for $f(x)$ in terms of e^{x}.

b) Hence by using the substitution $u=\mathrm{e}^{x}$, or otherwise, show that

$$\int_{\frac{1}{2}\ln 3}^{\ln 3}\frac{1}{f(x)}\,dx=\frac{\pi}{18}.$$

$$\boxed{f(x)=\tfrac{1}{2}\mathrm{e}^{x}+\tfrac{9}{2}\mathrm{e}^{-x}}$$

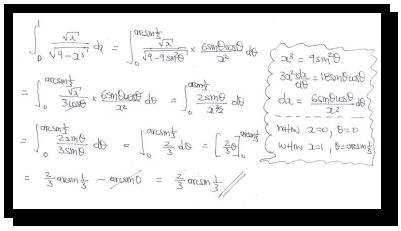


Question 36 **(***+)**

By using the substitution $x^{3}=9\sin^{2}\theta$, or otherwise, find the exact value of

$$\int_{0}^{1}\frac{\sqrt{x}}{\sqrt{9-x^{3}}}\,dx.$$

$$\boxed{\frac{2}{3}\arcsin\left(\frac{1}{3}\right)}$$



Question 37 **(***+)**

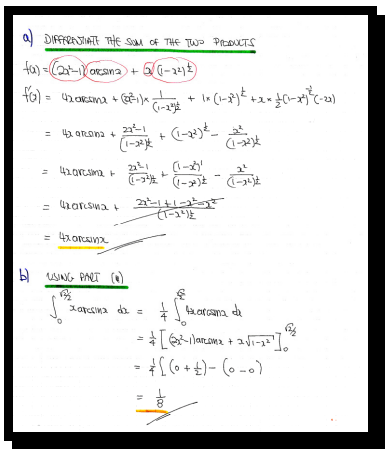
$$f(x)\equiv\left(2x^2-1\right)\arcsin x+x\sqrt{1-x^2}\ ,\ -1\le x\le 1\,.$$

a) Find a simplified expression for $f'(x)$.

b) Hence find

$$\int_0^{\frac{\sqrt{2}}{2}} x\arcsin x\ \ dx\,.$$

$\boxed{}$, $\boxed{f'(x)=4x\arcsin x}$, $\boxed{\frac{1}{8}}$

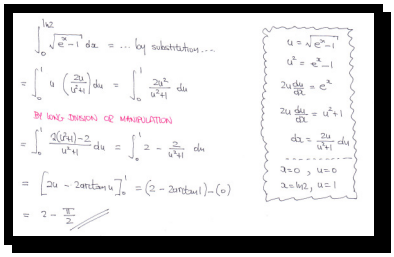


Question 38 (*+)**

By using the substitution $u=\sqrt{\mathrm{e}^{x}-1}$, or otherwise, find the exact value of

$$\int_{0}^{\ln 2}\sqrt{\mathrm{e}^{x}-1}\ dx\,.$$

$$\boxed{2-\frac{\pi}{2}}$$

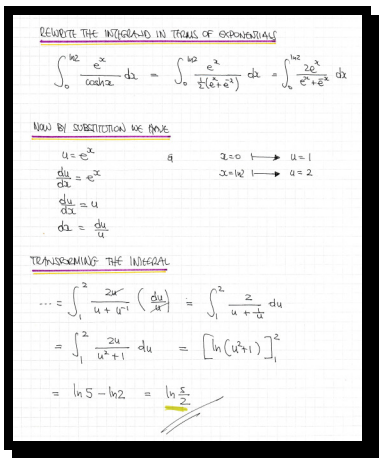


Question 39 (*+)**

Find in exact simplified form the value of

$$\int_{0}^{\ln 2}\frac{\mathrm{e}^{x}}{\cosh x}\ dx\,.$$

$$\boxed{\quad}\ ,\ \boxed{\ln\frac{5}{2}}$$



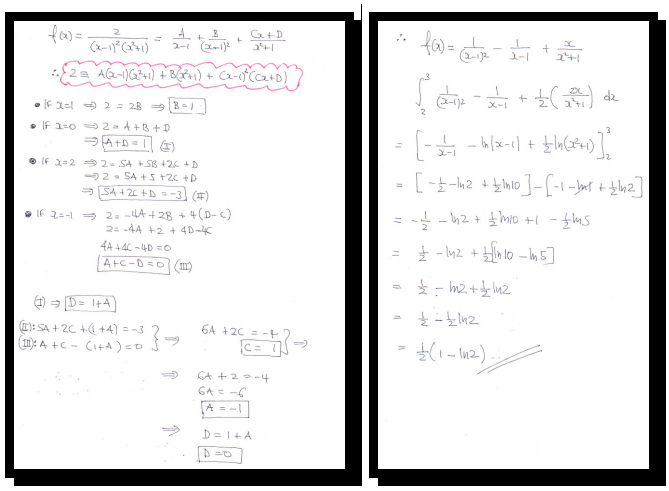
Question 40 (*+)**

$$f(x)=\frac{2}{(x-1)^2\left(x^2+1\right)},\ x\neq 0.$$

Use partial fractions to show that

$$\int_2^3 f(x)\ dx=\frac{1}{2}(1-\ln 2).$$

proof

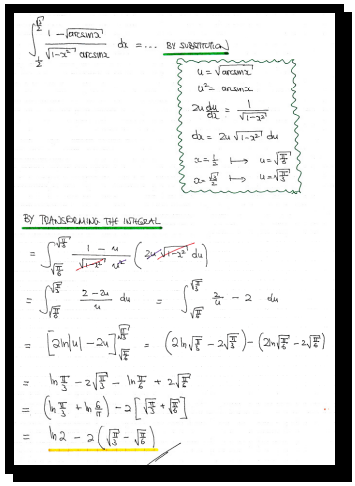


Question 41 (*+)**

Use an appropriate substitution to find an exact value for the following integral.

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1-\sqrt{\arcsin x}}{\sqrt{1-x^2}\,\arcsin x}\,dx\,.$$

$\square$, $\ln 2 - 2\left(\sqrt{\frac{1}{3}\pi}-\sqrt{\frac{1}{6}\pi}\right)$

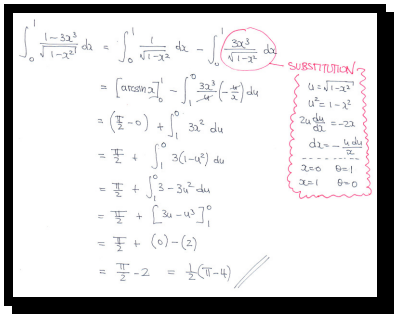


Question 42 **(****)**

Find the exact value of

$$\int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}}\,dx\,.$$

$$\boxed{\frac{1}{2}(\pi-4)}$$

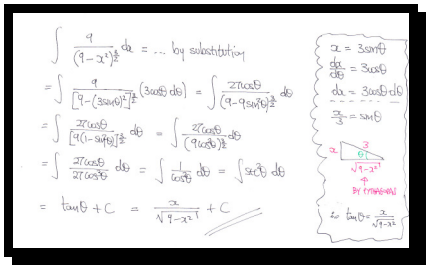


Question 43 **(****)**

Use a suitable trigonometric substitution to find an integrated expression for

$$\int \frac{9}{\left(9-x^2\right)^{\frac{3}{2}}}\,dx\,.$$

$$\boxed{\frac{x}{\sqrt{9-x^2}}+C}$$

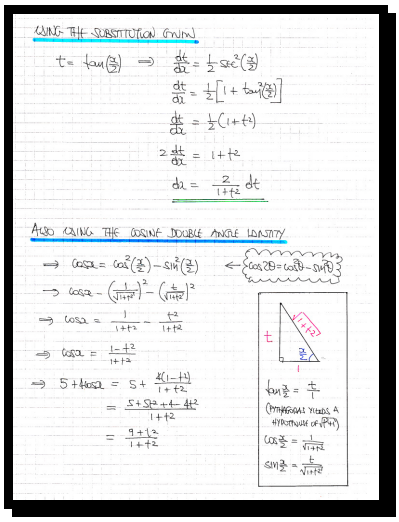


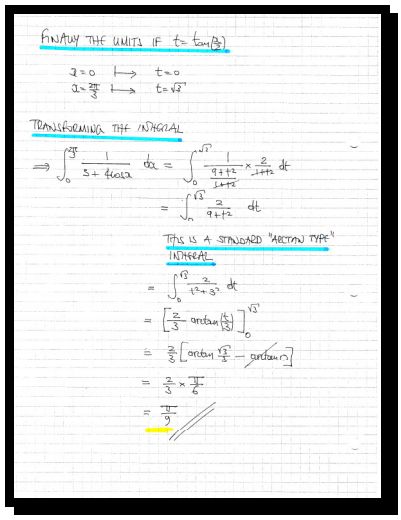
Question 44 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\, dx\,.$$

$\square$, $\boxed{\frac{\pi}{9}}$





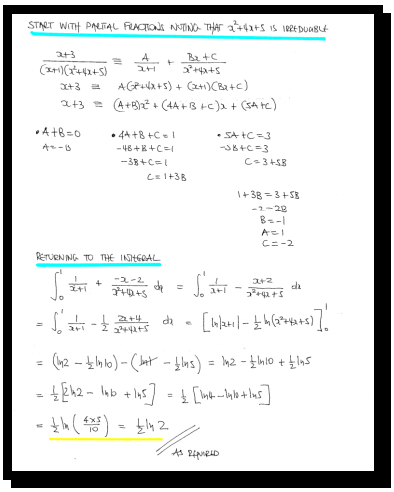
Question 45 **(****)**

Show that the exact value of the following integral

$$\int_0^1 \frac{x+3}{(x+1)\left(x^2+4x+5\right)}\,dx$$

is $\frac{1}{2}\ln 2$.

[ ], [proof]



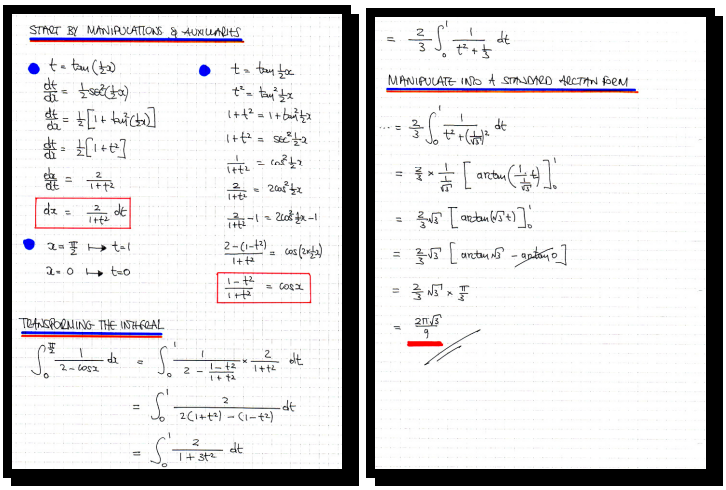
Question 46 **(****)**

Use the substitution $t=\tan\left(\frac{1}{2}x\right)$ to find an exact simplified value for

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\,dx\,.$$

Any trigonometric identities to convert $\cos x$ in terms of t must be derived.

$\square$, $\boxed{\frac{2\pi\sqrt{3}}{9}}$



Question 47 **(****)**

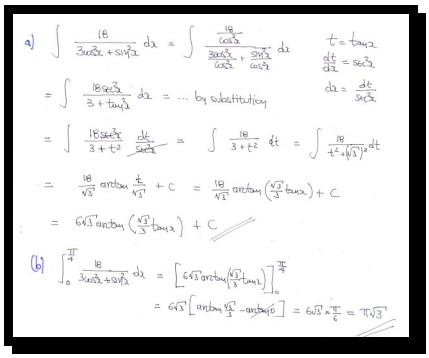
$$I=\int \frac{18}{3\cos^2 x+\sin^2 x}\,dx\,.$$

a) By using the substitution $t=\tan x$, or otherwise, show clearly that

$$I=6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}\tan x\right)+\text{constant}\,.$$

b) Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x+\sin^2 x}\,dx$.

$$\boxed{\pi\sqrt{3}}$$

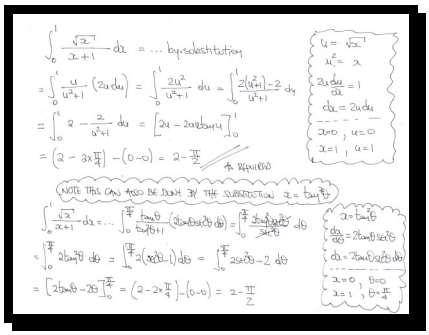


Question 48 (****)

By using the substitution $u=\sqrt{x}$, or otherwise, find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1}\,dx\,.$$

$$\boxed{2-\frac{\pi}{2}}$$

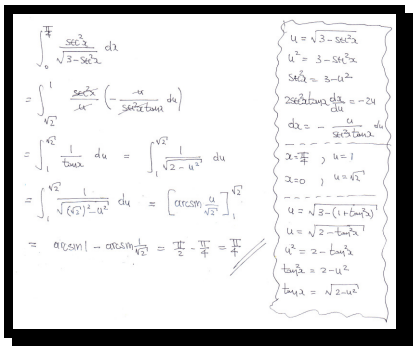


Question 49 (****)

By using the substitution $u=\sqrt{3-\sec^2 x}$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3-\sec^2 x}}\,dx\,.$$

$$\boxed{\frac{\pi}{4}}$$

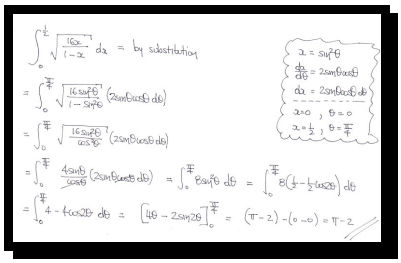


Question 50 (****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}}\ dx = \pi - 2\,.$$

proof

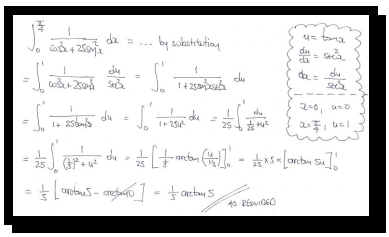


Question 51 (****)

By using the substitution $u = \tan x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25\sin^2 x}\ dx = \frac{1}{5}\arctan 5\,.$$

proof

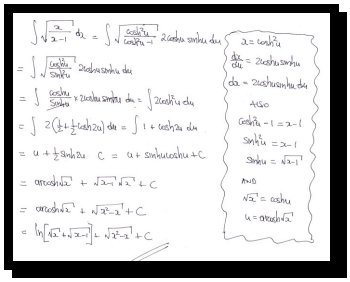


Question 52 (****)

By using the substitution $x = \cosh^2 u$, or otherwise, show that

$$\int \sqrt{\frac{x}{x-1}}\, dx = \ln\left(\sqrt{x} + \sqrt{x-1}\right) + \sqrt{x^2 - x} + \text{constant}.$$

proof



Question 53 **(****)**

$$\sin 2x \equiv \frac{2\tan x}{1+\tan^2 x}.$$

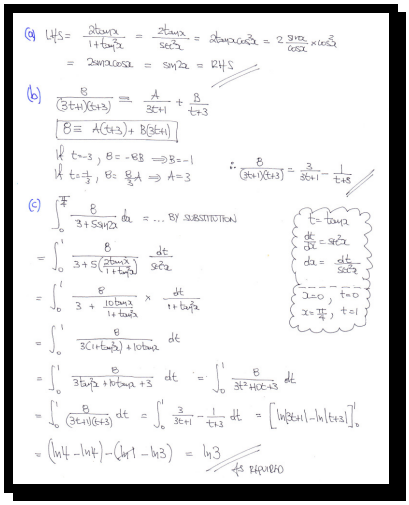
a) Prove the validity of the above trigonometric identity.

b) Express $\dfrac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t=\tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx = \ln 3.$$

$$\boxed{\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}}$$



Question 54 **(****)**

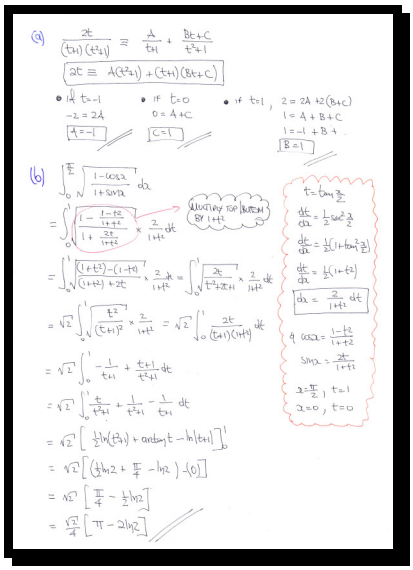
$$\frac{2t}{(t+1)\left(t^2+1\right)} \equiv \frac{A}{t+1}+\frac{Bt+C}{t^2+1}.$$

a) Determine the values of A, B and C in the above identity.

b) Hence find an value for

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}}\ dx.$$

$\boxed{}$, $\boxed{A=-1,\ B=1,\ C=1}$, $\boxed{\frac{\sqrt{2}}{2}(\pi-2\ln 2)}$

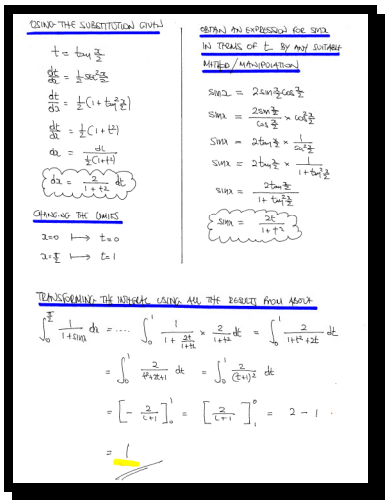


Question 55 (****)

Use the substitution $t=\tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx\,.$$

$\boxed{}$, $\boxed{1}$

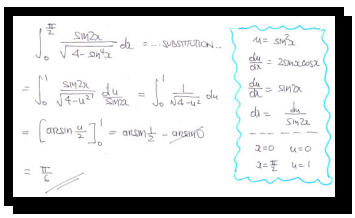


Question 56 (****)

Use suitable substitution to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4-\sin^4 x}}\,dx\,.$$

$\boxed{\frac{\pi}{6}}$



Question 57 **(****)**

$$I = \int \sqrt{\frac{x}{1-x}}\, dx$$

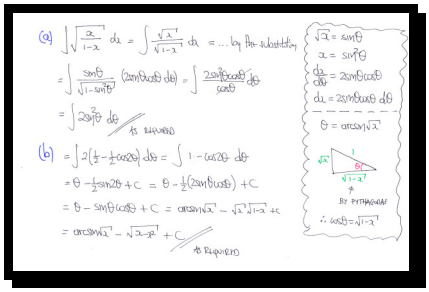
a) Use the substitution $\sqrt{x} = \sin\theta$ to show that

$$I = \int 2\sin^2\theta \, d\theta .$$

b) Hence show further that

$$I = \arcsin\sqrt{x} - \sqrt{x - x^2} + \text{constant}$$

proof



Question 58 (****)

The curve with the following equation is defined in the largest real domain.

$$y=(4x-3)\sqrt{-8\left(2x^2-3x+1\right)}+\arcsin(4x-3).$$

a) Show that

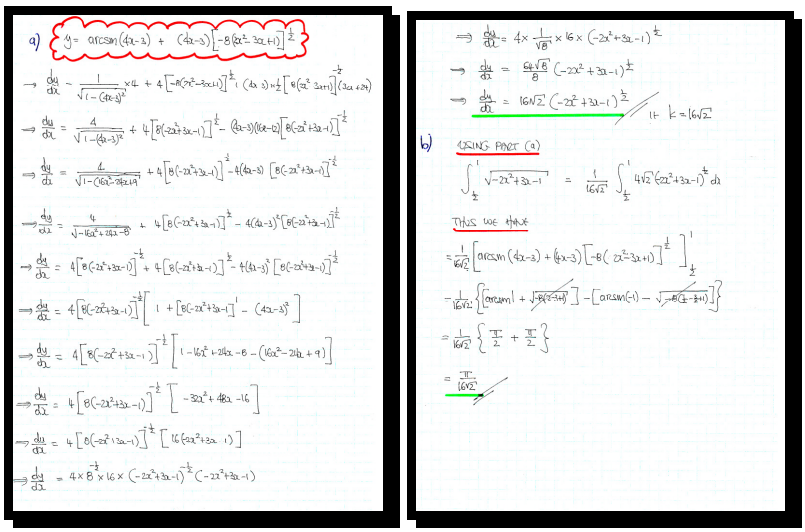
$$\frac{dy}{dx}=k\sqrt{-2x^2+3x-1},$$

where k is an exact constant to be found.

b) Hence find the exact value of the following integral.

$$\int_{\frac{1}{2}}^{1}\sqrt{-2x^2+3x-1}\ dx.$$

$\square$, $\boxed{k=16\sqrt{2}}$, $\boxed{\dfrac{\pi}{16\sqrt{2}}}$

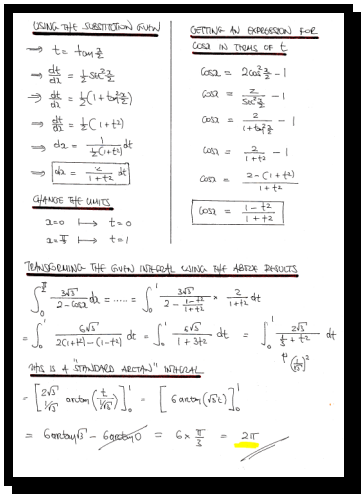


Question 59 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\, dx .$$

$\boxed{}$, $\boxed{2\pi}$

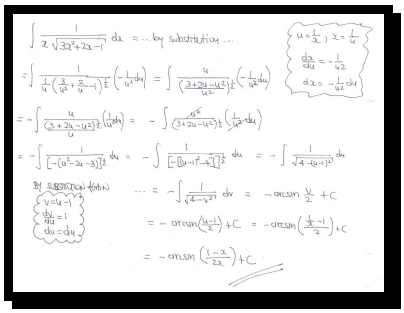


Question 60 (****)

Use the substitution $u = \dfrac{1}{x}$ to find

$$\int \frac{1}{x\sqrt{3x^2+2x-1}}\,dx\,.$$

$$\boxed{-\arcsin\left(\frac{1-x}{2x}\right)+C}$$



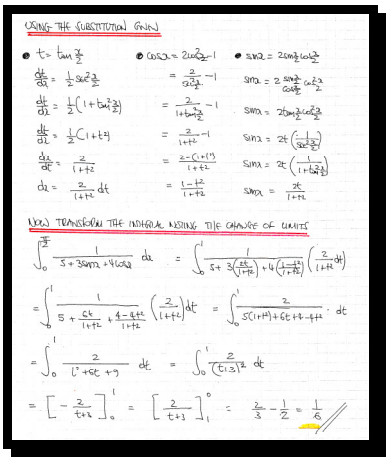
Question 61 (**+)**

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{5++3\sin x+4\cos x}\, dx\,.$$

All relevant results used in this evaluation must be carefully derived.

$\square$, $\boxed{\frac{1}{6}}$



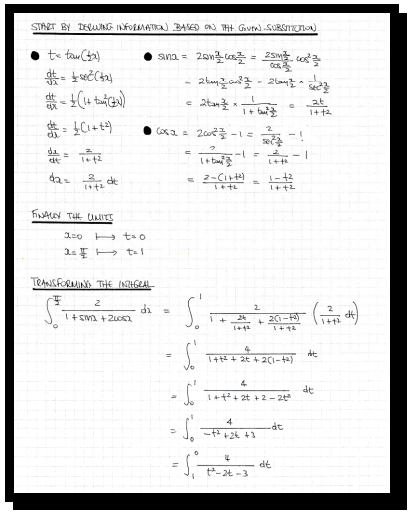
Question 62 (**+)**

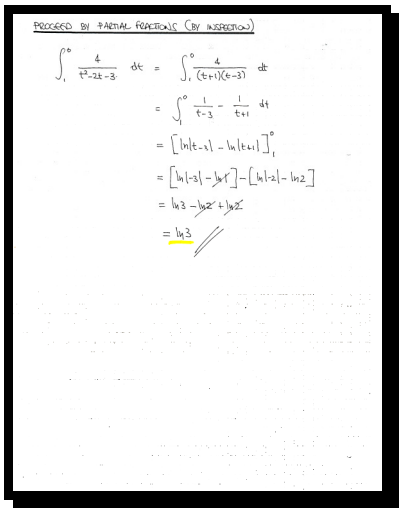
Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find the exact value for the integral

$$\int_0^{\frac{1}{2}\pi} \frac{2}{1+\sin x+2\cos x}\, dx$$

All relevant results used in this evaluation must be carefully derived.

$\boxed{}$, $\boxed{\ln 3}$





Question 63 (**+)**

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,.$$

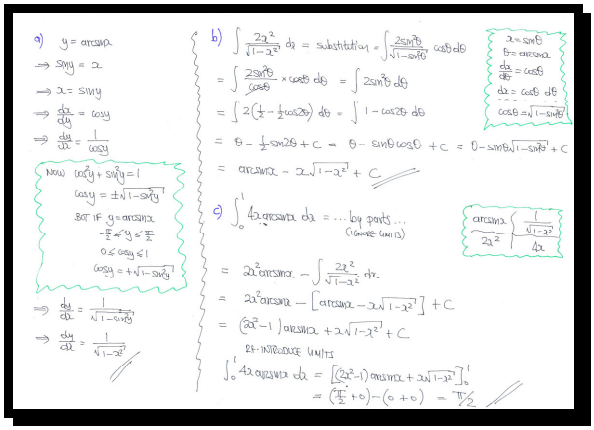
b) Use the substitution $x = \sin\theta$ to find

$$\int \frac{2x^2}{\sqrt{1-x^2}}\,dx\,.$$

c) Hence find an exact value for

$$\int_0^1 4x \arcsin x\ \ dx$$

$$\boxed{\arcsin x - x\sqrt{1-x^2} + C}\,,\ \boxed{\frac{\pi}{2}}$$



Question 64 **(****+)**

It is given that

$$x=-2+\sqrt{3}\cosh\theta\,,\ \theta\geq 0\,.$$

a) Show clearly that …

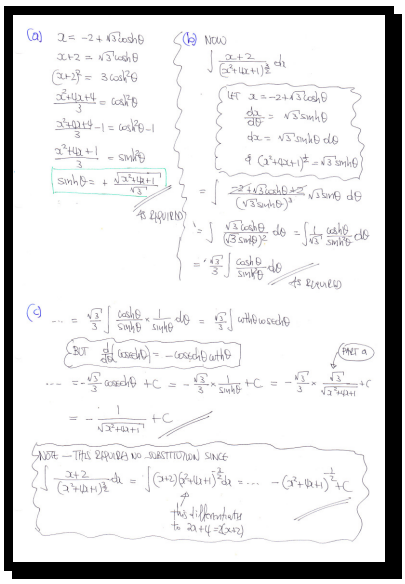
i. … $\sinh\theta=\dfrac{\sqrt{x^2+4x+1}}{\sqrt{3}}$.

ii. … $\displaystyle\int\frac{x+2}{\left(x^2+4x+1\right)^{\frac{3}{2}}}\,dx=\frac{\sqrt{3}}{3}\int\frac{\cosh\theta}{\sinh^2\theta}\,d\theta\,.$

b) By considering the derivative of $\operatorname{cosech}\theta$ find

$$\int\frac{x+2}{\left(x^2+4x+1\right)^{\frac{3}{2}}}\,dx$$

$$\boxed{-\left(x^2+4x+1\right)^{-\frac{1}{2}}+C}$$

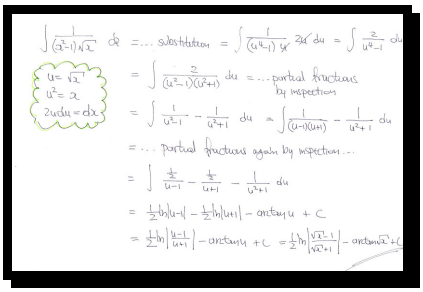


Question 65 (**+)**

Use the substitution $u=\sqrt{x}$ to find

$$\int \frac{1}{\left(x^2-1\right)\sqrt{x}}\,dx\,.$$

$$\boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|-\arctan\sqrt{x}+C}$$



Question 66 (**+)**

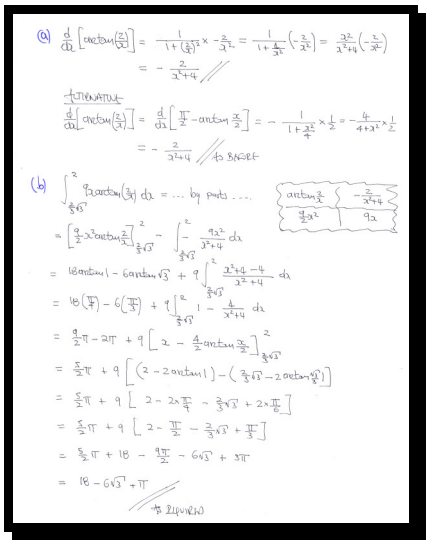
a) Find a simplified expression for

$$\frac{d}{dx}\left[\arctan\frac{2}{x}\right].$$

b) Hence show that

$$\int_{\frac{2}{3}\sqrt{3}}^{2} 9x\arctan\left(\frac{2}{x}\right)\,dx=\pi+18-6\sqrt{3}$$

$$\boxed{\frac{d}{dx}\left[\arctan\frac{2}{x}\right]=-\frac{2}{x^2+4}}$$



Question 67 **(****+)**

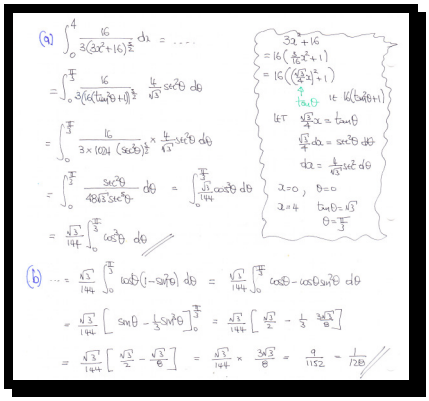
$$\int_0^4 \frac{16}{3\left(3x^2+16\right)^{\frac{5}{2}}}\,dx\,.$$

a) By using a suitable trigonometric substitution in terms of θ, show that the above integral can be transformed to

$$\frac{\sqrt{3}}{144}\int_0^{\frac{\pi}{3}} \cos^3\theta\ \,d\theta\,.$$

b) Hence evaluate the original integral.

$$\boxed{\frac{1}{128}}$$



Question 68 **(****+)**

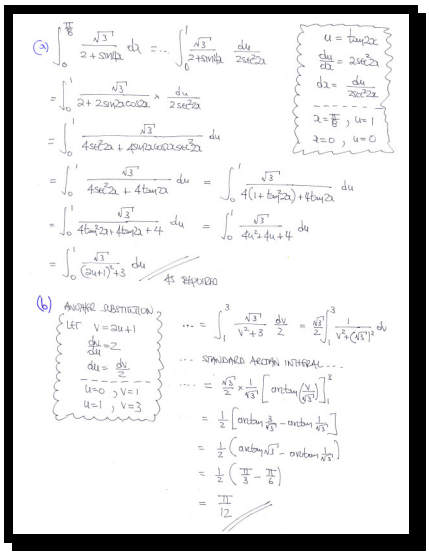
$$I=\int_0^{\frac{\pi}{8}}\frac{\sqrt{3}}{2+\sin 4x}\,dx\,.$$

a) Show that the substitution $u=\tan 2x$ transforms I into

$$J=\int_0^{1}\frac{\sqrt{3}}{(2u+1)^2+3}\,du\,.$$

b) Hence find the exact value of I, giving the answer in terms of π.

$$\boxed{I=\frac{\pi}{12}}$$



Question 69 **(****+)**

$$\sec x \equiv \frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}.$$

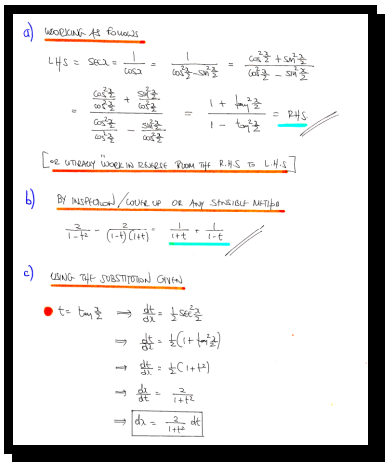
a) Prove the validity of the above trigonometric identity.

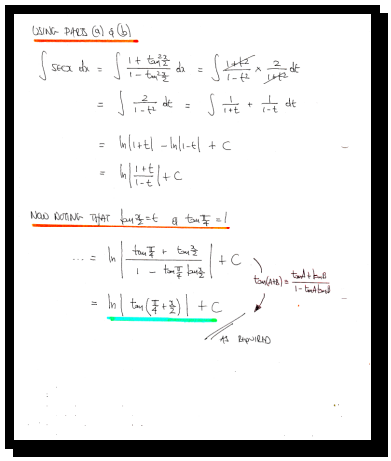
b) Express $\dfrac{2}{1-t^2}$ into partial fractions.

c) Hence use the substitution $t=\tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C.$$

$\square$, $\boxed{\dfrac{2}{1-t^2}=\dfrac{1}{1+t}+\dfrac{1}{1-t}}$



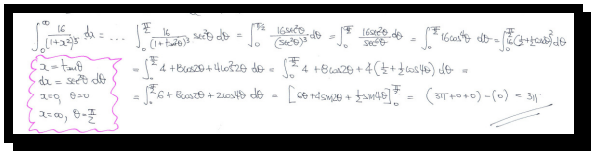


Question 70 **(****+)**

Find an exact value for

$$\int_0^{\infty} \frac{16}{\left(1+x^2\right)^3}\, dx\,.$$

3π



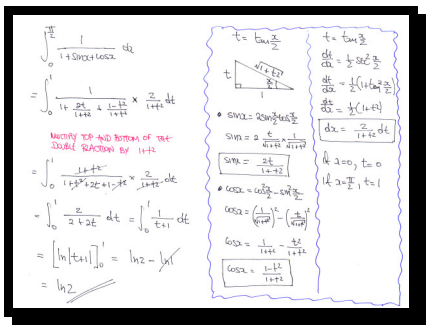
Question 71 **(****+)**

$$I=\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\, dx\,.$$

By using the substitution $t=\tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I=\ln 2\,.$$

proof

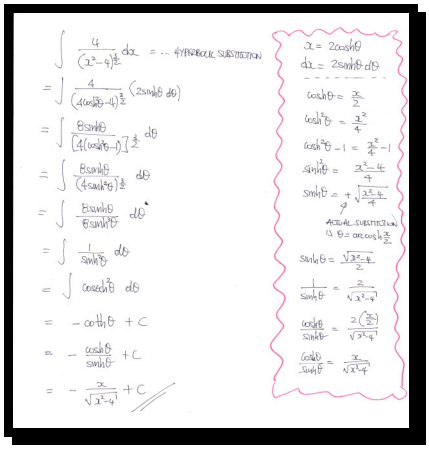


Question 72 (**+)**

Use the substitution $x = 2\cosh\theta$, to find a simplified expression for

$$\int \frac{6}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx\,.$$

$$\boxed{-\frac{x}{\sqrt{x^2-4}}+C}$$

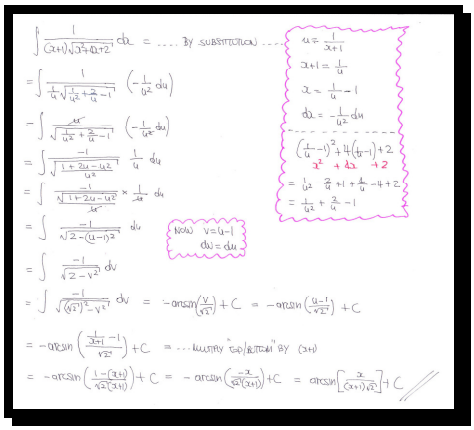


Question 73 (**+)**

Use the substitution $u=\dfrac{1}{x+1}$ to find

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}}\,dx\,.$$

$$\boxed{\arcsin\left[\frac{x}{(x+1)\sqrt{2}}\right]+C}$$



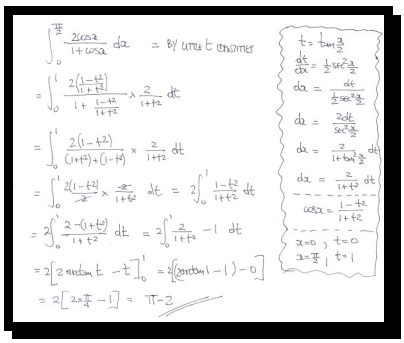
Question 74 (**+)**

$$I=\int_0^{\frac{\pi}{2}}\frac{2\cos x}{1+\cos x}\,dx\,.$$

By using the substitution $t=\tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I=\pi-2\,.$$

proof

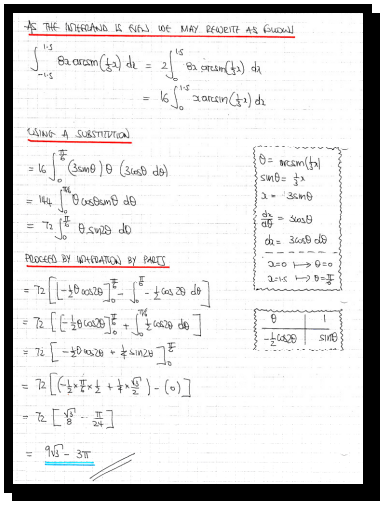


Question 75 (**+)**

Find an exact value for

$$\int_{-1.5}^{1.5} 8x\arcsin\left(\frac{1}{3}x\right)\,dx\,.$$

$\boxed{}$, $\boxed{9\sqrt{3}-3\pi}$



Question 76 (**+)**

$$I=\int \frac{\operatorname{sech} x}{\cosh x-\sinh x}\,dx\,.$$

a) By multiplying the numerator and denominator of the integrand by $\operatorname{sech} x$, show that

$$I=-\ln\left(1-\tanh x\right)+C\,,$$

where C in an arbitrary constant.

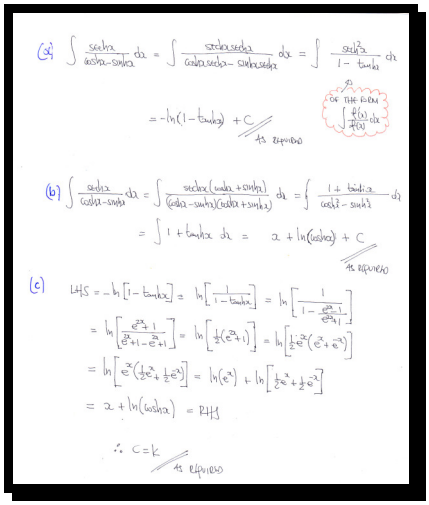
b) By multiplying the numerator and denominator of the integrand by $\left(\cosh x-\sinh x\right)$, show that

$$I=x+\ln\left(\cosh x\right)+K\,,$$

where K in an arbitrary constant.

c) Show clearly that $C=K$.

proof



Question 77 (**+)**

$$\sec x \equiv \frac{\cos x}{1-\sin^2 x}.$$

a) Prove the validity of the above trigonometric identity.

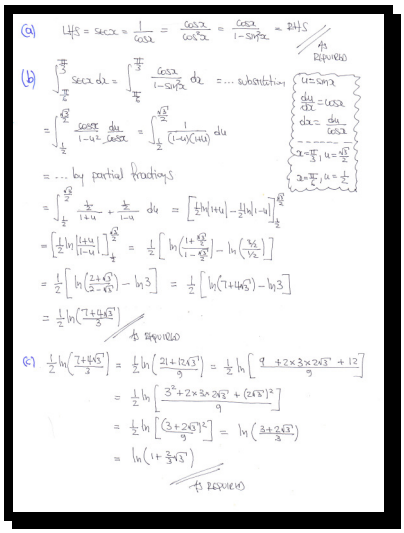
b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \ dx = \frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right).$$

c) Show clearly that

$$\frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right) = \ln\left(1+\frac{2}{3}\sqrt{3}\right)$$

proof



Question 78 **(****+)**

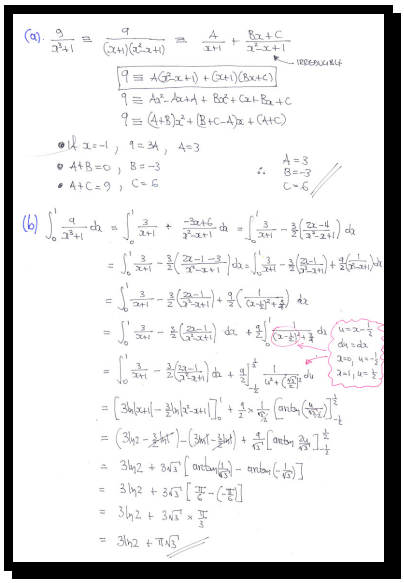
$$\frac{9}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

a) Find the value of each of the constants A, B and C in the above identity.

b) Hence find the exact value of

$$\int_0^1 \frac{9}{x^3-1}\,dx.$$

$\boxed{A=3}$, $\boxed{B=-3}$, $\boxed{C=6}$, $\boxed{3\ln 2+\pi\sqrt{3}}$

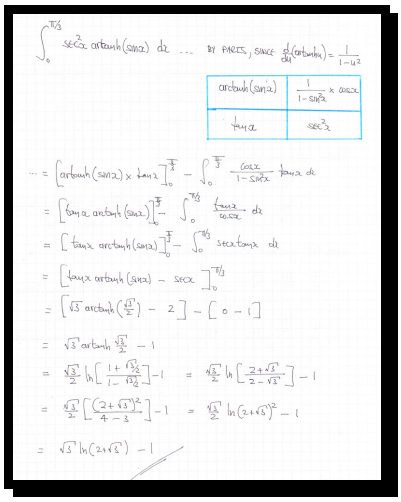


Question 79 **(****+)**

Show that

$$\int_0^{\frac{1}{3}\pi} \sec^2 x \ \operatorname{artanh}(\sin x) \ dx \ = \ -1+\sqrt{3}\ln\left(2+\sqrt{3}\right).$$

proof

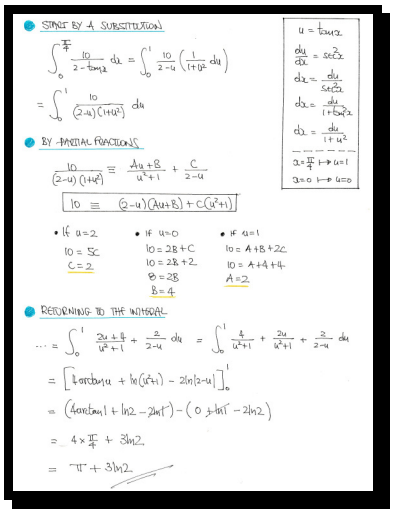


Question 80 **(****+)**

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\frac{1}{4}\pi} \frac{10}{2-\tan x}\,dx\,.$$

$\boxed{}$, $\boxed{\pi+3\ln 2}$

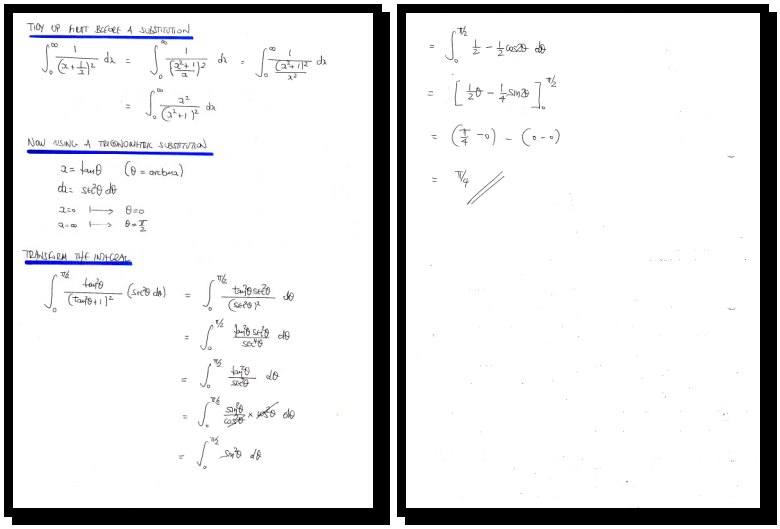


Question 81 (*****)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\infty} \frac{1}{\left(x+x^{-1}\right)^2}\,dx\,.$$

$\square$, $\boxed{\dfrac{\pi}{4}}$

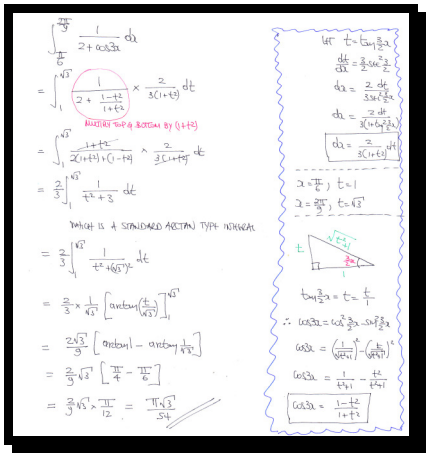


Question 82 **(*****)**

Use the substitution $t=\tan\left(\frac{3}{2}x\right)$ to find, in terms of π, the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2+\cos 3x}\,dx\,.$$

$$\boxed{\frac{\pi\sqrt{3}}{54}}$$

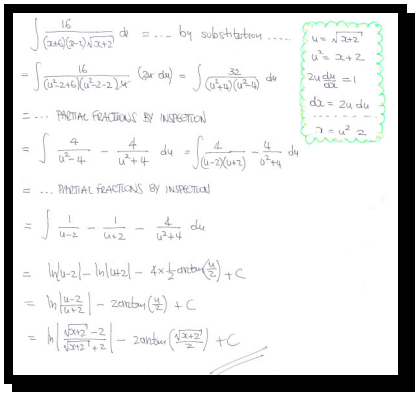


Question 83 **(*****)**

Use the substitution $u=\sqrt{x+2}$ to find

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}}\,dx\,.$$

$$\boxed{\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right|-2\arctan\left(\frac{\sqrt{x+2}}{2}\right)+C}$$

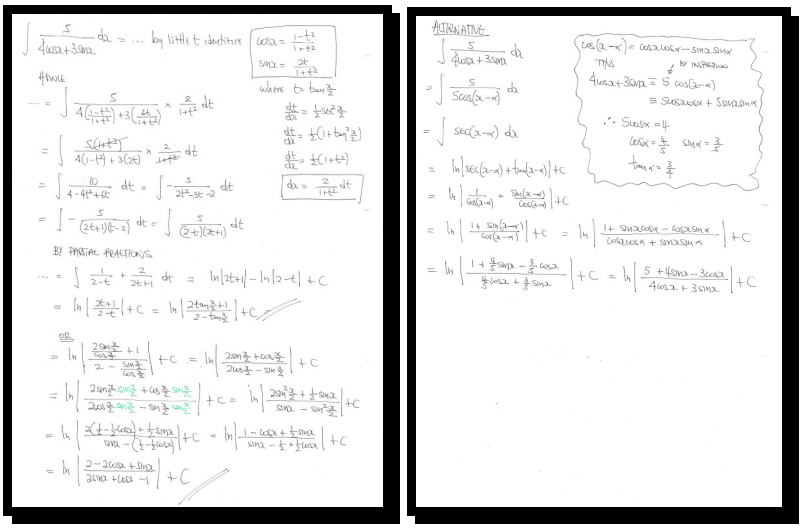


Question 84 **(*****)**

By using the substitution $t=\tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$\int \frac{5}{4\cos x+3\sin x}\,dx=\ln\left|\frac{2+\sin x-2\cos x}{2\sin x+\cos x-1}\right|+C\,.$$

proof

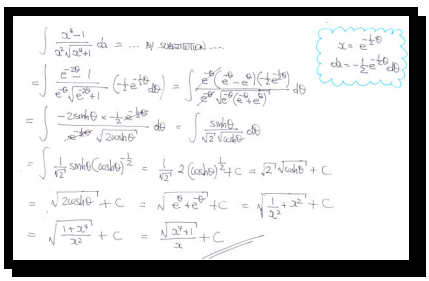


Question 85 **(*****)**

By using the substitution $x=\mathrm{e}^{-\frac{1}{2}u}$, or otherwise, find a simplified expression for

$$\int \frac{x^4-1}{x^2\sqrt{x^4+1}}\,dx\,.$$

$$\frac{\sqrt{x^4+1}}{x}+C$$

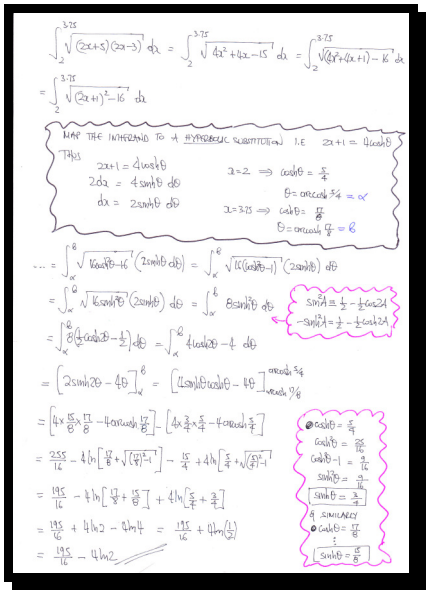


Question 86 (*****)

Use a suitable hyperbolic substitution to find the exact value of

$$\int_{2}^{3.75} \sqrt{(2x+5)(2x-3)}\ dx\ .$$

$$\boxed{\frac{195}{16}-4\ln 2}$$



Question 87 **(*****)**

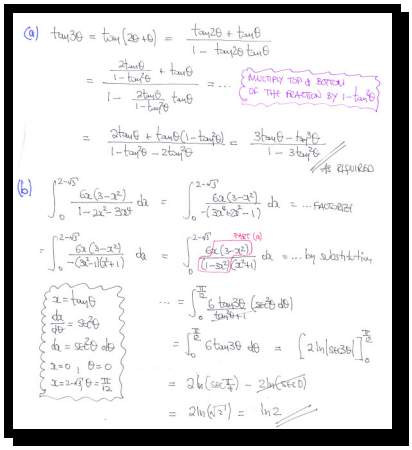
$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

a) Prove the validity of the above trigonometric identity by writing $\tan 3\theta$ as $\tan(2\theta+\theta)$.

b) Hence, show clearly that

$$\int_0^{2-\sqrt{3}} \frac{6x\left(3-x^2\right)}{1-2x^2-3x^4}\,dx = \ln 2.$$

proof



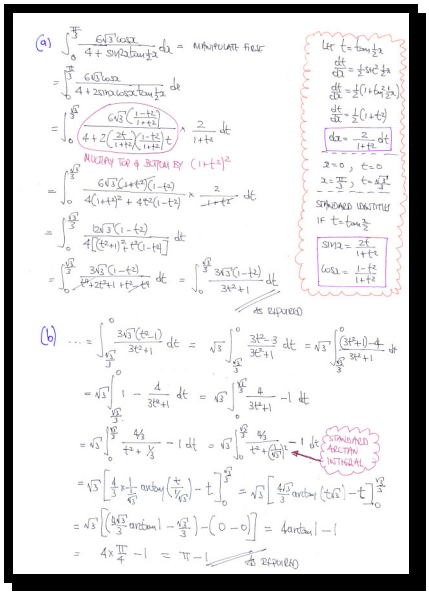
Question 88 **(*****)**

$$J=\int_0^{\frac{\pi}{3}}\frac{6\sqrt{3}\cos x}{4+\sin 2x\tan\left(\frac{1}{2}x\right)}\,dx\,.$$

Use the substitution $t=\tan\left(\frac{1}{2}x\right)$ to show that

$$J=\pi-1\,.$$

[ ] , [proof]

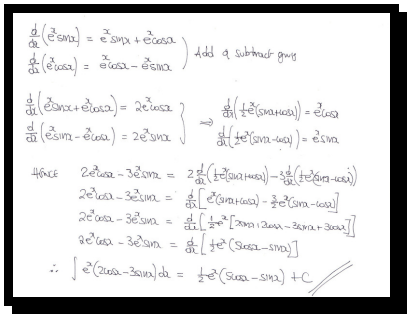


Question 89 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2\cos x - 3\sin x)\, dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}e^x(5\cos x - \sin x) + C}$

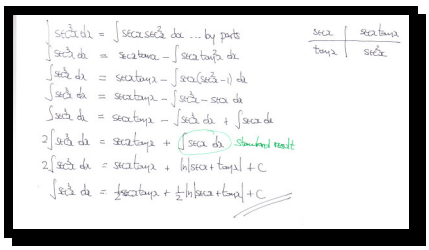


Question 90 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x\, dx .$$

$\boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$



Question 91 **(*****)**

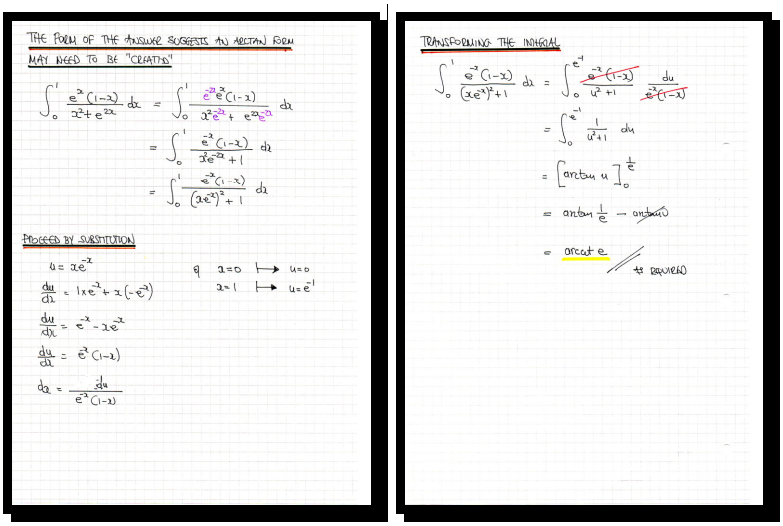
Show that the exact value of

$$\int_0^1 \frac{(1-x)\mathrm{e}^x}{x^2+\mathrm{e}^{2x}}\,dx,$$

can be written as

$$\operatorname{arccot}(\mathrm{e}).$$

, proof



Question 92 **(*****)**

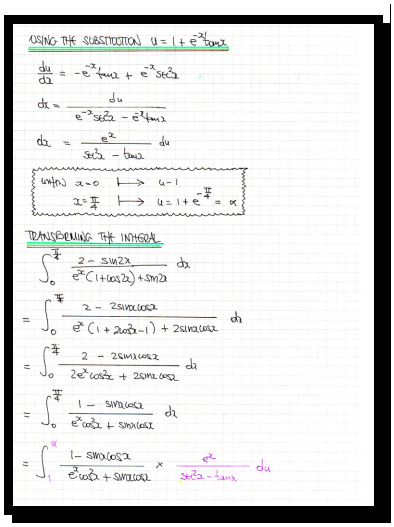
By using the substitution $u = 1 + \mathrm{e}^{-x} \tan x$, or otherwise, show that the exact value of

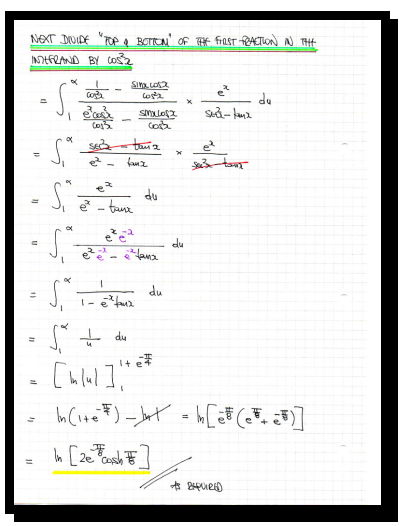
$$\int_0^{\frac{1}{4}\pi} \frac{2 - \sin 2x}{(1 + \cos 2x)\mathrm{e}^x + \sin 2x} \, dx ,$$

can be written as

$$\ln\left[2\mathrm{e}^{-\frac{1}{8}\pi} \cosh\left(\frac{1}{8}\pi\right) \right].$$

[ ], [proof]





Question 93 **(*****)**

The function f is a continuous function and a is a real constant.

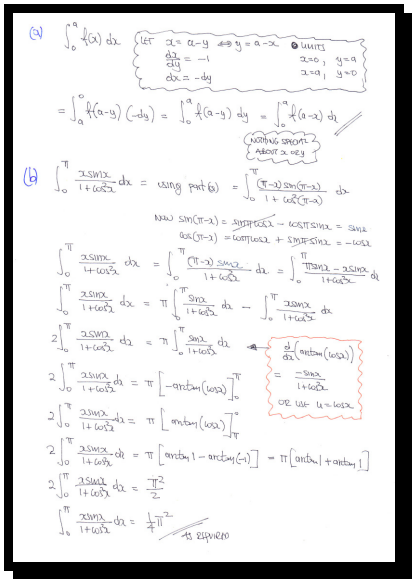
$$\int_0^a f(x)\ dx \equiv \int_0^a f(a-x)\ dx .$$

a) Prove the validity of the above identity.

b) Hence show clearly that

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\ dx = \frac{1}{4}\pi^2 .$$

proof

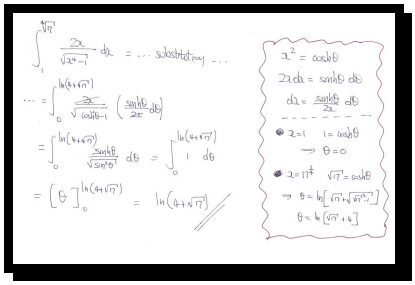


Question 94 (*****)

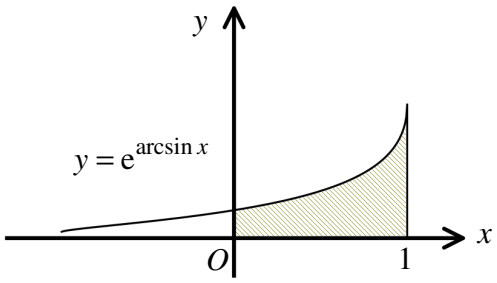
By using a suitable substitution, find the exact value of

$$\int_1^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4-1}}\, dx .$$

$\boxed{}$, $\boxed{\ln\left(4+\sqrt{17}\right)}$



Question 95 (*****)



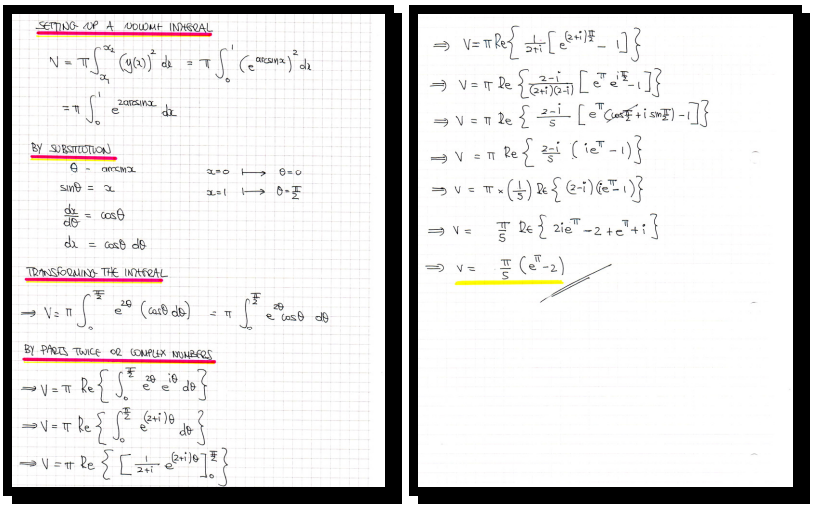
The figure above show the curve with equation

$$y = e^{\arcsin x}, \ x \in \mathbb{R}, \ |x| \leq 1.$$

The finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x = 1$, is fully revolved about the x axis.

Find, an exact simplified value, for the volume of the solid of revolution formed.

$\square$, $\boxed{\frac{1}{5}\pi\left(e^{\pi}-2\right)}$



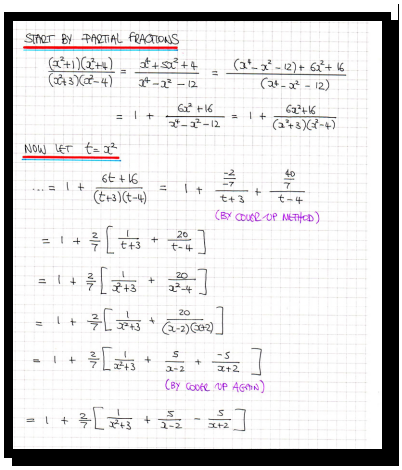
Question 96 **(*****)**

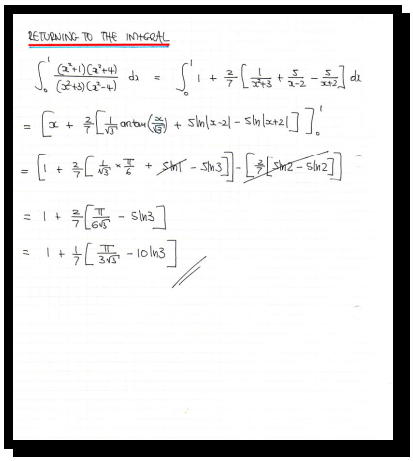
$$I=\int_0^1 \frac{\left(x^2+1\right)\left(x^2+4\right)}{\left(x^2+3\right)\left(x^2-4\right)}\,dx\,.$$

Use appropriate integration techniques to show that

$$I=1+\frac{2}{7}\left[\frac{\pi}{6\sqrt{3}}-5\ln 3\right].$$

[ ], [proof]



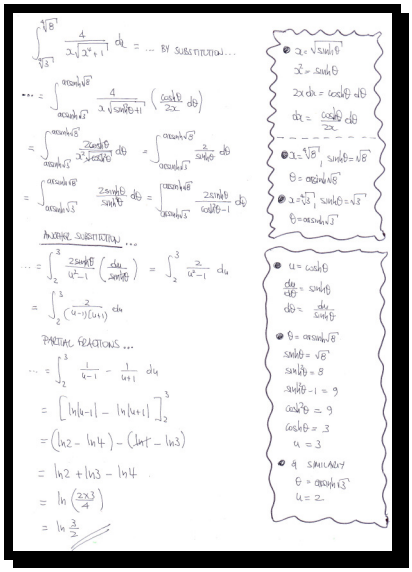


Question 97 (*****)

By using a suitable substitution, find the exact value of

$$\int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{2}{x\sqrt{x^4+1}}\,dx\,.$$

$\square$, $\boxed{\ln\left(\frac{3}{2}\right)}$



Question 98 **(*****)**

$$I=\int_0^{\arctan\left(\tanh\left(\ln 2\right)\right)}\frac{\sec^2 x\tan 2x}{\tan x-\tan^3 x}\,dx$$

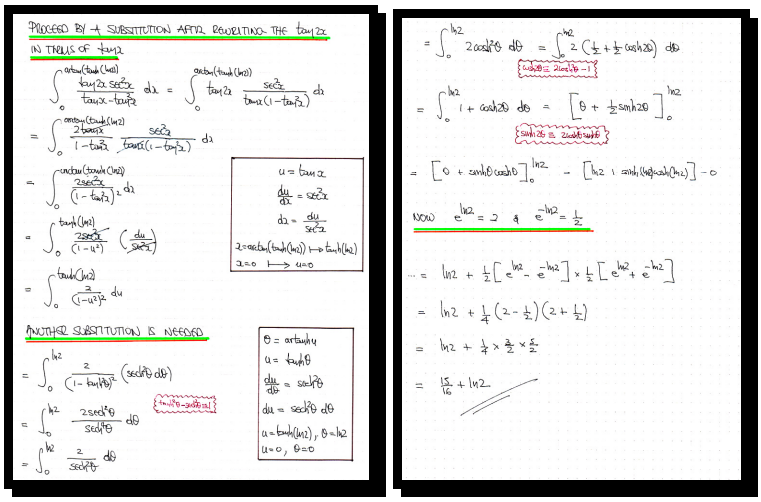
Use appropriate integration techniques to show that

$$I=k+\ln 2\,,$$

where k is a rational constant to be found.

You may assume that the limit of the integrand, as x tends to zero, exists.

$\square$, $\boxed{k=\frac{15}{16}}$

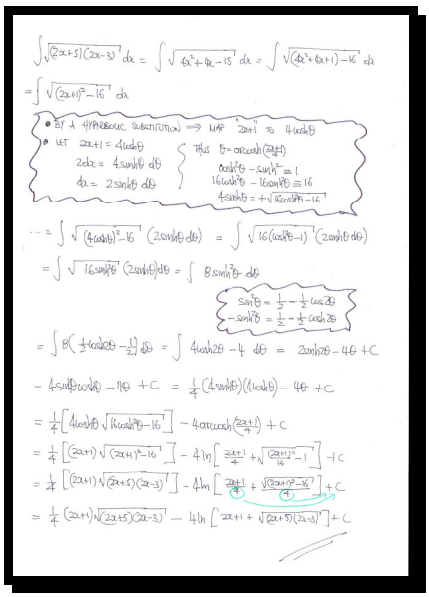


Question 99 (*****)

Use a suitable hyperbolic substitution to find a simplified expression for

$$\int \sqrt{(2x+5)(2x-3)}\,dx\,.$$

$\square$, $\boxed{\frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)}-4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right]+C}$



Question 100 (*****)

It is given that the following integral converges to a finite value L.

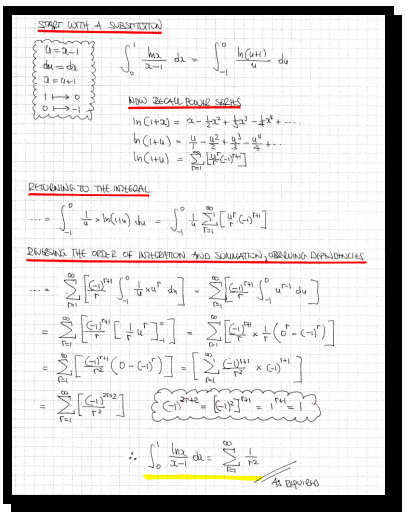
$$\int_0^1 \frac{\ln x}{x-1}\,dx\,.$$

Show, with details workings, that

$$L=\sum_{r=1}^{\infty}\left[\frac{1}{r^2}\right].$$

You may further assume that integration and summation commute.

[ ], proof



Question 101 (*****)

a) If $p\in(0,\infty)$, show that

$$\lim_{x\to 0^+}\left[x^p\ln x\right]=0,\quad x\in(0,\infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n\ln x\ dx,\quad n\in\mathbb{N}.$$

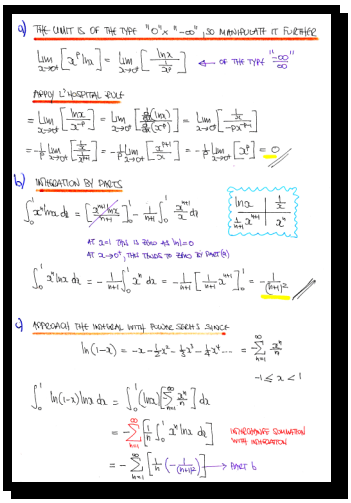
c) Hence, showing a detailed method, evaluate

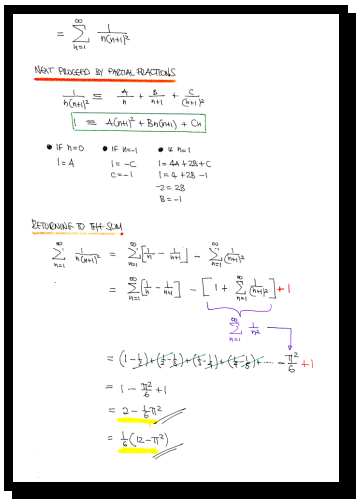
$$\int_0^1 \left[\ln(1-x)\right]\ln x\ dx.$$

You may assume without proof that

- the integral converges.
- integration and summation commute.
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{6}\pi^2,\ n\in\mathbb{N}.$

$$\square,\quad \boxed{-\frac{1}{(n+1)^2}},\quad \boxed{2-\frac{1}{6}\pi^2}$$





Question 102 **(*****)**

$$I=\int \cos\left(\ln x\right)\,dx \quad \text{and} \quad J=\int \sin\left(\ln x\right)\,dx$$

a) Use an appropriate method to find expressions for I and J.

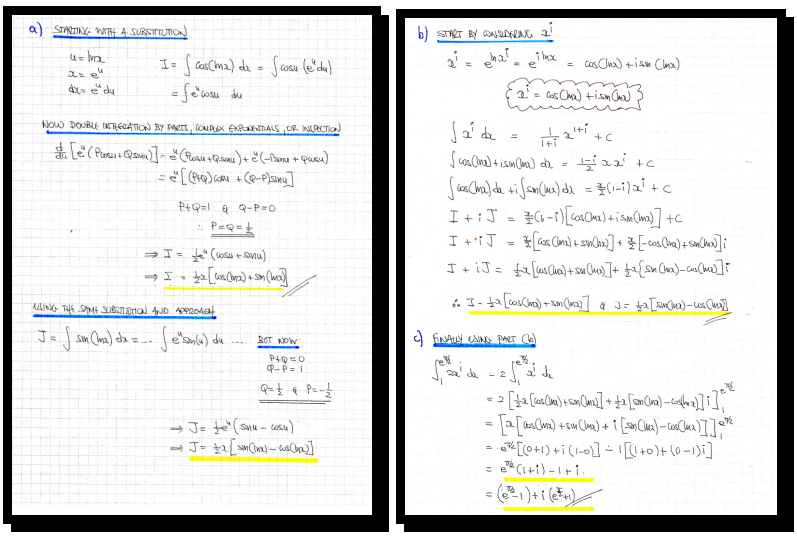
b) Use the integral $\displaystyle\int x^{\mathrm{i}}\,dx$, where i is the imaginary unit, to verify the answers given in part **(a)**.

c) Find an exact simplified value for

$$\int_{1}^{\mathrm{e}^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\,dx\,.$$

$\square$, $\boxed{I=\frac{1}{2}x\left[\sin\left(\ln x\right)+\cos\left(\ln x\right)\right]}$, $\boxed{J=\frac{1}{2}x\left[\sin\left(\ln x\right)-\cos\left(\ln x\right)\right]}$,

$$\boxed{\int_{1}^{\mathrm{e}^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\,dx=\left(\mathrm{e}^{\frac{1}{2}\pi}-1\right)+\left(\mathrm{e}^{\frac{1}{2}\pi}+1\right)\mathrm{i}}$$



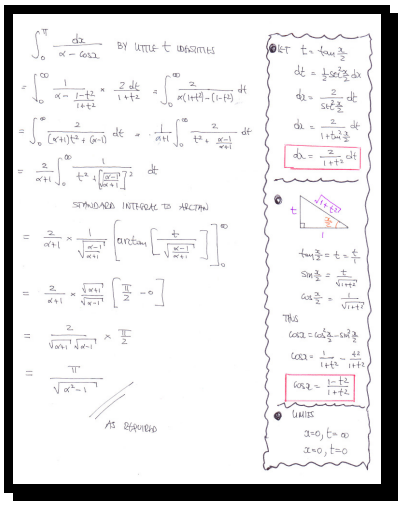
Question 103 (*****)

$$I(\alpha)=\int_0^{\pi}\frac{1}{\alpha-\cos x}\,dx\,,\ |\alpha|>1.$$

Use an appropriate method to show that

$$I(\alpha)=\frac{\pi}{\sqrt{\alpha^2-1}}.$$

$\boxed{}$, $\boxed{\text{proof}}$

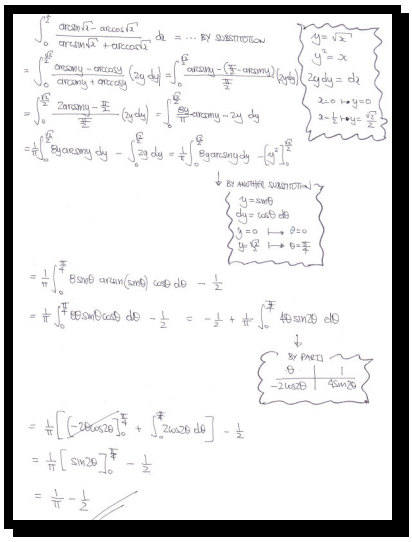


Question 104 (*****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}} \frac{\arcsin\sqrt{x}-\arccos\sqrt{x}}{\arcsin\sqrt{x}+\arccos\sqrt{x}}\,dx = \frac{1}{\pi}-\frac{1}{2}.$$

[ ], [proof]



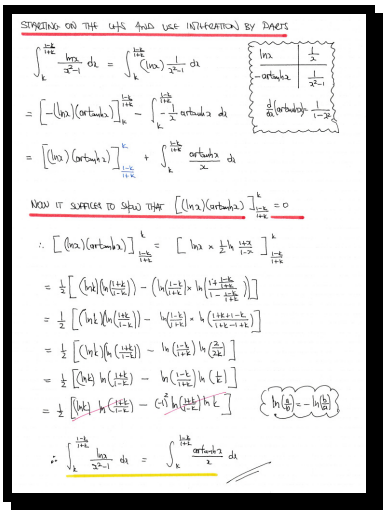
Question 105 **(*****)**

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1}\, dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x}\, dx .$$

You need not evaluate these integrals.

[ ], [proof]

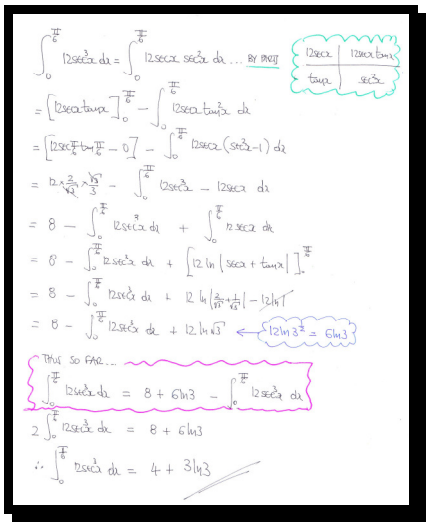


Question 106 (*****)

Use integration by parts and trigonometric identities to find the exact value of

$$\int_0^{\frac{\pi}{6}} 12\sec^3 x \ dx .$$

$\boxed{4+3\ln 3}$

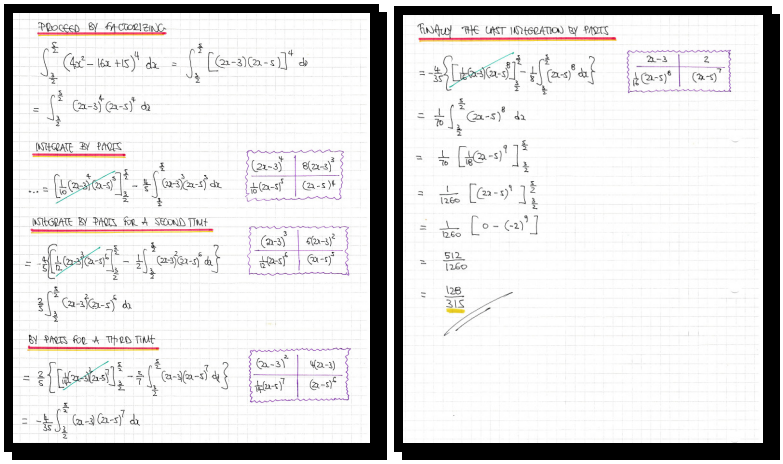


Question 107 (*****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} \left(4x^2-16x+15\right)^4 \, dx \, .$$

$\square$, $\dfrac{128}{315}$



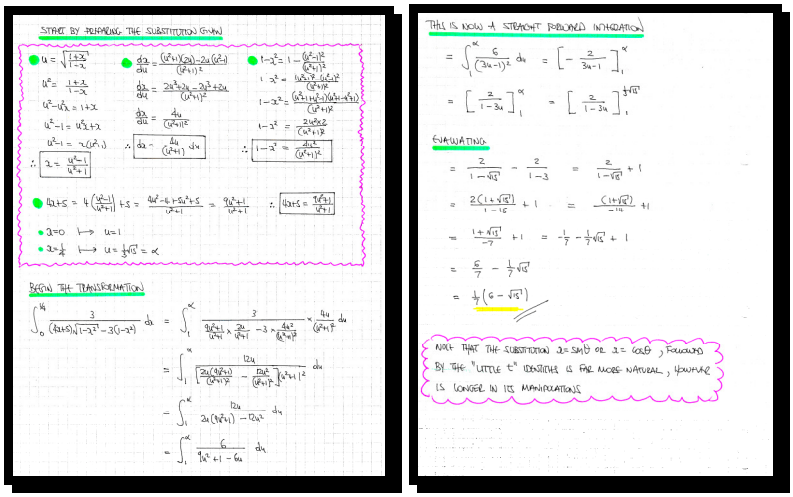
Question 108 **(*****)**

Use the substitution $u=\sqrt{\dfrac{1+x}{1-x}}$, to evaluate the following integral.

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2}-3\left(1-x^2\right)}\,dx\,.$$

Give the answer in the form $\frac{1}{7}\left(a+\sqrt{b}\right)$, where a and b are integers.

$\square$, $\boxed{\frac{1}{7}\left(6-\sqrt{15}\right)}$



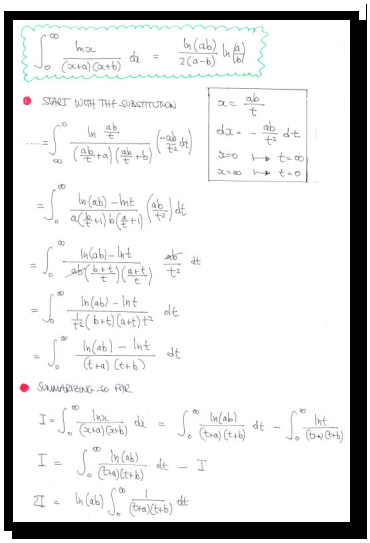
Question 109 **(*****)**

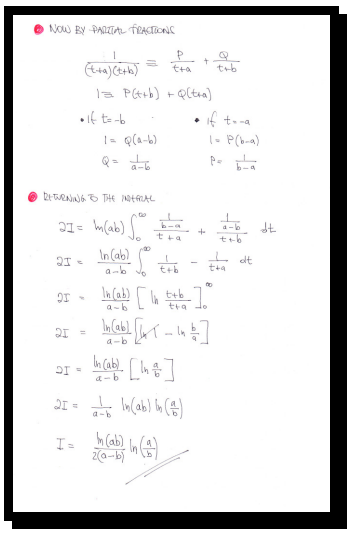
Use the substitution $x=\dfrac{ab}{t}$ to find the exact value of

$$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)}\,dx,$$

where a and b are real positive constants with $a>b$.

$$\boxed{\phantom{\frac{1}{1}}}\,,\quad \boxed{\frac{\ln(ab)}{2(a-b)}\ln\left(\frac{a}{b}\right)=\frac{(\ln a)^2-(\ln b)^2}{2(a-b)}}$$



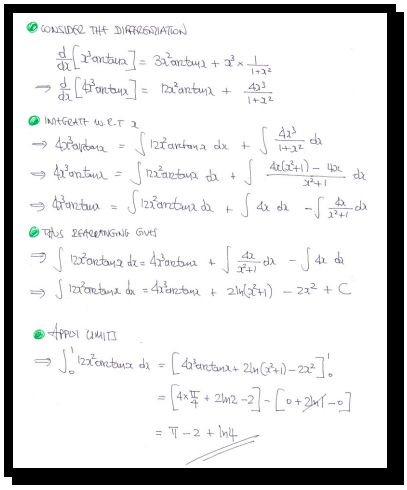


Question 110 (*****)

Use appropriate integration methods to show that

$$\int_0^1 12x^2 \arctan x \ \ dx = \pi - 2 + \ln 4 .$$

[ ], [proof]

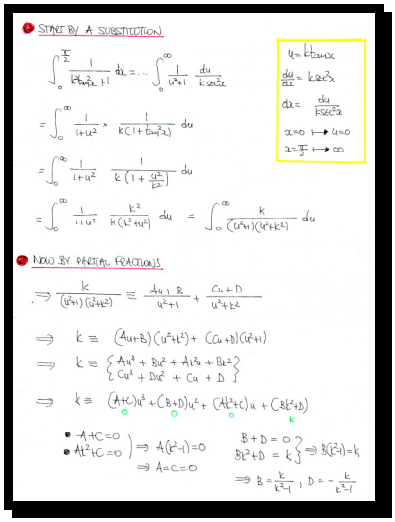


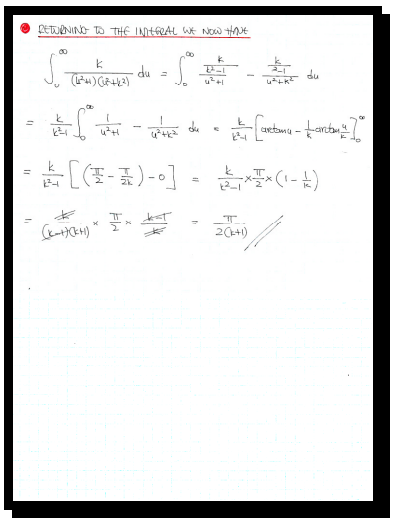
Question 111 (*****)

Use appropriate integration methods to find, in terms of k, a simplified expression for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\tan^2 x}\,dx, \quad |k|\neq 1.$$

$\square$, $\boxed{\dfrac{\pi}{2(k+1)}}$





Question 112 **(*****)**

$$I=\int_0^{\frac{1}{2}\ln 3}\operatorname{sech} x\ dx\,.$$

a) Use the substitution $u=\mathrm{e}^x$ to show that $I=\dfrac{\pi}{k}$, where k is a positive integer.

b) Given that $t=\tanh\left(\frac{1}{2}x\right)$ show that …

i. … $\dfrac{dt}{dx}=\frac{1}{2}\left(1-t^2\right)$.

ii. … if $x=\frac{1}{2}\ln 3$, then $t=2-\sqrt{3}$.

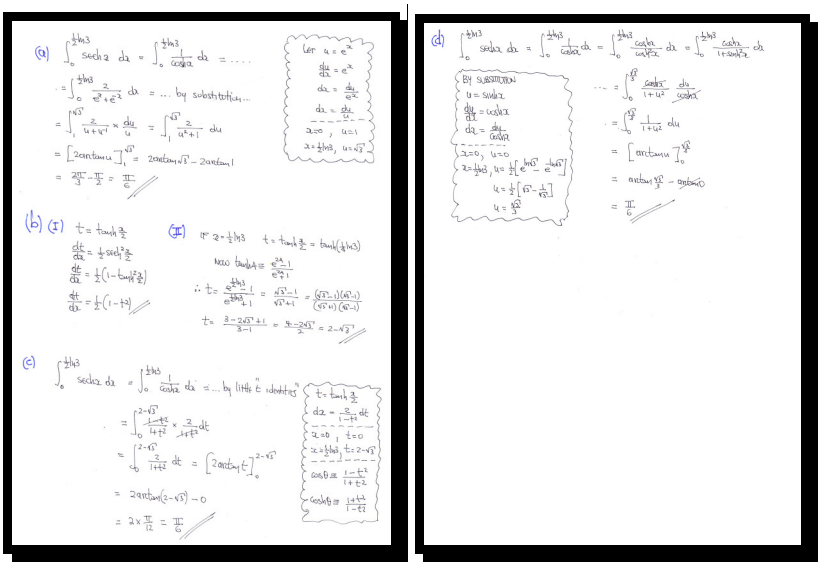
c) Use the results of part **(b)** to find again the exact value of I.

d) Show that I can be written as

$$\int_0^{\frac{1}{2}\ln 3}\frac{\cosh x}{1+\sinh^2 x}\ dx\,,$$

and hence obtain the exact value of I for a third time.

[ ], [proof]



Question 113 **(*****)**

$$I=\int_0^1 2\operatorname{arsinh}\sqrt{x}\ \ dx\,.$$

The value of I is to be found using two methods.

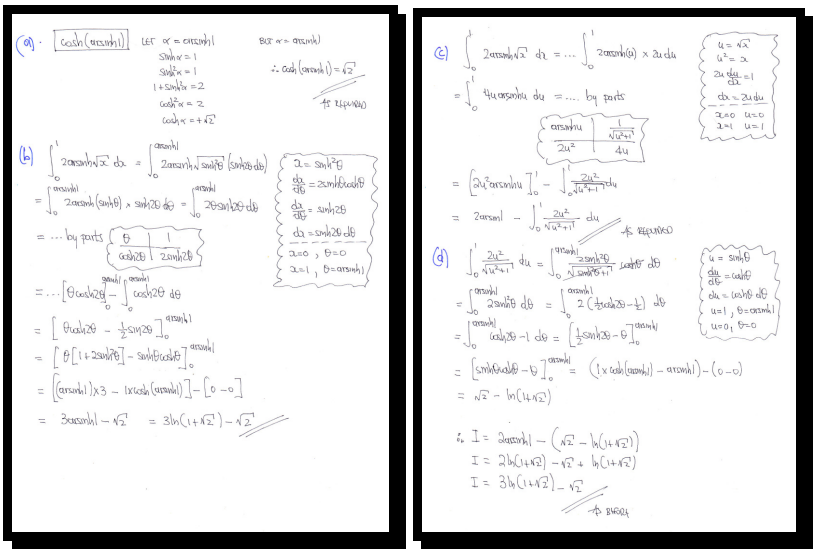
a) Use the substitution $x=\sinh^2\theta$ to show that

$$I=3\ln\left(1+\sqrt{2}\right)-\sqrt{2}\,.$$

A different approach is to be used to find the value of I.

b) Use the substitution $u=\sqrt{x}$, followed by a suitable hyperbolic substitution to to verify the answer of part **(a)**.

, proof

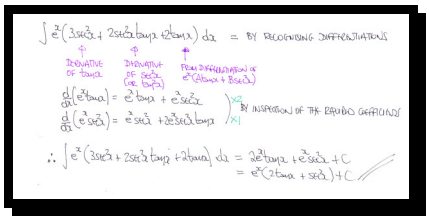


Question 114 (*****)

By considering the differentiation of products of appropriate functions, find

$$\int \mathrm{e}^{x}\left(3\sec^2 x+2\sec^2 x\tan x+2\tan x\right)\,dx\,.$$

$\square$, $\boxed{\mathrm{e}^{x}\left(2\tan x+\sec^2 x\right)+C}$

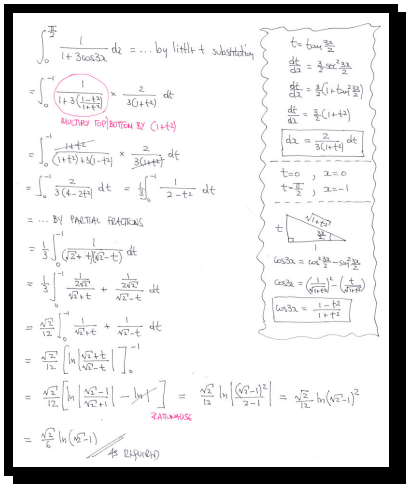


Question 115 (*****)

By using a trigonometric substitution or otherwise, find an exact simplified value for the following integral.

$$\int_{0}^{\frac{\pi}{2}}\frac{1}{1+3\cos 3x}\,dx\,.$$

$\square$, $\boxed{\dfrac{\sqrt{2}}{6}\ln\left(\sqrt{2}-1\right)}$



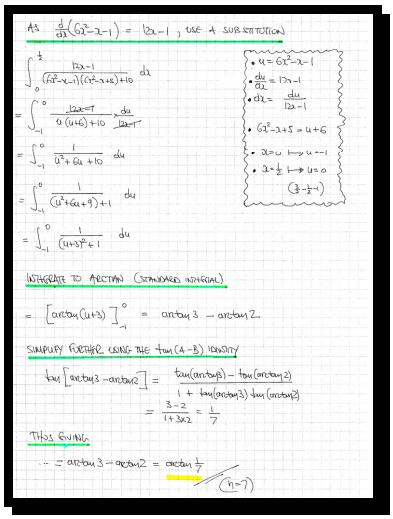
Question 116 (*****)

Find the value of the following definite integral.

$$\int_0^{\frac{1}{2}} \frac{12x-1}{\left(6x^2-x-1\right)\left(6x^2-x-5\right)+10}\,dx\ .$$

Give the answer in the form $\arctan\left(\frac{1}{n}\right)$, where n is a positive integer.

$\square$, $\boxed{n=7}$



Question 117 **(*****)**

$$I=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\sqrt{1+x^2}}{x^4}\,dx\,.$$

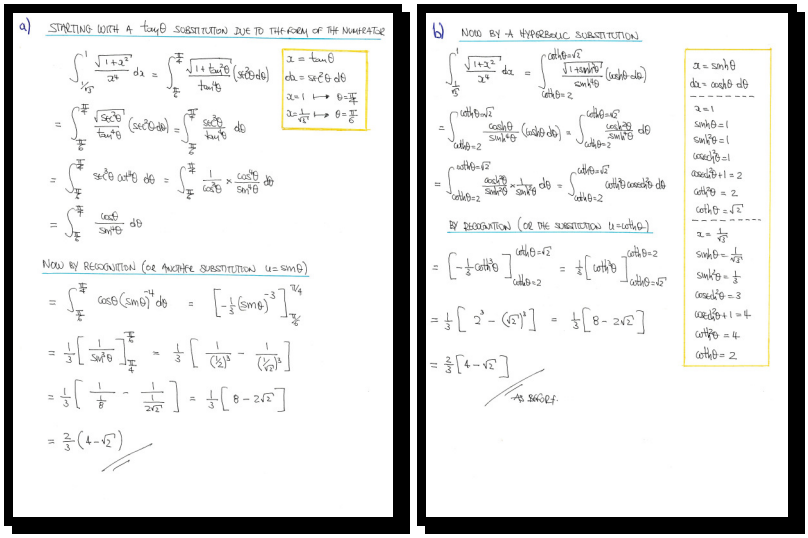
a) Use a trigonometric substitution to show that

$$I=\frac{2}{3}\left(a+b\sqrt{2}\right),$$

where a and b are integers to be found.

b) Use a hyperbolic substitution to verify the answer of part **(a)**.

$\boxed{}$, $\boxed{I=\frac{2}{3}\left(4-\sqrt{2}\right)}$



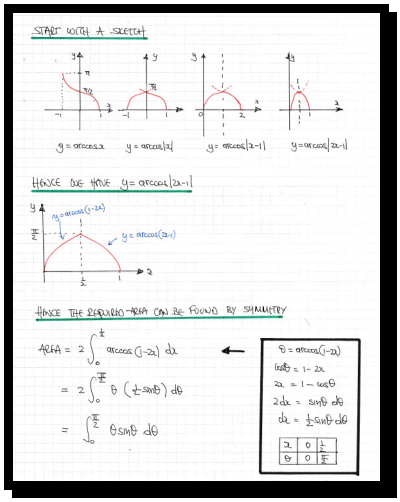
Question 118 **(*****)**

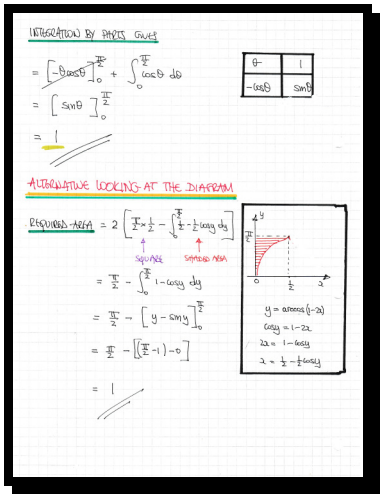
The function f is defined in the largest real domain by the equation

$$f(x) \equiv \arccos|2x-1|.$$

Determine the area of the finite region bounded by f and the coordinate axes.

$\boxed{}$, $\boxed{\text{area} = 1}$





Question 119 **(*****)**

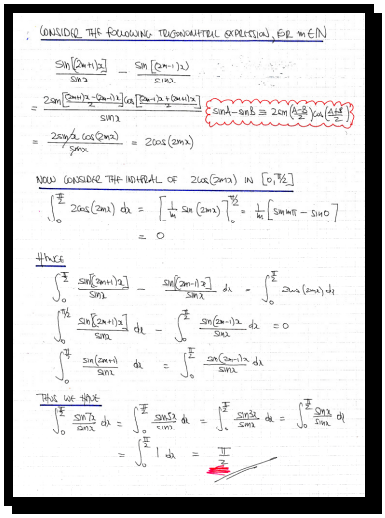
By considering

$$\frac{\sin\left[(2m+1)x\right]}{\sin x}-\frac{\sin\left[(2m-1)x\right]}{\sin x},\ m\in\mathbb{N},$$

determine the exact value of

$$\int_0^{\frac{1}{2}\pi}\frac{\sin 7x}{\sin x}\,dx.$$

$\square$, $\boxed{\frac{1}{2}\pi}$

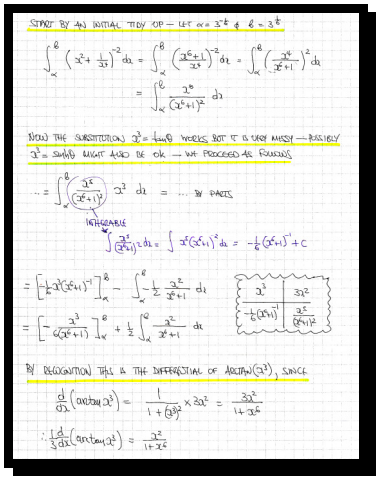


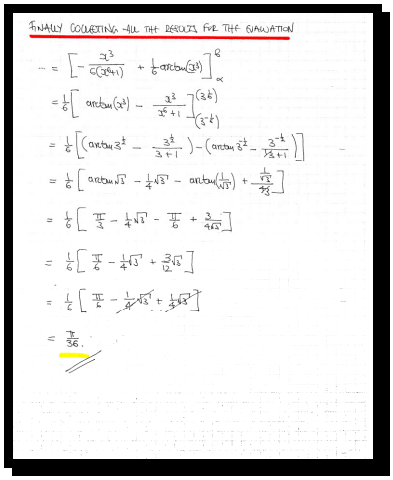
Question 120 (*****)

Find in exact simplified form the value of the following definite integral.

$$\int_{3^{-\frac{1}{6}}}^{3^{\frac{1}{6}}} \left(x^2 + \frac{1}{x^4} \right)^{-2} \, dx \ .$$

$\square$, $\boxed{\dfrac{\pi}{36}}$



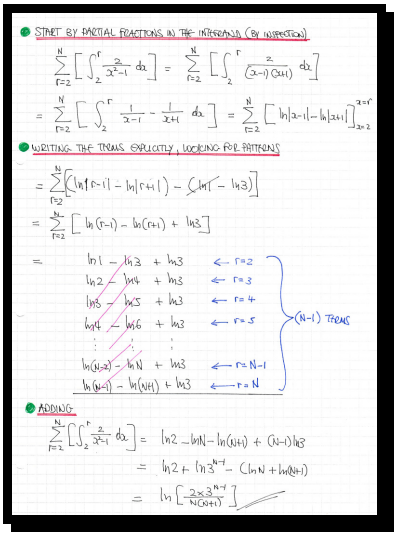


Question 121 (*****)

Determine a simplified expression, in the form $\ln\left[f(n)\right]$, for the following sum.

$$\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{x^2-1}\,dx\right].$$

$\square$, $\boxed{\ln\left[\dfrac{2\times 3^{N-1}}{N(N+1)}\right]}$

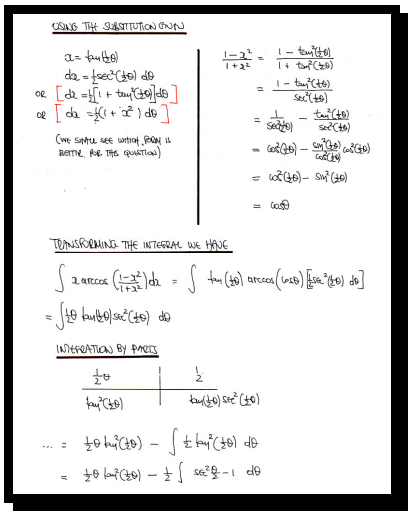


Question 122 (*****)

Use appropriate integration methods to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx .$$

$\boxed{}$, $\boxed{-x+\left(1+x^2\right)\arctan x+\text{constant}}$

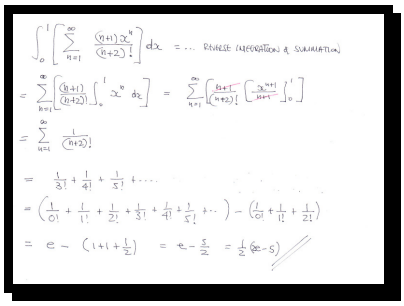


Question 123 (*****)

Find the exact value of

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}(2\mathrm{e}-5)}$



Question 124 **(*****)**

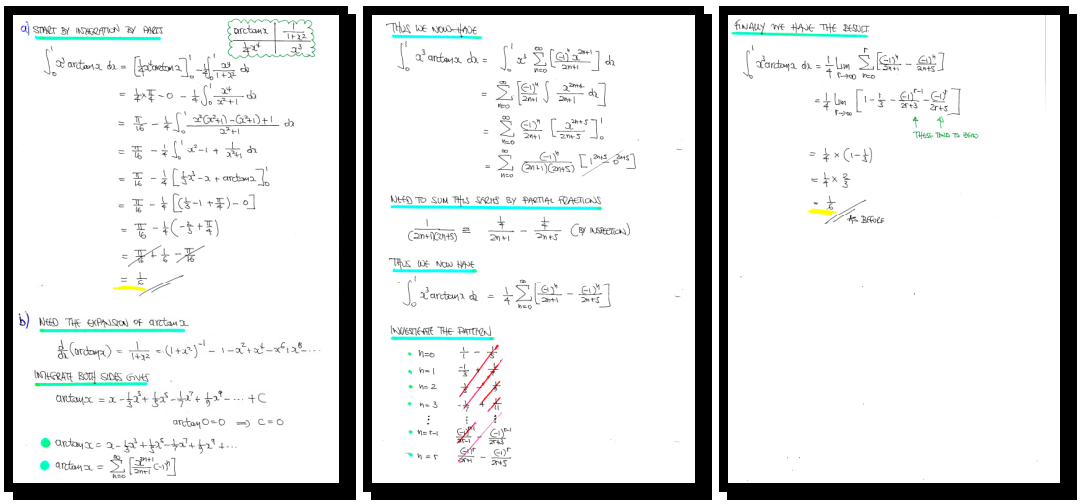
a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \ \ dx \, .$$

b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{6}}$

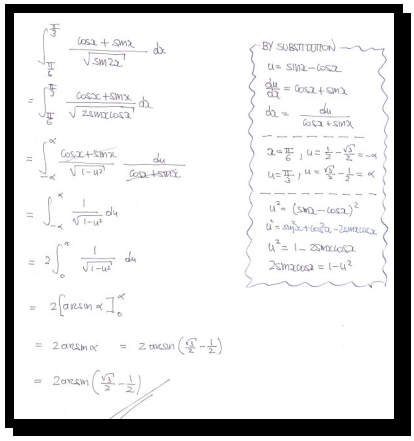


Question 125 (*****)

Find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x+\sin x}{\sqrt{\sin 2x}}\,dx\,.$$

$\square$, $\boxed{2\arcsin\left[\frac{\sqrt{3}-1}{2}\right]}$



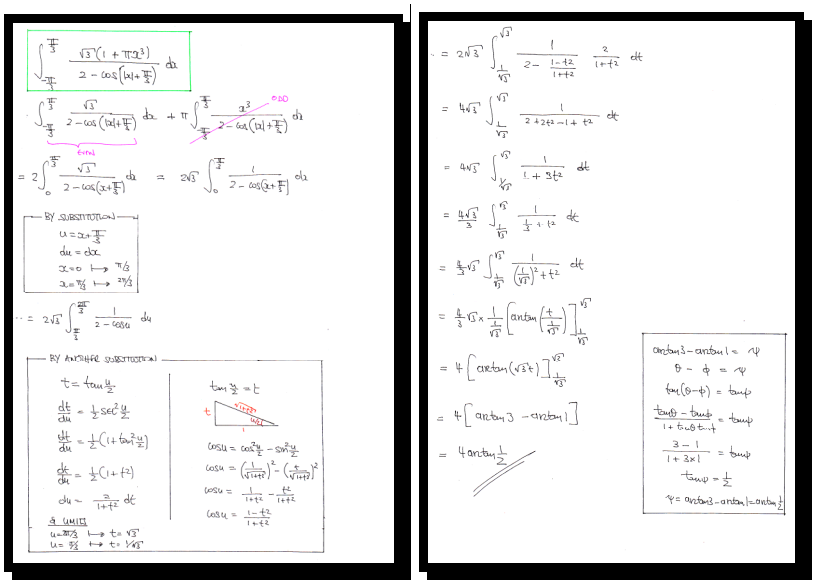
Question 126 **(*****)**

$$I=\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}\frac{\sqrt{3}\left(1+\pi x^3\right)}{2-\cos\left(|x|+\frac{1}{3}\pi\right)}\,dx\,.$$

Show that

$$I=4\arctan\frac{1}{2}\,.$$

[ ], [proof]

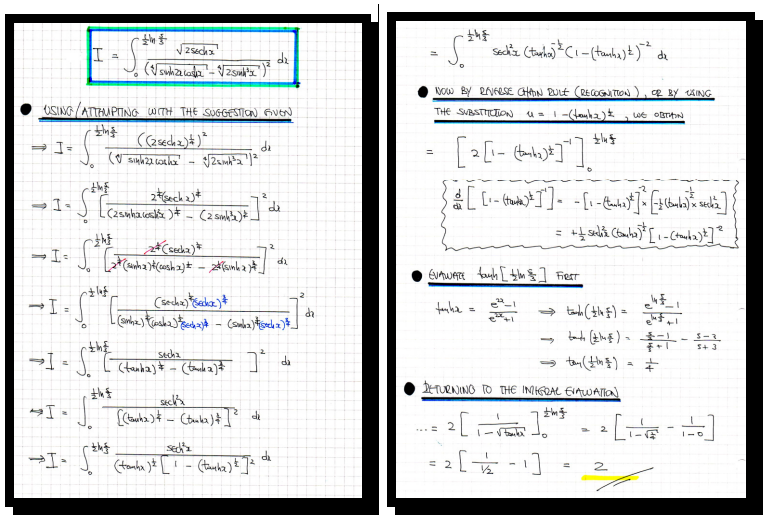


Question 127 (*****)

By expressing the integrand in the form $\operatorname{sech}^2 x\, f(\tanh x)$, or otherwise, find the value of the following integral.

$$\int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\sqrt{2\operatorname{sech} x}}{\sqrt[4]{\sinh 2x\cosh x}-\sqrt[4]{2\sinh^3 x}}\,dx\,.$$

$\square$, $\boxed{2}$

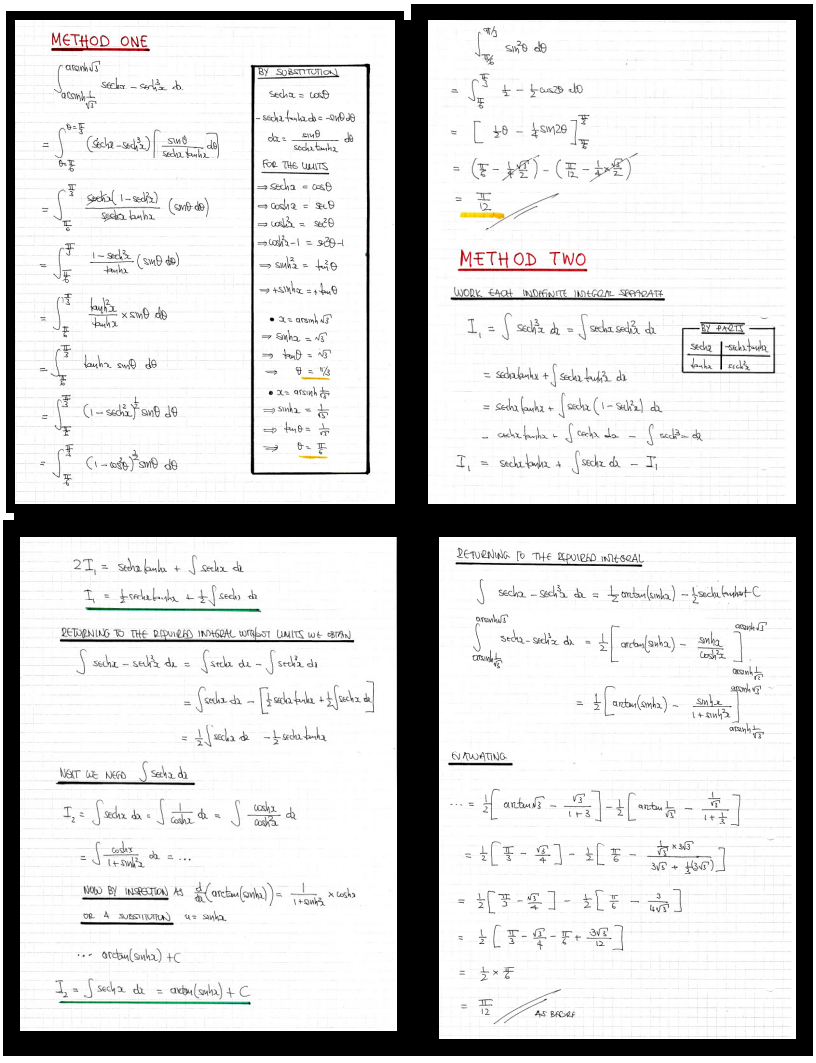


Question 128 **(*****)**

Use appropriate integration techniques to show that

$$\int_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}} \operatorname{sech} x\left(1-\operatorname{sech} x\right)\,dx \;=\; \frac{\pi}{12}.$$

[ ], [proof]



Question 129 **(*****)**

The function f is defined as.

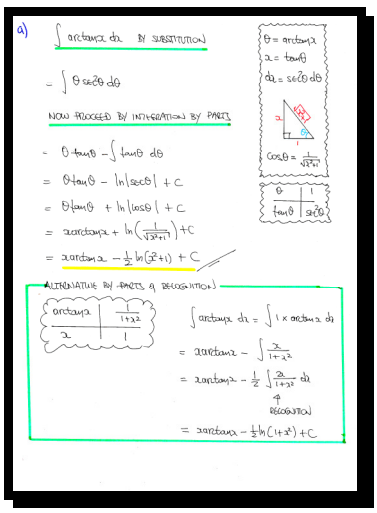
$$f(x)=\arctan x\,,\qquad x\in\left(-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi\right).$$

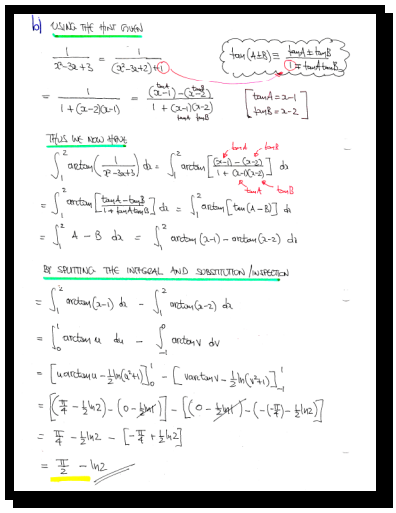
a) Find a simplified expression for $\displaystyle\int f(x)\,dx$.

b) By considering the tangent compound angle identity, or otherwise, find an exact simplified value for

$$\int_1^2 \arctan\left[\frac{1}{x^2-3x+3}\right]dx\,.$$

$\boxed{}$, $\boxed{x\arctan x-\frac{1}{2}\ln\left(x^2+1\right)}$, $\boxed{\frac{1}{2}\pi-\ln 2}$





Question 130 (*****)

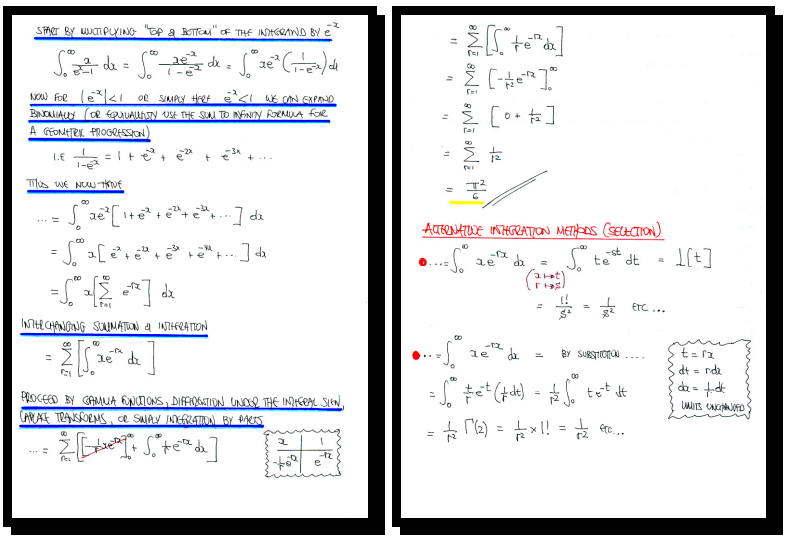
Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \frac{x}{\mathrm{e}^x - 1}\, dx\,.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

$\square$, $\boxed{\dfrac{\pi^2}{6}}$



Question 131 **(*****)**

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

a) Show, with a detailed method, that

$$\frac{d}{dx}\left[x\left(x^{\phi}+1\right)^{1-\phi} \right] = \left(x^{\phi}+1\right)^{-\phi}.$$

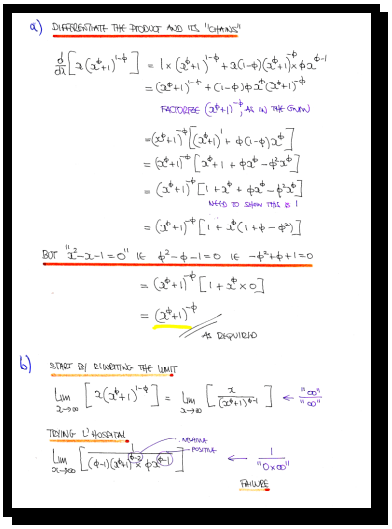
b) Show, with full justification, that

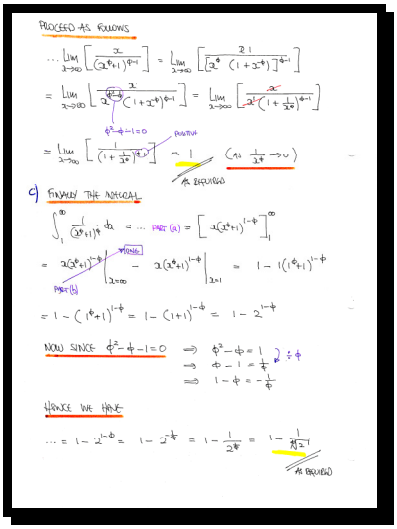
$$\lim_{x\to\infty}\left[x\left(x^{\phi}+1\right)^{1-\phi} \right] = 1.$$

c) Show further that

$$1-\frac{1}{\sqrt[\phi]{2}} = \int_1^{\infty} \frac{1}{\left(x^{\phi}+1\right)^{\phi}}\, dx.$$

[ ], [proof]





Question 132 (*****)

It is given that

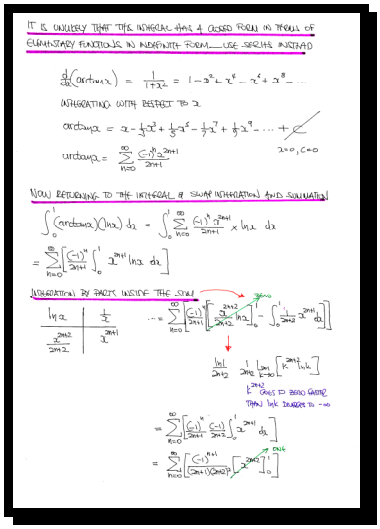
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ \ \ldots=\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2$

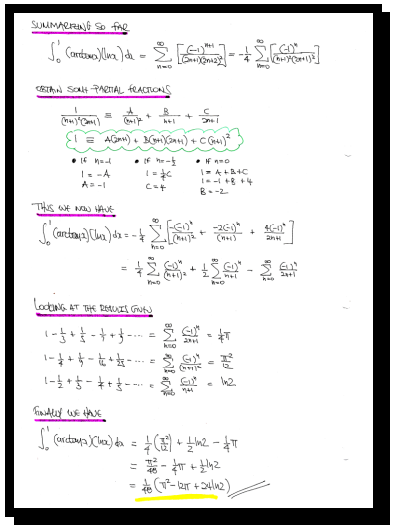
Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx\,.$$

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$





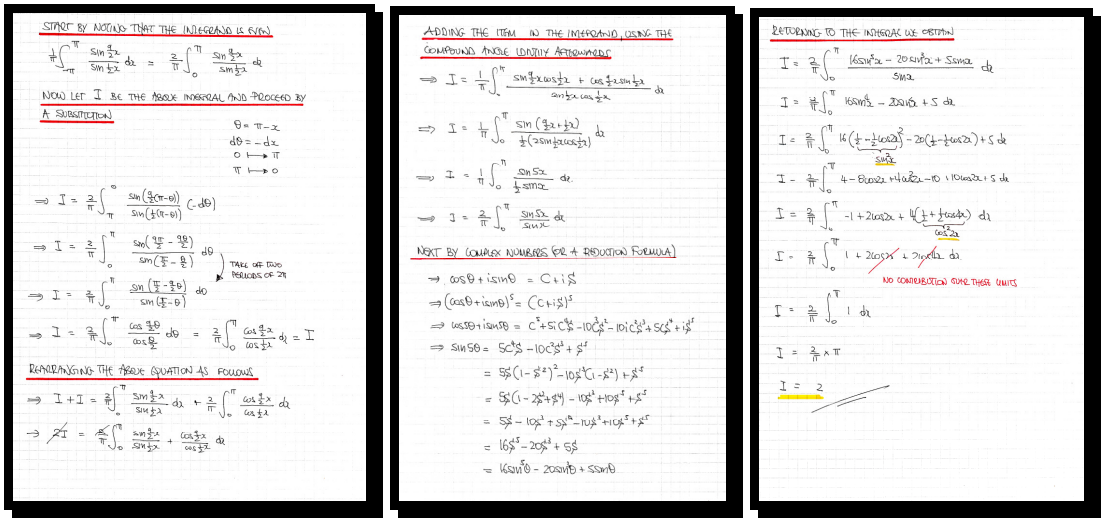
Question 133 (*****)

Find the value of

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin\frac{9}{2}x}{\sin\frac{1}{2}x}\,dx\,.$$

You may assume that the integrand is continuous at $x=0$.

$\boxed{}$, $\boxed{2}$



Question 134 (*****)

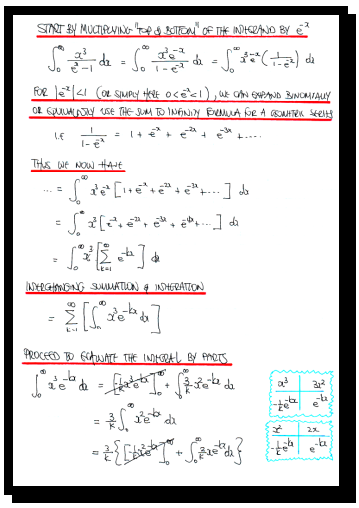
Use appropriate integration techniques to find an exact simplified value for the following improper integral.

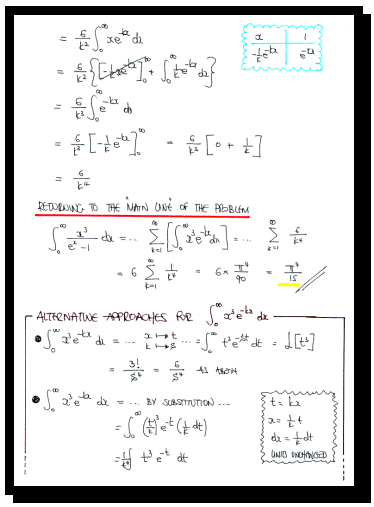
$$\int_0^{\infty} \frac{x^3}{\mathrm{e}^x-1}\,dx\,.$$

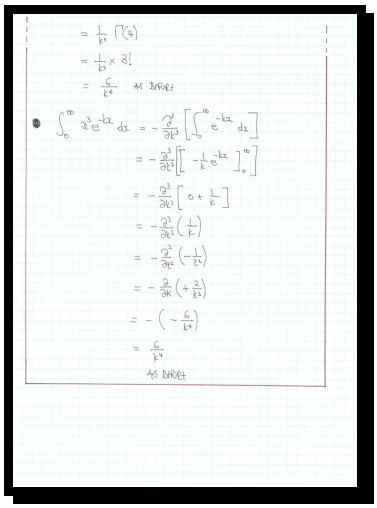
You may assume without proof that

$$\sum_{r=1}^{\infty}\frac{1}{r^4}=\frac{\pi^4}{90}\,.$$

$\square$, $\boxed{\dfrac{\pi^4}{15}}$







Question 135 **(*****)**

The function f is defined as

$$f(x)=\arctan\left(\frac{1}{2x^2}\right), \quad x\in(-\infty,\infty).$$

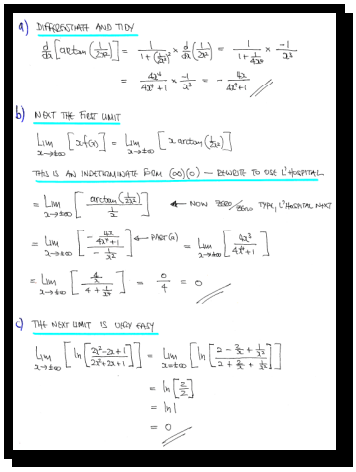
a) Find a simplified expression for $f'(x)$.

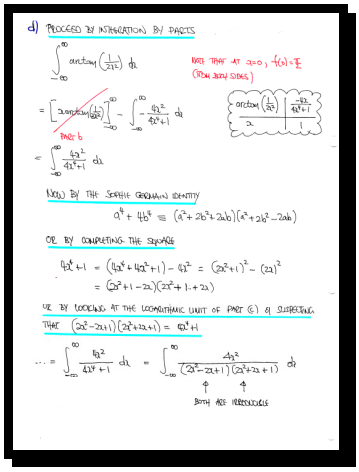
b) Show that $\lim_{x\to\pm\infty}\left[x\,f(x)\right]=0$.

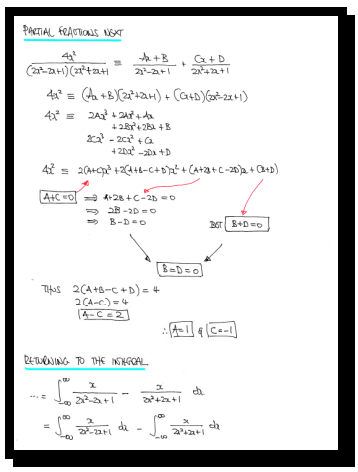
c) Determine the value of $\lim_{x\to\pm\infty}\left[\ln\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]\right]$.

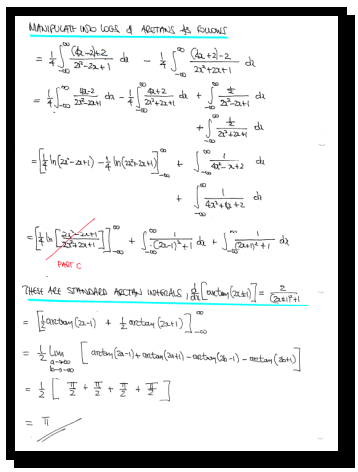
d) Hence find the value of $\int_{-\infty}^{\infty} f(x)\,dx$.

$\square$, $\boxed{f'(x)=-\frac{4x}{4x^4+1}}$, $\boxed{\lim_{x\to\pm\infty}\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]=0}$, $\boxed{\int_{0}^{\infty} f(x)\,dx=\frac{1}{2}\pi}$









Question 136 **(*****)**

A family of functions $f_n(x)$, where $n = 0, 1, 2, 3, 4, \ldots$, satisfies the equation

$$\sum_{n=0}^{\infty}\left[t^n f_n(x)\right] = \left(1-2xt+t^2\right)^{-\frac{1}{2}}.$$

By integrating both sides of the above equation with respect to t, from 0 to 1, show that

$$\sum_{n=0}^{\infty}\left[\frac{f_n(\cos\theta)}{n+1}\right] = \ln\left[1+\operatorname{cosec}\left(\tfrac{1}{2}\theta\right)\right].$$

You may assume in this question that integration and summation commute.

[ ], [proof]

