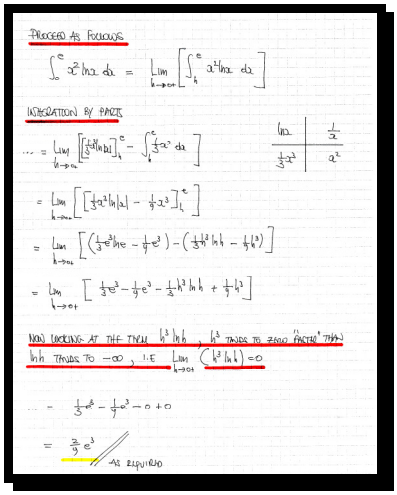
IMPROPER INTEGRALS

Question 1 **(***)**

Indicating clearly the limiting processes used, show that

$$\int_0^{\mathrm{e}} x^2 \ln x \ dx = \frac{2}{9}\mathrm{e}^3 .$$

[ ], proof

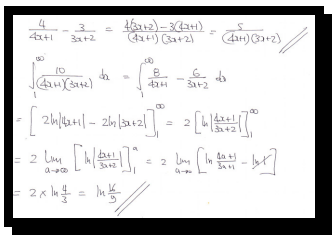


Question 2 **(***)**

Indicating clearly the limiting processes used, show that

$$\int_1^{\infty} \frac{10}{(3x+2)(4x+1)} \ dx = \ln\frac{16}{9} .$$

proof



Question 3 **(***)**

$$I = \int 16x\mathrm{e}^{-4x}\,dx\,.$$

a) Show that

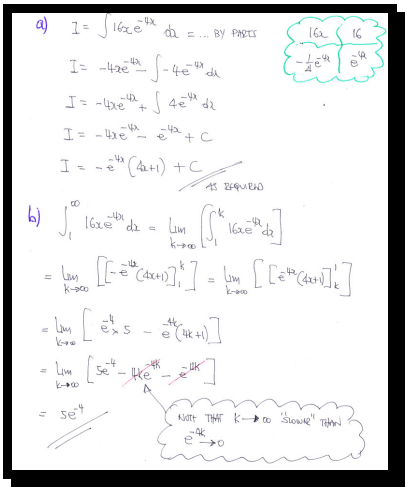
$$I = -\mathrm{e}^{-4x}\left(4x+1\right)+\text{constant}\,.$$

b) Hence find

$$\int_1^{\infty} 16x\mathrm{e}^{-4x}\,\,dx\,,$$

showing clearly the limiting process used.

$$\boxed{5\mathrm{e}^{-4}}$$

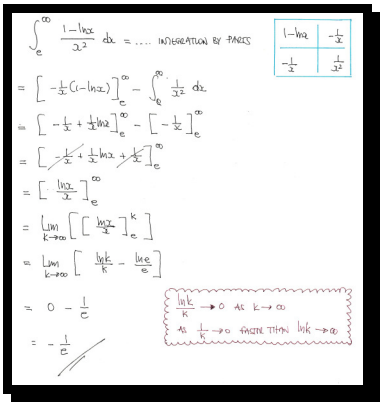


Question 4 (***)

Find the exact value of the following integral.

$$\int_{\mathrm{e}}^{\infty} \frac{1-\ln x}{x^2}\, dx .$$

$\boxed{}$, $\boxed{-\mathrm{e}^{-1}}$



Question 5 (*+)**

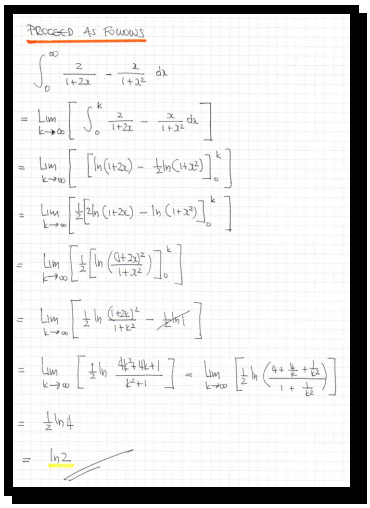
Evaluate the integral

$$\int_0^{\infty} \frac{2}{1+2x} - \frac{x}{1+x^2}\, dx\,,$$

showing clearly the limiting processes used.

Give the answer in the form $\ln N$, where N is a positive integer.

$\boxed{}$, $\boxed{\ln 2}$



Question 6 (*+)**

The function f is defined by.

$$f(x)\equiv\frac{ax+b}{\mathrm{e}^{x}},\ x\in\mathbb{R},$$

where a and b are non zero constants.

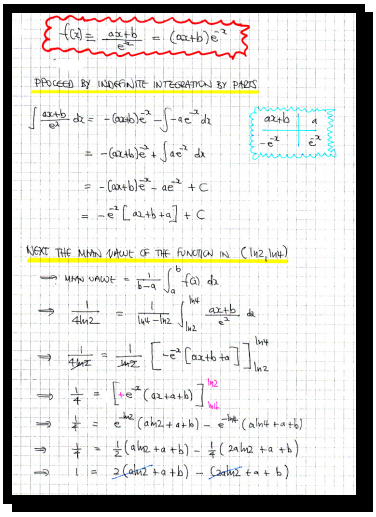
The mean value of f in the interval $(\ln 2,\ln 4)$ is $\dfrac{1}{4\ln 2}$.

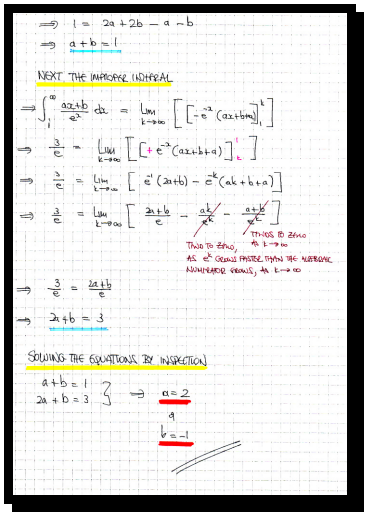
Given further that

$$\int_{1}^{\infty}f(x)\ dx=\frac{3}{\mathrm{e}},$$

determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=-1}$





Question 7 (*+)**

$$f(y)\equiv\frac{4y}{y^4-1},\ y\in\mathbb{R},\ |y|\neq 1.$$

a) Express $f(y)$ into three partial fractions.

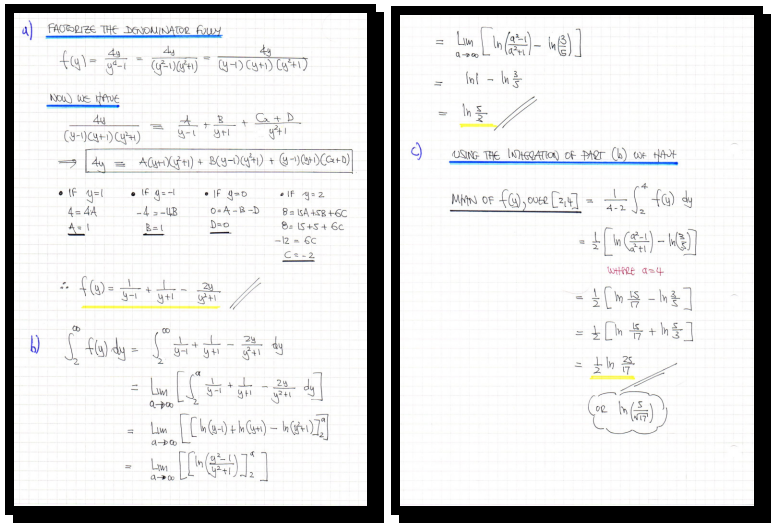
b) Hence evaluate the improper integral

$$\int_2^{\infty} f(y)\ dy,$$

showing clearly the limiting processes used.

c) Find, in exact form, the mean value of f, in the interval $\{y\in\mathbb{R}:\ 2\leq y\leq 4\}$.

$\square$, $\boxed{f(y)=\frac{1}{y+1}+\frac{1}{y-1}-\frac{2y}{y^2+1}}$, $\boxed{\ln\left(\frac{5}{3}\right)}$, $\boxed{\frac{1}{2}\ln\left(\frac{25}{17}\right)=\ln\left(\frac{5}{\sqrt{17}}\right)}$



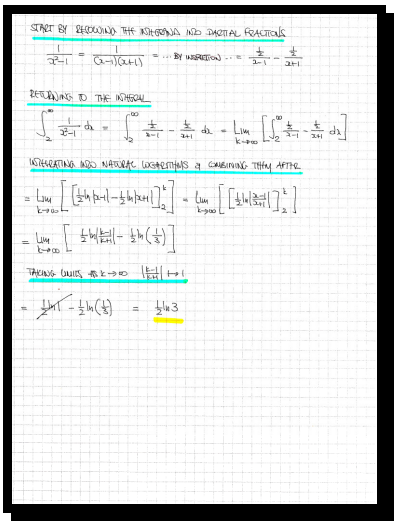
Question 8 **(***+)**

By showing formally all the limiting processes evaluate the following integral

$$\int_2^{\infty} \frac{1}{x^2-1}\,dx \ .$$

Give the answer in the form $k\ln 3$, where k is a positive constant to be found.

$\square$, $\boxed{k=\frac{1}{2}}$



Question 9 (****)

$$\int \frac{\ln x^2}{x^3} dx\,,\ x \neq 0\,.$$

a) Show that the substitution $y = \dfrac{1}{x}$ transforms the above integral into

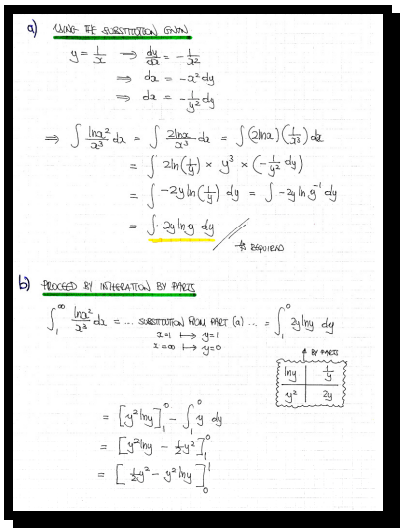
$$\int 2y \ln y \, dy\,.$$

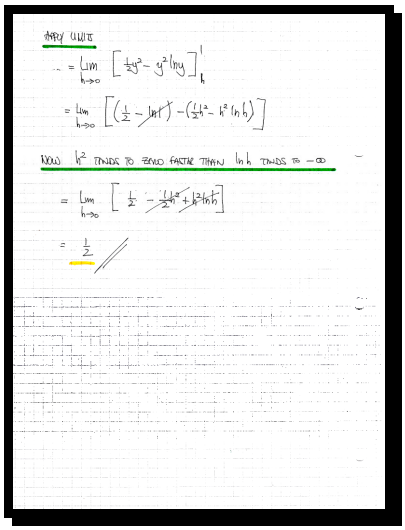
b) Hence evaluate

$$\int_1^{\infty} \frac{\ln x^2}{x^3} \, dx\,,$$

showing clearly the limiting process used.

$\boxed{}$, $\boxed{\dfrac{1}{2}}$





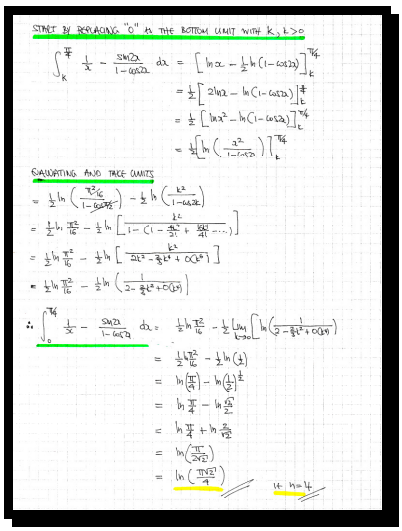
Question 10 (****)

By showing formally all the limiting processes evaluate the following integral

$$\int_0^{\frac{1}{4}\pi} \frac{1}{x}-\frac{\sin 2x}{1-\cos 2x}\,dx\ .$$

Give the answer in the form $\ln\left[\dfrac{\pi\sqrt{2}}{n}\right]$, where n is a positive integer to be found.

$\boxed{}$, $\boxed{n=4}$



Question 11 (*****)

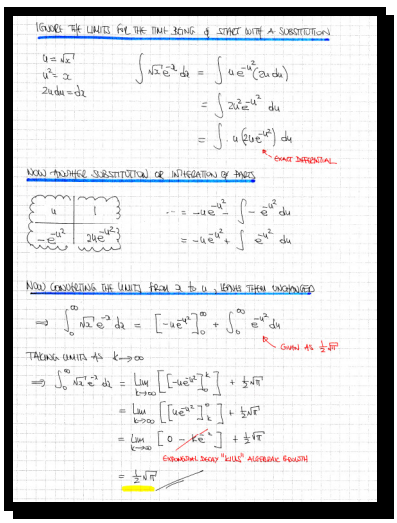
By showing formally all the limiting processes, find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \sqrt{x}\,\mathrm{e}^{-x}\ dx\,.$$

You may assume without proof that

$$\int_0^{\infty} \mathrm{e}^{-x^2}\ dx = \frac{1}{2}\sqrt{\pi}\,.$$

$\square$, $\boxed{\frac{1}{2}\sqrt{\pi}}$



Question 12 **(*****)**

The function f is defined as

$$f(x)=\arctan\left(\frac{1}{2x^2}\right), \quad x\in(-\infty,\infty).$$

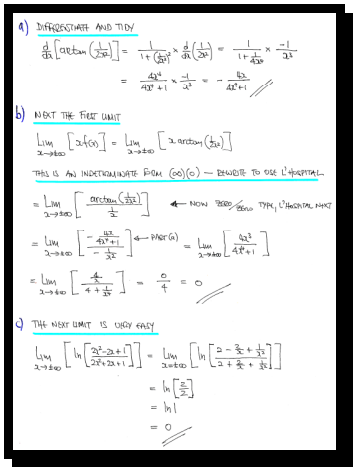
a) Find a simplified expression for $f'(x)$.

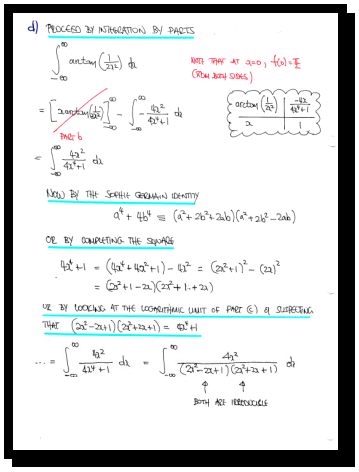
b) Show that $\lim_{x\to\pm\infty}\left[x\,f(x)\right]=0$.

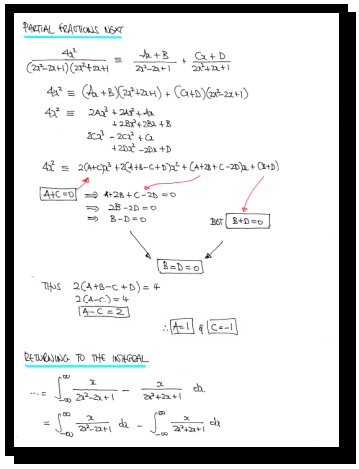
c) Determine the value of $\lim_{x\to\pm\infty}\left[\ln\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]\right]$.

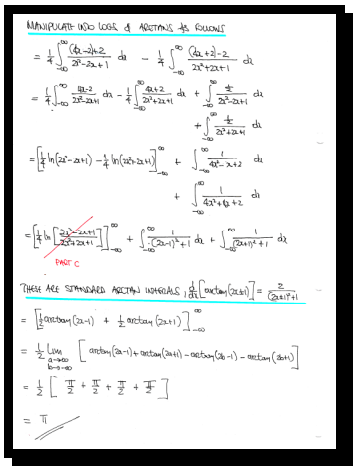
d) Hence find the value of $\int_{-\infty}^{\infty} f(x)\,dx$.

$\square$, $f'(x)=-\dfrac{4x}{4x^4+1}$, $\lim_{x\to\pm\infty}\left[\dfrac{2x^2-2x+1}{2x^2+2x+1}\right]=0$, $\int_{-\infty}^{\infty} f(x)\,dx=\pi$









Created by T. Madas