HYPERBOLIC FUNCTIONS

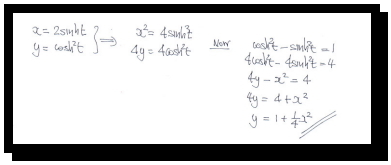
Question 1 (**)

A curve is given parametrically by the equations

$$x = 2\sinh t\,,\ \ y = \cosh^2 t\,,\ \ t \in \mathbb{R}\,.$$

Find a Cartesian equation of the curve, in the form $y = f(x)$.

$$\boxed{y = 1 + \frac{1}{4}x^2}$$



Question 2 (**)

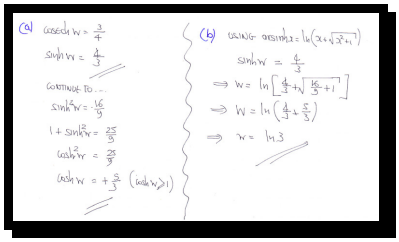
It is given that

$$\operatorname{cosech} w = \frac{3}{4}\,.$$

a) Use hyperbolic identities to find the exact values of $\sinh w$ and $\cosh w$.

b) Hence find the exact value of w, in terms of natural logarithms.

$$\boxed{\sinh w = \frac{4}{3}}\,,\ \ \boxed{\cosh w = \frac{5}{3}}\,,\ \ \boxed{w = \ln 3}$$



Question 3 (**)

$$f(x)=\operatorname{artanh} x\,,\ x\in\mathbb{R}\,,\ -1<x<1\,.$$

a) Show clearly that

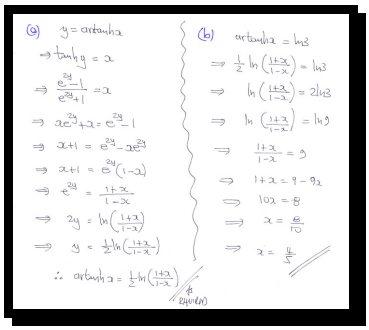
$$f(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ x\in\mathbb{R}\,,\ -1<x<1\,.$$

b) Without the use of any calculating aid solve the equation

$$\operatorname{artanh} x=\ln 3\,,$$

showing clearly all the relevant steps in the calculation.

$$\boxed{x=\frac{4}{5}}$$

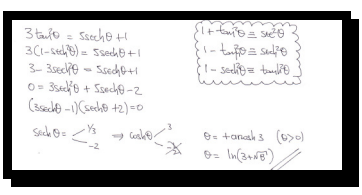


Question 4 (+)**

Find, in exact logarithmic form, the positive root of the equation

$$3\tanh^2\theta=5\operatorname{sech}\theta+1\,,\ \theta\in\mathbb{R}\,.$$

$$\boxed{\theta=\ln\left(3+\sqrt{8}\right)}$$



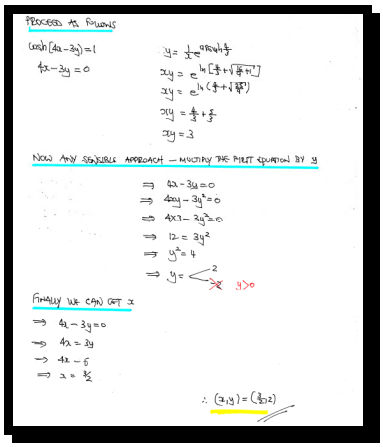
Question 5 (+)**

Given that $x>0$ and $y>0$, solve the simultaneous equations

$$\cosh\left(4x-3y\right)=1$$

$$y=\frac{1}{x}\mathrm{e}^{\operatorname{arsinh}\frac{4}{3}}.$$

$\boxed{}$, $\boxed{x=\frac{3}{2},\ y=2}$



Question 6 (+)**

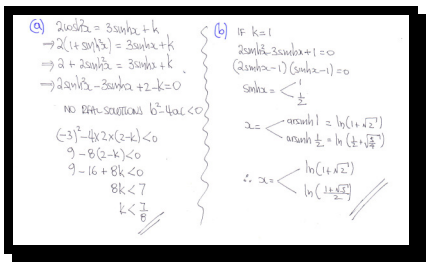
Consider the following hyperbolic equation, given in terms of a constant k.

$$2\cosh^2 x = 3\sinh x + k .$$

a) Find the range of values of k for which the above equation has no real solutions.

b) Given further that $k = 1$, find in exact logarithmic form, the solutions of the above equation.

$$\boxed{k < \frac{7}{8}}, \quad \boxed{x = \ln\left(1+\sqrt{2}\right), \ln\left(\frac{1+\sqrt{5}}{2}\right)}$$



Question 7 (+)**

$$f(x)=\text{artanh}\,x\,,\ x\in\mathbb{R}\,,\ -1<x<1\,.$$

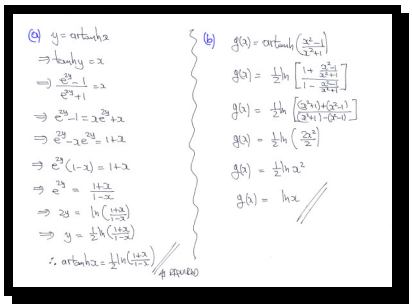
a) Show clearly that

$$f(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ x\in\mathbb{R}\,,\ -1<x<1\,.$$

b) Hence simplify fully

$$g(x)=\text{artanh}\left(\frac{x^2-1}{x^2+1}\right),\ x>0\,.$$

$$\boxed{g(x)=\ln x}$$

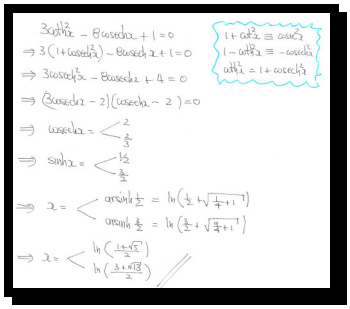


Question 8 **(**+)**

Solve the following equation, giving each of the answers in exact simplified form, in terms of natural logarithms.

$$3\coth^2 x - 8\operatorname{cosech} x + 1 = 0 .$$

$$\boxed{x = \ln\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right], \quad x = \ln\left[\frac{1}{2}\left(3+\sqrt{13}\right)\right]}$$

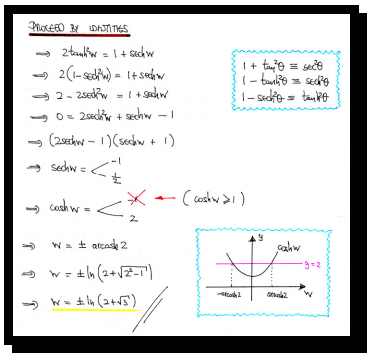


Question 9 **(**+)**

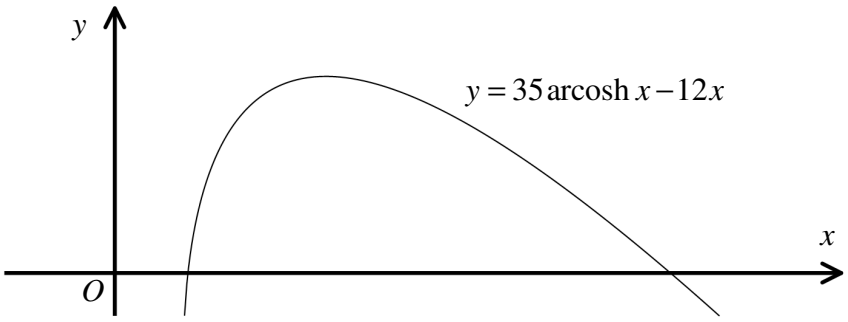
Solve the following equation, giving the solutions as exact simplified natural logarithms.

$$2\tanh^2 w = 1 + \operatorname{sech} w , \quad w \in \mathbb{R} .$$

$$\boxed{}, \quad \boxed{w = \pm \ln\left(2+\sqrt{3}\right)}$$



Question 10 (+)**



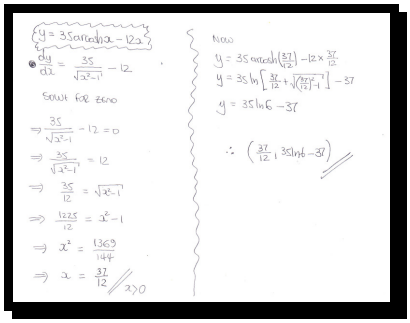
The figure above shows the graph of the curve with equation

$$y = 35\operatorname{arcosh} x - 12x, \ x \in \mathbb{R}, \ x \geq 1.$$

The curve has a single stationary point with coordinates $\left(\frac{a}{b}, c\ln 6 - d\right)$, where a, b, c and d are positive integers.

Determine the values of a, b, c and d.

$\boxed{a = 37}$, $\boxed{b = 12}$, $\boxed{c = 35}$, $\boxed{d = 37}$



Question 11 (+)**

$$f(x)=3-\cosh x,\ x\in\mathbb{R}.$$

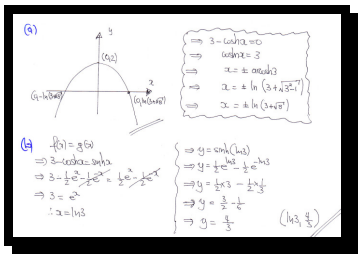
a) Sketch the graph of $f(x)$.

The graph must include the coordinates of any points where the graph meets the coordinate axes.

$$g(x)=\sinh x,\ x\in\mathbb{R}.$$

b) Find the exact coordinates of the point of intersection between the graphs of $f(x)$ and $g(x)$.

$$\left(\ln 3,\frac{4}{3}\right)$$



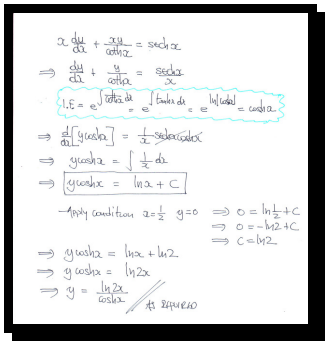
Question 12 (+)**

$$x\frac{dy}{dx}+\frac{xy}{\coth x}=\operatorname{sech} x\ ,\ x>0\,.$$

Given that $y=0$ at $x=\frac{1}{2}$, show that the solution of the above differential equation is

$$y=\frac{\ln 2x}{\cosh x}\,.$$

proof

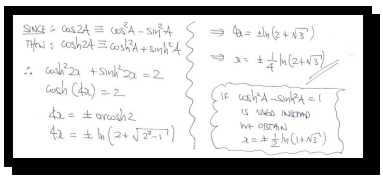


Question 13 (+)**

Find in exact logarithmic form the solutions of the following equation.

$$\cosh^2 2x+\sinh^2 2x=2\,.$$

$$\boxed{x=\pm\frac{1}{4}\ln\left(2+\sqrt{3}\right)=\pm\frac{1}{2}\ln\left(1+\sqrt{3}\right)}$$

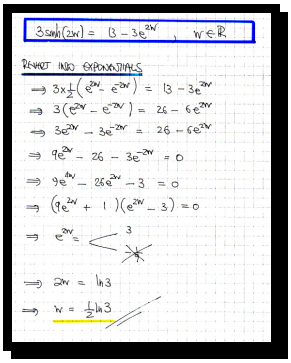


Question 14 **(**+)**

Find, in exact logarithmic form, the solution of the following equation.

$$3\sinh(2w) = 13-3e^{2w}, \quad w\in\mathbb{R}.$$

$\square$, $w=\frac{1}{2}\ln 3$



Question 15 (***)

It is given that

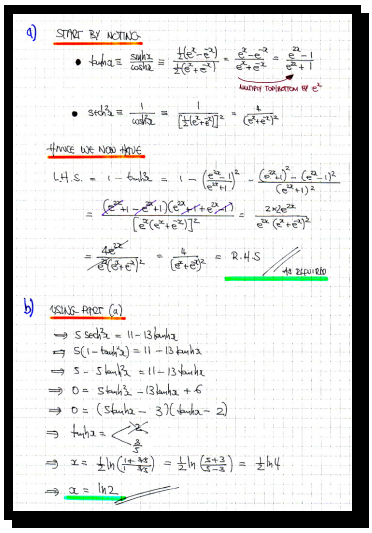
$$1-\tanh^2 x \equiv \operatorname{sech}^2 x .$$

a) Use the definitions of hyperbolic functions, in terms of exponentials, to prove the validity of the above identity.

b) Hence find in exact logarithmic form the solution of the following equation.

$$5\operatorname{sech}^2 x = 11-13\tanh x , \quad x \in \mathbb{R} .$$

$\boxed{}$, $\boxed{x=\ln 2}$



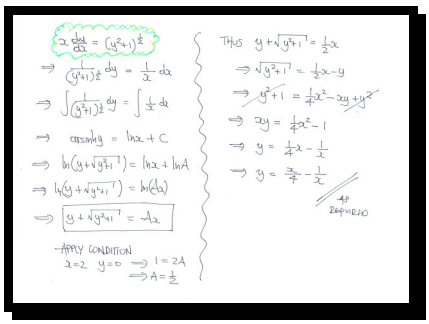
Question 16 (***)

$$x\frac{dy}{dx}=\sqrt{y^2+1}\ ,\ x>0\,.$$

Given that $y=0$ at $x=2$, show that the solution of the above differential equation is

$$y=\frac{x}{4}-\frac{1}{x}\,.$$

proof



Question 17 (***)

The curves C_1 and C_2 have respective equation

$$y=\sinh x \text{ and } y=\frac{1}{2}\operatorname{sech} x .$$

a) Sketch in the same diagram the graphs of C_1 and C_2.

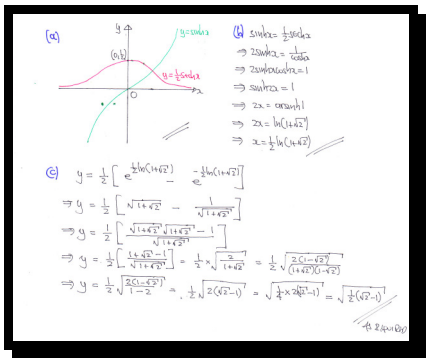
The two graphs intersect at the point P.

b) Find the x coordinates of P.

c) Hence show that the y coordinates of P is.

$$\sqrt{\frac{1}{2}\left(\sqrt{2}-1\right)}.$$

$$\boxed{x=\frac{1}{2}\ln\left(1+\sqrt{2}\right)}$$



Question 18 (***)

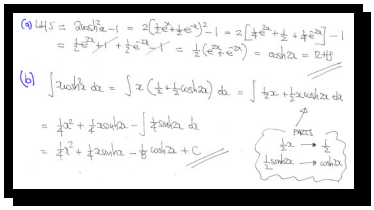
$$2\cosh^2 x - 1 \equiv \cosh 2x .$$

a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence find

$$\int x\cosh^2 x \, dx .$$

$$\boxed{\frac{1}{4}x^2 + \frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x + C}$$

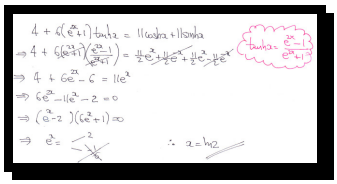


Question 19 (***)

Solve the hyperbolic equation

$$4 + 6\left(\mathrm{e}^{2x} + 1\right)\tanh x = 11\cosh x + 11\sinh x .$$

$$\boxed{x = \ln 2}$$



Question 20 (***)

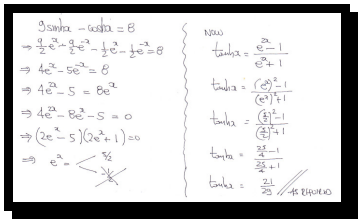
Given that

$$9\sinh x-\cosh x=8$$

show clearly that

$$\tanh x=\frac{21}{29}.$$

proof



Question 21 (***)

$$\cosh^2 x-\sinh^2 x\equiv 1.$$

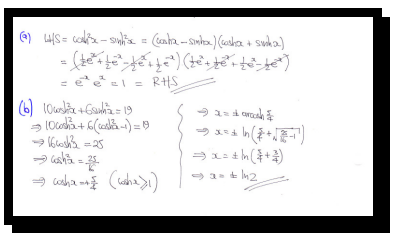
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence solve the equation

$$10\cosh^2 x+6\sinh^2 x=19$$

giving the answers as exact natural logarithms.

$x=\pm\ln 2$



Question 22 (***)

$$2\cosh 3x\cosh x \equiv \cosh 4x+\cosh 2x\,.$$

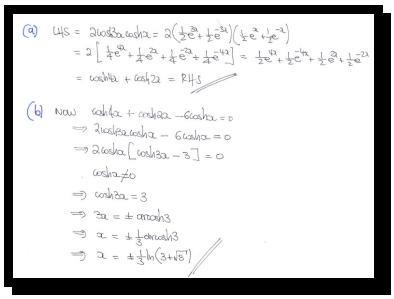
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ in terms of exponentials.

a) Hence solve the equation

$$\cosh 4x+\cosh 2x-6\cosh x=0$$

giving the answer as an expression involving exact natural logarithms.

$$\boxed{x=\pm\tfrac{1}{3}\ln\left(3+\sqrt{8}\right)}$$

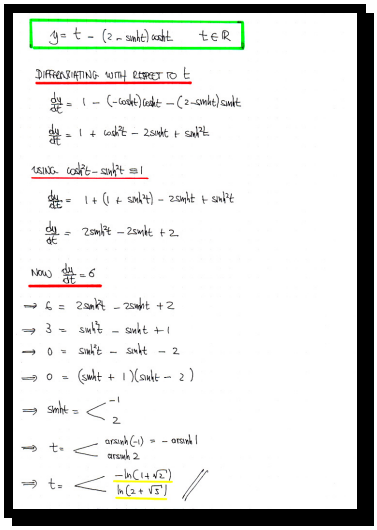


Question 23 **(***)**

$$y = t - (2 - \sinh t)\cosh t\,,\ t \in \mathbb{R}\,.$$

Determine the values of t for which $\dfrac{dy}{dt} = 6$, giving the answers as exact simplified natural logarithms.

$\square$, $\boxed{t = -\ln\left(1+\sqrt{2}\right) \ \cup \ t = \ln\left(2+\sqrt{5}\right)}$



Question 24 **(***)**

$$\cosh\left(A-B\right)\equiv\cosh A\cosh B-\sinh A\sinh B\,.$$

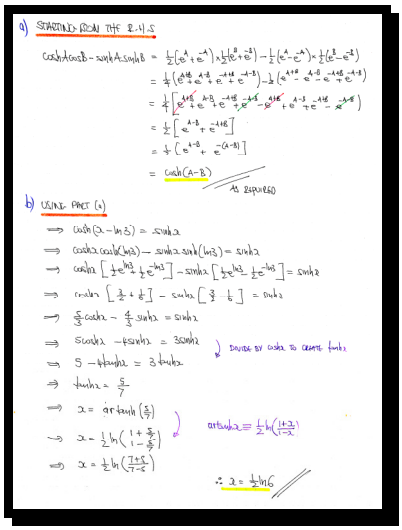
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence solve the equation

$$\cosh(x-\ln 3)=\sinh x$$

giving the answer as an exact natural logarithm.

$\boxed{}$, $\boxed{x=\frac{1}{2}\ln 6}$



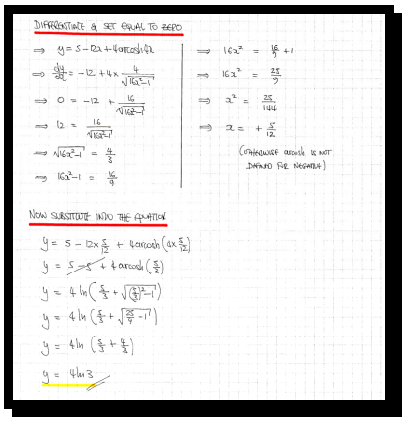
Question 25 **(***)**

Find, in exact simplified logarithmic form, the y coordinate of the stationary point of the curve with equation

$$y=5-12x+4\operatorname{arcosh}(4x).$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{4\ln 3}$

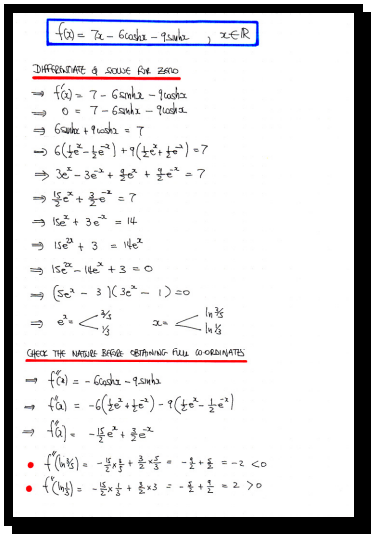


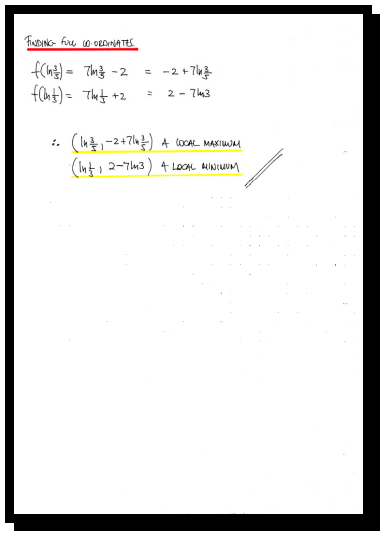
Question 26 **(***)**

$$f(x)\equiv 7x-6\cosh x-9\sinh x,\ x\in\mathbb{R}.$$

Find the exact coordinates of the stationary points of $f(x)$, and determine their nature. Give the coordinates in terms of simplified natural logarithms.

$$\boxed{\quad},\ \boxed{\left[\ln\left(\tfrac{3}{5}\right),-2+7\ln\left(\tfrac{3}{5}\right)\right]\ \cup\ \left[\ln\left(\tfrac{1}{3}\right),2-7\ln 3\right]}$$



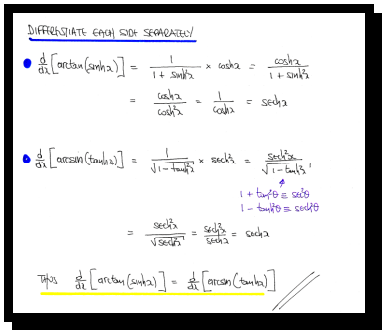


Question 27 **(***)**

Show with detailed workings that

$$\frac{d}{dx}\left[\arctan\left(\sinh x\right)\right]\ =\ \frac{d}{dx}\left[\arcsin\left(\tanh x\right)\right]$$

$$\boxed{\quad},\ \boxed{\text{proof}}$$



Question 28 **(***)**

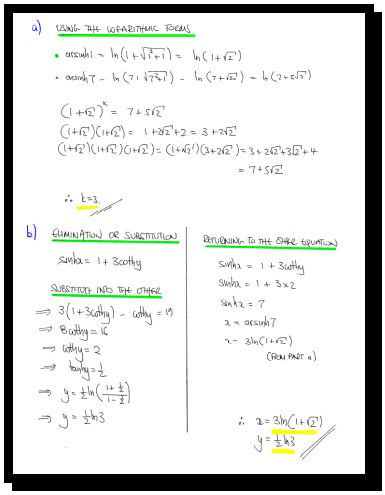
a) Given that $\operatorname{arsinh} 7 = k \operatorname{arsinh} 1$ determine the value of k.

b) Solve the following simultaneous equations.

$$\sinh x - 3\coth y = 1$$
$$3\sinh x - \coth y = 19$$

Give the answers in simplified logarithmic form.

$\boxed{}$, $\boxed{k=3}$, $\boxed{[x,y]=\left[3\ln\left(1+\sqrt{2}\right),\ \frac{1}{2}\ln 3\right]}$

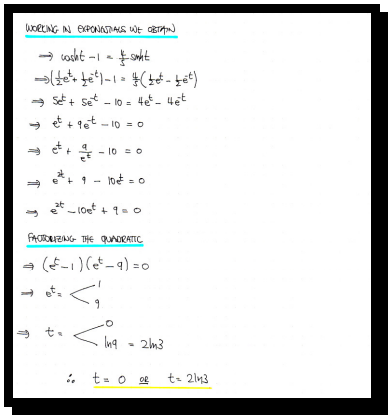


Question 29 (***)

Solve the following equation, giving the answers as exact logarithms where appropriate.

$$\cosh t - 1 = \frac{4}{5}\sinh t .$$

$\square$, $\boxed{t = 0, \quad t = 2\ln 3}$



Question 30 (***)

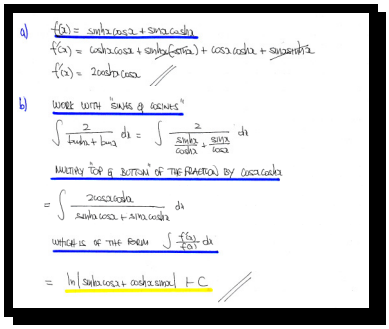
$$f(x)=\sinh x\cos x+\sin x\cosh x\,,\ \ x\in\mathbb{R}\,.$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part (a) to find

$$\int\frac{2}{\tanh x+\tan x}\,dx\,.$$

$\boxed{}$, $\boxed{f'(x)=2\cosh x\cos x}$, $\boxed{\ln\left|\sinh x\cos x+\sin x\cosh x\right|+C}$



Question 31 (***)

It is given that for all real x

$$\cosh 2x \equiv 1+2\sinh^2 x \, .$$

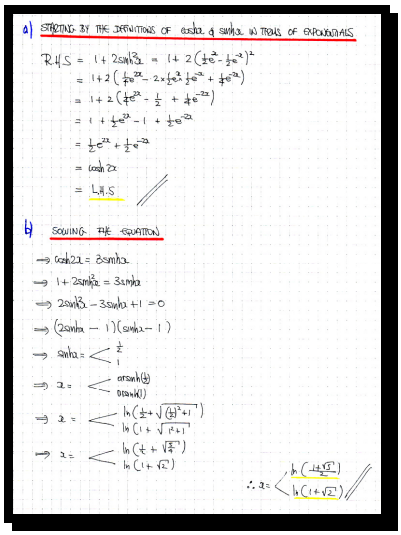
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$\cosh 2x = 3\sinh x \, ,$$

giving the final answers as exact simplified natural logarithms.

$\square$, $\boxed{x=\ln\left(1+\sqrt{2}\right) \quad \cup \quad x=\ln\left(\frac{1+\sqrt{5}}{2}\right)}$



Question 32 (***)

It is given that for all real x

$$\cosh 2x \equiv 2\cosh^2 x - 1 .$$

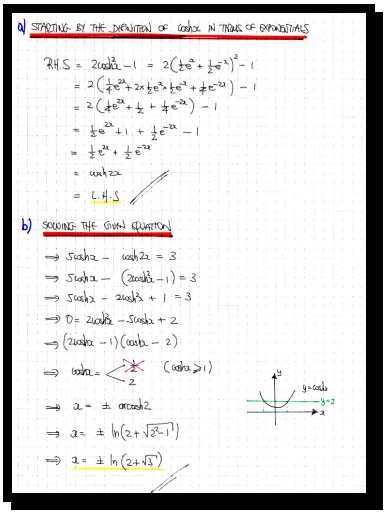
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$5\cosh x - \cosh 2x = 3 ,$$

giving the final answers as exact simplified natural logarithms.

$\boxed{}$, $\boxed{x = \pm \ln\left(2+\sqrt{3}\right)}$



Question 33 (*+)**

It is given that for all real x

$$\cosh 3x \equiv 4\cosh^3 x - 3\cosh x .$$

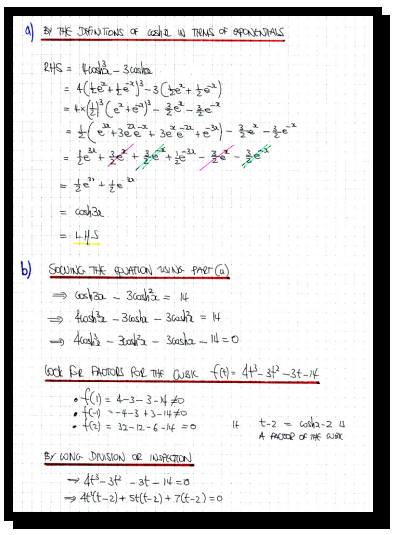
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$\cosh 3x - 3\cosh^2 x = 14 ,$$

giving the final answers as exact simplified natural logarithms.

$\square$, $\boxed{x = \pm \ln\left(2+\sqrt{3}\right)}$


a) BY THE DEFINITIONS OF $\cosh x$ IN TERMS OF EXPONENTIALS

LHS $= 4\cosh^3 x - 3\cosh x$

$= 4\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^3 - 3\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)$

$= 4\times\left(\frac{1}{2}\right)^3\left(e^x + e^{-x}\right)^3 - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$

$= \frac{1}{2}\left(e^{3x} + 3e^{2x}e^{-x} + 3e^{x}e^{-2x} + e^{-3x}\right) - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$

$= \frac{1}{2}e^{3x} + \frac{3}{2}e^{x} + \frac{3}{2}e^{-x} + \frac{1}{2}e^{-3x} - \frac{3}{2}e^{x} - \frac{3}{2}e^{-x}$

$= \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$

$= \cosh 3x$

$= $ L.H.S

b) SOLVING THE EQUATION USING PART (a)

$\Rightarrow \cosh 3x - 3\cosh^2 x = 14$

$\Rightarrow 4\cosh^3 x - 3\cosh x - 3\cosh^2 x = 14$

$\Rightarrow 4\cosh^3 x - 3\cosh^2 x - 3\cosh x - 14 = 0$

LOOK FOR FACTORS FOR THE CUBIC $f(t) = 4t^3 - 3t^2 - 3t - 14$

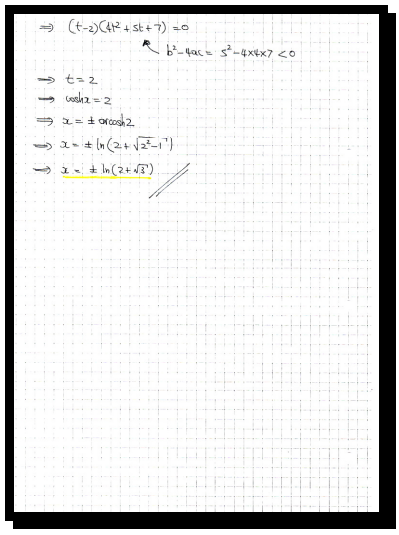
- $f(1) = 4 - 3 - 3 - 14 \neq 0$
- $f(-1) = -4 - 3 + 3 - 14 \neq 0$
- $f(2) = 32 - 12 - 6 - 14 = 0$

$\Rightarrow t - 2 = \cosh x - 2$ IS A FACTOR OF THE CUBIC

BY LONG DIVISION OR INSPECTION

$\Rightarrow 4t^3 - 3t^2 - 3t - 14 = 0$

$\Rightarrow 4t^2(t-2) + 5t(t-2) + 7(t-2) = 0$


$\Rightarrow (t-2)(4t^2 + 5t + 7) = 0$

$b^2 - 4ac = 5^2 - 4\times4\times7 < 0$

$\Rightarrow t = 2$

$\Rightarrow \cosh x = 2$

$\Rightarrow x = \pm \operatorname{arccosh} 2$

$\Rightarrow x = \pm \ln\left(2 + \sqrt{2^2 - 1}\right)$

$\Rightarrow x = \pm \ln\left(2 + \sqrt{3}\right)$

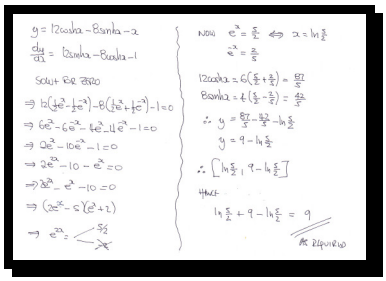
Question 34 (*+)**

A curve C has equation

$$y = 12\cosh x - 8\sinh x - x,\ x \in \mathbb{R}.$$

Show that the sum of the coordinates of the turning point of C is 9.

proof



Question 35 (*+)**

$$y = \operatorname{artanh} x\,,\ -1 < x < 1$$

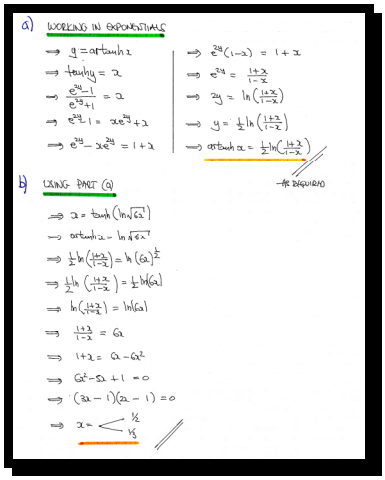
a) By using the definitions of hyperbolic functions in terms of exponentials prove that

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

b) Hence solve the equation

$$x = \tanh\left(\ln\sqrt{6x}\right).$$

$\square$, $\boxed{x = \frac{1}{2}, \frac{1}{3}}$



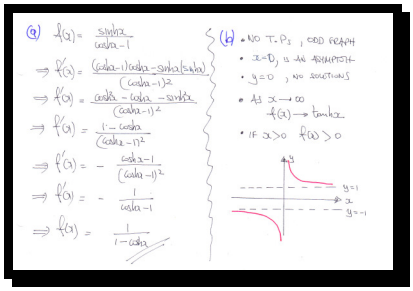
Question 36 (*+)**

$$f(x)=\frac{\sinh x}{\cosh x-1},\ x\in\mathbb{R},\ x\neq 0.$$

a) Find a simplified expression for $f'(x)$.

b) Sketch the graph of $f(x)$.

$$\boxed{f'(x)=\frac{1}{1-\cosh x}}$$



Question 37 (*+)**

$$y = \text{arsinh}\, x\,,\ x \in \mathbb{R}\,.$$

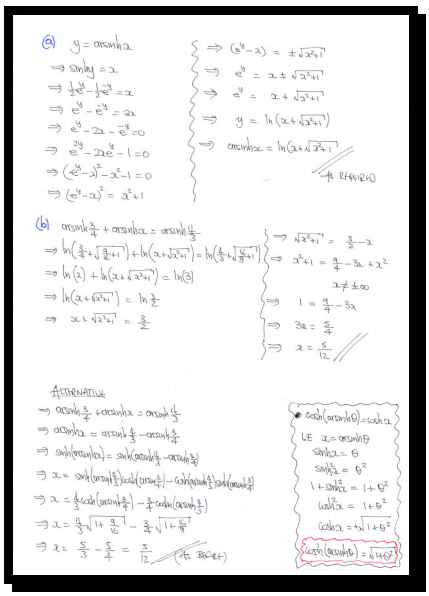
a) Show that

$$\text{arsinh}\, x = \ln\left[x + \sqrt{x^2+1} \right].$$

b) Solve the equation

$$\text{arsinh}\, \frac{3}{4} + \text{arsinh}\, x = \text{arsinh}\, \frac{4}{3}\,.$$

$$\boxed{x = \frac{5}{12}}$$



Question 38 (*+)**

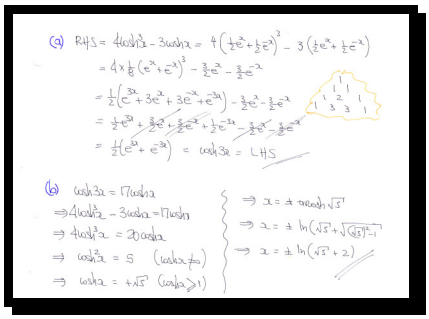
$$\cosh 3x \equiv 4\cosh^3 x - 3\cosh x .$$

a) Prove the validity of the above hyperbolic identity by using the definition of $\cosh x$ in terms of exponential functions.

b) Hence find in exact logarithmic form the solutions of the equation

$$\cosh 3x = 17\cosh x .$$

$$\boxed{x = \pm\ln\left(2+\sqrt{5}\right) = \mp\ln\left(-2+\sqrt{5}\right)}$$



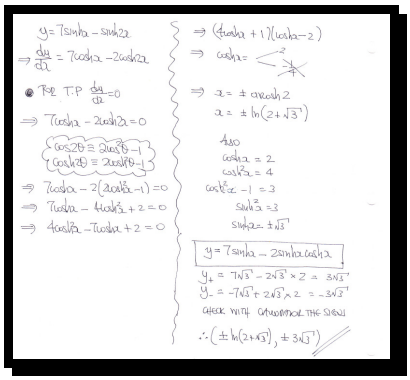
Question 39 (*+)**

The curve C has equation

$$y = 7\sinh x - \sinh 2x, \ x \in \mathbb{R}.$$

Find in terms of natural logarithms and/or surds the exact coordinates of the stationary points of C.

$$\boxed{\pm\left(\ln\left(2+\sqrt{3}\right), 3\sqrt{3}\right)}$$



Question 40 (*+)**

The curves C_1 and C_2 have respective equations

$$y = 18\cosh x,\ x \in \mathbb{R} \qquad \text{and} \qquad y = 12 + 14\sinh x,\ x \in \mathbb{R}.$$

a) Find the exact coordinates of the points of intersection between C_1 and C_2.

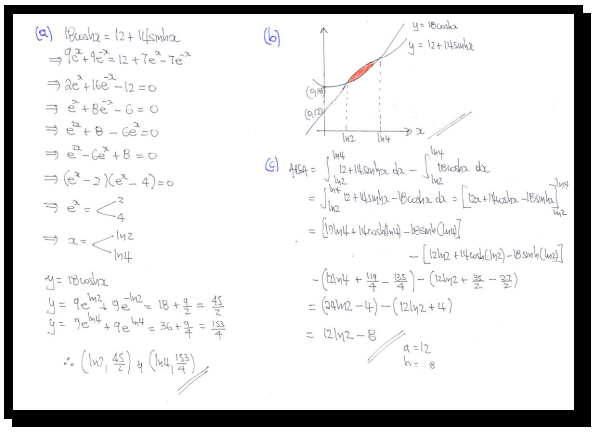
b) Sketch in the same diagram the graph of C_1 and the graph of C_2.

c) Show that the finite region bounded by the graphs of C_1 and C_2 has an area of

$$a\ln 2 + b,$$

where a and b are integers to be found.

$$\boxed{\left(\ln 2, \frac{45}{2}\right) \ \& \ \left(\ln 4, \frac{153}{4}\right)}, \qquad \boxed{12\ln 2 - 8}$$



Question 41 (*+)**

It is given that

$$\cosh\left(A+B\right)\equiv\cosh A\cosh B+\sinh A\sinh B\,.$$

a) Prove the validity of the above hyperbolic identity by using the definitions of the hyperbolic functions in terms of exponential functions.

It is now given that

$$5\cosh x+4\sinh x\equiv R\cosh\left(x+\alpha\right),$$

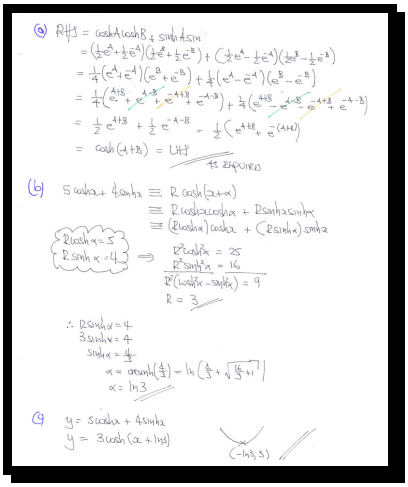
where R and α are positive constants.

b) Determine, in terms of natural logarithms where appropriate, the exact values of R and α.

c) Hence state the coordinates of the minimum point on the graph of

$$y=5\cosh x+4\sinh x\,.$$

$\boxed{R=3}$, $\boxed{\alpha=\ln 3}$, $\boxed{\left(-\ln 3,3\right)}$



Question 42 **(***+)**

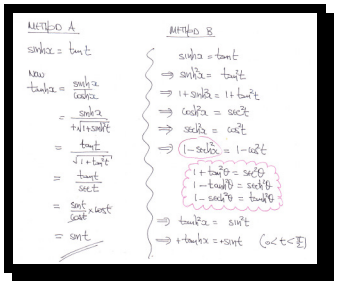
Given that

$$\sinh x = \tan t\,,\ 0 < t < \frac{\pi}{2},$$

show clearly that

$$\tanh x = \sin t\,.$$

proof



Question 43 **(***+)**

$$f(x)\equiv \operatorname{artanh} x \ , \ x\in\mathbb{R}, \ |x|<1$$

a) Use the definition of the hyperbolic tangent to prove that

$$f(x)\equiv \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right].$$

b) Use a method involving complex numbers and the trigonometric identity

$$1+\tan^2 x\equiv \sec^2 x,$$

to obtain the hyperbolic equivalent

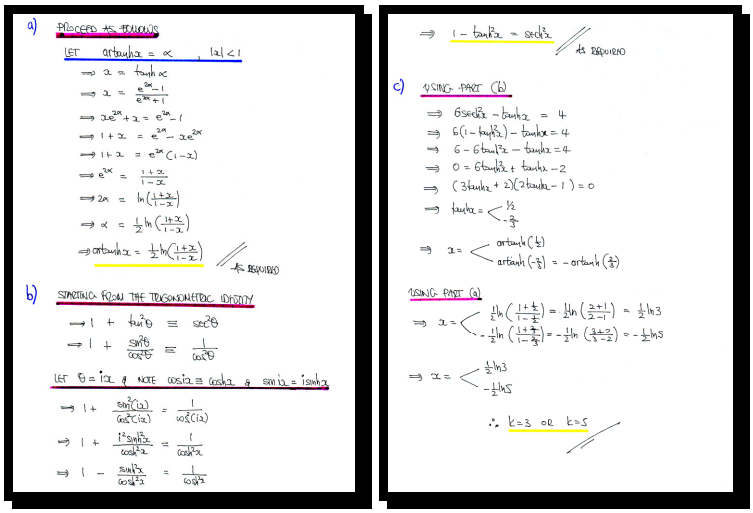
$$1-\tanh^2 x\equiv \operatorname{sech}^2 x.$$

c) Hence solve the equation

$$6\operatorname{sech}^2 x-\tanh x=4,$$

giving the two solutions in the form $\pm\frac{1}{2}\ln k$, where k are two distinct integers.

$$\boxed{}, \ \boxed{x=\frac{1}{2}\ln 3}, \ \boxed{x=-\frac{1}{2}\ln 5}$$



Question 44 **(***+)**

a) Sketch a detailed graph of the curve with equation

$$y = \text{artanh}\, x\,,$$

defined in the largest real domain.

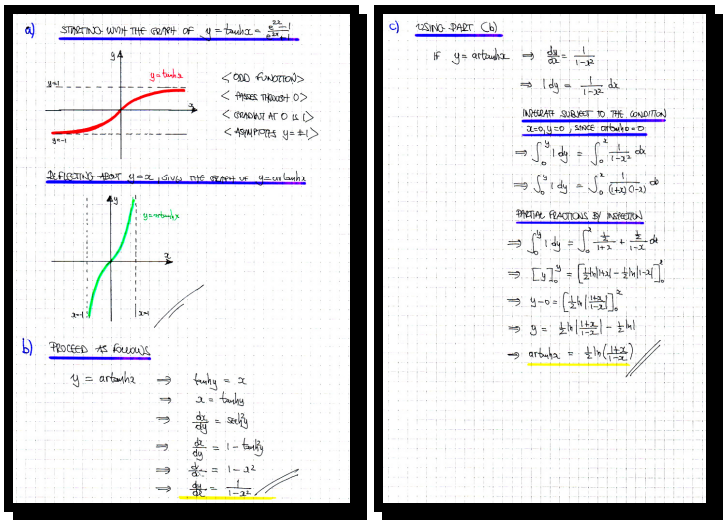
b) Obtain a simplified expression for $\dfrac{dy}{dx}$, in terms of x only.

c) Use integration and the answer of part **(b)** to show that

$$\text{artanh}\, x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right].$$

No credit will be given for any alternative methods used in part ***(c)****.*

$\square$, $\boxed{\dfrac{dy}{dx} = \dfrac{1}{1-x^2}}$



Question 45 **(***+)**

a) Starting from the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials, show that

$$\cos(\mathrm{i}\varphi)\equiv\cosh(\varphi) \quad \text{and} \quad \sin(\mathrm{i}\varphi)\equiv\mathrm{i}\sinh(\varphi).$$

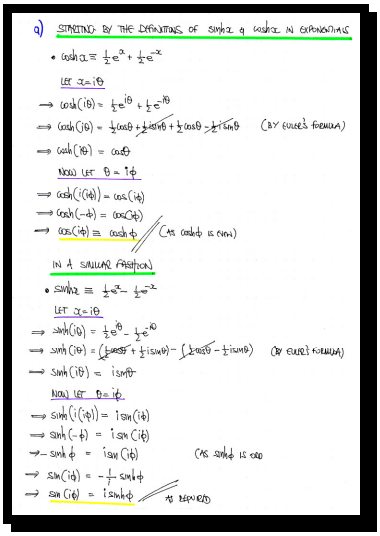
b) Use the results of part **(a)** to deduce

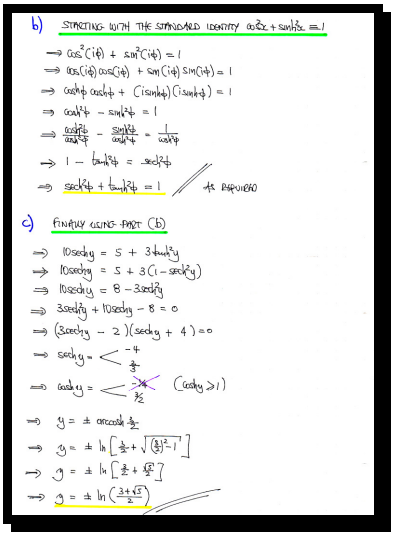
$$\operatorname{sech}^2\varphi+\tanh^2\varphi\equiv 1.$$

c) Hence find, in exact logarithmic form, the solutions of the following equation.

$$10\operatorname{sech} y=5+3\tanh^2 y.$$

$\square$, $\boxed{y=\pm\ln\left(\frac{3+\sqrt{5}}{2}\right)}$





Question 46 (*+)**

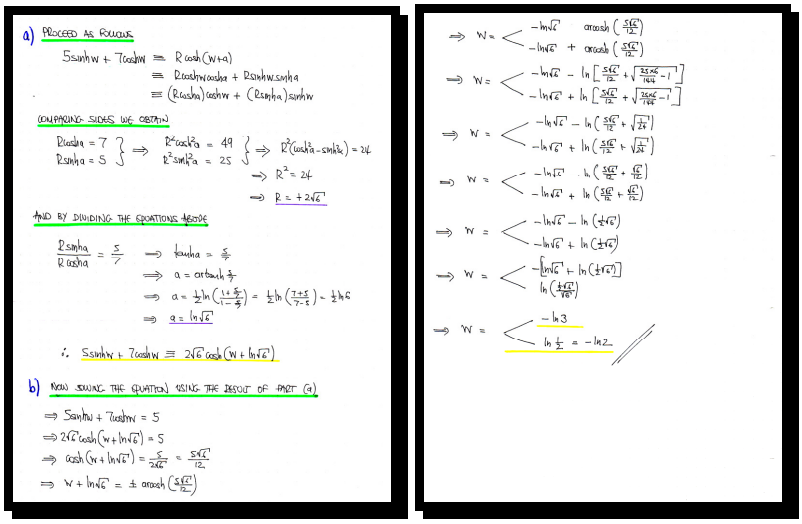
$$f(w)\equiv 5\sinh w+7\cosh w,\ w\in\mathbb{R}$$

a) Express $f(w)$ in the form $R\cosh(w+a)$, where R and a are exact constants with $R>0$.

b) Use the result of part **(a)** to find, in exact logarithmic form, the solutions of the following equation.

$$5\sinh w+7\cosh w=5\,.$$

$\boxed{}$, $\boxed{R=\sqrt{24}=2\sqrt{6}}$, $\boxed{a=\frac{1}{2}\ln 6=\ln\sqrt{6}}$, $\boxed{w=-\ln 2\ \cup\ w=-\ln 3}$

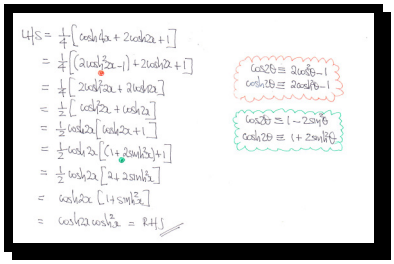


Question 47 (*+)**

By using suitable hyperbolic identities, or otherwise, show that

$$\frac{1}{4}\left[\cosh 4x+2\cosh 2x+1\right]\equiv \cosh 2x\cosh^2 x\,.$$

proof



Question 48 (****)

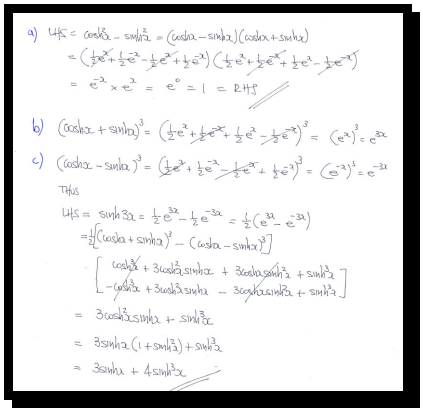
a) By expressing $\cosh x$ and $\sinh x$ in terms of exponentials, show that

$$\cosh^2 x-\sinh^2 x\equiv 1\,.$$

b) Simplify $\left(\cosh x+\sinh x\right)^3$, writing the final answer as a single exponential.

c) Hence express $\sinh 3x$ in terms of $\sinh x$

$\left(\cosh x+\sinh x\right)^3=\mathrm{e}^{3x}$, $\sinh 3x=3\sinh x+4\sinh^4 x$



Question 49 (****)

The curve C has equation

$$y=\cosh\left(2\operatorname{arsinh}x\right),\ x\in\mathbb{R}.$$

a) Find an expression for $\dfrac{dy}{dx}$.

b) Show clearly that

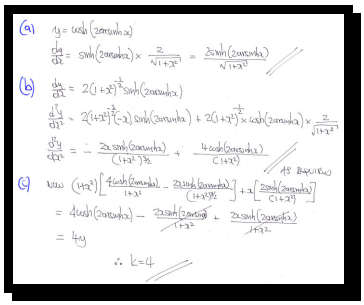
$$\frac{d^2y}{dx^2}=\frac{4}{1+x^2}\cosh\left(2\operatorname{arsinh}x\right)-\frac{2x}{\left(1+x^2\right)^{\frac{3}{2}}}\sinh\left(2\operatorname{arsinh}x\right)$$

c) Hence show further that

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-ky=0,$$

for some value of the constant k.

$$\boxed{\frac{dy}{dx}=\frac{2\sinh\left(2\operatorname{arsinh}x\right)}{\sqrt{1+x^2}}},\quad \boxed{k=4}$$



Question 50 (****)

A function is defined in terms of exponentials by

$$f(x)=\frac{2}{\mathrm{e}^{x}+\mathrm{e}^{-x}},\ x\in\mathbb{R}.$$

a) Sketch the graph of $f(x)$.

b) Show clearly that

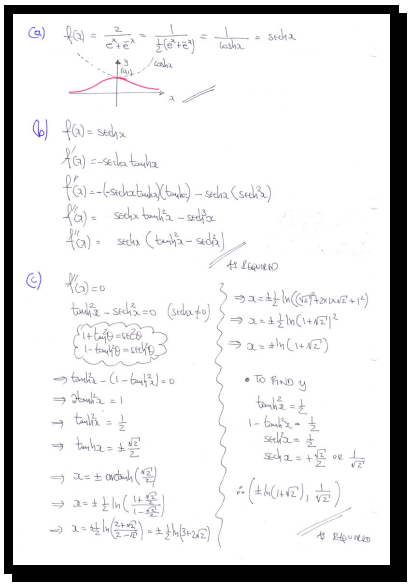
$$f''(x)=\operatorname{sech}x\left(\tanh^{2}x-\operatorname{sech}^{2}x\right).$$

It is given that the graph of $f(x)$ has two points of inflection.

c) Show further that the coordinates of these points are

$$\left(\pm\ln\left(1+\sqrt{2}\right),\frac{1}{\sqrt{2}}\right).$$

proof



Question 51 (****)

It is given that

$$\cosh\left(A+B\right)\equiv\cosh A\cosh B+\sinh A\sinh B\,.$$

a) Prove the validity of the above hyperbolic identity by using the definitions of the hyperbolic functions in terms of exponential functions.

It is now given that

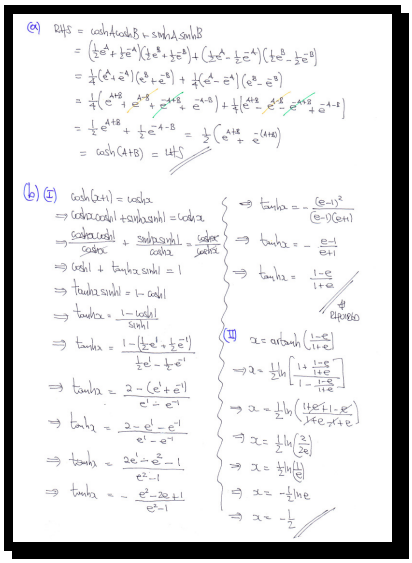
$$\cosh\left(x+1\right)=\cosh x\,,$$

b) Show clearly that …

i. … $\tanh x=\dfrac{1-\mathrm{e}}{1+\mathrm{e}}$.

ii. … $x=-\dfrac{1}{2}$.

proof

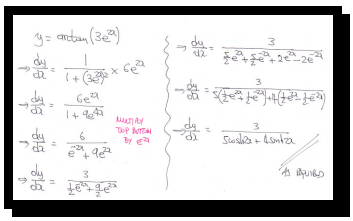


Question 52 (****)

Given that $y=\arctan\left(3\mathrm{e}^{2x}\right)$, show clearly that

$$\frac{dy}{dx}=\frac{3}{5\cosh 2x+4\sinh 2x}$$

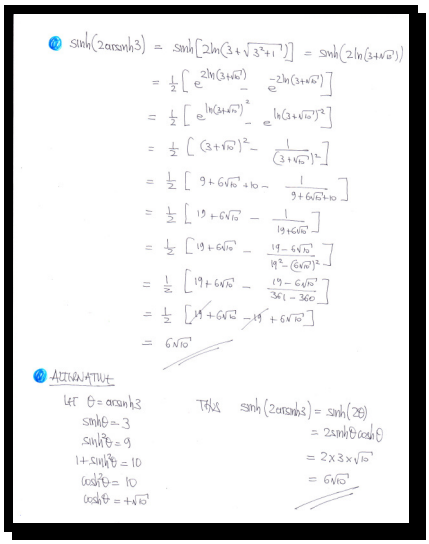
proof



Question 53 (****)

Find in exact simplified form the value of $\sinh\left(2\operatorname{arsinh}3\right)$.

$6\sqrt{10}$



Question 54 **(****)**

$$\cosh 2x \equiv 2\cosh^2 x - 1$$

a) Prove the validity of the above identity by using the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials.

The curve C has equation

$$y = \cosh x - 1, \ x \in \mathbb{R}.$$

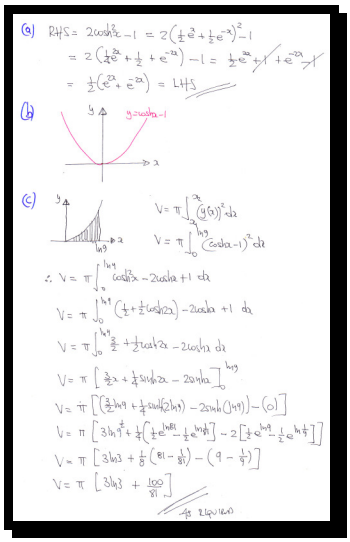
b) Sketch the graph of C.

The region bounded by C, the x axis and the line with equation $x = \ln 9$ is rotated through 2π radians about the x axis to form a volume of revolution S.

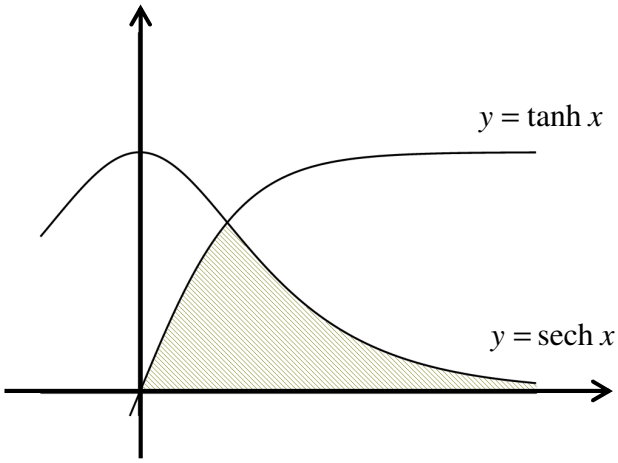
c) Show that the volume S is

$$\pi\left(3\ln 3 + \frac{100}{81}\right).$$

proof



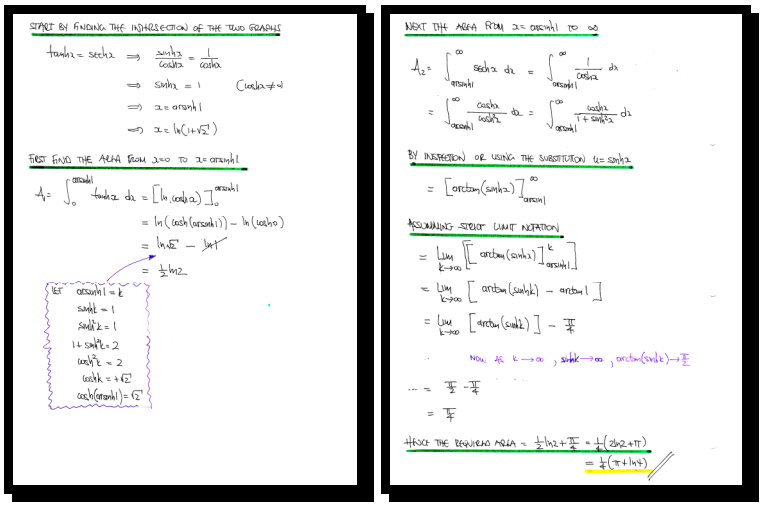
Question 55 **(****)**



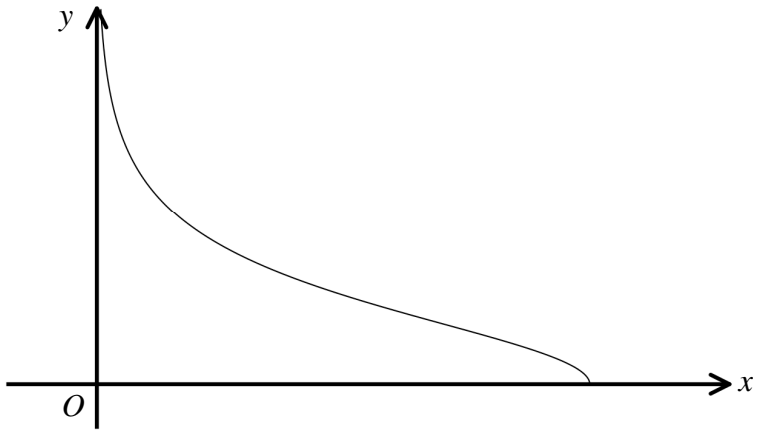
The figure above shows the graphs of $y = \tanh x$ and $y = \operatorname{sech} x$, in the first quadrant.

Show that the area shown shaded in the figure for which $x \geq 0$ is exactly $\frac{1}{4}[\pi + \ln 4]$.

, proof



Question 56 (****)



The figure above shows the graph of $y = \text{arsech}\, x$, $0 < x \leq 1$.

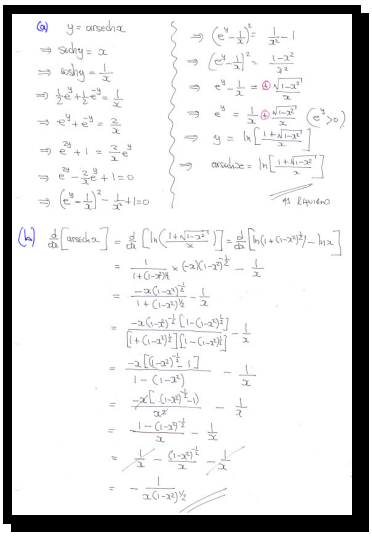
a) Show clearly that

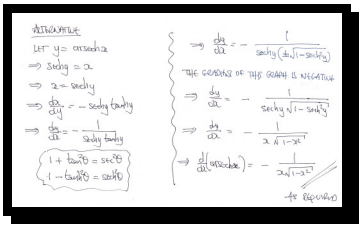
$$\text{arsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

b) Show further that

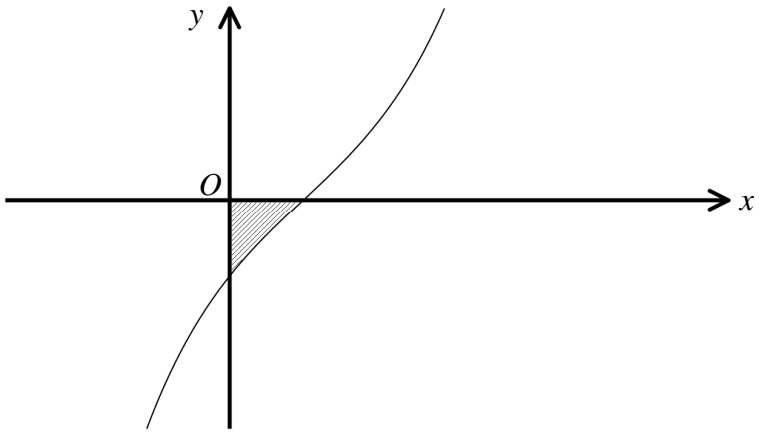
$$\frac{d}{dx}\left(\text{arsech}\, x\right) = -\frac{1}{x\sqrt{1-x^2}}.$$

proof





Question 57 **(****)**



The figure above shows the graph of the curve with equation

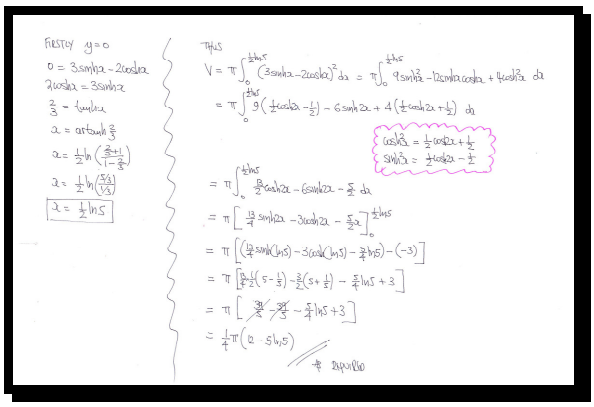
$$y = 3\sinh x - 2\cosh x\,,\ \ x \in \mathbb{R}\,.$$

The finite region bounded by the curve and the coordinate axes, shown shaded in the figure above, is revolved by 2π about the x axis to form a solid S.

Show that the volume of S is

$$\frac{1}{4}\pi\left(12 - 5\ln 5\right).$$

proof



Question 58 **(****)**

a) Sketch the graph of $y=\text{arsech}\,x$, defined for $0<x\leq 1$.

b) Show clearly that

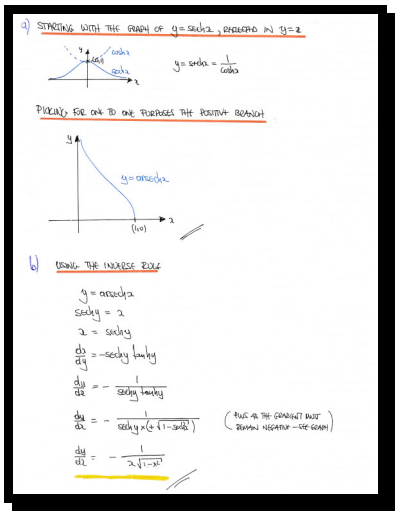
$$\frac{dy}{dx}=-\frac{1}{x\sqrt{1-x^2}}.$$

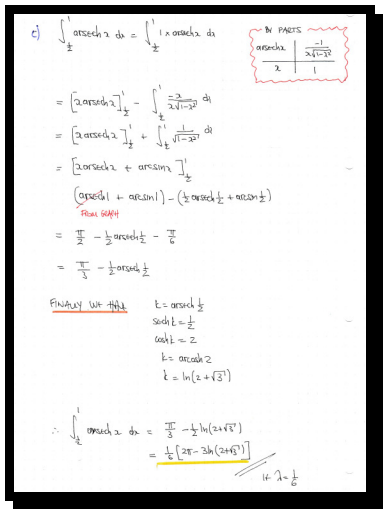
c) Hence evaluate

$$\int_{\frac{1}{2}}^{1}\text{arsech}\,x\ \ dx.$$

Give the answer in the form $\lambda\left[2\pi-3\ln\left(2+\sqrt{3}\right)\right]$, where λ is a rational number to be found.

$\square$, $\boxed{\lambda=\frac{1}{6}}$





Question 59 (****)

It is given that for all real x

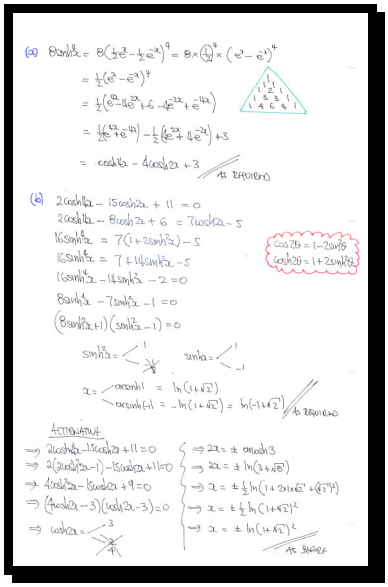
$$8\sinh^2 x \equiv \cosh 4x - 4\cosh 2x + 3 .$$

a) Prove the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence, or otherwise, show that $x = \pm \ln\left(1+\sqrt{2}\right)$ are the solutions of the equation

$$2\cosh 4x - 15\cosh 2x + 11 = 0 .$$

proof



Question 60 (****)

A curve C has equation

$$y=\cosh 2x+\sinh x,\ x\in\mathbb{R}.$$

a) Show that the x coordinate of the turning point of C is

$$-\ln\left(\frac{1+\sqrt{17}}{4}\right).$$

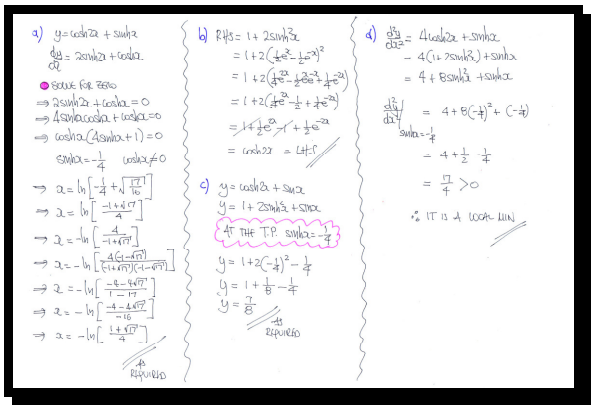
b) Using the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials, prove that

$$\cosh 2x\equiv 1+2\sinh^2 x.$$

c) Hence show that the y coordinate of the turning point of C is $\frac{7}{8}$.

d) Determine the nature of the turning point.

min



Question 61 **(****)**

It is given that

$$A\cosh x + B\sinh x \equiv R\cosh\left(x+\alpha\right),$$

where the A, B, R and α are constants with $A > B > 0$, $R > 0$.

a) Show clearly that …

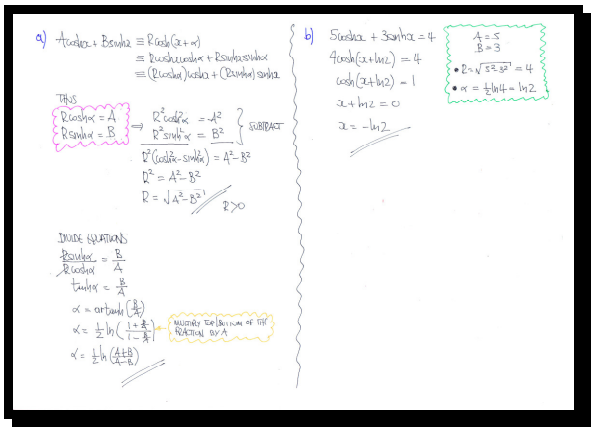
i. … $\alpha = \dfrac{1}{2}\ln\left(\dfrac{A+B}{A-B}\right)$.

ii. … $R = \sqrt{A^2 - B^2}$.

b) Use the above result to determine the exact solution of the equation

$$5\cosh x + 3\sinh x = 4.$$

$\boxed{x = -\ln 2}$



Question 62 (****)

$$f(x)\equiv \cosh 2x-8\cosh x\,,\ x\in\mathbb{R}\,.$$

a) Determine, in exact logarithmic form, the solutions of the equation

$$f(x)=-1\,.$$

b) If k is a real constant, determine the value, values or range of values of k, so that the equation $f(x)=k$ has…

i. … one repeated real root.

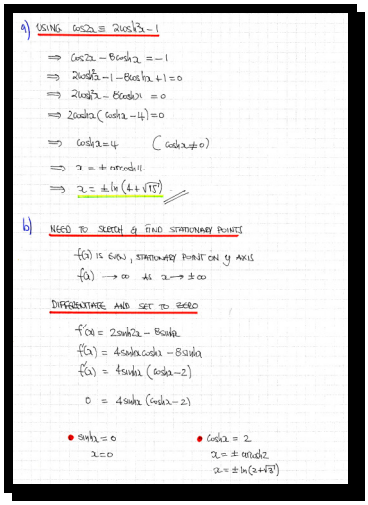
ii. … more than one repeated real root.

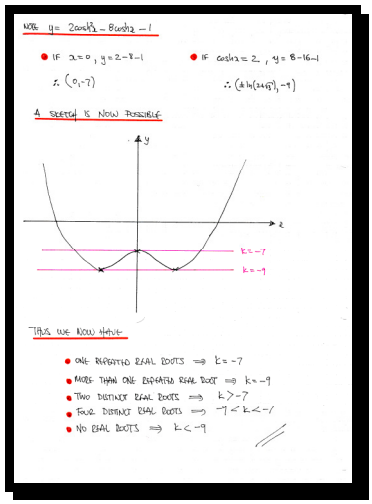
iii. … two distinct real roots.

iv. … four distinct real roots.

v. … no real roots.

$\square$, $\boxed{x=\pm\ln\left(4+\sqrt{15}\right)}$





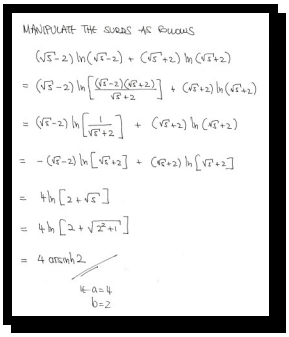
Question 63 **(****)**

Show that

$$\left(\sqrt{5}-2\right)\ln\left(\sqrt{5}-2\right)+\left(\sqrt{5}+2\right)\ln\left(\sqrt{5}+2\right),$$

can be written in the form $a\,\text{arsinh}\,b$, where a and b are positive integers to be found.

$\square$, $\boxed{4\,\text{arsinh}\left(2\right)}$

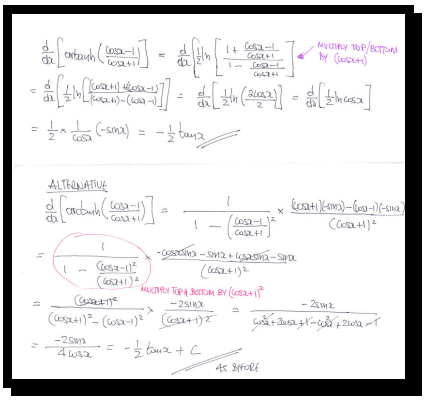


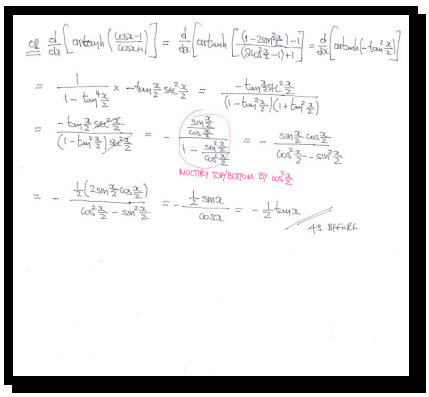
Question 64 **(****+)**

Show clearly that

$$\frac{d}{dx}\left[\text{artanh}\left(\frac{\cos x+1}{\cos x-1}\right)\right]=-\frac{1}{2}\tan x\,.$$

$\boxed{\text{proof}}$





Question 65 **(****)**

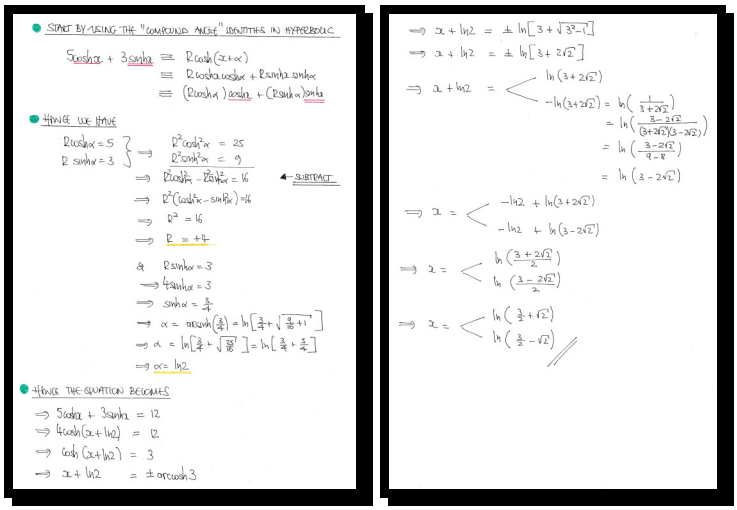
$$5\cosh x+3\sinh x=12$$

Express the left side of the above equation in the form $R\cosh\left(x+\alpha\right)$, where R and α are positive constants, and use it to show that

$$x=\ln\left(A\pm\sqrt{B}\right),$$

where A and B are constants to be found.

$\boxed{}$, $\boxed{x=\ln\left(\frac{3}{2}\pm\sqrt{2}\right)}$



Question 66 (**+)**

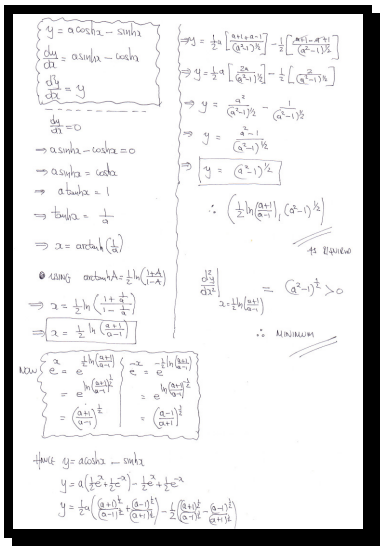
The curve C has equation

$$y = a\cosh x - \sinh x \text{, where } a > 1 .$$

Show that C has a minimum turning point with coordinates

$$\left(\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right), \sqrt{a^2+1}\right).$$

proof

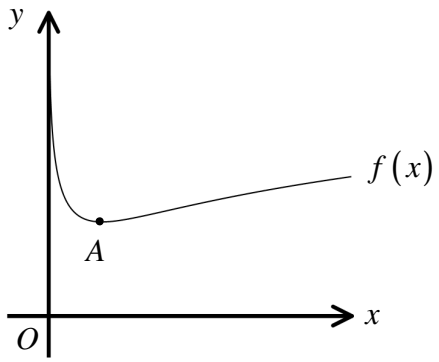


Question 67 **(****+)**

$$f(x)=\operatorname{arsinh}x+\operatorname{arsinh}\left(\frac{1}{x}\right),\ x\in\mathbb{R},\ x\neq 0.$$

a) Show clearly that $f'(x)=\dfrac{x^2-|x|}{x^2\sqrt{x^2+1}}$.

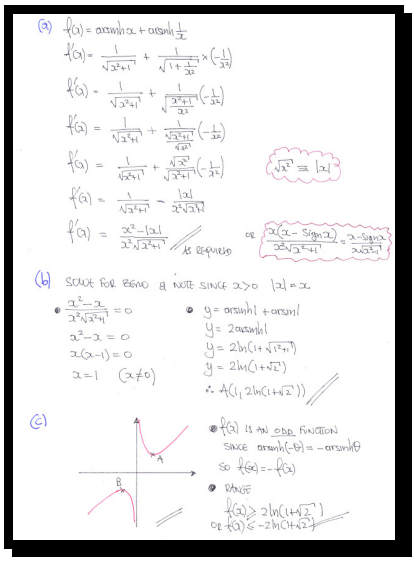
The graph of $f(x)$, for $x>0$ is shown in the figure below.



b) Determine, in terms of natural logarithms where appropriate, the coordinates of the stationary point of $f(x)$, labelled as point A in the figure.

c) Sketch the graph of $f(x)$, fully justifying its shape for $x<0$, and state its range.

$$\boxed{A\left[1,2\ln\left(1+\sqrt{2}\right)\right]},\quad \boxed{f(x)\geq 2\ln\left(1+\sqrt{2}\right)\cup f(x)\leq -2\ln\left(1+\sqrt{2}\right)}$$



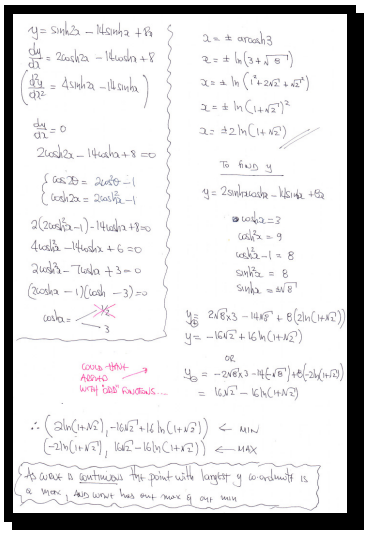
Question 68 (**+)**

The curve C has equation

$$y=\sinh 2x-14\sinh x+8x\,.$$

Find the exact coordinates of the turning points of C and determine their nature.

$$\left[2\ln\left(1+\sqrt{2}\right),-16\sqrt{2}+16\ln\left(1+\sqrt{2}\right)\right],\left[-2\ln\left(1+\sqrt{2}\right),16\sqrt{2}-16\ln\left(1+\sqrt{2}\right)\right]$$

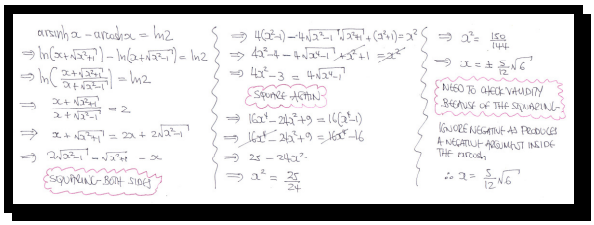


Question 69 (**+)**

Find, in exact surd form the solution of the equation

$$\operatorname{arsinh}x-\operatorname{arcosh}x=\ln 2\,.$$

$$x=\frac{5}{12}\sqrt{6}$$



Question 70 (**+)**

$$\cosh x \equiv \frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right) \quad \text{and} \quad \sinh x \equiv \frac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right).$$

a) Use the above definitions to show that …

i. … $\cosh^2 x-\sinh^2 x\equiv 1$.

ii. … $4\cosh^3 x-3\cosh x\equiv\cosh 3x$.

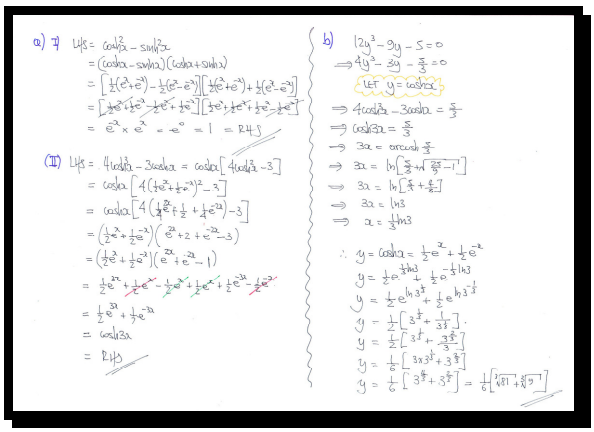
b) Hence show that the real root of the equation

$$12y^3-9y-5=0,$$

can be written as

$$\frac{1}{6}\left(\sqrt[3]{81}+\sqrt[3]{9}\right).$$

proof

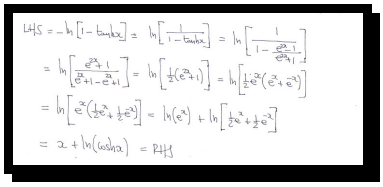


Question 71 (**+)**

Show clearly that

$$-\ln\left(1-\tanh x\right)\equiv x+\ln\left(\cosh x\right).$$

proof



Question 72 (**+)**

A curve C has equation

$$y=3\sinh x-2\cosh x\,,\ \ x\in\mathbb{R}\,.$$

Sketch the graph of C.

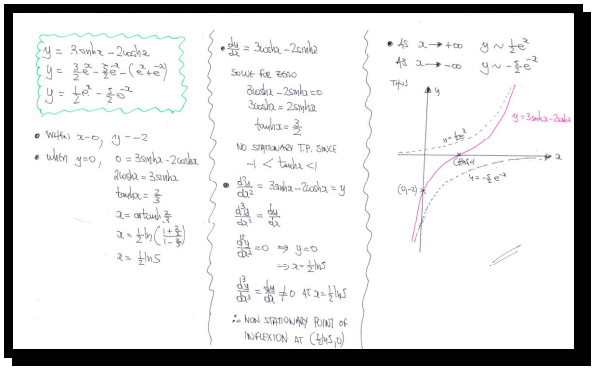
The sketch must include …

… the coordinates of any points where the graph of C meets the coordinates axes.

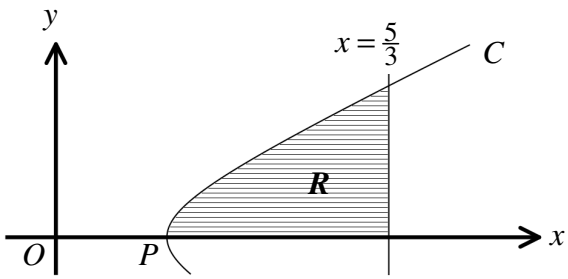
… the coordinates of any stationary or non stationary turning points.

… the behaviour of the curve for large positive and large negative values of x

graph



Question 73 **(****+)**



The figure above shows part of the curve C with parametric equations

$$x=t+\frac{1}{4t}\,,\quad y=t-\frac{1}{4t}\quad,\quad t>0\,.$$

The curve crosses the x axis at P.

a) Determine the coordinates of P.

b) By considering $x+y$ and $x-y$ find a Cartesian equation for C.

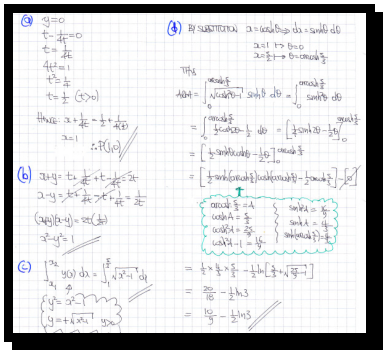
The region R bounded by C, the straight line with equation $x=\frac{5}{3}$ and the x axis is shown shaded in the figure.

c) Show that the area of R is given by

$$\int_1^{\frac{5}{3}}\sqrt{x^2-1}\ dx\,.$$

d) Hence calculate an exact value for the area of R.

$\boxed{P(1,0)}$, $\boxed{x^2-y^2=1}$, $\boxed{\text{Area}=\frac{10}{9}-\frac{1}{2}\ln 3}$



Question 74 **(****+)**

The function f is defined

$$f(t) \equiv \ln(1+\sin t), \quad \sin t \neq \pm 1$$

a) Show clearly that …

i. … $f(t)-f(-t)=2\ln(\sec t+\tan t)$.

ii. … $2\ln(\sec t+\tan t)=-2\ln(\sec t-\tan t)$

A curve C is given parametrically by

$$x=f(t)+f(-t), \quad y=f(t)-f(-t).$$

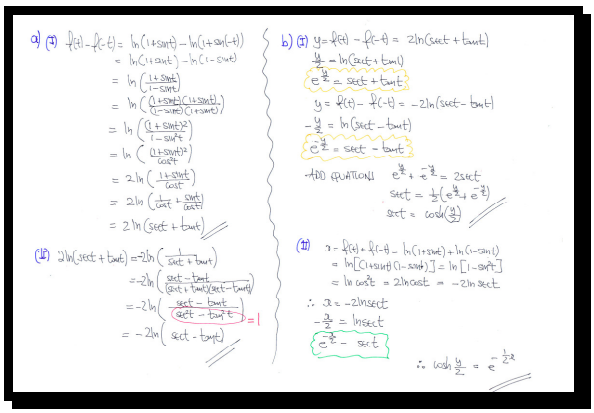
b) Show further that …

i. … $\sec t=\cosh\frac{y}{2}$

ii. … a Cartesian equation of C can be written as

$$\cosh\frac{y}{2}=\mathrm{e}^{-\frac{1}{2}x}$$

proof



Question 75 (**+)**

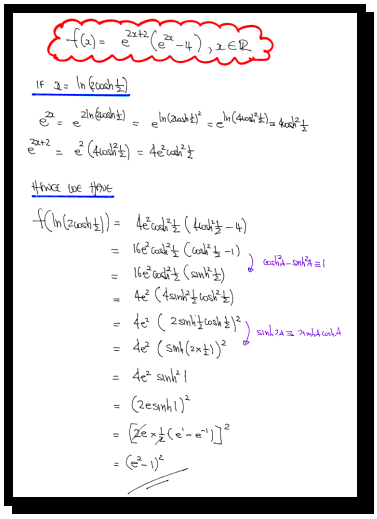
The function f is given by

$$f(x)\equiv \mathrm{e}^{2x+2}\left(\mathrm{e}^{2x}-4\right),\quad x\in\mathbb{R}.$$

Show that

$$f\left[\ln\left(2\cosh\frac{1}{2}\right)\right]=\left(\mathrm{e}^{2}-1\right)^{2}.$$

[ ], [proof]



Question 76 (**+)**

It is given that for suitable values of x

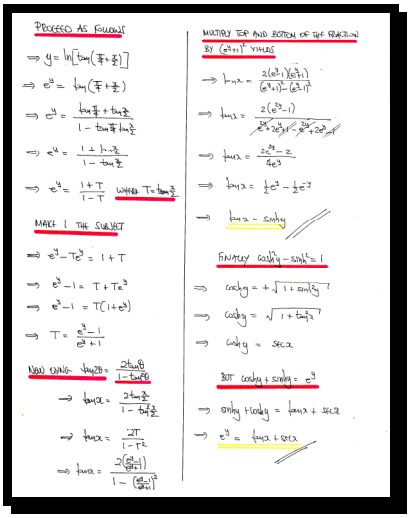
$$y=\ln\left[\tan\left(\frac{1}{4}\pi+\frac{1}{2}x\right)\right] \quad .$$

Show, with detailed workings, that

$$\sinh y=\tan x\,,$$

and hence deduce a simplified expression for e^{y} in terms of x.

$\boxed{}$, $\boxed{\mathrm{e}^{y}=\tan x\ +\ \sec x}$

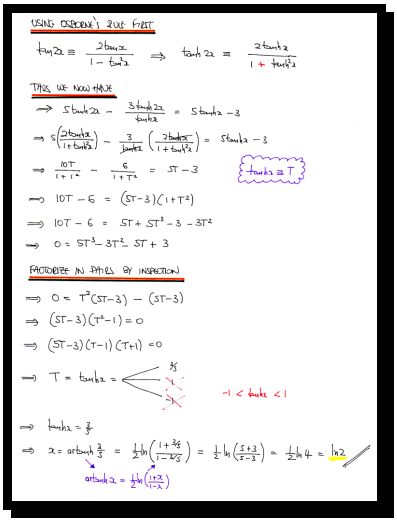


Question 77 (**+)**

$$5\tanh 2x-\frac{3\tan 2x}{\tanh x}=5\tanh x-3\,.$$

Find, as an exact natural logarithm, the real solution of the above equation.

$\boxed{}$, $\boxed{x=\ln 2}$



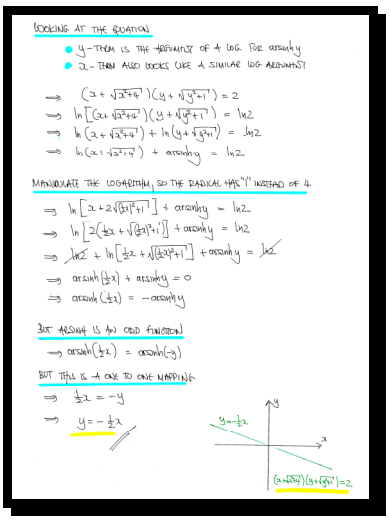
Question 78 (*****)

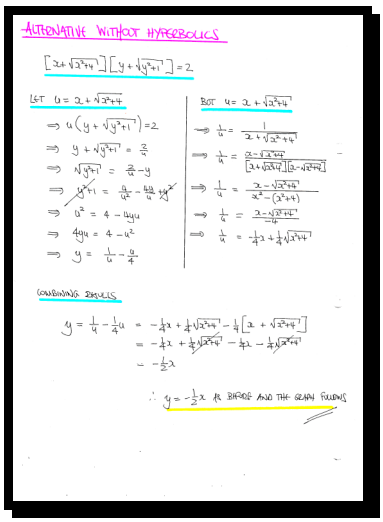
Sketch the graph of

$$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2, \quad x\in(-\infty,\infty), \ y\in(-\infty,\infty)$$

You must show a detailed method in this question

, proof



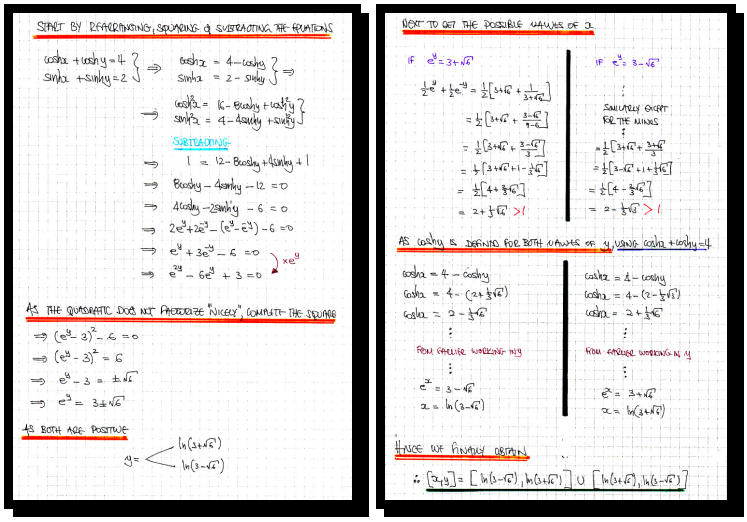


Question 79 (*****)

Determine, as exact simplified natural logarithms, the solutions of the following simultaneous equations

$$\cosh x+\cosh y=4 \qquad \text{and} \qquad \sinh x+\sinh y=2\,.$$

$\square$, $\boxed{[x,y]=\left[\ln\left(3-\sqrt{6}\right),\ln\left(3+\sqrt{6}\right)\right]=\left[\ln\left(3+\sqrt{6}\right),\ln\left(3-\sqrt{6}\right)\right]}$



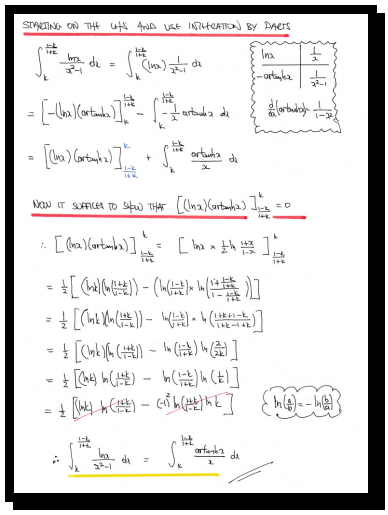
Question 80 **(*****)**

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1}\, dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x}\, dx .$$

You need not evaluate these integrals.

, proof

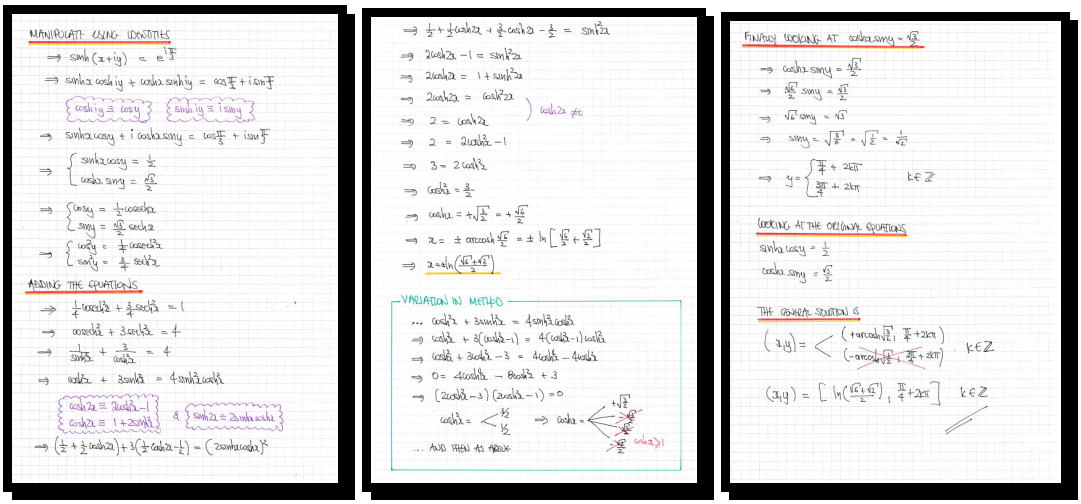


Question 81 (*****)

Determine the general solution of the following equation.

$$\sinh\left(x+\mathrm{i}y\right)=\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}, \quad x\in\mathbb{R}, \quad y\in\mathbb{R}.$$

$\square$, $\boxed{(x,y)=\left[\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right),\frac{\pi}{4}+2k\pi\right],\ k\in\mathbb{Z}}$



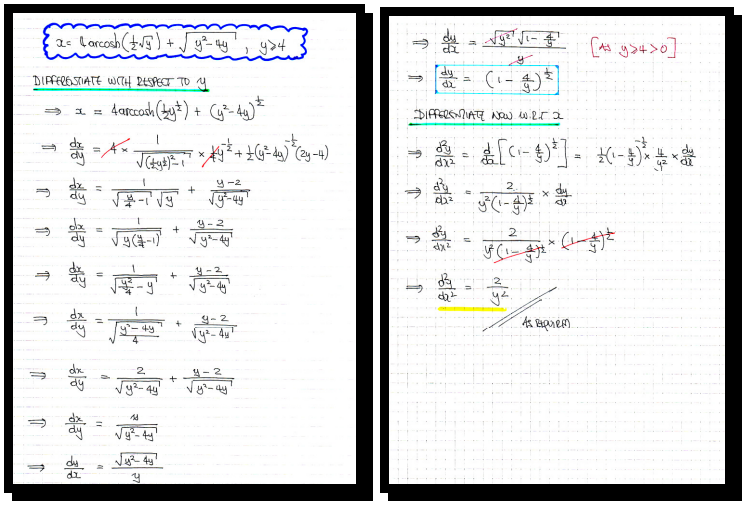
Question 82 **(*****)**

$$x = 4\operatorname{arcosh}\left(\frac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}\,, \quad y \geq 4\,.$$

Use differentiation to show that

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}\,.$$

[ ], [proof]

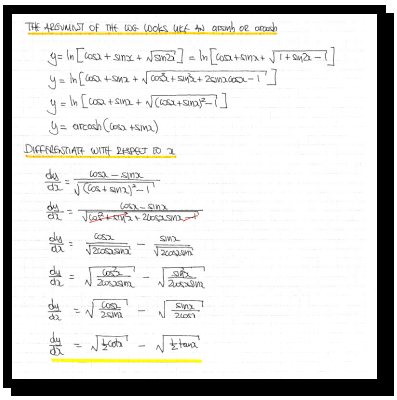


Question 83 **(*****)**

Use inverse hyperbolic functions to show that

$$\frac{d}{dx}\left[\ln\left(\cos x+\sin x+\sqrt{\sin 2x}\right)\right] \equiv \sqrt{\frac{1}{2}\cot x}-\sqrt{\frac{1}{2}\tan x}\,.$$

, proof

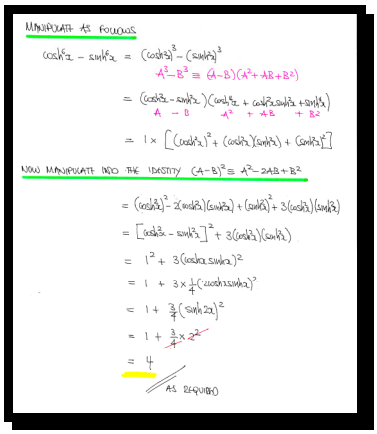


Question 84 **(*****)**

Show, with detailed workings, that

$$\sinh 2x=2 \quad \Rightarrow \quad \cosh^6 x-\sinh^6 x=4$$

, proof

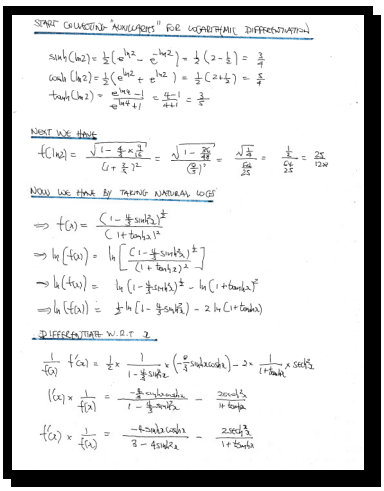


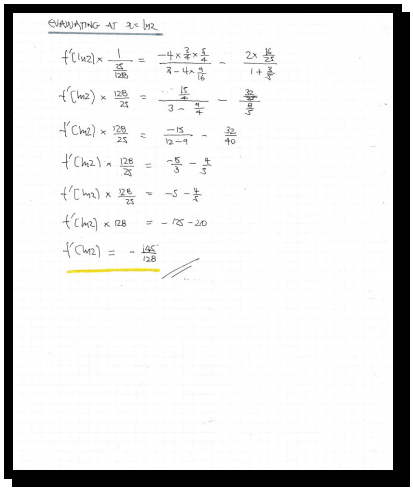
Question 85 (*****)

$$f(x)\equiv\frac{\sqrt{1-\frac{4}{3}\sinh^2 x}}{(1+\tanh x)^2}.$$

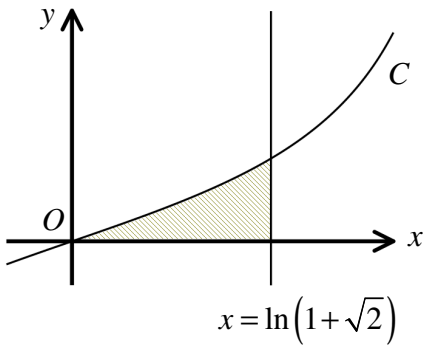
Determine the value of $f'(\ln 2)$.

V, [ ], $\boxed{f'(\ln 2)=-\frac{145}{256}}$





Question 86 **(*****)**

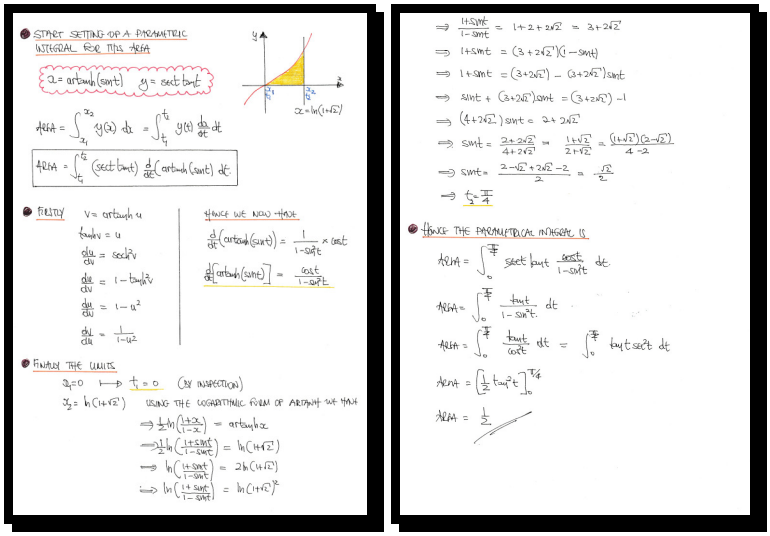


The figure above shows the curve C whose parametric equations are

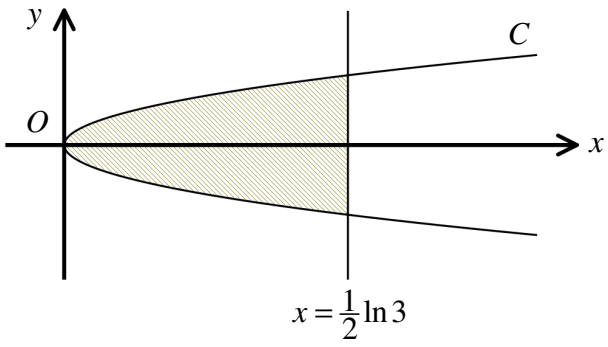
$$x=\operatorname{artanh}\left(\sin t\right),\quad y=\sec t\,\tan t,\quad -\tfrac{1}{2}\pi<t<\tfrac{1}{2}\pi.$$

Find the area of the finite region bounded by the x axis, the curve and the straight line with equation $x=\ln\left(1+\sqrt{2}\right)$.

$\square$, $\boxed{\text{area}=\frac{1}{2}}$



Question 87 **(*****)**



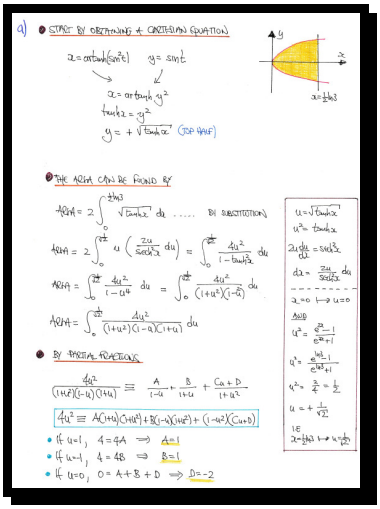
The figure above shows the curve C whose parametric equations are

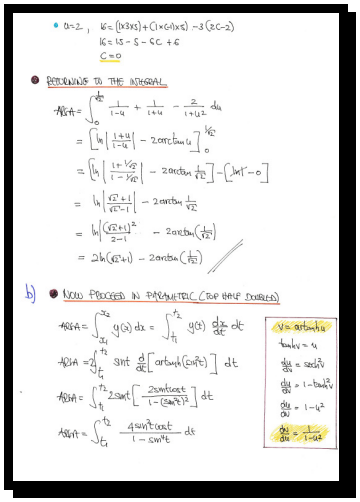
$$x=\operatorname{artanh}\left(\sin^2 t\right),\quad y=\sin t,\quad -\tfrac{1}{2}\pi<t<\tfrac{1}{2}\pi.$$

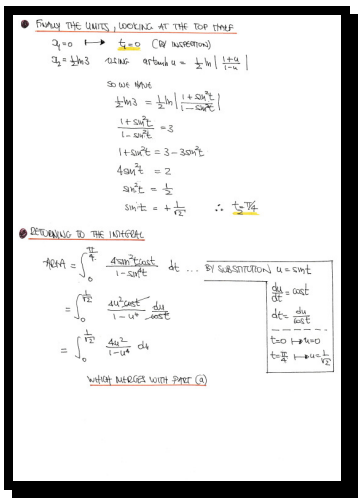
a) Use integration in Cartesian coordinates to find the exact area of the finite region bounded by the curve and the straight line with equation $x=\frac{1}{2}\ln 3$.

b) Use integration in parametric to verify the validity of the result of part **(a)**.

$\square$, $\boxed{\text{area}=2\ln\left(1+\sqrt{2}\right)-2\arctan\left(\frac{1}{\sqrt{2}}\right)}$







Question 88 **(*****)**

Given that p and q are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\operatorname{artanh}\left(x^2\right)$ useful in this question.

$\boxed{}$, proof

