DIFFERENTIATION

from first principles

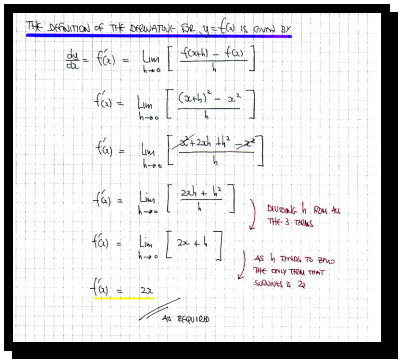
Question 1 (**)

$$f(x)=x^2,\ x\in\mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=2x.$$

[ ], [proof]



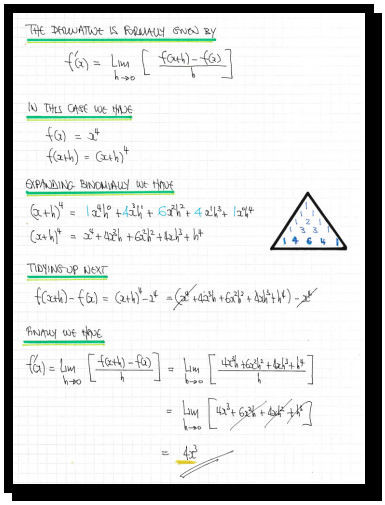
Question 2 (**)

$$f(x)=x^4,\ x\in\mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=4x^3.$$

[ ], [proof]



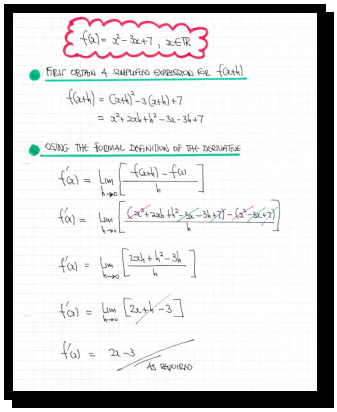
Question 3 (**)

$$f(x)=x^2-3x+7\,,\ x\in\mathbb{R}\,.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=2x-3\,.$$

$\boxed{}$, proof



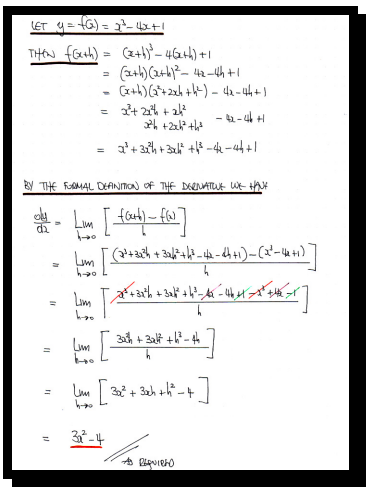
Question 4 (+)**

$$y = x^3 - 4x + 1, \ x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$\frac{dy}{dx} = 3x^2 - 4.$$

[ ], [proof]



Question 5 **(**+)**

$$f(x)=x^3+2,\ x\in\mathbb{R}.$$

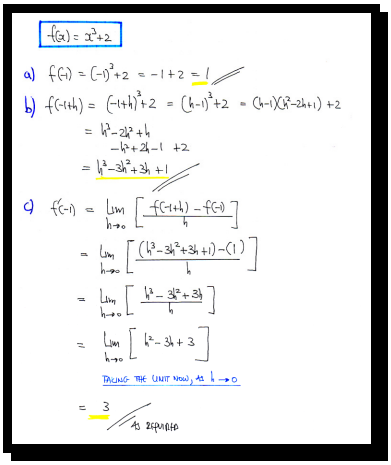
a) State the value of $f(-1)$.

b) Find a simplified expression for $f(-1+h)$.

c) Use the formal definition of the derivative as a limit, to show that

$$f'(-1)=3.$$

$\boxed{}$, $\boxed{f(-1)=1}$, $\boxed{f(-1+h)=1+3h-3h^2+h^3}$



Question 6 **(***)**

$$f(x)=x^4-4x\,,\ x\in\mathbb{R}\,.$$

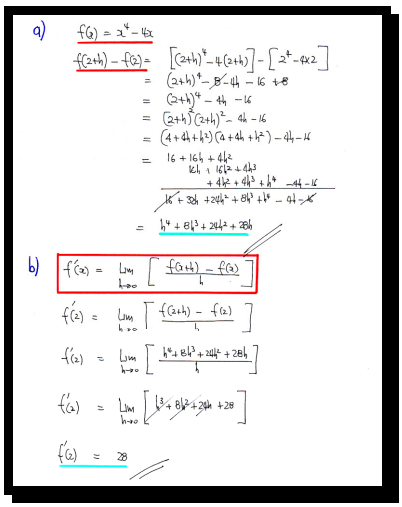
a) Find a simplified expression for

$$f(2+h)-f(2)\,.$$

b) Use the formal definition of the derivative as a limit, to show that

$$f'(2)=28\,.$$

$\square$, $\boxed{f(2+h)-f(2)=28h+24h^2+8h^3+h^4}$



Question 7 (***)

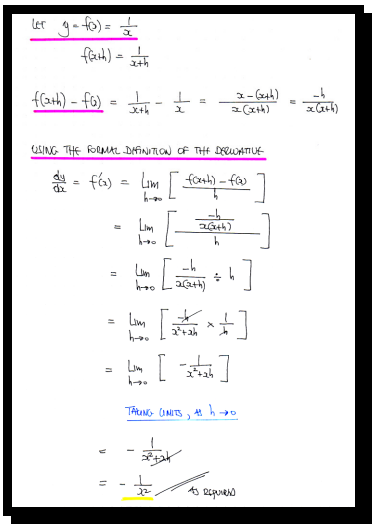
A reciprocal curve has equation

$$y=\frac{1}{x},\ x\neq 0\,.$$

Use the formal definition of the derivative as a limit, to show that

$$\frac{dy}{dx}=-\frac{1}{x^2}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

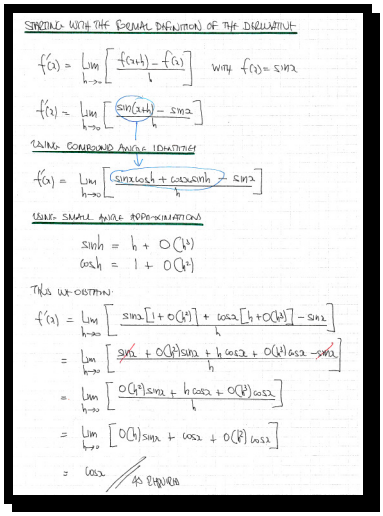


Question 8 **(***)**

$$\frac{d}{dx}(\sin x)=\cos x\,.$$

Prove by first principles the validity of the above result by using the small angle approximations for $\sin x$ and $\cos x$.

[ ], [proof]



Question 9 **(***)**

If x is in radians

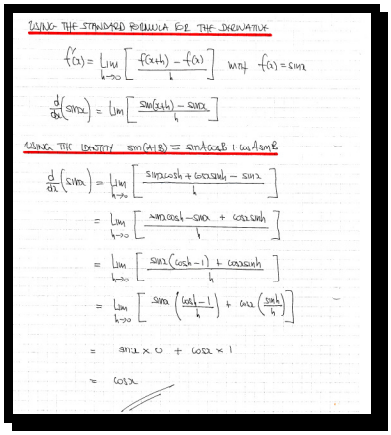
$$\frac{d}{dx}(\sin x)=\cos x\,.$$

Prove the validity of the above result from first principles.

You may assume that if h is small and measured in radians, then as $h\to\infty$

$$\frac{\cos(h)-1}{h}\to 0 \quad \text{and} \quad \frac{\sin(h)}{h}\to 1\,.$$

[ ], [proof]



Question 10 **(***+)**

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

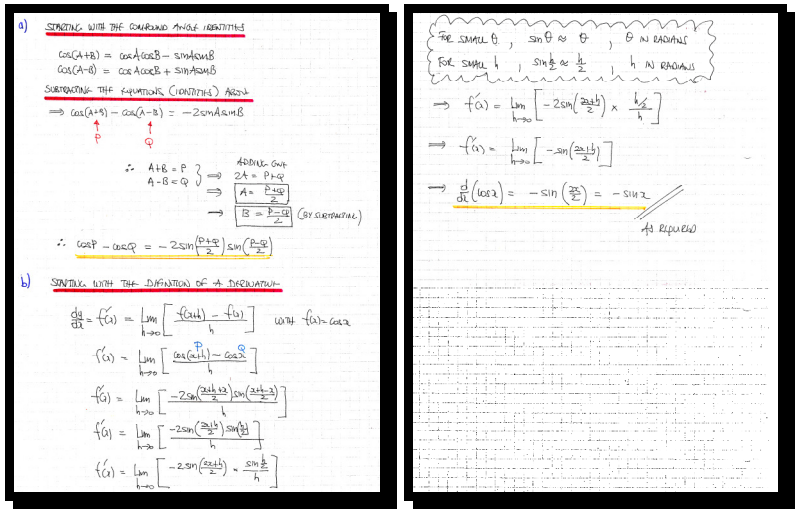
a) By using the above identities show that

$$\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right).$$

b) Hence, prove by first principles that

$$\frac{d}{dx}(\cos x) = -\sin x$$

, proof

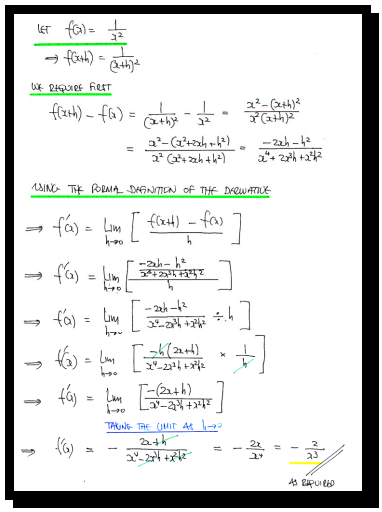


Question 11 (*+)**

Differentiate from first principles

$$\frac{1}{x^2},\ x\neq 0.$$

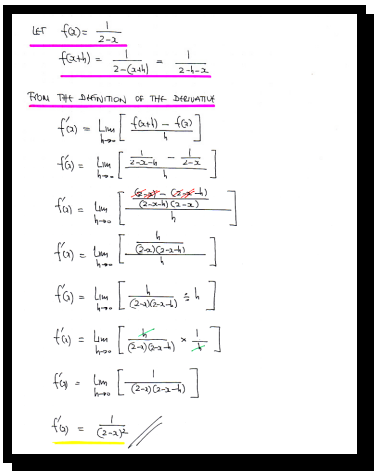
$$\boxed{},\ \boxed{-\frac{2}{x^3}}$$



Question 12 (*+)**

Differentiate $\dfrac{1}{2-x}$ from first principles.

$\square$, $\boxed{\dfrac{d}{dx}\left(\dfrac{1}{2-x}\right)=\dfrac{1}{(2-x)^2}}$



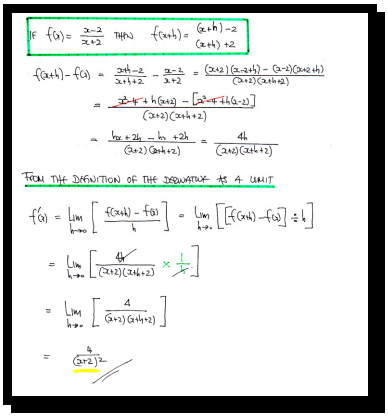
Question 13 **(***+)**

$$f(x)=\frac{x-2}{x+2},\ x\in\mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=\frac{4}{(x+2)^2}.$$

[ ], [proof]

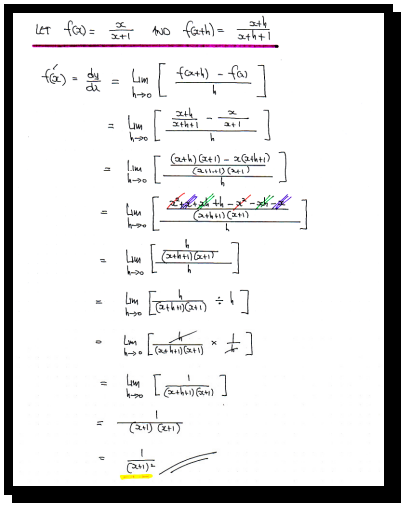


Question 14 (*+)**

Differentiate from first principles

$$\frac{x}{x+1},\ x\neq -1.$$

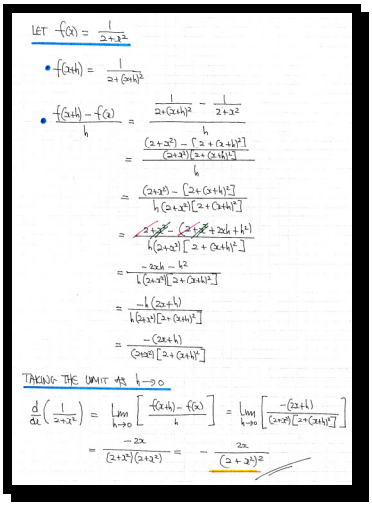
$$\boxed{},\ \boxed{\frac{1}{(x+1)^2}}$$



Question 15 **(***+)**

Differentiate $\dfrac{1}{2+x^2}$ from first principles.

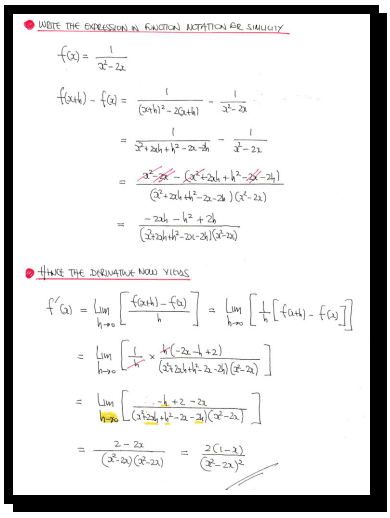
$\square$, $\boxed{\dfrac{d}{dx}\left(\dfrac{1}{2+x^2}\right)=-\dfrac{2x}{\left(2+x^2\right)^2}}$



Question 16 (*+)**

Differentiate $\dfrac{1}{x^2-2x}$ from first principles.

$\square$, $\boxed{\dfrac{d}{dx}\left(\dfrac{1}{x^2-2x}\right)=\dfrac{2(1-x)}{\left(x^2-2x\right)^2}}$



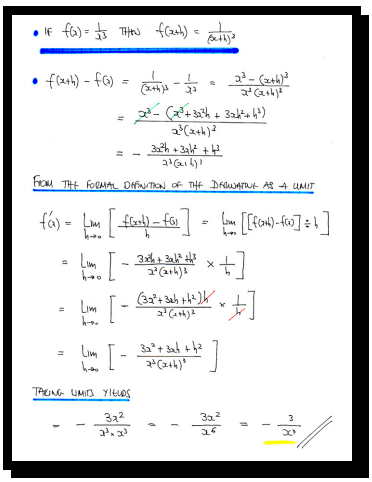
Question 17 **(***+)**

$$f(x)=\frac{1}{x^3},\ x\in\mathbb{R},\ x\neq 0.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{3}{x^4}.$$

$\boxed{}$, $\boxed{\text{proof}}$



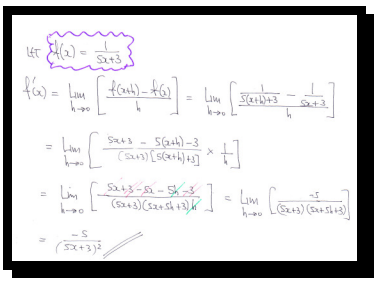
Question 18 **(***+)**

$$f(x)=\frac{1}{5x+3},\ x\in\mathbb{R},\ x\neq-\frac{3}{5}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{5}{(5x+3)^2}.$$

proof



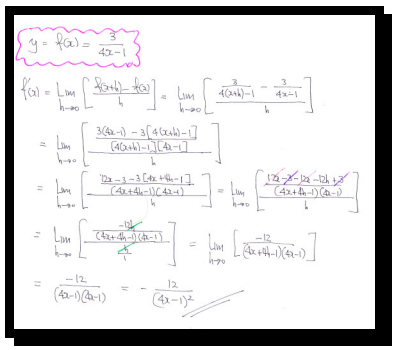
Question 19 **(***+)**

$$f(x)=\frac{3}{4x-1},\ x\in\mathbb{R},\ x\neq-\frac{1}{4}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{12}{(4x-1)^2}.$$

proof



Question 20 (*+)**

$$\frac{d}{dx}(\sin x)=\cos x\,.$$

Prove the validity of the above result by …

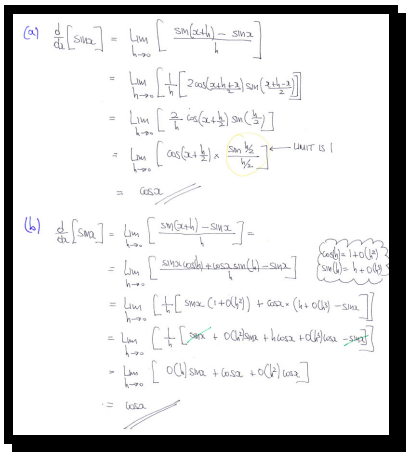
a) … using $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$

and the trigonometric identity

$$\sin A-\sin B\equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\,.$$

b) … using small angle approximations for $\sin x$ and $\cos x$.

proof



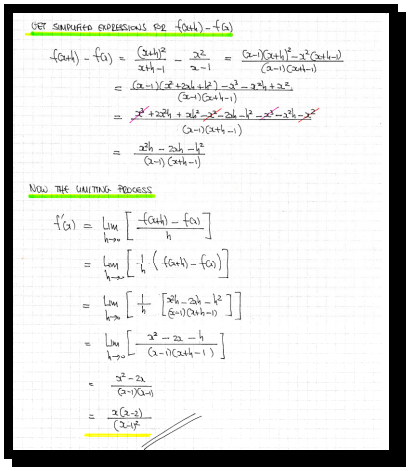
Question 21 (*+)**

$$f(x)\equiv\frac{x^2}{x-1},\ x\in\mathbb{R},\ x\neq-\frac{1}{4}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=\frac{x(x-2)}{(x-1)^2}.$$

[ ], [proof]

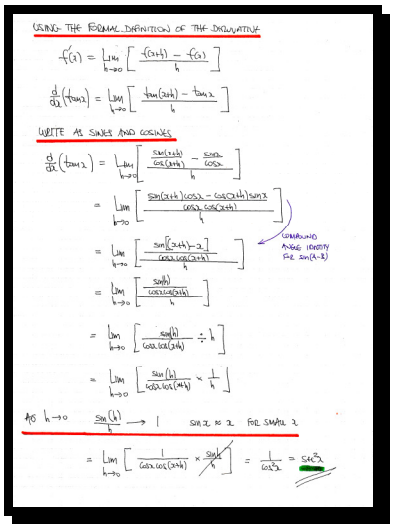


Question 22 **(****)**

Prove by first principles, and by using the small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}(\tan x)=\sec^2 x\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

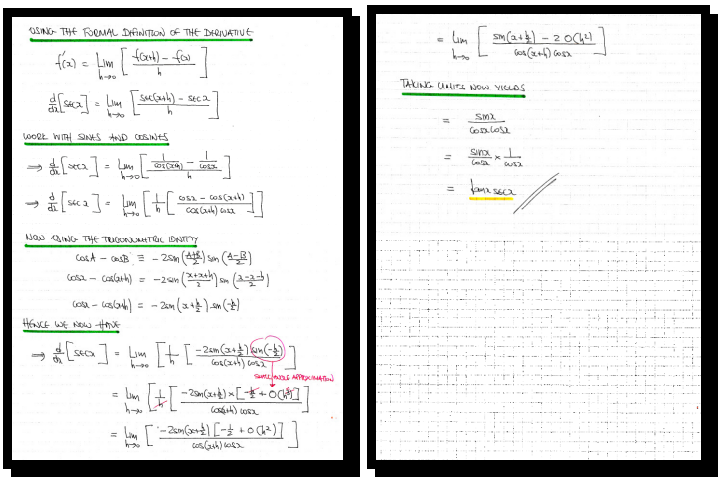


Question 23 **(****)**

Prove by first principles, and by using the small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}(\sec x) = \sec x \tan x .$$

[ ], [proof]



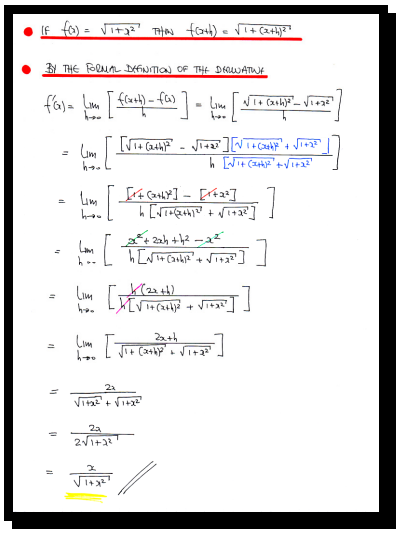
Question 24 **(****+)**

$$f(x)=\sqrt{1+x^2}\ ,\ x\in\mathbb{R}\ .$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=\frac{x}{\sqrt{1+x^2}}\ .$$

[ ] , [proof]



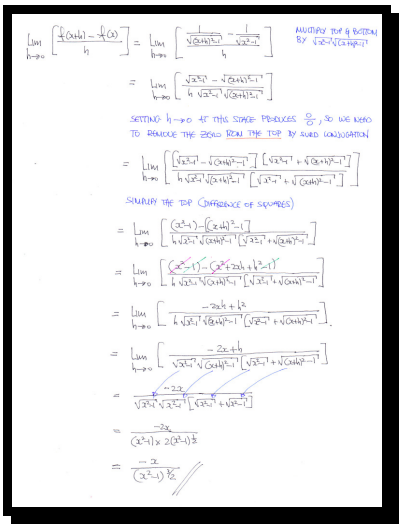
Question 25 **(****+)**

$$f(x)=\frac{1}{\sqrt{x^2-1}},\ x\in\mathbb{R},\ |x|>1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{x}{\left(x^2-1\right)^{\frac{3}{2}}}.$$

proof



Question 26 **(****+)**

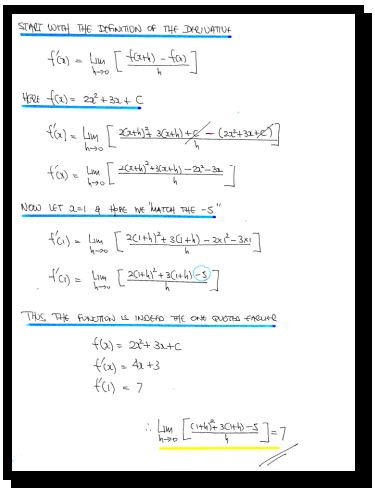
The limit expression shown below represents a student's evaluation for $f'(x)$, for a specific value of x.

$$\lim_{h\to 0}\left[\frac{2(1+h)^2+3(1+h)-5}{h}\right].$$

Determine an expression for $f(x)$ and once obtained, **differentiate it directly** to find the value of $f'(x)$, for the specific value of x the student was evaluating.

No credit will be given for evaluating the limit directly.

$\boxed{}$, $\boxed{f(x)=2x^2+3x}$, $\boxed{f'(1)=7}$



Question 27 **(*****)**

Use the formal definition of the derivative to prove that if

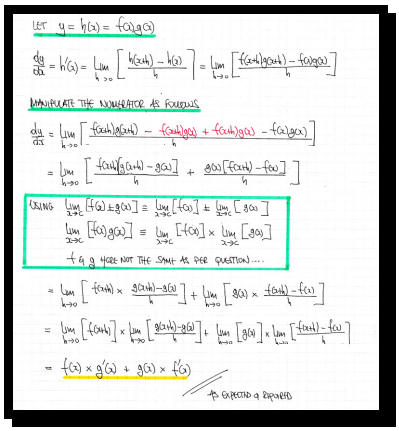
$$y = f(x)\, g(x),$$

then $\dfrac{dy}{dx} = f'(x)\, g(x) + f(x)\, g'(x)$

You may assume that

- $\lim\limits_{x \to c}\left[f(x) + g(x) \right] = \lim\limits_{x \to c}\left[f(x) \right] + \lim\limits_{x \to c}\left[g(x) \right]$
- $\lim\limits_{x \to c}\left[f(x) \times g(x) \right] = \lim\limits_{x \to c}\left[f(x) \right] \times \lim\limits_{x \to c}\left[g(x) \right]$

V, ☐, proof



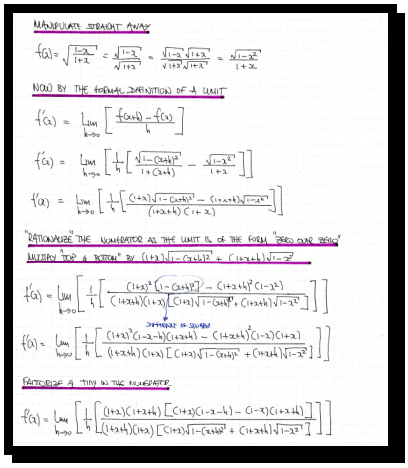
Question 28 **(*****)**

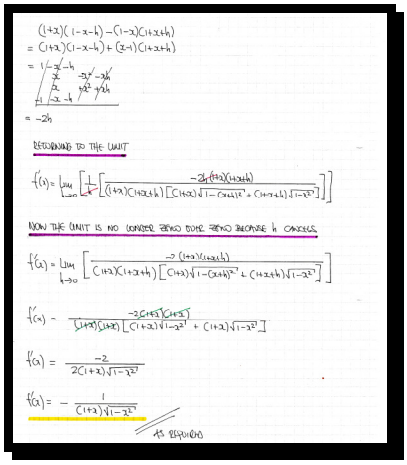
$$f(x)=\sqrt{\frac{1-x}{1+x}},\ x\in\mathbb{R},\ |x|<1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{1}{(1+x)\sqrt{1-x^2}}.$$

[ ], [proof]





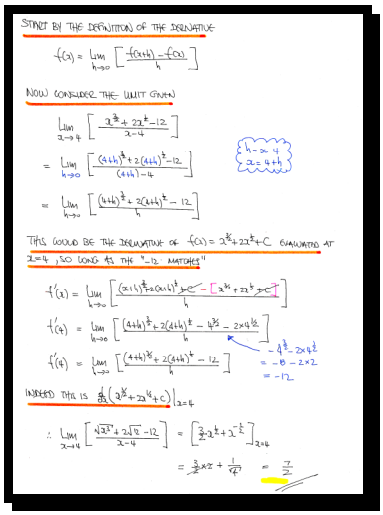
Question 29 **(*****)**

Use the formal definition of the derivative of a suitable expression, to find the value for the following limit

$$\lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right].$$

No credit will be given for using L'Hospital's rule.

$\boxed{}$, $\boxed{\frac{7}{2}}$



Created by T. Madas