CONIC SECTIONS

CIRCLE

Question 1 (*+)**

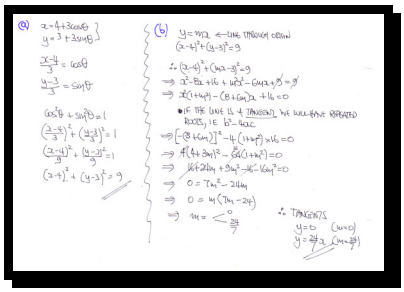
A circle is given parametrically by the equations

$$x = 4 + 3\cos\theta, \ y = 3 + 3\sin\theta, \ 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation for the circle.

b) Find the equations of the two tangents to the circle, which pass through the origin O.

$\boxed{(x-4)^2 + (y-3)^2 = 9}$, $\boxed{y = 0 \quad \text{and} \quad y = \frac{24}{7}x}$



Question 2 **(****+)**

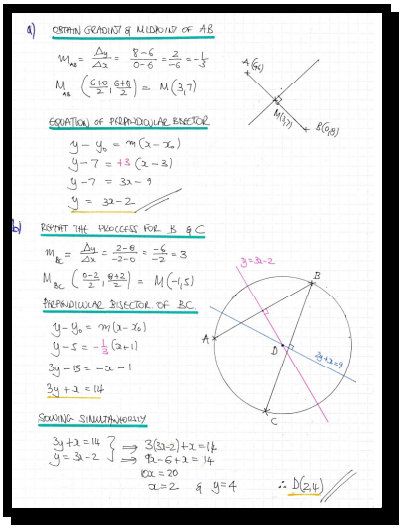
The points A, B and C have coordinates $(6,6)$, $(0,8)$ and $(-2,2)$, respectively.

a) Find an equation of the perpendicular bisector of AB.

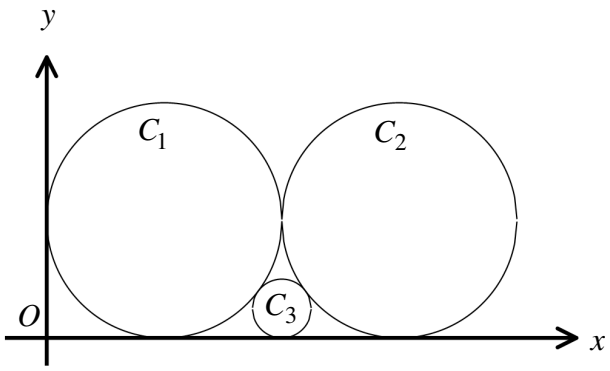
The points A, B and C lie on the circumference of a circle whose centre is located at the point D.

b) Determine the coordinates of D.

$\boxed{}$, $\boxed{y=3x-2}$, $\boxed{D(2,4)}$



Question 3 (**+)**



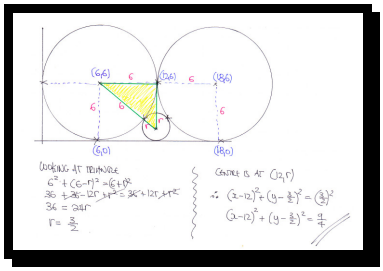
The figure above shows three circles C_1, C_2 and C_3.

The coordinates of the centres of all three circles are positive.

- The circle C_1 has centre at $(6,6)$ and **touches** both the x axis and the y axis.
- The circle C_2 has the same size radius as C_1 and **touches** the x axis.
- The circle C_3 **touches** the x axis and **both** C_1 and C_2.

Determine an equation of C_3.

$\square$, $\boxed{(x-12)^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4}}$



Question 4 (**+)**

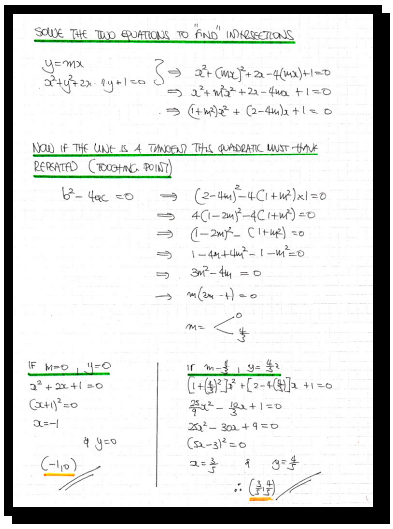
A circle C has equation

$$x^2+y^2+2x-4y+1=0$$

The straight line L with equation $y=mx$ is a tangent to C.

Find the possible values of m and hence determine the possible coordinates at which L meets C.

$\boxed{}$, $\boxed{m=0,\ m=\frac{4}{3}}$, $\boxed{(-1,0),\ \left(\frac{3}{5},\frac{4}{5}\right)}$



Question 5 (**+)**

A circle C has equation

$$x^2+y^2+4x-10y+9=0\,.$$

a) Find the coordinates of the centre of C and the size of its radius.

A tangent to the circle T, passes through the point with coordinates $(0,-1)$ and has gradient m, where $m<0$.

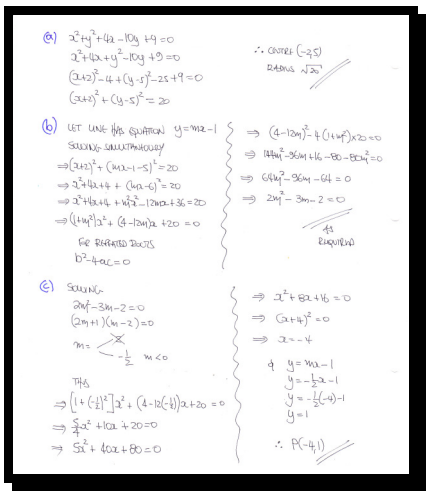
b) Show that m is a solution of the equation

$$2m^2-3m-2=0\,.$$

The tangent T meets C at the point P.

c) Determine the coordinates of P.

$\boxed{(-2,5),\ r=\sqrt{20}}$, $\boxed{P(-4,1)}$



Question 6 **(****+)**

A circle has equation

$$x^2+y^2-4x-2y=13\,.$$

a) Find the coordinates of the centre of the circle and the size of its radius.

The points A and B lie on the circle such that the length of AB is 6 units.

b) Show that $\measuredangle ACB=90°$, where C is the centre of the circle.

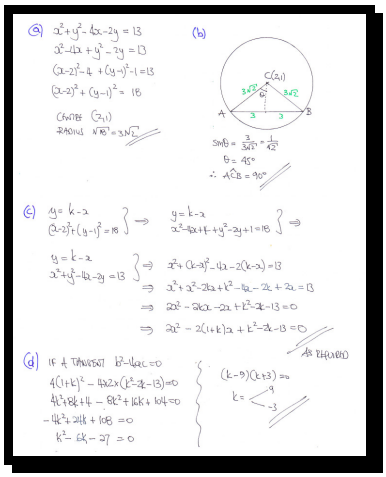
A tangent to the circle has equation $y=k-x$, where k is a constant.

c) Show clearly that

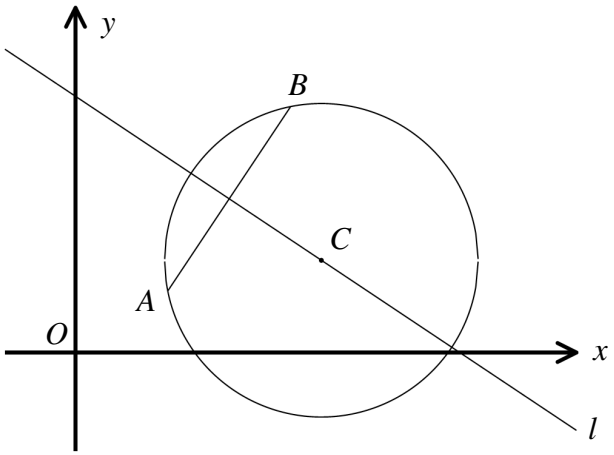
$$2x^2+2(1-k)x+k^2-2k-13=0\,.$$

d) Determine the possible values of k.

$\boxed{(2,1),\ r=\sqrt{18}}$, $\boxed{k=-3,\ 9}$



Question 7 (**+)**



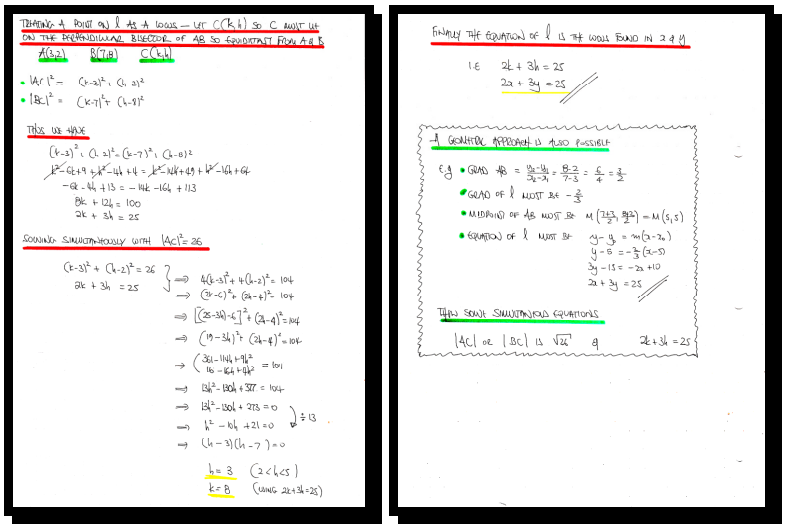
The figure above shows a circle whose centre is located at the point $C(k,h)$, where k and h are constants such that $2<h<5$.

The points $A(3,2)$ and $B(7,8)$ lie on this circle.

The straight line l passes through C and the midpoint of AB.

Given that the radius of the circle is $\sqrt{26}$, find an equation for l, the value of k and the value of h.

$\boxed{}$, $\boxed{2x+3y=25}$, $\boxed{k=8}$, $\boxed{h=3}$



Question 8 (**+)**

The straight line passing through the points $P(1,9)$ and $Q(5,5)$ is a tangent to a circle with centre at $C(6,8)$.

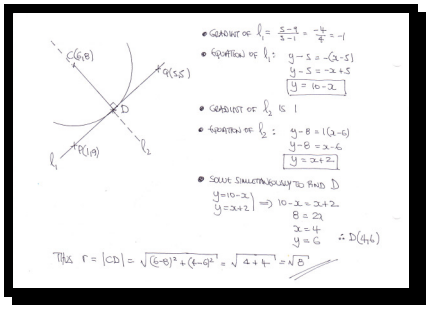
Determine, in exact surd form, the radius of the circle.

In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

… any form of calculus.

$$\boxed{r=\sqrt{8}}$$



Question 9 **(****+)**

The straight line with equation $y=2x-3$ is a tangent to a circle with centre at the point $C(2,-3)$.

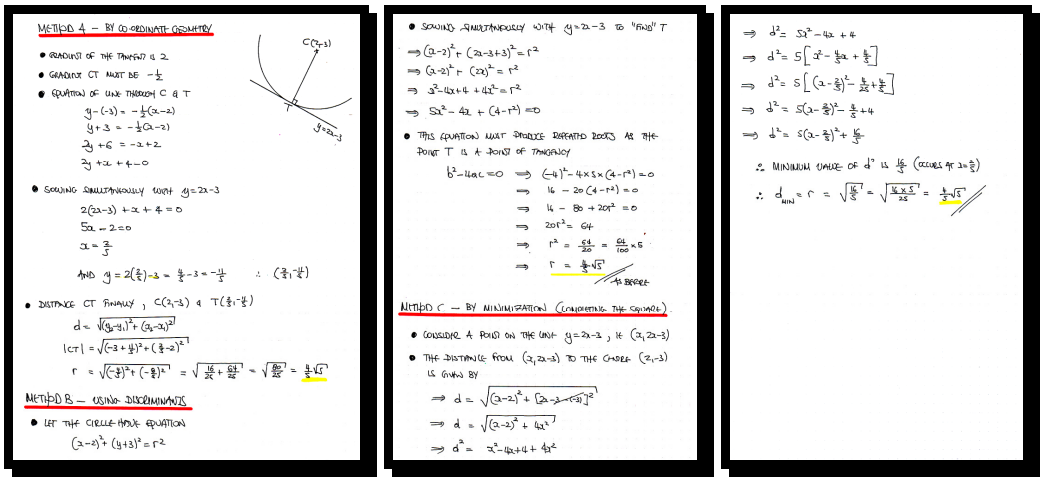
Determine, in exact surd form, the radius of the circle.

In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

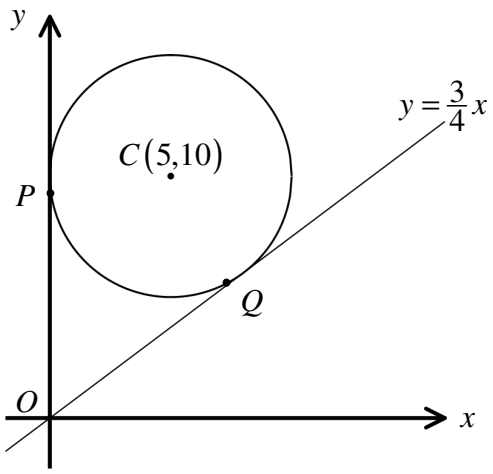
… any form of calculus.

$\boxed{}$, $\boxed{r=\frac{4}{5}\sqrt{5}}$



Question 10 (*****)

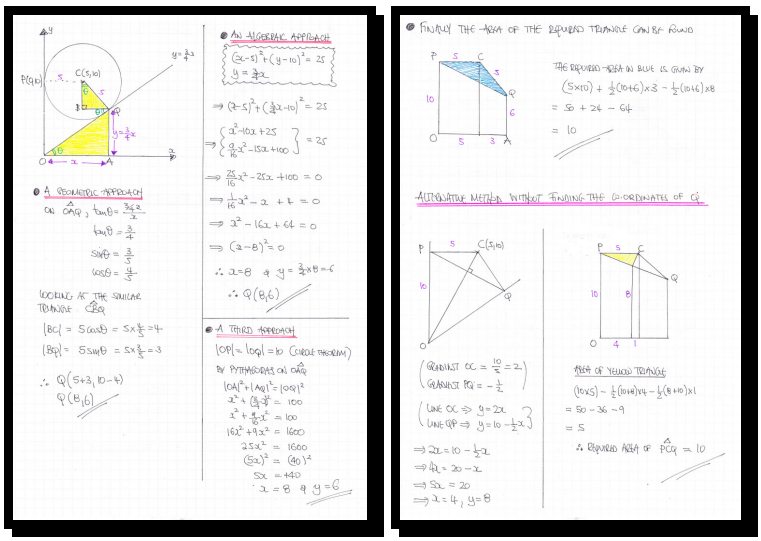
The figure below shows the circle with centre at $C(5,10)$ and radius 5.



The straight lines with equations, $x=0$ and $y=\frac{3}{4}x$ are tangents to the circle at the points P and Q respectively.

Show that the area of the triangle PCQ is 10 square units.

[ ], proof



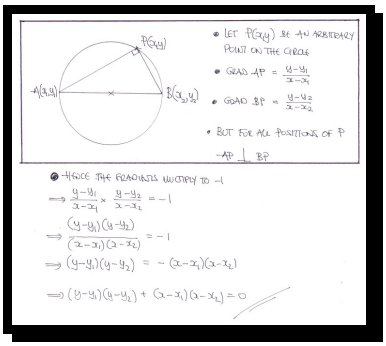
Question 11 (*****)

A circle passes through the points $A(x_1, y_1)$ and $A(x_2, y_2)$.

Given that AB is a diameter of the circle, show that the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

[ ], proof

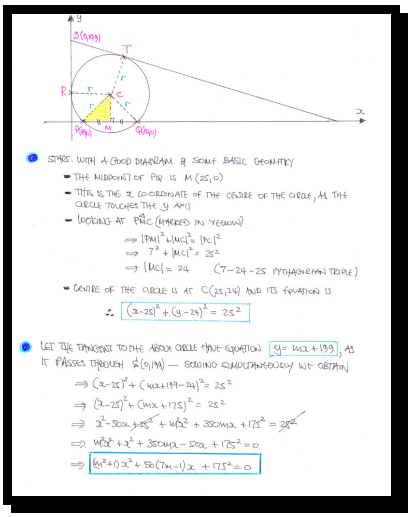


Question 12 (*****)

A circle passes through the points $P(18,0)$ and $Q(32,0)$. A tangent to this circle passes through the point $S(0,199)$ and touches the circle at the point T.

Given that the y axis is a tangent to this circle, determine the coordinates of T

[ ], $(49,31)$



Question 13 **(*****)**

The circle C_1 has equation

$$x^2+y^2-4x-4y+6=0\,.$$

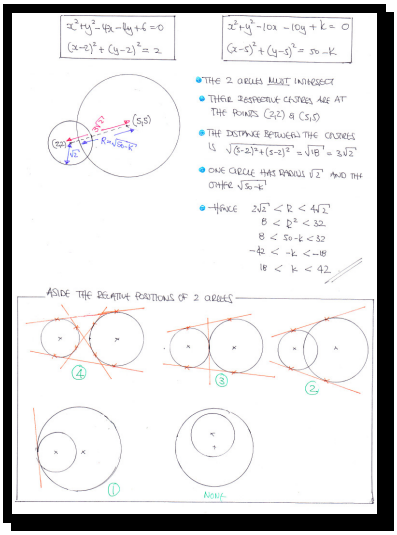
The circle C_2 has equation

$$x^2+y^2-10x-10y+k=0\,,$$

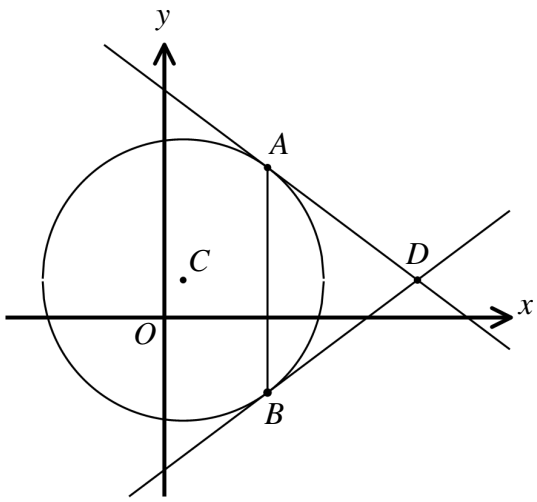
where k is a constant.

Given that C_1 and C_2 have exactly two common tangents, determine the range of possible values of k.

$\boxed{}$, $\boxed{18<k<42}$



Question 14 (*****) non calculator



The figure above shows the circle with equation

$$x^2+y^2-4x-8y=205,$$

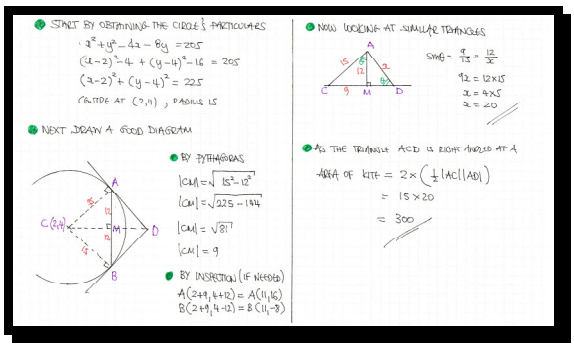
with centre at the point C and radius r.

The straight line AB is parallel to the y axis and has length 24 units.

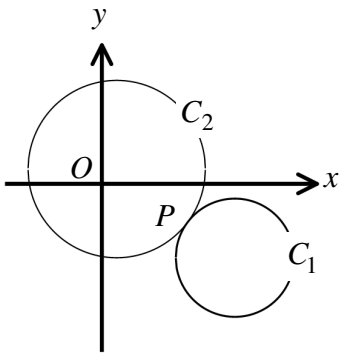
The tangents to the circle at A and B meet at the point D.

Find the length of AD and hence deduce the area of the kite $CADB$.

$\boxed{}$, $\boxed{|AD|=20}$, $\boxed{\text{area}=300}$



Question 15 **(*****)**



The figure above shows a circle C_1 with equation

$$x^2+y^2-18x+ky+90=0,$$

where k is a positive constant.

a) Determine, in terms of k, the coordinates of the centre of C_1 and the size of its radius.

Another circle C_2 has equation

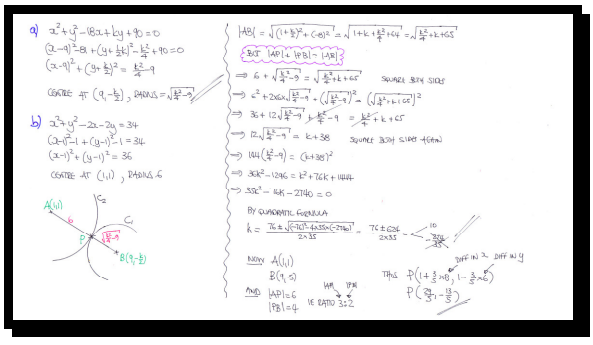
$$x^2+y^2-2x-2y=34.$$

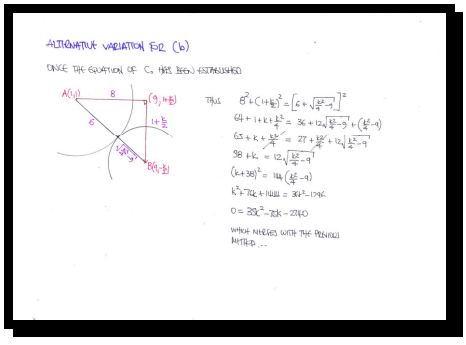
b) Given that C_1 and C_2 are **touching externally** at the point P, find …

i. … the value of k.

ii. … the coordinates of P.

$\square$, $\boxed{\left(9,-\frac{1}{2}k\right),\ r=\sqrt{\frac{k^2}{4}-9}}$, $\boxed{k=10}$, $\boxed{P\left(\frac{29}{5},-\frac{13}{5}\right)}$





Question 16 (*****)

The curve C has equation

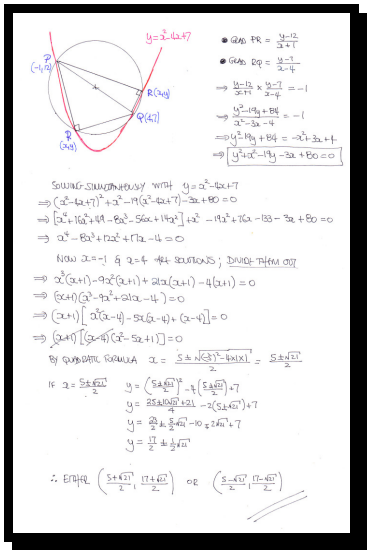
$$y = x^2 - 4x + 7 .$$

The points $P(-1,12)$ and $Q(4,7)$ lie on C.

The point R also lies on C so that $\angle PRQ = 90°$.

Determine, as exact surds, the possible coordinates of R.

$$\square, \boxed{\left(\frac{5+\sqrt{21}}{2}, \frac{17+\sqrt{21}}{2}\right) \text{ or } \left(\frac{5-\sqrt{21}}{2}, \frac{17-\sqrt{21}}{2}\right)}$$



Question 17 **(*****)**

A circle C is centred at (a,a) and has radius a, where a is a positive constant.

The straight line L has equation

$$4x-3y+4=0.$$

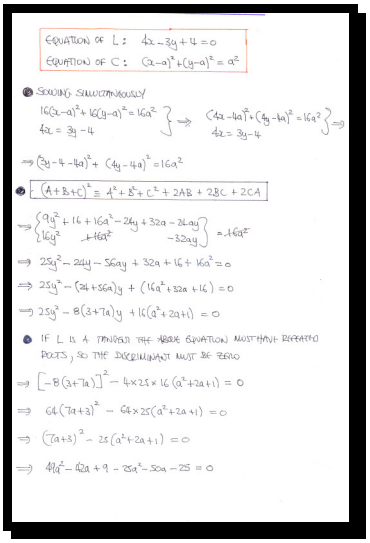
Given that L is tangent to C at the point P, determine …

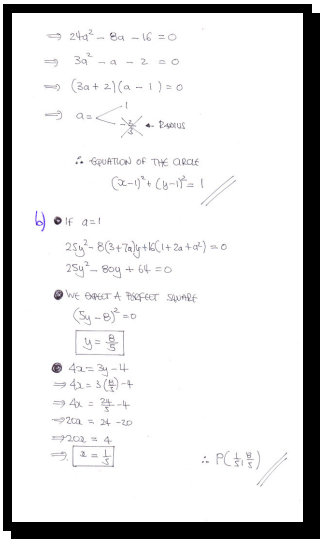
a) … an equation of C.

b) … the coordinates of P.

*You may **not** use a formula which determines the shortest distance of a point from a straight line in this question.*

$\boxed{}$, $\boxed{(x-1)^2+(y-1)^2=1}$, $\boxed{P\left(\frac{1}{5},\frac{8}{5}\right)}$





Question 18 (*****)

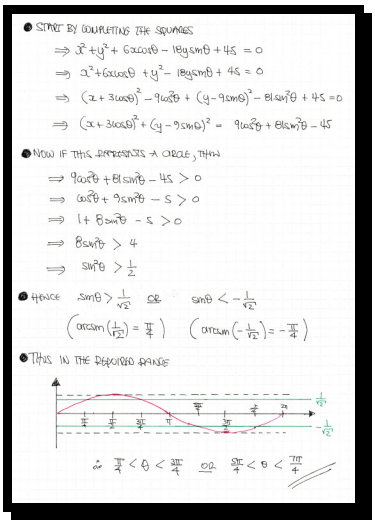
A curve in the x-y plane has equation

$$x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0,$$

where θ is a parameter such that $0 \le \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ.

$\square$, $\left\{\frac{1}{4}\pi < \theta < \frac{3}{4}\pi\right\} \cup \left\{\frac{5}{4}\pi < \theta < \frac{7}{4}\pi\right\}$



Question 19 **(*****)**

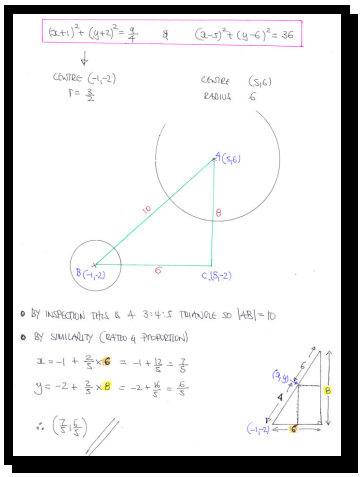
The circles C_1 and C_2 have respective equations

$$(x+1)^2+(y+2)^2=\frac{9}{4} \qquad \text{and} \qquad (x-5)^2+(y-6)^2=36.$$

The point P lies on C_2 so that the distance of P from C_1 is least.

Determine the exact coordinates of P.

$\square$, $\boxed{P\left(\frac{7}{5},\frac{6}{5}\right)}$



Question 20 (*****)

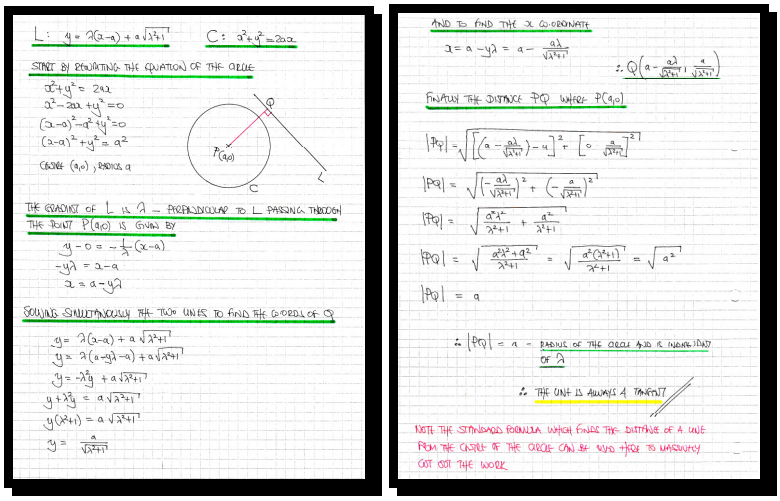
The straight line L and the circle C, have respective equations

$$L:\ y=\lambda(x-a)+a\sqrt{\lambda^2+1} \quad \text{and} \quad C:\ x^2+y^2=2ax,$$

where a is a positive constant and λ is a parameter.

Show that for all values of λ, L is a tangent to C.

$\boxed{}$, proof



Question 21 (*****)

The straight line with equation

$$y=t(x-2),$$

where t is a parameter,

crosses the circle with equation

$$x^2+y^2=1$$

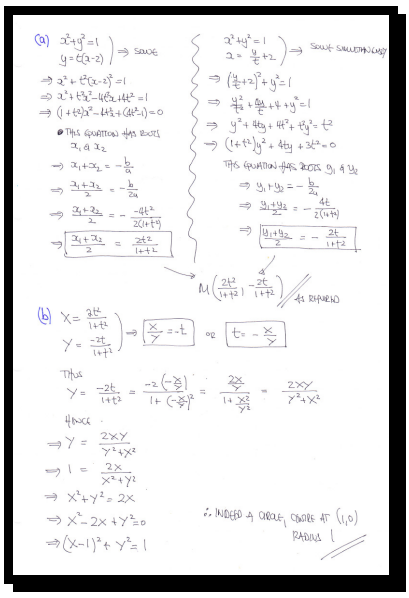
at two distinct points A and B.

a) Show that the coordinates of the midpoint of AB are given by

$$M\left(\frac{2t^2}{1+t^2},-\frac{2t}{1+t^2}\right).$$

b) Hence show that the locus of M as t varies is a circle, stating its radius and the coordinates of its centre.

$$\boxed{(x-1)^2+y^2=1}$$



Question 22 **(*****)**

Two parallel straight lines, L_1 and L_2, have respective equations

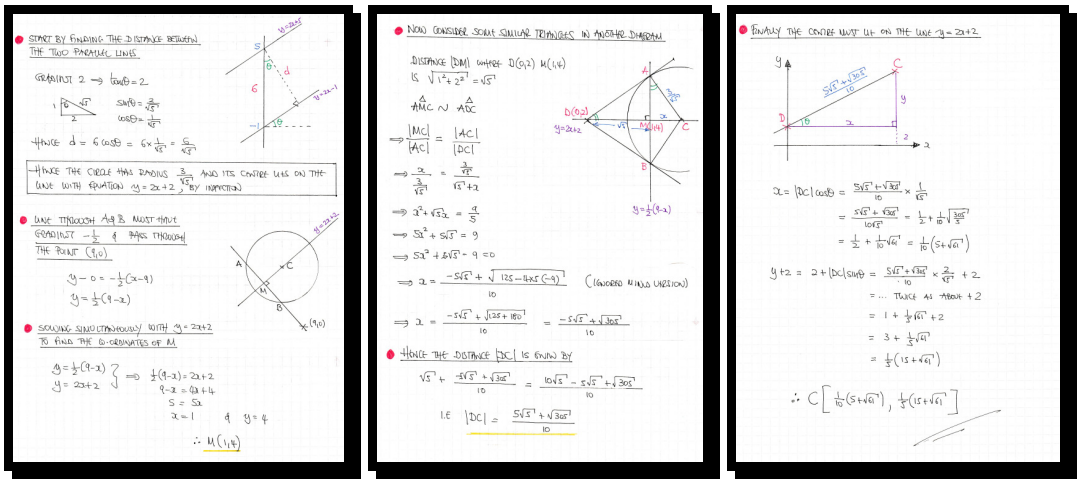
$$y=2x+5 \qquad \text{and} \qquad y=2x-1 \ .$$

L_1 and L_2, are tangents to a circle centred at the point C .

A third line L_3 is perpendicular to L_1 and L_2, and meets the circle in two distinct points, A and B.

Given that L_3 passes through the point $(9,0)$, find, in exact simplified surd form, the coordinates of C.

$$\boxed{}, \quad \boxed{C\left[\frac{1}{10}\left(5+\sqrt{61}\right), \frac{1}{5}\left(15+\sqrt{61}\right)\right]}$$



Question 23 (*****)

Two circles, C_1 and C_2, have respective radii of 4 units and 1 unit and are touching each other externally at the point A.

The coordinates axes are tangents to C_1, whose centre P lies in the first quadrant.

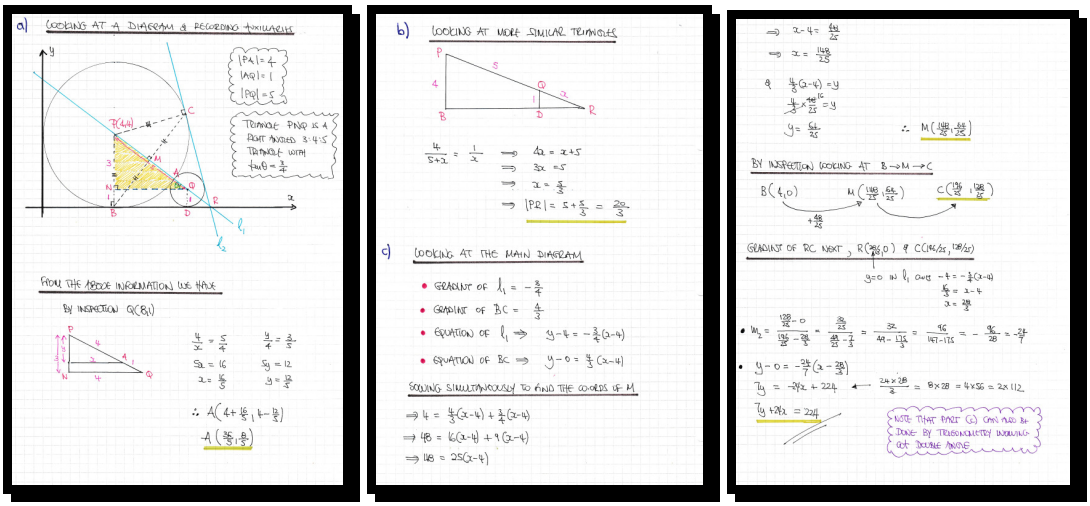
The x axis is a tangent to C_2, whose centre Q also lies in the first quadrant.

The straight line l_1, passes through P and Q, and meets the x axis at the point R.

The straight line l_2 has negative gradient, passes through R and is a common tangent to C_1 and C_2.

Determine, in any order and in exact form where appropriate, the coordinates of A, the length of PR and an equation of l_2.

$$\boxed{}\,,\quad \boxed{A\left(\frac{36}{5},\frac{8}{5}\right)}\,,\quad \boxed{|PR|=\frac{20}{3}}\,,\quad \boxed{24x+7y=224}$$



Question 24 (*****)

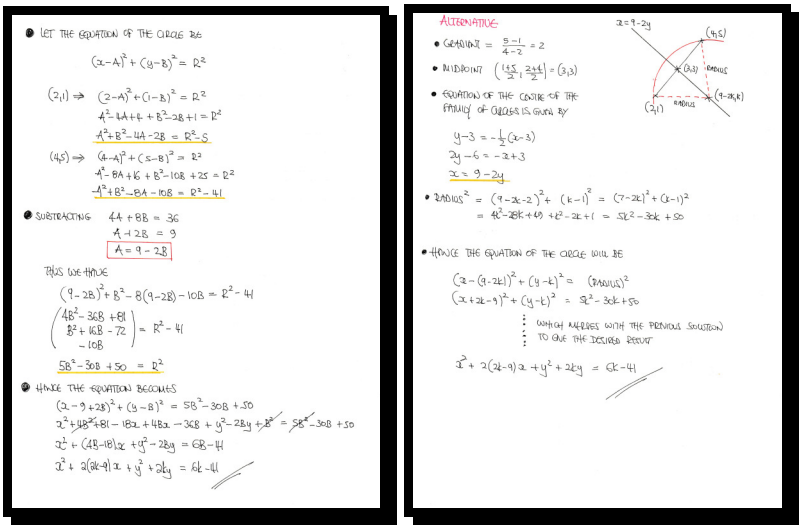
A family of circles is passing through the points with coordinates $(2,1)$ and $(4,5)$

Show that the equation of every such circle has equation

$$x^2+y^2+2x(2k-9)+2ky=6k-41,$$

where k is a parameter.

, proof



Question 25 (*****)

Three circles, C_1, C_2 and C_3, have their centres at A, B and C, respectively, so that $|AB|=5$, $|AC|=4$ and $|BC|=3$.

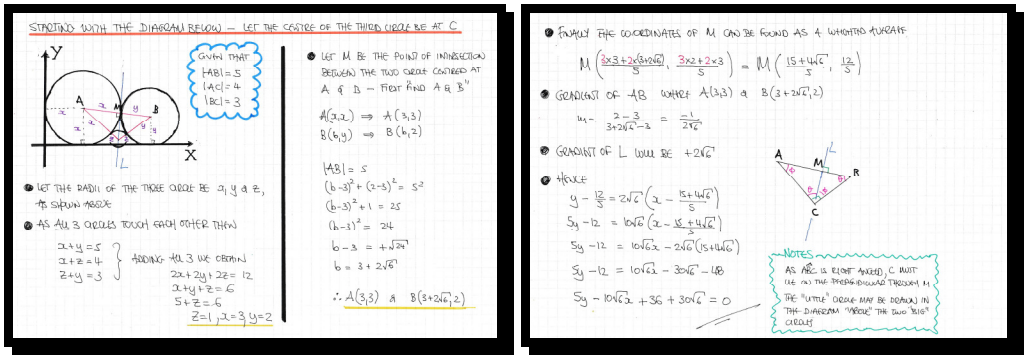
The positive x and y axis are tangents to C_1.

The positive x axis is a tangent to C_2.

C_1 and C_2 touch each other externally at the point M.

Given further that C_3 touches externally both C_1 and C_2, find, in exact simplified form, an equation of the straight line which passes through M and C.

$\boxed{}$, $\boxed{5y-10\sqrt{6}\,x+36+30\sqrt{6}=0}$



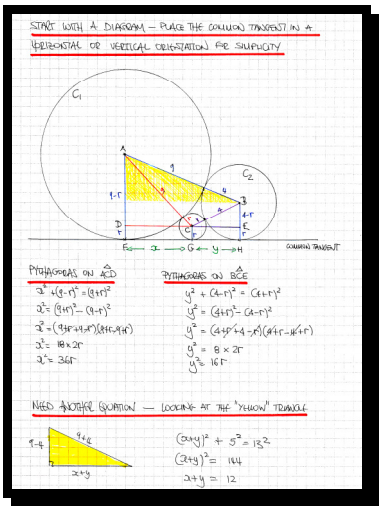
Question 26 (*****)

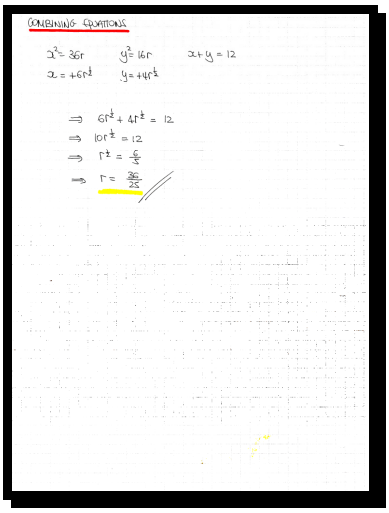
Two circles, C_1 and C_2, are touching each other **externally**, and have respective radii of 9 and 4 units.

A third circle C_3, of radius r, touches C_1 **and** C_2 **externally**.

Given further that all three circles have a common tangent, determine the value of r.

$\square$, $\boxed{r=\dfrac{36}{25}=1.44}$





Question 27 **(*****)**

The point $A(6,-1)$ lies on the circle with equation

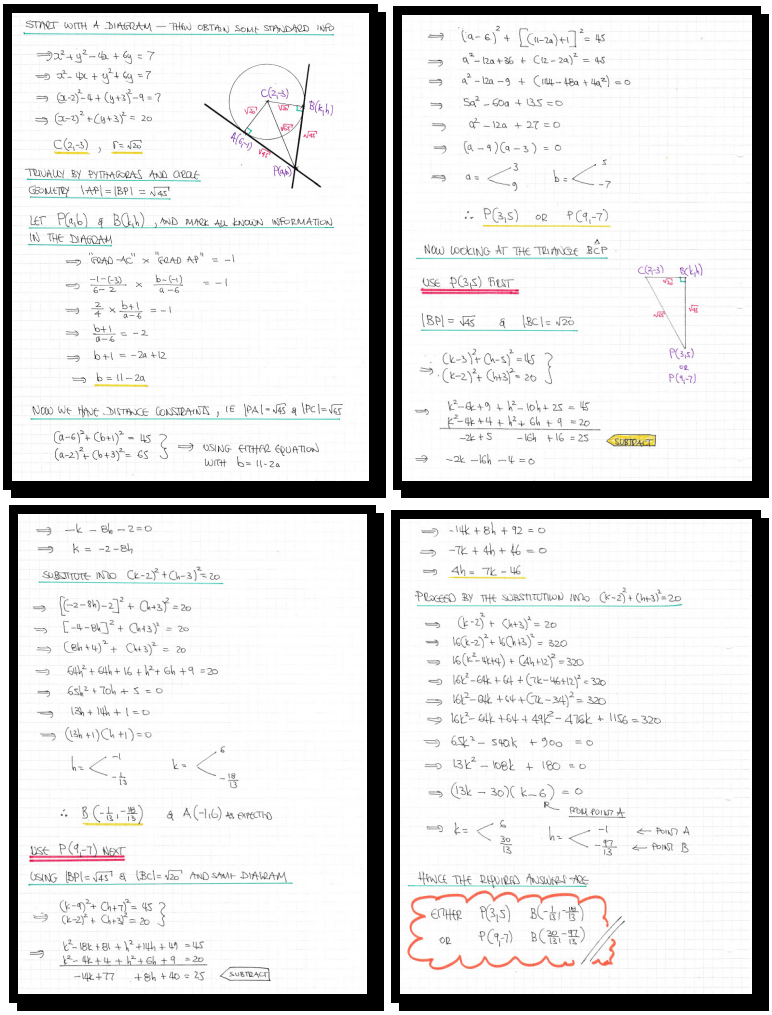
$$x^2+y^2-4x+6y=7.$$

The tangent to the circle at A passes through the point P, so that the distance of P from the centre of the circle is $\sqrt{65}$.

Another tangent to the circle, at some point B, also passes through P.

Determine in any order the two sets of the possible coordinates of P and B.

$$\boxed{\quad}, \quad \boxed{P(3,5)\cap B\left(-\frac{1}{13},-\frac{18}{13}\right) \quad \cup \quad P(9,-7)\cap B\left(\frac{30}{13},-\frac{97}{13}\right)}$$



PARABOLA

Question 1 (**)

The general point $P\left(9t^2,18t\right)$, where t is a parameter, lies on the parabola with Cartesian equation

$$y^2=36x\,.$$

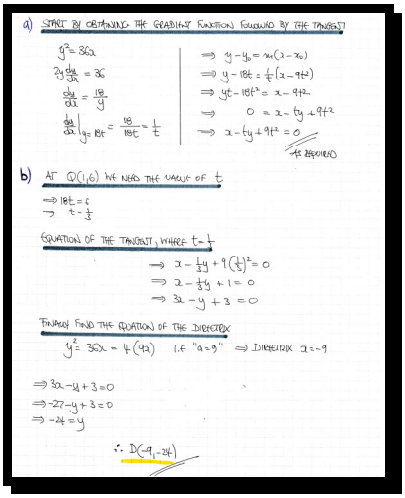
a) Show that the equation of a tangent at the point P is given by

$$x-ty+9t^2=0\,.$$

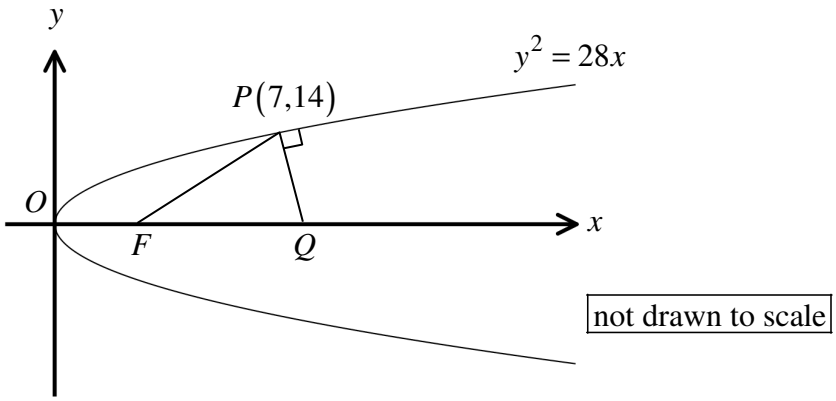
The tangent to the parabola $y^2=36x$ at the point $Q\left(1,6\right)$ crosses the directrix of the parabola at the point D.

b) Find the coordinates of D.

$\boxed{}$, $\boxed{D\left(-9,-24\right)}$



Question 2 (**)



The figure above shows the graph of the parabola with equation

$$y^2 = 28x\,,\ x \in \mathbb{R}\,,\ x \geq 0\,.$$

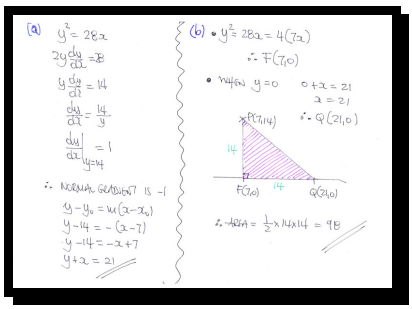
The point $P(7,14)$ lies on the parabola.

a) Find an equation of the normal to the parabola at P.

This normal meets the x axis at the point Q and F is the focus of the parabola.

b) Determine the area of the triangle PQF.

$\boxed{x+y=21}$, $\boxed{\text{area}=98}$



Question 3 **(**+)**

A parabola H has Cartesian equation

$$y^2 = 12x\,,\ x \geq 0\,.$$

The point $P\left(3t^2, 6t\right)$, where t is a parameter, lies on H.

a) Show that the equation of a tangent to the parabola at P is given by

$$yt = x + 3t^2\,.$$

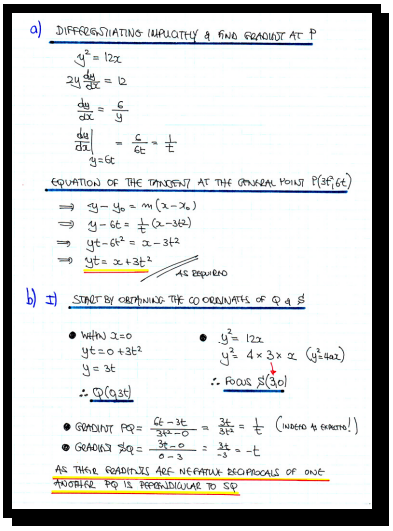
The tangent to the parabola at P meets the y axis at the point Q and the point S is the focus of the parabola.

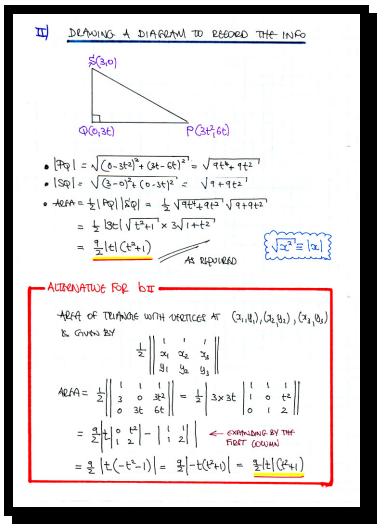
b) Show further that …

i. … PQ is perpendicular to SQ.

ii. … the area of the triangle PQS is $\frac{9}{2}|t|\left(1+t^2\right)$.

[ ], [proof]





Question 4 (***)

The general point $P\left(3t^2,6t\right)$ lies on a parabola.

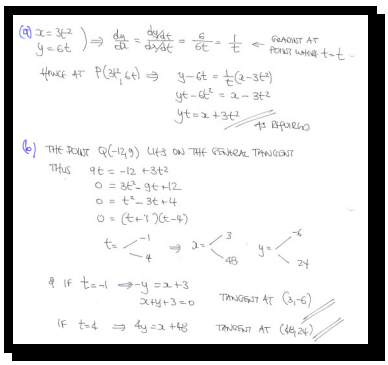
a) Show that the equation of a tangent at P is given by

$$ty=x+3t^2.$$

The point $Q\left(-12,9\right)$ does not lie on the parabola.

b) Find the equations of the two tangents to the parabola which pass through Q and deduce the coordinates of their corresponding points of tangency.

$$\boxed{x+y+3=0,\ \ (3,-6)},\ \boxed{4y=x+48,\ \ (48,24)}$$



Question 5 (***)

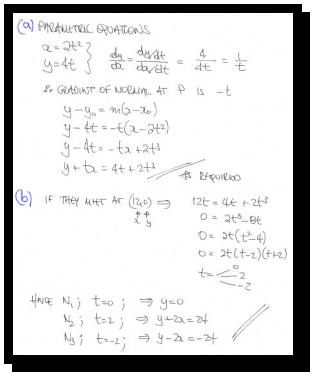
The general point $P\left(2t^2,4t\right)$ lies on a parabola.

a) Show that the equation of a normal at P is given by

$$y+tx=4t+2t^3 .$$

b) Find the equation of each of the three normals to the parabola that meet at the point with coordinates $\left(12,0\right)$.

$$\boxed{y=0,\quad y+2x=24,\quad y-2x=-24}$$



Question 6 (***)

A parabola is defined parametrically by

$$x = at^2, \quad y = 2at, \quad t \in \mathbb{R},$$

where a is a positive constant and t is a parameter.

a) Show that an equation of a normal to the parabola at the point P, where $t = p$, $p \neq 0$, is given by

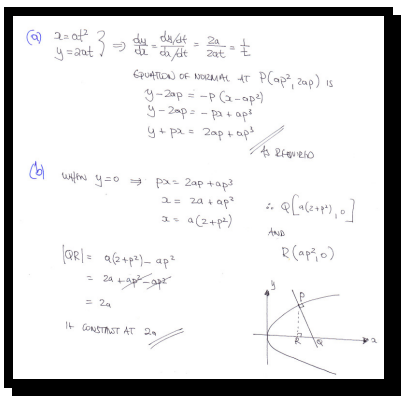
$$y + px = 2ap + ap^3.$$

The normal at P meets the x axis at the point Q.

The point R, lies on the x axis, so that PR is parallel to the y axis.

b) Show that the distance QR remains constant for all values of the parameter, and state this distance.

$$\boxed{|QR| = 2a}$$



Question 7 **(***+)**

The point $P\left(4p^2,8p\right)$, $p\geq 0$, lies on the parabola with equation

$$y^2=16x\,,\ x\geq 0\,.$$

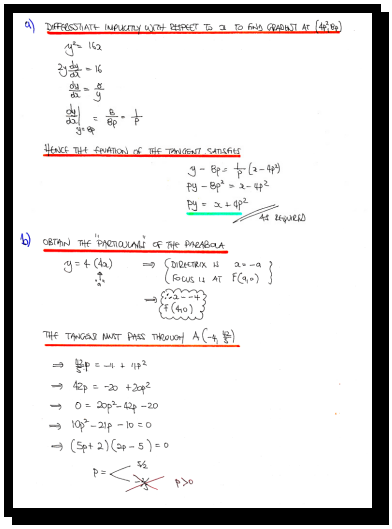
a) Show that the equation of the tangent to the parabola at P is given by

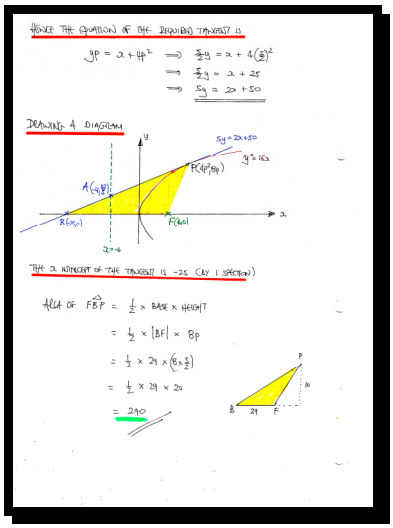
$$yp=x+4p^2\,.$$

The tangent to the parabola at P meets the directrix of the parabola at the point A and the x axis at the point B. The point F is the focus of the parabola.

b) Given that the y coordinate of A is $\frac{42}{5}$, find the area of the triangle FBP.

$\boxed{}$, $\boxed{\text{area}=290}$





Question 8 **(***+)**

The point $P\left(3p^2,6p\right)$, $p>0$, lies on the parabola with equation

$$y^2=12x\,,\ x\geq 0\,.$$

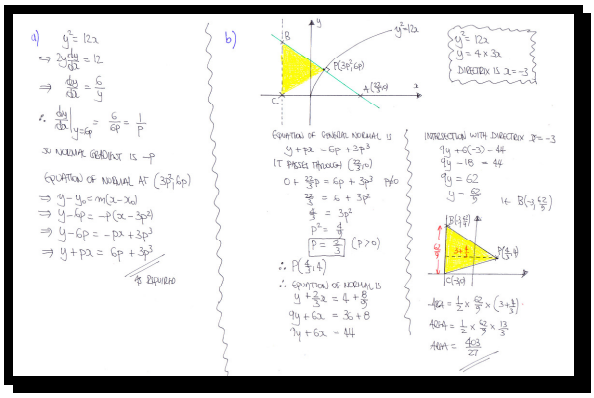
a) Show that the equation of the normal to the parabola at P is given by

$$y+px=6p+3p^3\,.$$

The normal to the parabola at P meets the x axis at the point A and the directrix of the parabola at the point B. The point C is the point of intersection of the directrix of the parabola with the x axis.

b) Given that the coordinates of A are $\left(\frac{22}{3},0\right)$, find as an exact simplified fraction the area of the triangle BCP.

$$\boxed{\text{area}=\frac{403}{27}}$$



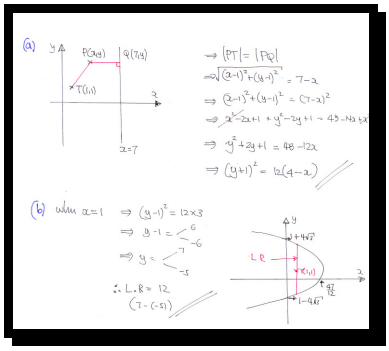
Question 9 **(***+)**

A parabola has its focus at $T(1,1)$ and its directrix has equation $x-7=0$.

a) Find an equation for the parabola.

b) Sketch the parabola and show that its latus rectum is 12 units.

$$\boxed{(y-1)^2=12(4-x)}$$



Question 10 (*+)**

A parabola is given parametrically by the equations

$$x=4-t^2,\ y=1-t,\ t\in\mathbb{R}.$$

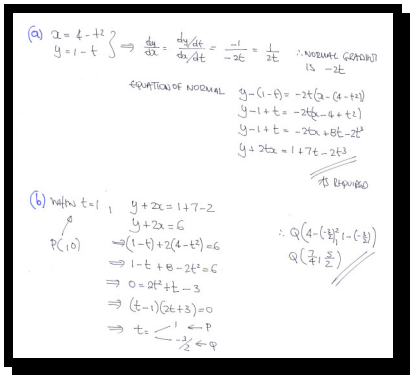
a) Show that an equation of the normal at the general point on the parabola is

$$y+2tx=1+7t-2t^3.$$

The normal to parabola at $P(3,0)$ meets the parabola again at the point Q.

b) Find the coordinates of Q.

$$\boxed{Q\left(\frac{7}{4},\frac{5}{2}\right)}$$



Question 11 (*+)**

The points P and Q lie on the parabola with equation

$$y^2 = 2x\,,$$

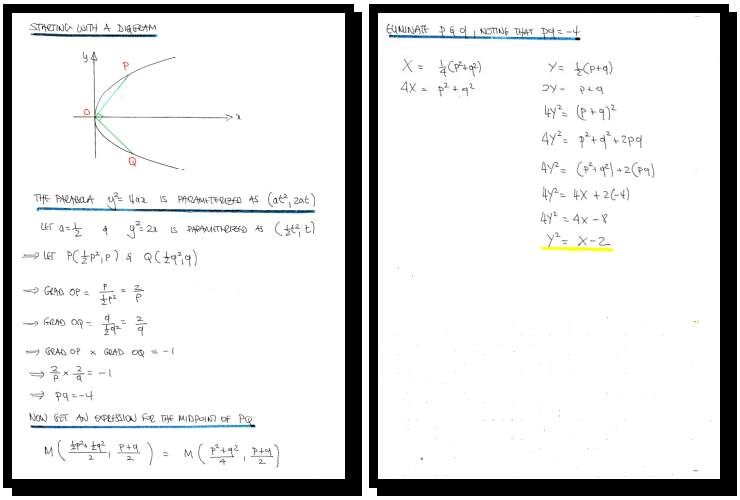
so that OP is perpendicular to OQ, where O is the origin.

The point M is the midpoint of PQ.

Show that the Cartesian locus of M lies on the curve with equation

$$y^2 = x - 2\,.$$

[ ], [proof]



Question 12 (*+)**

The point P has coordinates

$$P\left(at^2, 2at\right),$$

where a is a positive constant and t is a real parameter.

The point P traces a parabola.

a) Show that the equation of a normal at the point P is given by

$$y+tx=2at+at^3.$$

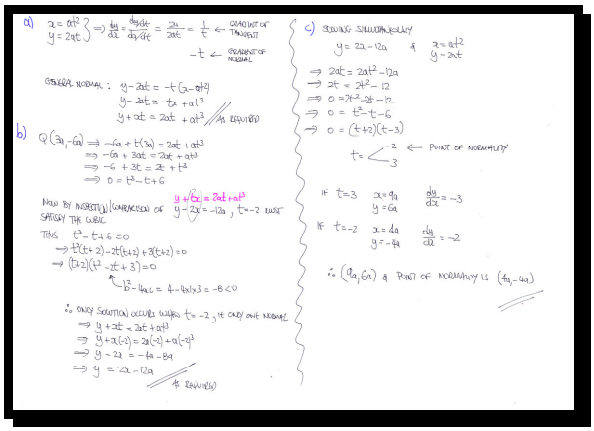
b) Show that the straight line with equation

$$y=2x-12a,$$

is the only normal to the parabola passing through the point $Q\left(3a,-6a\right)$.

c) Determine the coordinates of the two points of intersection between this normal and the parabola, indicating clearly which point of intersection represents the point of normality.

$\boxed{\left(9a,6a\right),\ \text{normal at } \left(4a,-4a\right)}$



Question 13 (*+)**

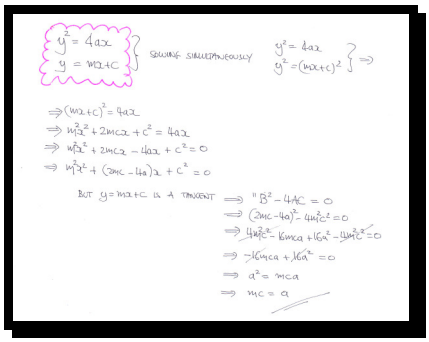
A straight line L is a tangent to the parabola with equation

$$y^2 = Ax$$

where A is a positive constant.

Given that L does not pass through the origin O, show that the product of the gradient and the y intercept of L equals the x coordinate of the focus of the parabola.

proof



Question 14 (*+)**

Sketch the parabola with equation

$$y^2-4y-2x=2\,.$$

The sketch must include the …

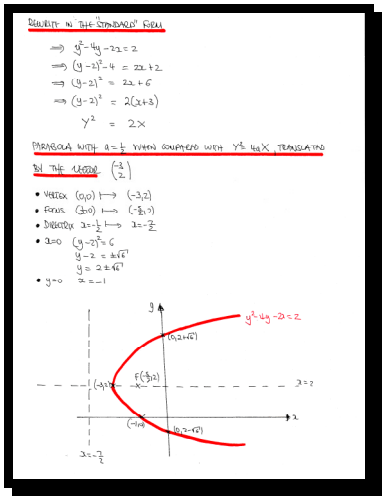
a) … coordinates of points of intersection with the coordinate axes.

b) … coordinates of the vertex of the parabola.

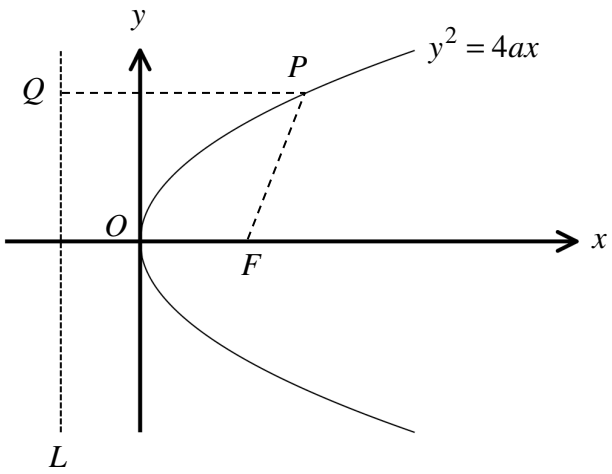
c) … coordinates of the focus of the parabola.

d) … equation of the directrix of the parabola.

[ ], graph



Question 15 (*+)**



The figure above shows the sketch of the parabola with equation

$$y^2 = 4ax,$$

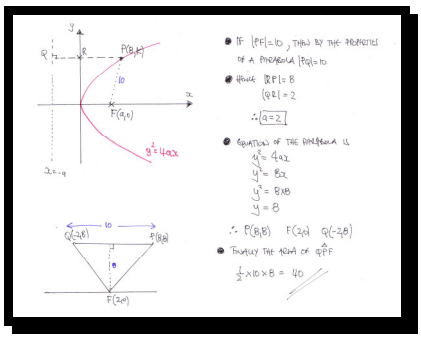
where a is a positive constant.

The straight line L and the point F are the directrix and the focus of the parabola, respectively.

The point $P(8, y)$, $y > 0$, lies on the parabola. The point Q lies on L, so that QP is parallel to the x axis.

Given further that $|PF| = 10$, determine the area of the triangle FPQ.

$\boxed{\text{area} = 40}$



Question 16 (*+)**

The point $T\left(at^2, 2at\right)$, lies on the parabola with equation

$$y^2 = 4ax, \quad a > 0, \ x \geq 0.$$

a) Show clearly that an equation of a normal to the parabola at the point $P\left(ap^2, 2ap\right)$, $p \neq 0$, can be written as

$$y + px = 2ap + ap^3.$$

The normal at P meets the x axis at the point Q.

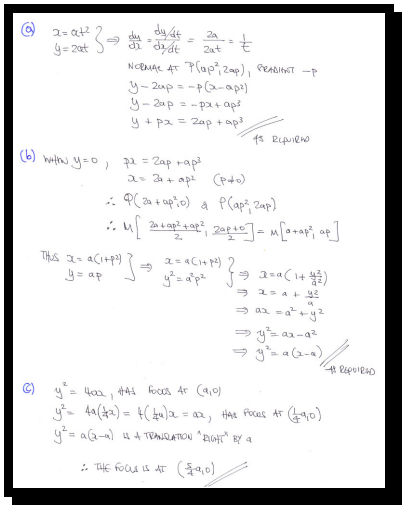
The midpoint of PQ is M.

b) Show that the locus of M as p varies is the parabola with equation

$$y^2 = a\left(x - a\right).$$

c) Find the coordinates of the focus of $y^2 = a\left(x - a\right)$.

$$\left(\frac{5}{4}a, 0\right)$$



Question 17 (*+)**

The point $T\left(at^2, 2at\right)$, where a is a positive constant and t is a real parameter, lies on the parabola with equation

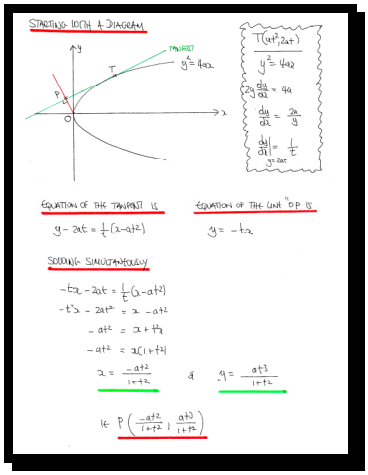
$$y^2 = 4ax .$$

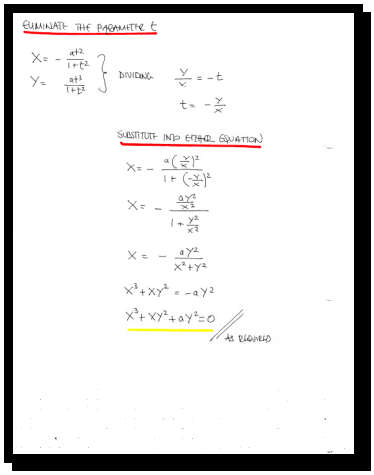
A straight line passing through the origin, intersects at right angles the tangent to the parabola at T, at the point P.

Show that as t varies, the Cartesian locus of P is

$$x^3 + xy^2 + ay^2 = 0 .$$

[ ], proof



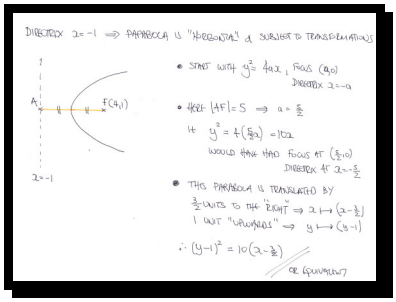


Question 18 (****)

A parabola has its focus at the point with coordinates $(4,1)$ and its directrix has equation $x=-1$.

Determine a Cartesian equation of the parabola.

$$\boxed{(y-1)^2=10\left(x-\frac{3}{2}\right)}$$



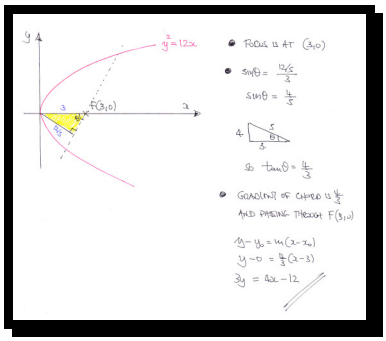
Question 19 (****)

A straight line L passes through the focus of the parabola with equation

$$y^2=12x.$$

Given further that the shortest distance of L from the origin O is $\frac{12}{5}$, determine an equation for L.

$$\boxed{3y=4x-12}$$



Question 20 (****)

A parabola P has Cartesian equation

$$y^2-4y-8x+28=0\,.$$

a) Determine …

i. … the coordinates of the vertex of P.

ii. … the coordinates of the focus of P.

iii. … the equation of the directrix of P.

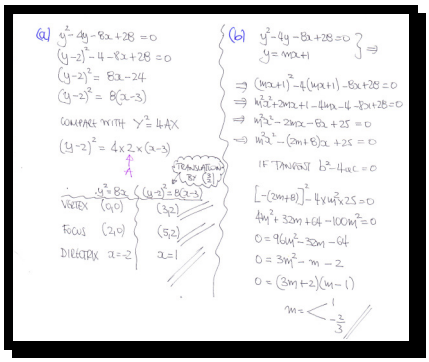
The line with equation

$$y=mx+1,\ \text{where } m \text{ is a constant,}$$

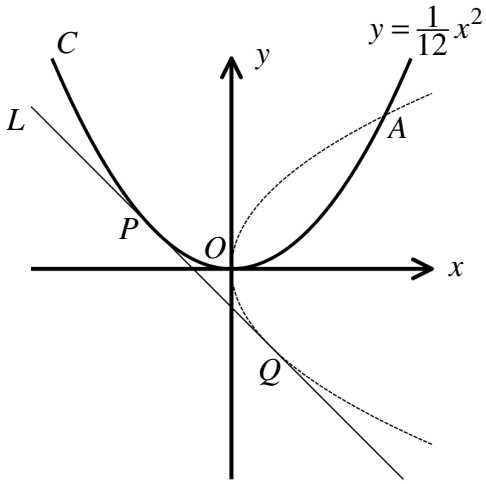
is a tangent at some point of P.

b) Find the possible values of m.

$$\boxed{\text{vertex at } (3,2),\ \ \text{focus at } (5,2),\ \ \text{directrix } x=1},\ \boxed{m=-\frac{2}{3},1}$$



Question 21 **(****)**



The figure above shows the parabola C with equation $y=\frac{1}{12}x^2$.

The dotted line in the figure is the reflection of C in the line $y=x$.

a) Find the exact distance between the focus of C and the focus of its reflection.

The parabola intersects its reflection at the origin and at the point A.

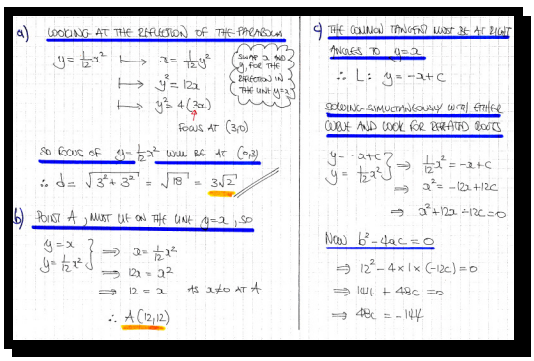
b) Determine the coordinates of A.

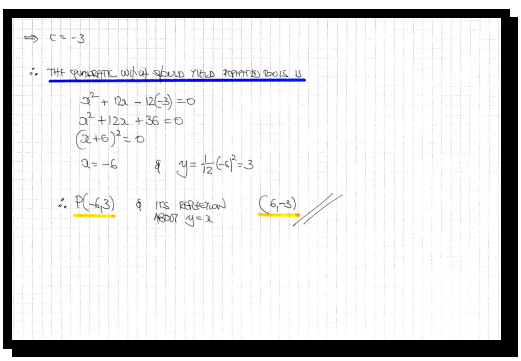
The straight line L is a common tangent to both C and the reflection of C.

L touches C at the point P and the reflection of C at the point Q.

c) Determine the coordinates of P and Q.

$\boxed{}$, $\boxed{3\sqrt{2}}$, $\boxed{A(12,12)}$, $\boxed{P(-6,3)}$, $\boxed{Q(3,-6)}$





Question 22 (****)

The point $T\left(at^2, 2at\right)$, lies on the parabola with equation

$$y^2 = 4ax, \quad a > 0, \ x \geq 0 .$$

a) Show clearly that an equation of a normal to the parabola at the point $P\left(ap^2, 2ap\right)$, $p \neq 0$, can be written as

$$y + px = 2ap + ap^3 .$$

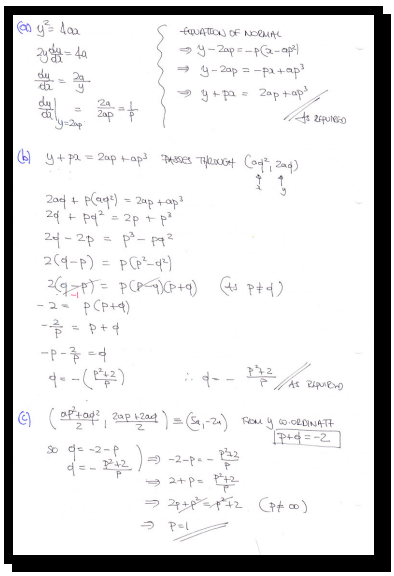
The normal at P re-intersects the parabola at the point $Q\left(aq^2, 2aq\right)$.

b) Show that

$$q = -\frac{p^2 + 2}{p} .$$

c) Given that the midpoint of PQ has coordinates $\left(5a, -2a\right)$, find the value of p.

$$\boxed{p = 1}$$



Question 23 **(****)**

The point $P\left(2t^2,4t\right)$, lies on the parabola with equation

$$y^2=8x,\ x\geq 0.$$

a) Show that an equation of a tangent to the parabola at P, can be written as

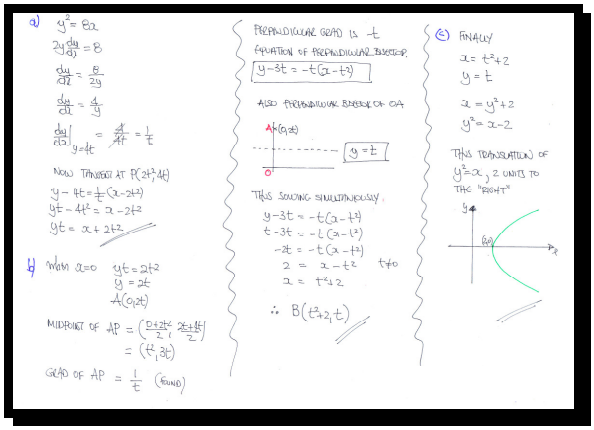
$$yt=x+2t^2,\ t\neq 0.$$

The tangent to the parabola at P meets the y axis at the point A. The perpendicular bisectors of the straight line segments AP and OA, meets at the point B.

b) Find the coordinates of B, in terms of t.

c) Sketch the locus of B as t varies.

$$\boxed{B\left(t^2+2,t\right)}$$



Question 24 (****)

A parabola C has Cartesian equation

$$y^2+4y-16x+36=0\,.$$

a) Describe the transformations that map the graph of the curve with equation $y^2=16x$ onto the graph of C.

b) Determine the coordinates of the focus of C.

c) Show that …

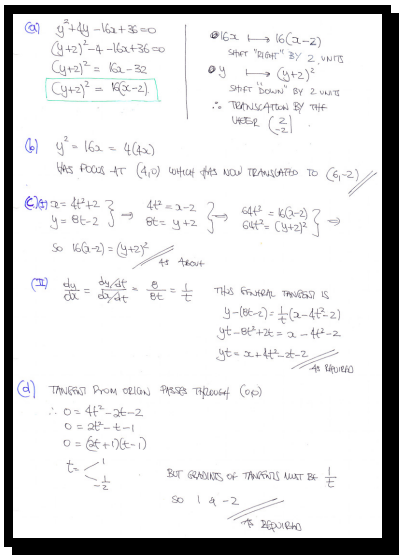
i. … the point $P\left(4t^2+2,8t-2\right)$, lies on the parabola.

ii. … the equation of a tangent to the parabola at the point P, is

$$yt=x+4t^2-2t-2\,.$$

d) Hence show that the gradients of the two tangents from the origin to the parabola have gradients -2 and 1.

translation by vector $\begin{pmatrix}2\\-2\end{pmatrix}$, $(6,-2)$



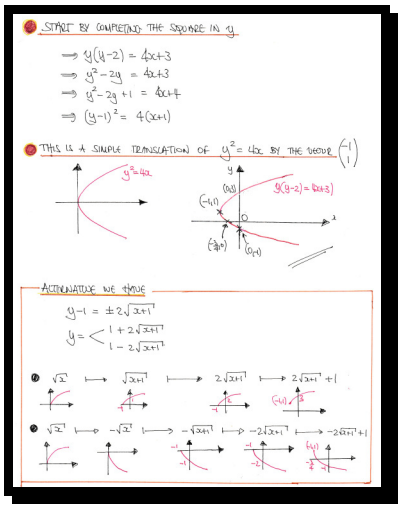
Question 25 (****)

Sketch the graph of the parabola with equation

$$y(y-2)=4x+3\,.$$

The sketch must include the coordinates of any intersections with the axes and the coordinates of the vertex of the parabola.

graph



Question 26 (****)

The point $P\left(ap^2,2ap\right)$, where p is a parameter, lies on the parabola, with Cartesian equation

$$y^2=4ax,$$

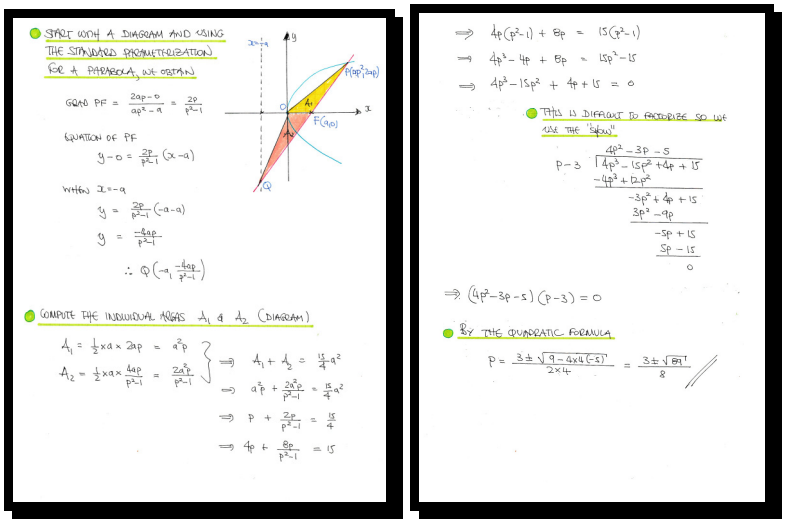
where a is a positive constant.

The point F is the focus of the parabola and O represents the origin.

The straight line which passes through P and F meets the directrix of the parabola at the point Q, so that the area of the triangle OPQ is $\frac{15}{4}a^2$.

Show that one of the possible values of p is 3 and find in exact surd form the other 2 possible values.

$\boxed{}$, $\boxed{p=\frac{1}{8}\left(3\pm\sqrt{89}\right)}$



Question 27 **(****+)**

A parabola P has focus $S(6,0)$ and directrix the line $x=0$.

a) Show that a Cartesian equation for P is $y^2=12(x-3)$.

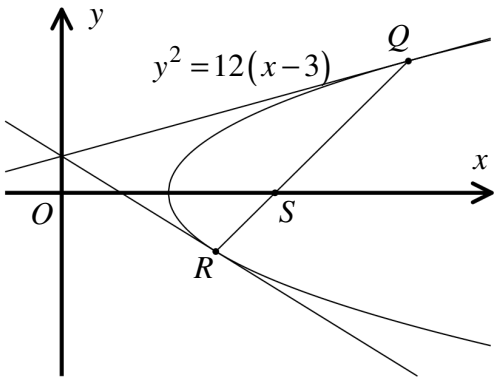
b) Verify that the parametric equations of P are

$$x=3t^2+3,\ y=6t.$$

c) Show that the equation of the tangent at the point $Q(3q^2+3,6q)$ is

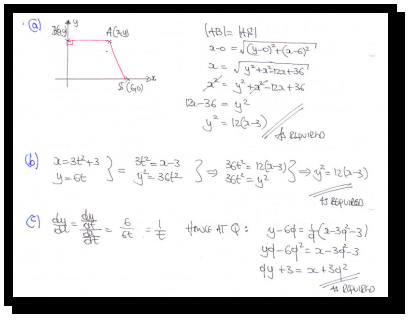
$$qy+3=x+3q^2$$

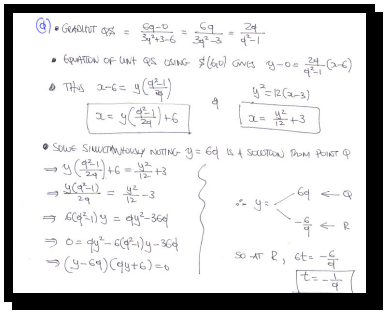
The diagram below shows the parabola and its tangents at the points Q and R. The point R lies on the parabola so that QSR is a straight line.

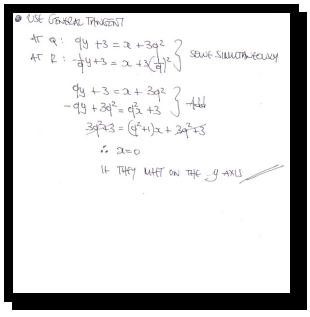


d) Show that the tangents to the parabola at Q and at R, meet on the y axis.

proof







Question 28 (**+)**

A parabola C has parametric equations

$$x=-2t^2,\quad y=4t$$

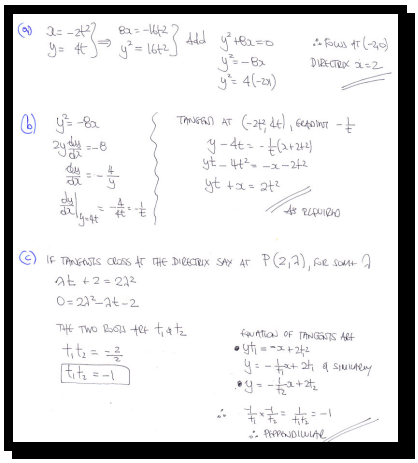
a) Determine the coordinates of the focus and the equation of directrix of C.

b) Show that an equation of the tangent to C, at the general point $T\left(-2t^2,4t\right)$ is

$$yt+x=2t^2$$

c) By considering the product of the roots of a suitable quadratic equation, show that any two tangents that meet on the directrix of C, are perpendicular.

$\boxed{F\left(-2,0\right)}$, $\boxed{x=2}$



Question 29 (**+)**

The point $P\left(at^2, 2at\right)$ lies on the parabola with equation

$$y^2 = 4ax,$$

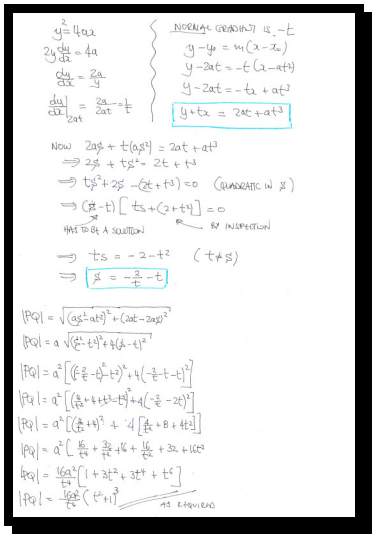
where a is a positive constant and t is a real parameter.

The normal to the parabola at P, meets the parabola again at the point $Q\left(as^2, 2as\right)$.

Show that

$$|PQ| = \frac{16a^2}{t^4}\left(t^2+1\right)^3.$$

proof



Question 30 (**+)**

A parabola C has Cartesian equation

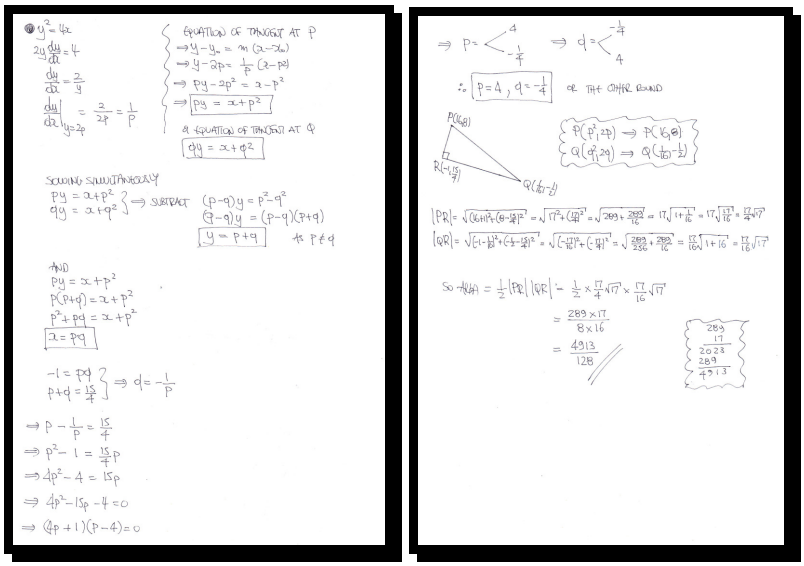
$$y^2 = 4x \ , \ x \in \mathbb{R} , \ x \geq 0 .$$

The points $P\left(p^2, 2p\right)$ and $Q\left(q^2, 2q\right)$ are distinct and lie on C.

The tangent to C at P and the tangent to C at Q meet at $R\left(-1, \frac{15}{4}\right)$.

Calculate as an exact simplified fraction the area of the triangle PQR.

$$\boxed{\text{area} = \frac{4913}{128}}$$



Question 31 (**+)**

A parabola C has Cartesian equation

$$y^2 = 4ax \ ,$$

where a is a positive constant.

The points $P\left(ap^2, 2ap\right)$ and $Q\left(aq^2, 2aq\right)$ are distinct and lie on C.

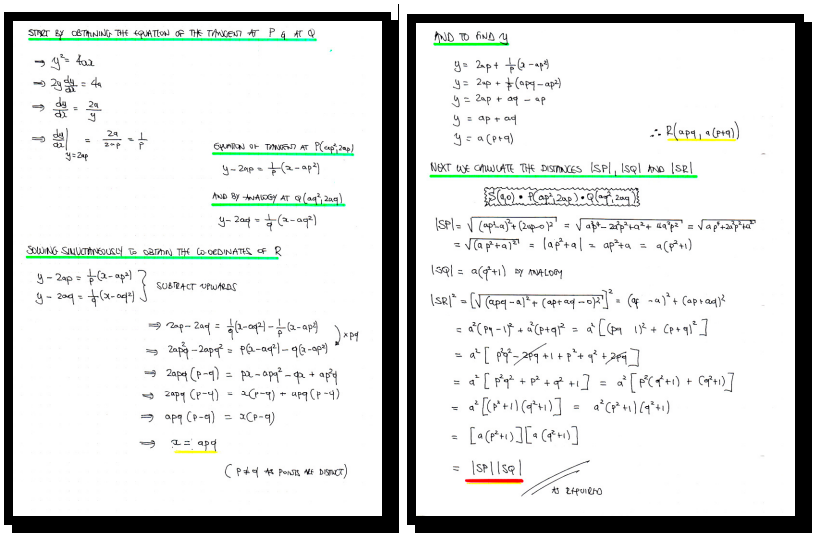
The tangent to C at P and the tangent to C at Q meet at the point R.

Show that

$$\left|SR\right|^2 = \left|SP\right|\left|SQ\right|,$$

where S is the focus of the parabola.

, proof



Question 32 (**+)**

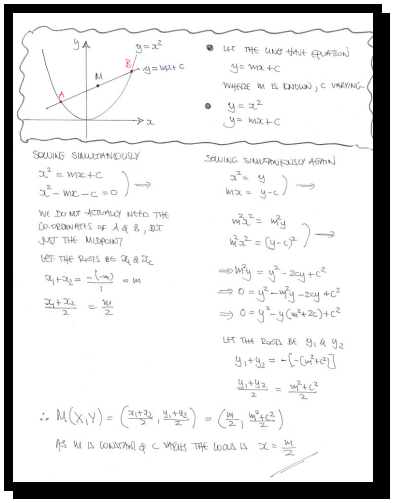
A parabola has Cartesian equation

$$y = x^2 \ , \ x \in \mathbb{R} \, .$$

A chord of the parabola is defined as the straight line segment joining any two distinct points on the parabola.

Find the equation of the locus of the midpoints of parallel chords of the parabola whose gradient is m.

$\boxed{}$, $\boxed{x = \frac{1}{2}m}$



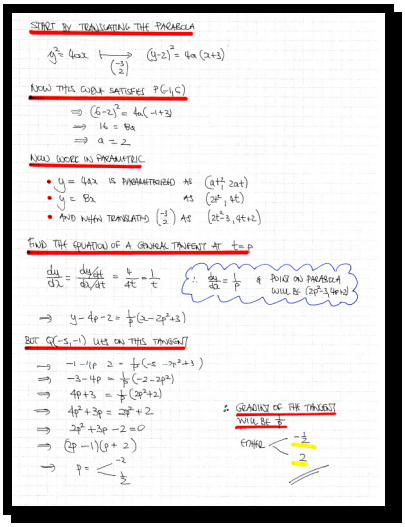
Question 33 (*****)

The points P and Q have respective coordinates $(-1,6)$ and $(-5,-1)$.

When the parabola with equation $y=4ax$, where a is a constant, is translated by the vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ it passes through the point P.

Find the possible values of the gradient of the straight line which passes through Q and is a tangent to the **translated** parabola.

$\square$, $\boxed{m=-\frac{1}{2} \cup m=2}$



Question 34 (*****)

A parabola has Cartesian equation

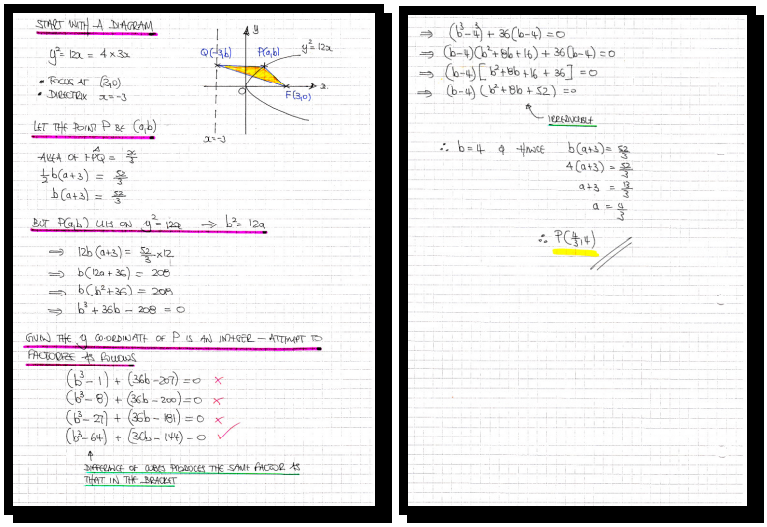
$$y^2 = 12x\,, \quad x \geq 0\,.$$

The point P lies on the parabola and the point Q lies on the directrix of the parabola so that PQ is parallel to the x axis.

The area of the triangle PQF is $8\frac{2}{3}$ square units, where the point F represents the focus of the parabola.

Determine the coordinates of P, given further that the y coordinate of P is a positive integer.

$\square$, $\boxed{P\left(\frac{4}{3},4\right)}$



Question 35 (*****)

The point $P\left(2p, p^2\right)$, where p is a parameter, lies on the parabola, with Cartesian equation

$$x^2 = 4y\,.$$

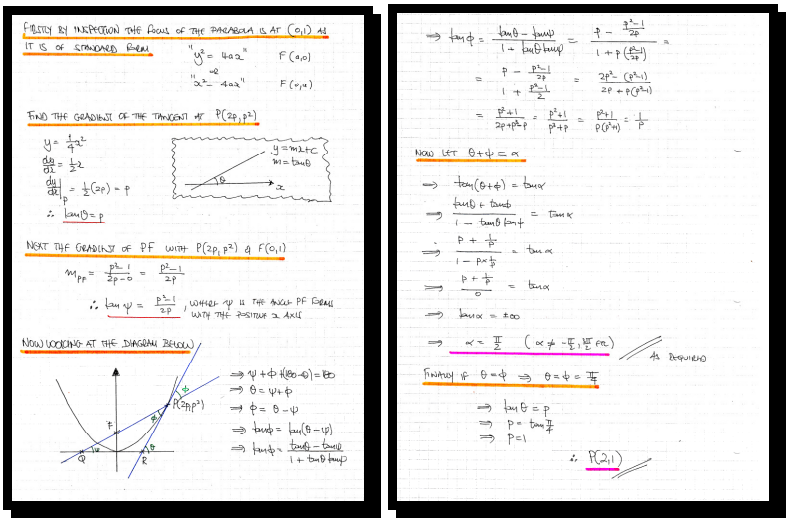
The point F is the focus of the parabola and O represents the origin.

The tangent to the parabola at P forms an angle θ with the positive x axis.

The straight line which passes through P and F forms an acute angle φ with the tangent to the parabola at P.

Show that $\theta + \varphi = \frac{1}{2}\pi$ and hence state the coordinates of P if $\theta = \varphi$.

$\boxed{}$, $\boxed{P(2,1)}$



Question 36 (*****)

A parabola has Cartesian equation

$$y=\frac{1}{2}x^2, \quad x\in\mathbb{R}.$$

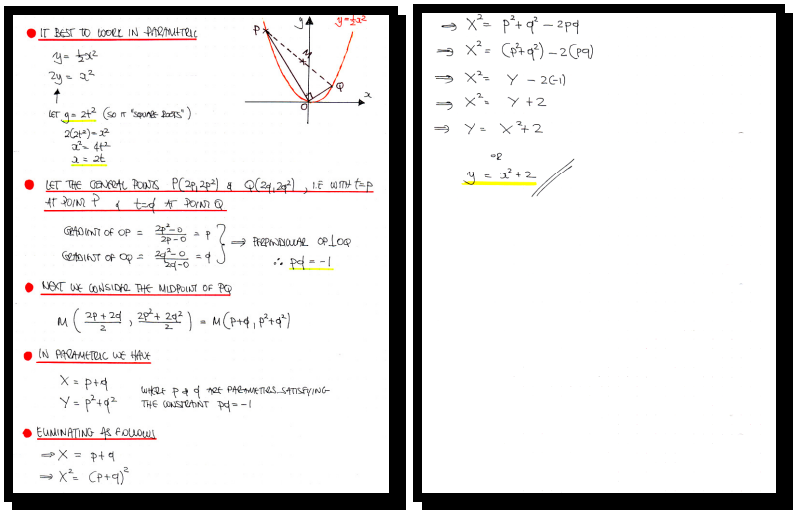
The points P and Q both lie on the parabola so that POQ is a right angle, where O is the origin.

The point M represents the midpoint of PQ.

Show that as the position of P varies along the parabola, the locus of M is the curve with equation

$$y=x^2-2.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 37 (*****)

The cubic equation

$$x^3 + px + q = 0,$$

has 2 distinct real roots.

a) Show that $27q^2 + 4p^3 < 0$.

A parabola has Cartesian equation

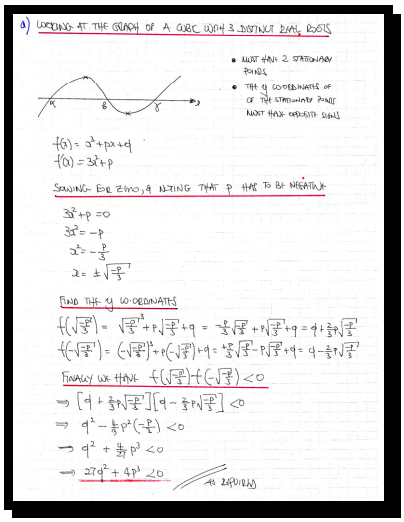
$$y = x^2, \quad x \in \mathbb{R}.$$

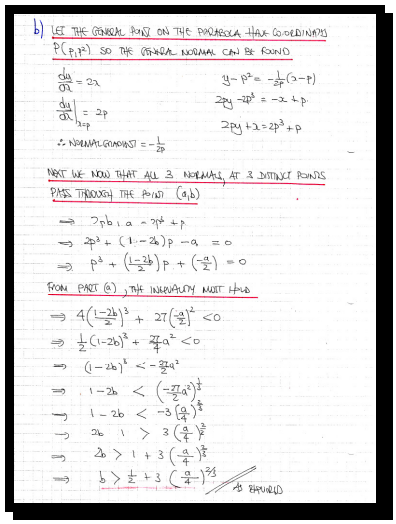
Three distinct normals to this parabola pass through the point, which does not lie on the parabola, whose coordinates are (a,b).

b) Show further that

$$b > \frac{1}{2} + 3\left(\frac{1}{4}a\right)^{\frac{2}{3}}.$$

, proof





Question 38 (*****)

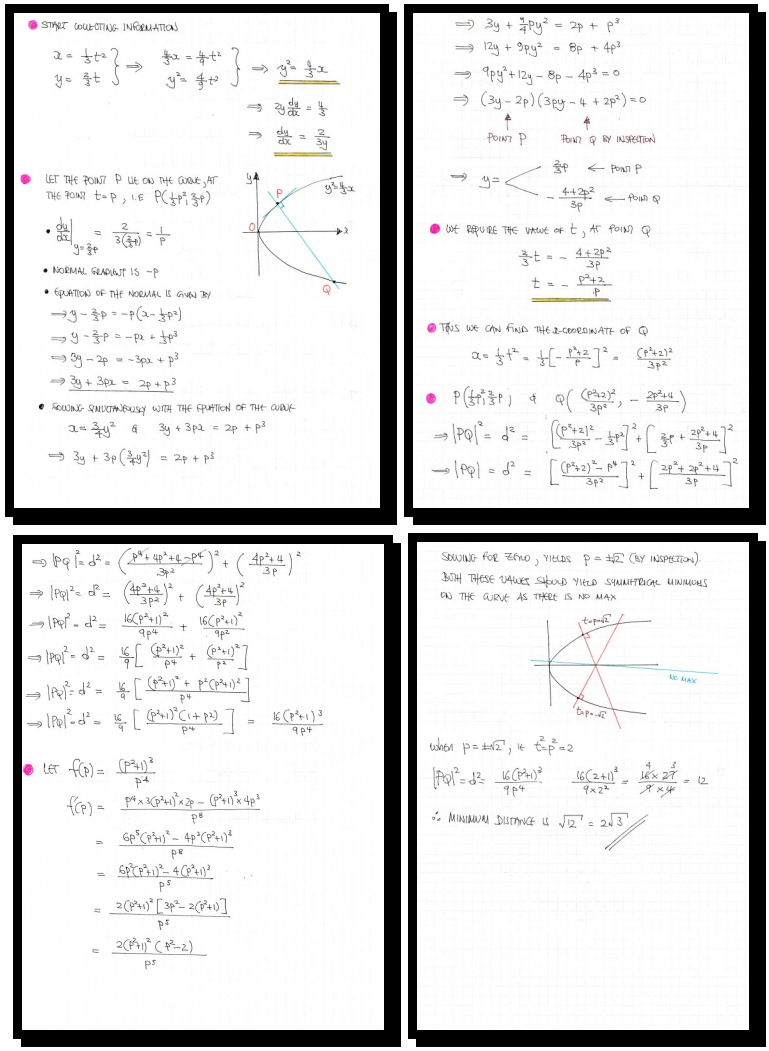
A parabola is given parametrically by

$$x=\frac{1}{3}t^2, \quad y=\frac{2}{3}t, \quad t\in\mathbb{R}.$$

The normal to the parabola at the point P meets the parabola again at the point Q.

Show that the minimum value of $|PQ|$ is $\sqrt{12}$.

[ ], proof



ELLIPSE

Question 1 (**)

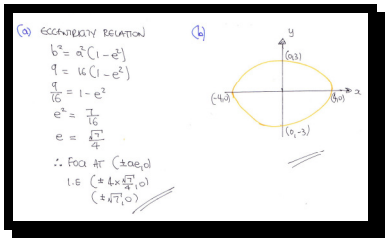
An ellipse has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{9}=1\,.$$

a) Find the coordinates of its foci.

b) Sketch the ellipse.

$\boxed{(\pm\sqrt{7},0)}$



Question 2 (**)

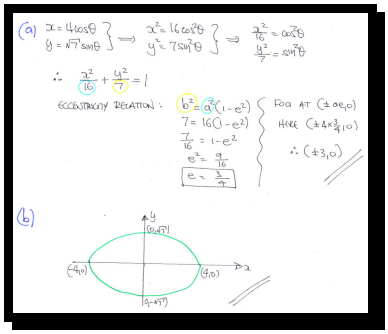
An ellipse has parametric equations

$$x=4\cos\theta,\quad y=\sqrt{7}\sin\theta,\quad 0\le\theta<2\pi\,.$$

a) Find the coordinates of its foci.

b) Sketch the ellipse.

$\boxed{(\pm3,0)}$

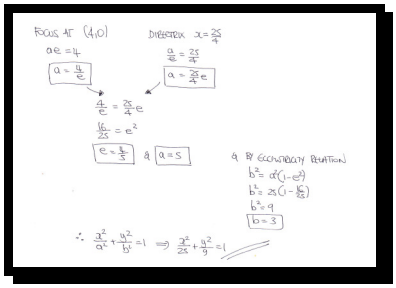


Question 3 (**)

An ellipse has a focus at $(4,0)$ and the associated directrix has equation $x=\dfrac{25}{4}$.

Determine a Cartesian equation of the ellipse.

$$\boxed{\frac{x^2}{25}+\frac{y^2}{9}=1}$$



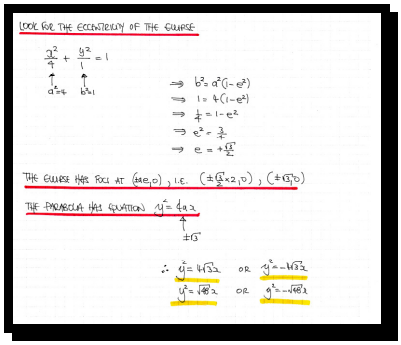
Question 4 (**)

$$\frac{x^2}{4}+y^2=1\,.$$

The ellipse with Cartesian equation above and a parabola with vertex at the origin share the same focal point.

Find the possible Cartesian equation for the parabola.

$$\boxed{}\,,\qquad \boxed{y^2=\pm\sqrt{48}x}$$



Question 5 (+)**

An ellipse E is given parametrically by the equations

$$x=\cos t\,,\ y=2\sin t\,,\ 0\le t<2\pi\,.$$

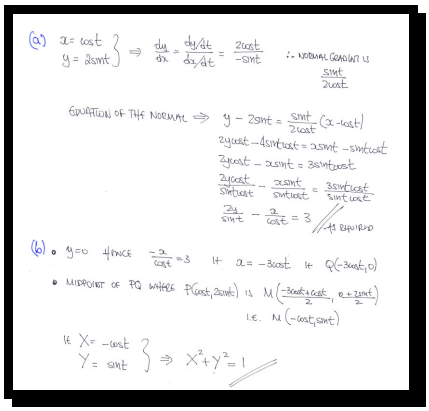
a) Show that an equation of the normal to E at the general point $P(\cos t,2\sin t)$ can be written as

$$\frac{2y}{\sin t}-\frac{x}{\cos t}=3\,.$$

The normal to E at P meets the x axis at the point Q. The midpoint of PQ is M.

b) Find the equation of the locus of M as t varies.

$$\boxed{x^2+y^2=1}$$



Question 6 **(***)**

$$\frac{x^2}{9}+\frac{y^2}{4}=1\,.$$

The ellipse with the Cartesian equation given above, has foci S and S'.

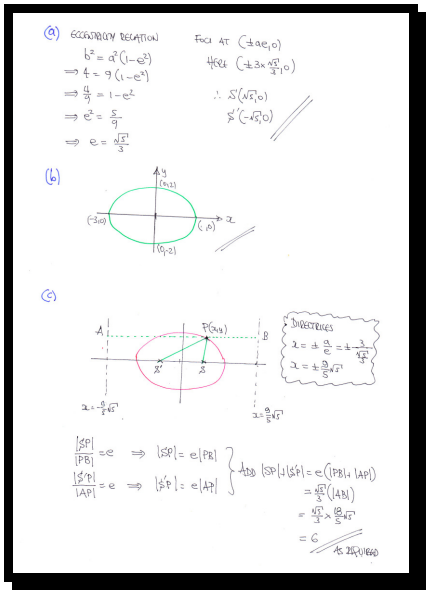
a) Find the coordinates S and S'.

b) Sketch the ellipse.

c) Show that for every point P on this ellipse,

$$|SP|+|S'P|=6\,.$$

$\boxed{(\pm\sqrt{5},0)}$



Question 7 (***)

An ellipse E has Cartesian equation

$$\frac{x^2}{289}+\frac{y^2}{64}=1\,.$$

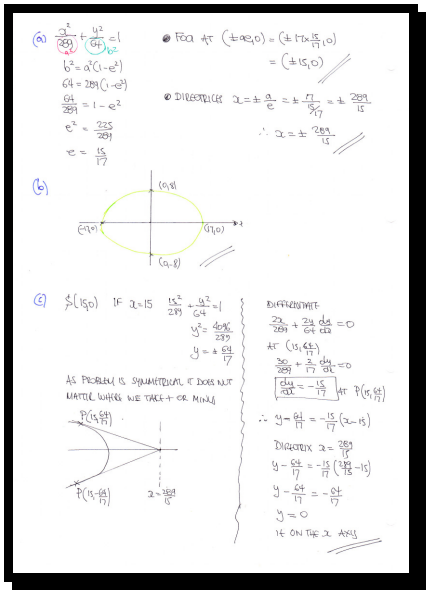
a) Find the coordinates of the foci of E, and the equations of its directrices.

b) Sketch the ellipse.

The point P lies on E so that PS is vertical, where S is the focus of the ellipse with positive x coordinate.

c) Show that the tangent to the ellipse at the point P meets one the directrices of the ellipse on the x axis.

$\boxed{(\pm 15,0)}$, $\boxed{x=\pm\frac{289}{15}}$



Question 8 (***)

The point $P(5\cos\theta, 4\sin\theta)$ lies on the an ellipse E with Cartesian equation

$$16x^2 + 25y^2 = 400 .$$

a) Find the coordinates of the foci of E.

b) Show that an equation of the normal to the ellipse at P is

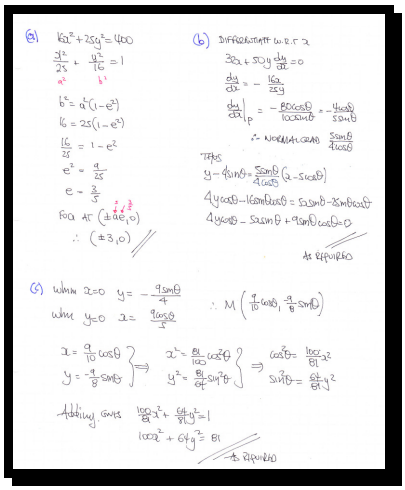
$$4y\cos\theta - 5x\sin\theta + 9\sin\theta\cos\theta = 0 .$$

The normal to the ellipse intersects the coordinate axes at the points A and B, and the point M is the midpoint of AB.

c) Show that the locus of M, as θ varies, is the ellipse with equation

$$100x^2 + 64y^2 = 81 .$$

$\boxed{(\pm 3, 0)}$



Question 9 (***)

The ellipse E has parametric equations

$$x = a\cos\theta \ , \ y = b\sin\theta , \ \ 0 \le \theta < 2\pi$$

where a and b are positive constants.

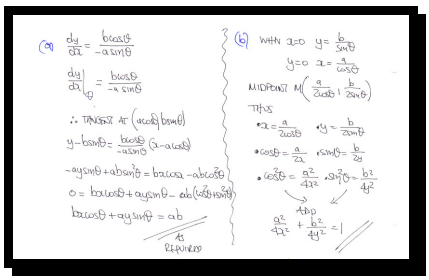
a) Show that an equation of the tangent at a general point on E is

$$bx\cos\theta + ay\sin\theta = ab .$$

This tangent to E intersects the coordinate axes at the points A and B, and the point M is the midpoint of AB.

b) Find a Cartesian locus of M, as θ varies.

$$\boxed{\frac{a^2}{ax^2} + \frac{b^2}{4y^2} = 1}$$



Question 10 (***)

An ellipse has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1\,.$$

The general point $P\left(4\cos\theta,2\sin\theta\right)$ lies on the ellipse.

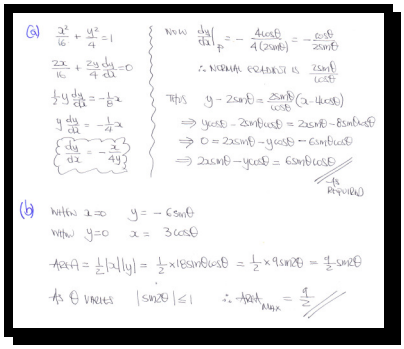
a) Show that the equation of the normal to the ellipse at P is

$$2x\sin\theta-y\cos\theta=6\sin\theta\cos\theta\,.$$

The normal to the ellipse at P meets the x axis at the point Q and O is the origin.

b) Show clearly that as θ varies, the maximum area of the triangle OPQ is $4\frac{1}{2}$.

proof



Question 11 (*+)**

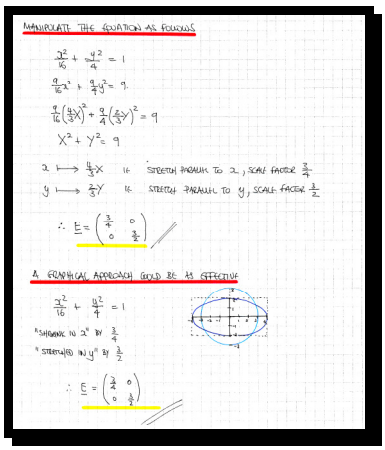
An ellipse with equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

is transformed by the enlargement matrix $\mathbf{E}$ into a circle of radius 3, with centre at the origin.

Determine the elements of $\mathbf{E}$.

$\boxed{}$, $\boxed{\mathbf{E}=\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{3}{2}\end{pmatrix}}$



Question 12 (*+)**

An ellipse has Cartesian equation

$$2x^2+3y^2-4x+12y+8=0\,.$$

Determine …

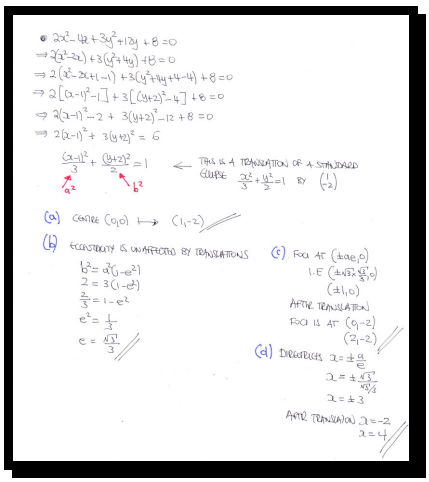
a) … the coordinates of the centre of the ellipse.

b) … the eccentricity of the ellipse.

c) … the coordinates of the foci of the ellipse.

d) … the equations of the directrices of the ellipse.

$$\boxed{(1,-2)}\,,\quad \boxed{e=\frac{\sqrt{3}}{3}}\,,\quad \boxed{(0,-2),(2,-2)}\,,\quad \boxed{x=-2,\ x=4}$$



Question 13 (*+)**

An ellipse has Cartesian equation

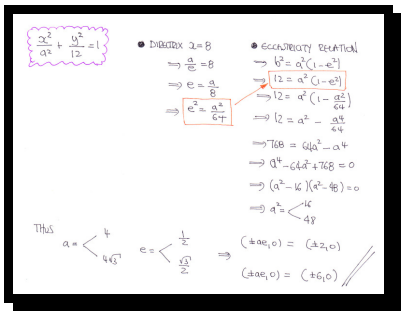
$$\frac{x^2}{a^2}+\frac{y^2}{12}=1,$$

where a is a positive constant.

The straight line with equation $x=8$ is a directrix for the ellipse.

Determine the possible set of coordinates for the foci of the ellipse.

$$\boxed{(\pm 2,0) \text{ or } (\pm 6,0)}$$



Question 14 (*+)**

An ellipse has equation

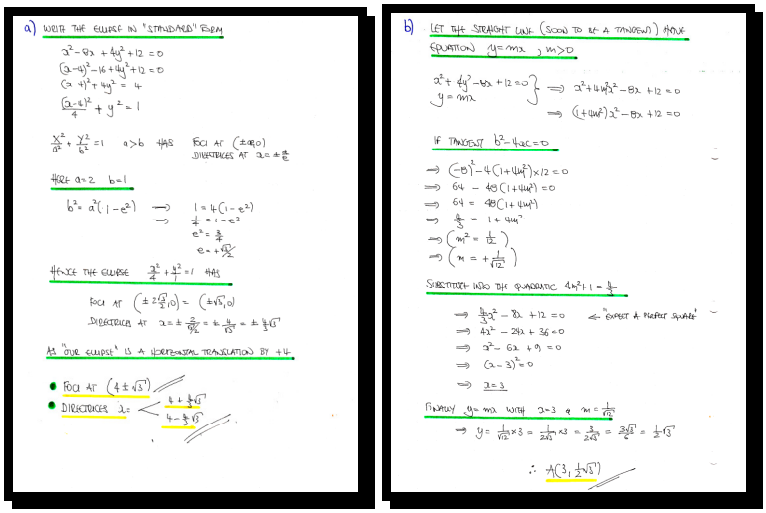
$$x^2-8x+4y^2+12=0\,.$$

a) Determine the coordinates of the foci and the equations of the directrices of the ellipse.

A straight line with positive gradient passes through the origin O and t**ouches** the ellipse at the point A.

b) Find the coordinates of A.

$$\boxed{},\ \boxed{\left(4-\sqrt{3},0\right),\ \left(4+\sqrt{3},0\right)},\ \boxed{x=4-\frac{4}{3}\sqrt{3},\ x=4+\frac{4}{3}\sqrt{3}},\ \boxed{\left(3,\frac{1}{3}\sqrt{2}\right)}$$



Question 15 **(***+)**

A point P lies on the ellipse with Cartesian equation

$$\frac{x^2}{64}+\frac{y^2}{16}=1\,.$$

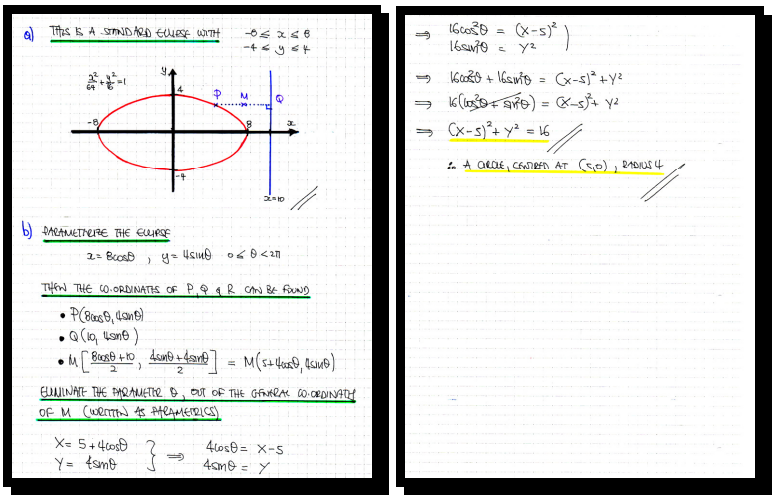
The point Q is the foot of the perpendicular from the point P to the straight line with equation $x=10$.

a) Sketch in the same diagram the ellipse, the straight line with equation $x=12$ and the straight line segment PQ.

The point M is the midpoint of PQ.

b) Determine a Cartesian equation for the locus of M as the position of P varies, further describing this locus geometrically.

$\boxed{}$, $\boxed{(x-5)^2+y^2=16}$



Question 16 **(***+)**

An ellipse E has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1.$$

a) Show that an equation of the tangent to E at the point $A\left(4\cos\theta,2\sin\theta\right)$ is given by

$$2y\sin\theta+x\cos\theta=4.$$

The point $B\left(4\cos\theta,4\sin\theta\right)$ lies on the circle with Cartesian equation

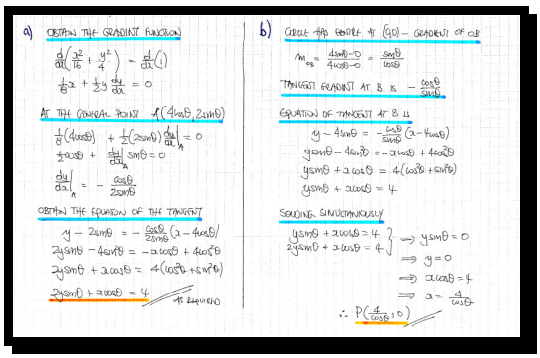
$$x^2+y^2=16.$$

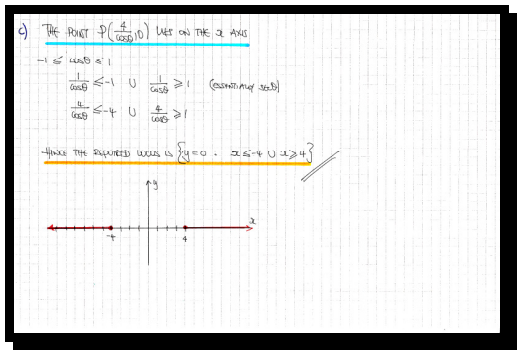
The tangent to the circle at the point B meets the tangent to the ellipse at the point A at the point P.

b) Determine the coordinates of P, in terms of θ.

c) Describe mathematically the locus of P as θ varies.

$\boxed{}$, $\boxed{P\left(4\sec\theta,0\right)}$, $\boxed{\text{the } x \text{ axis, so that } x\in\left(-\infty,-4\right]\cup\left[4,\infty\right)}$





Question 17 (****)

An ellipse has Cartesian equation

$$\frac{x^2}{2}+y^2=1.$$

A straight line L has equation $y=mx+c$, where m and c are positive constants.

a) Show that the x coordinates of the points of intersection between L and the ellipse satisfy the equation

$$\left(2m^2+1\right)x^2+4mcx+2\left(c^2-1\right)=0.$$

b) Given that L is a tangent to the ellipse, show that $c^2=2m^2+1$.

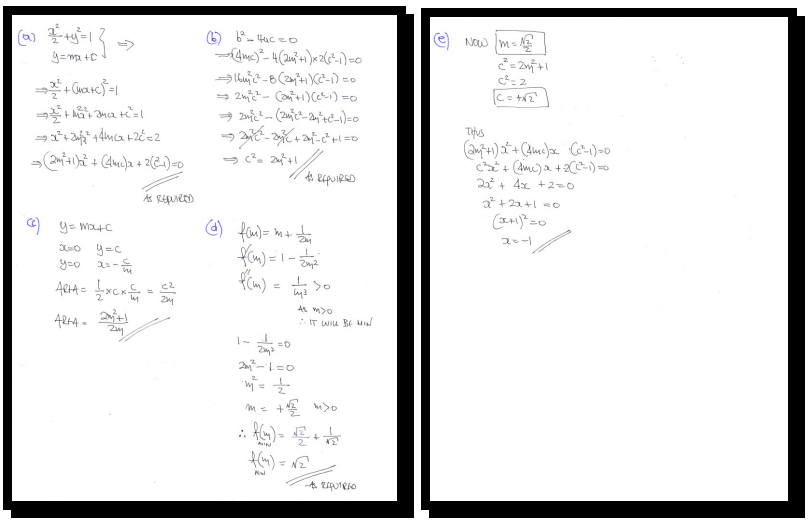
The line L meets the negative x axis and the positive y axis at the points X and Y respectively. The point O is the origin.

c) Find the area of the triangle OXY, in terms of m

d) Show that as m varies, the minimum area of the triangle OXY is $\sqrt{2}$.

e) Find the x coordinate of the point of tangency between the line L and the ellipse when the area of the triangle is minimum.

$$\boxed{\text{area}=m+\frac{1}{2m}}\,,\qquad \boxed{x=-1}$$



Question 18 (****)

The point $P(x, y)$ lies on an ellipse with foci at $A(2,0)$ and $B(6,0)$.

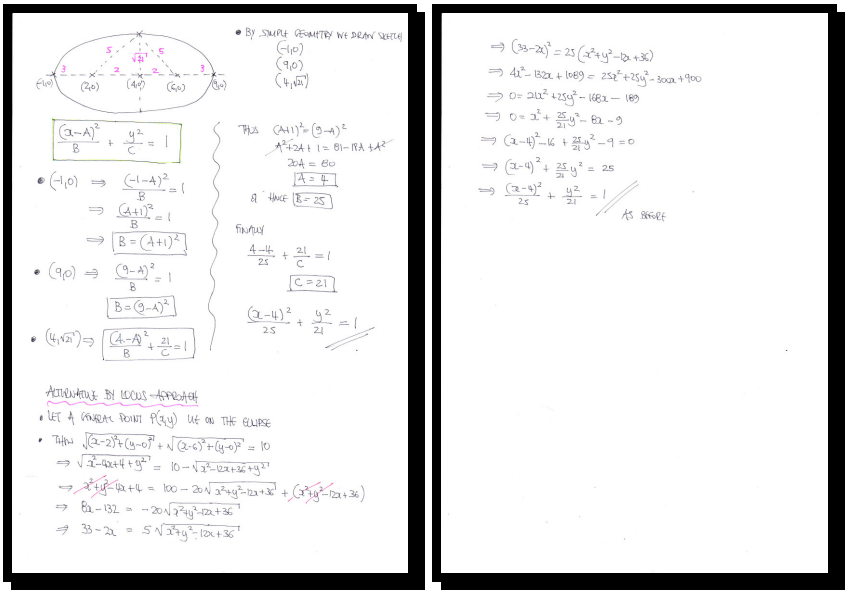
Given further that

$$|AP|+|BP|=10,$$

determine a simplified Cartesian equation for the ellipse, giving the final answer in the form

$$f(x, y)=1.$$

$$\boxed{\frac{(x-4)^2}{25}+\frac{y^2}{21}=1}$$



Question 19 (****)

An ellipse has Cartesian equation

$$\frac{x^2}{25}+\frac{y^2}{9}=1\,.$$

The general point $P\left(5\cos\theta,3\sin\theta\right)$ lies on the ellipse.

a) Show that the equation of the normal to the ellipse at P is

$$3y\cos\theta-5x\sin\theta+16\sin\theta\cos\theta=0\,.$$

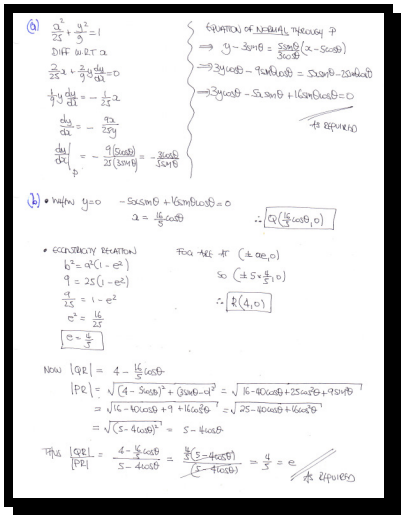
The normal to the ellipse at P meets the x axis at the point Q and R is one of the foci of the ellipse.

b) Show clearly that

$$\frac{|QR|}{|PR|}=e\,,$$

where e is the eccentricity of the ellipse.

proof



Question 20 (**+)**

An ellipse is given, in terms of a parameter θ, by the equations

$$x = 3\sqrt{2}\cos\theta, \quad y = 4\sin\theta, \quad 0 \le \theta < 2\pi .$$

a) Determine …

i. … the coordinates of the foci of the ellipse.

ii. … the equations of the directrices of the ellipse.

b) Show that an equation of the tangent at a general point on the ellipse is

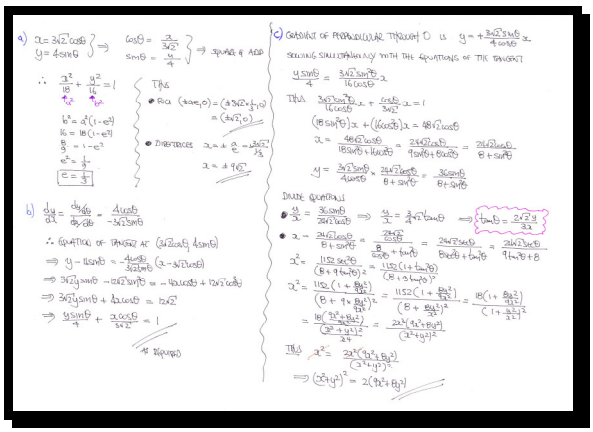
$$\frac{y\sin\theta}{4} + \frac{x\cos\theta}{3\sqrt{2}} = 1 .$$

A straight line passes through the origin and meets the general tangent whose equation is given in part (b), at the point P.

c) Show that, as θ varies, P traces the curve with equation

$$\left(x^2 + y^2\right)^2 = 2\left(9x^2 + 8y^2\right).$$

$\boxed{F\left(\pm\sqrt{2},0\right)}$, $\boxed{x = \pm 9\sqrt{2}}$



Question 21 (**+)**

The equation of an ellipse is given by

$$\frac{x^2}{4}+\frac{y^2}{3}=1\,.$$

a) Determine the coordinates of the foci of the ellipse, and the equation of each of its two directrices.

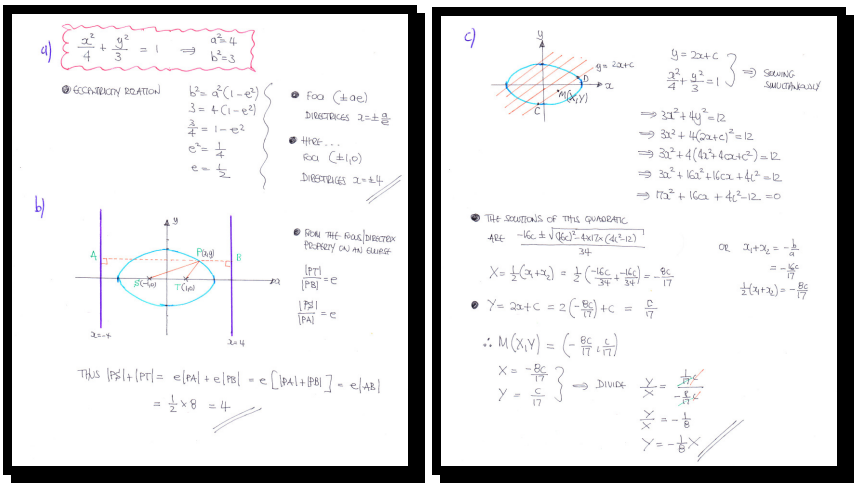
b) Show that

$$|SP|+|TP|=4\,.$$

A chord of the ellipse is defined as the straight line segment joining any two distinct points on the ellipse.

c) Find the equation of the locus of the midpoints of parallel chords of the ellipse whose gradient is 2.

$$\boxed{(\pm 1,0)}\,,\quad \boxed{x=\pm 4}\,,\quad \boxed{y=-\frac{1}{8}x}$$

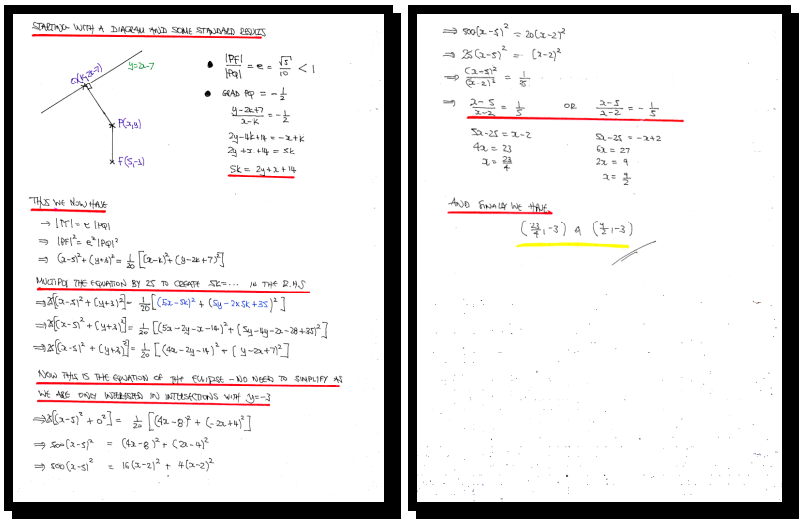


Question 22 (*****)

An ellipse has a focus at $(5,-3)$ and directrix with equation $y=2x-7$.

Given that the eccentricity of the ellipse is $\dfrac{\sqrt{5}}{10}$, find the coordinates of the points of intersection of the ellipse with the straight line with equation $y=-3$.

V, ☐, $\left(\frac{23}{4},-3\right),\ \left(\frac{9}{2},-3\right)$



Question 23 (*****)

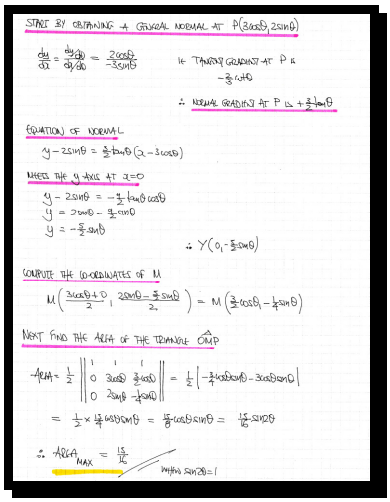
The point P lies on the ellipse with parametric equations

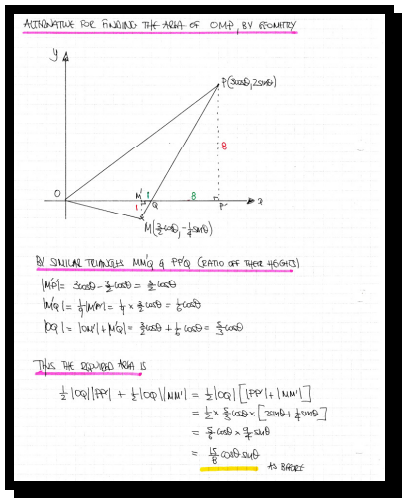
$$x = 3\cos\theta \qquad y = 2\sin\theta \qquad 0 \le \theta \le \frac{1}{2}\pi .$$

The point M is the midpoint of PY, where Y is the point where the normal to ellipse at P meets the y axis.

If O represents the origin, determine the maximum value of the area of the triangle OMP, as θ varies.

V, $\square$, $\text{Area}_{\max} \frac{15}{16}$





Question 24 (*****)

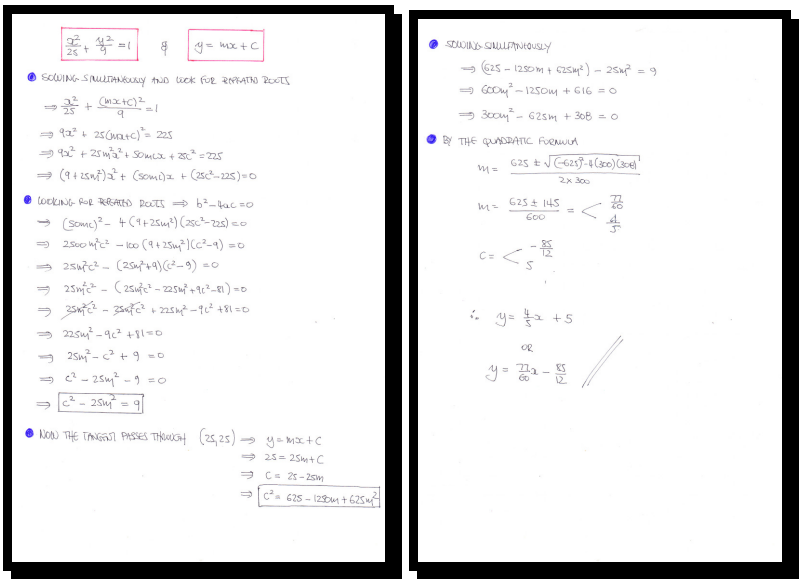
The straight line L with equation $y = mx + c$, where m and c are constants, passes through the point $(25, 25)$.

Given further that L is a tangent to the ellipse with equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1,$$

determine the possible equations of L.

$$\boxed{y = \frac{4}{5}x + 5}, \quad \boxed{y = \frac{77}{60}x - \frac{85}{12}}$$



Question 25 **(*****)**

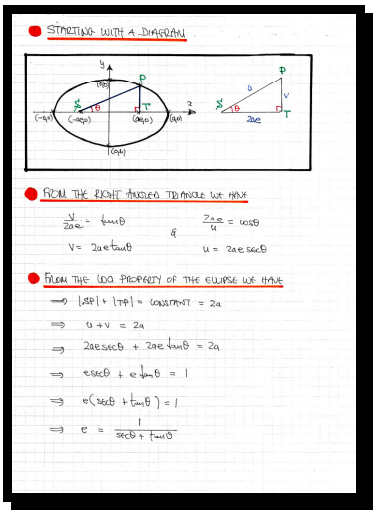
The point P lies on an ellipse whose foci are on the x axis at the points S and T.

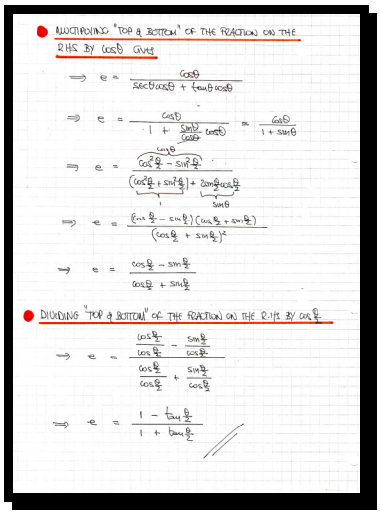
Given further that the triangle STP is right angled at T, show that

$$e=\frac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta},$$

where e is the eccentricity of the ellipse, and θ is the angle PST.

, proof





Question 26 **(*****)**

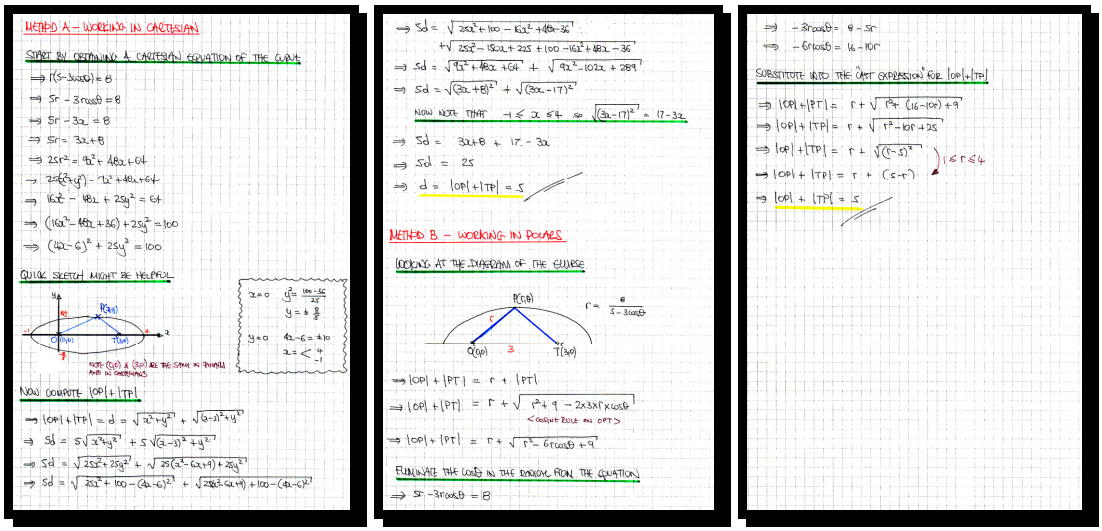
The point P lies on the ellipse with polar equation

$$r(5-3\cos\theta)=8,\ 0\le\theta<2\pi.$$

The ellipse has foci at $O(0,0)$ and at $T(3,0)$.

Show that $|OP|+|PT|$ is constant for all positions of P.

$\boxed{}$, $\boxed{|OP|+|PT|=5}$



RECTANGULAR HYPERBOLA

Question 1 (**)

The rectangular hyperbola H has Cartesian equation

$$xy = 9, \quad x \neq 0, \quad y \neq 0.$$

The point $P\left(3t, \frac{3}{t}\right)$, $t \neq 0$, where t is a parameter, lies on H.

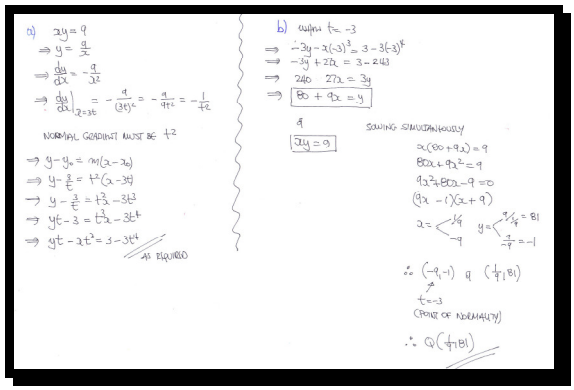
a) Show that the equation of a normal to H at P is given by

$$yt - xt^3 = 3 - 3t^4.$$

The normal to H at the point where $t = -3$ meets H again at the point Q.

b) Determine the coordinates of Q.

$$\boxed{Q\left(\frac{1}{9}, 81\right)}$$



Question 2 (**)

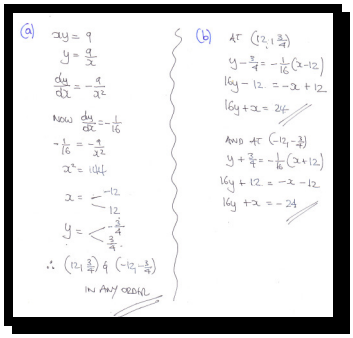
The tangents to the hyperbola with equation $xy=9$, at two distinct points A and B, have gradient $-\dfrac{1}{16}$.

Determine in any order …

a) … the coordinates of A and B.

b) … the equation of each of the two tangents.

$$\boxed{A\left(12,\frac{3}{4}\right),\ B\left(-12,-\frac{3}{4}\right)},\quad \boxed{x+16y=24,\ x+16y=-24}$$



Question 3 (+)**

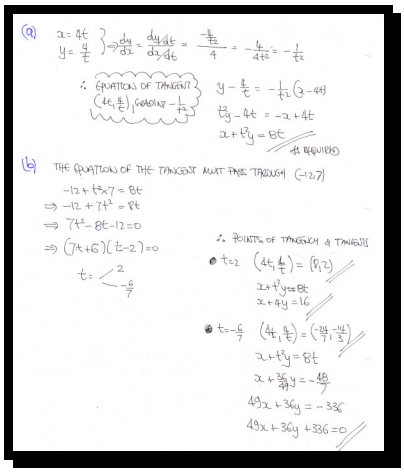
The general point $P\left(4t,\frac{4}{t}\right)$, $t\neq 0$, where t is a parameter, lies on a hyperbola H.

a) Show that the equation of a tangent at the point P is given by

$$x+t^2y=8t\,.$$

b) Find the equation of each of the two tangents to H which pass through the point $Q(-12,7)$, and further deduce the coordinates of their corresponding points of tangency.

$$\boxed{x+4y=16,\ (8,2)}\,,\quad \boxed{49x+36y+336=0,\ \left(-\frac{24}{7},-\frac{14}{3}\right)}$$



Question 4 (***)

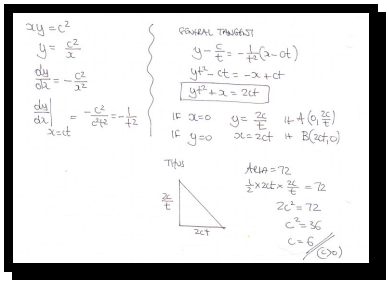
The general point $P\left(ct,\frac{c}{t}\right),\ c>0,\ t>0$, lies on a hyperbola H with Cartesian equation

$$xy=c^2 .$$

The tangent to H at P meets the coordinate axes at the points A and B.

Given the area of the triangle BOA is 72 square units, find the value of c.

$\boxed{c=6}$



Question 5 (***)

The general point $P\left(3t,\frac{3}{t}\right)$, $t\neq 0$, where t is a parameter, lies on a hyperbola H.

a) Show that the equation of a tangent at the point P is given by

$$x+t^2y=6t\,.$$

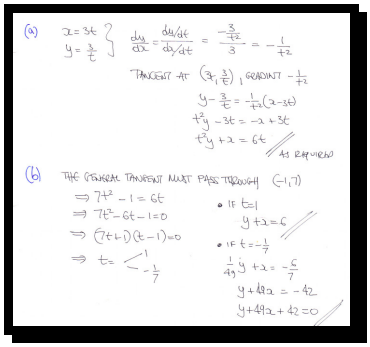
The tangents to the hyperbola at points A and B intersect at the point $Q(-1,7)$.

b) Determine in any order …

i. …the coordinates of A and B.

ii. … the equation of each of the two tangents.

$$\boxed{A(3,3),\ \ B\left(-\frac{3}{7},-21\right)},\qquad \boxed{x+y=6,\ \ 49x+y+42=0}$$



Question 6 **(***+)**

The point $P\left(ap,\frac{a}{p}\right)$ lies on the rectangular hyperbola H, with Cartesian equation

$$xy=a^2,$$

where a is a positive constant and p is a parameter.

a) Show that the equation of a tangent at the point P is given by

$$x+p^2y=2ap.$$

The point $Q\left(aq,\frac{a}{q}\right)$ also lies on H, where q is a parameter, so that $q\neq p$.

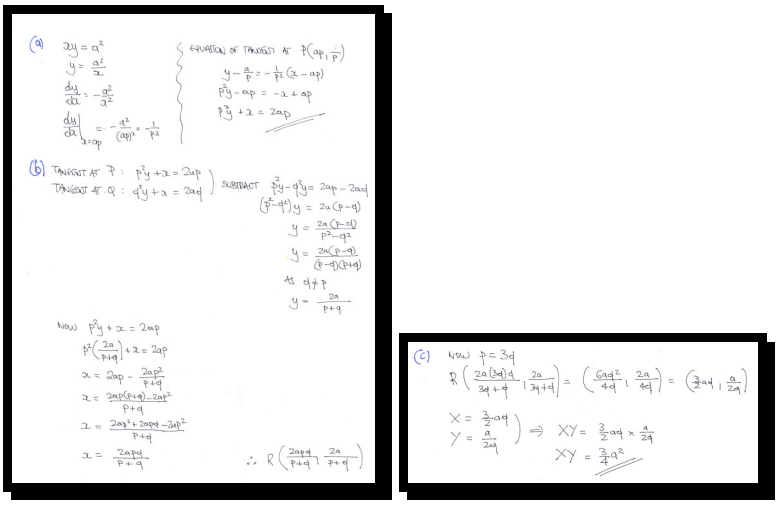
The tangent at P and the tangent at Q intersect at the point R.

b) Find simplified expressions for the coordinates of R.

The values of p and q are such so that $p=3q$.

c) Find a Cartesian locus of R as p varies.

$$\boxed{R\left(\frac{2apq}{p+q},\frac{2a}{p+q}\right)},\quad \boxed{xy=\frac{3}{4}a^2}$$



Question 7 (****)

The general point $P\left(cp,\frac{c}{p}\right)$, $p\neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy=c^2,$$

where c is a positive constant.

a) Show that an equation of the tangent to the hyperbola at P is given by

$$yp^2+x=2cp.$$

Another point $Q\left(cq,\frac{c}{q}\right)$, $p\neq\pm q$ also lies on the hyperbola.

The tangents to the hyperbola at P and Q meet at the point R.

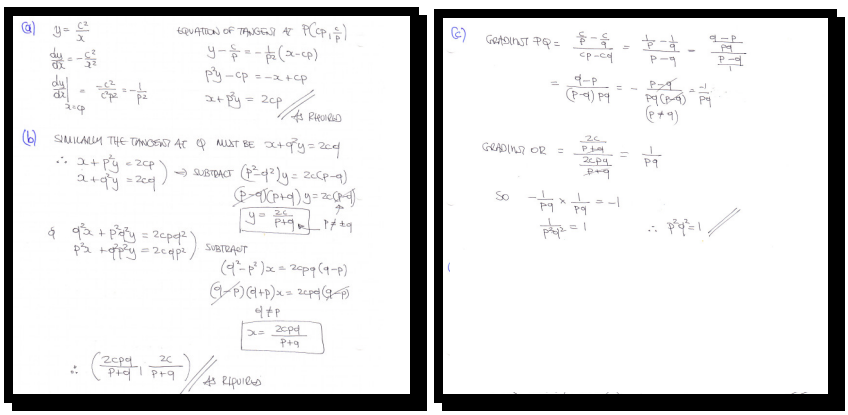
b) Show that the coordinates of R are given by

$$\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right).$$

c) Given that PQ is perpendicular to OR, show that

$$p^2q^2=1.$$

proof



Question 8 (****)

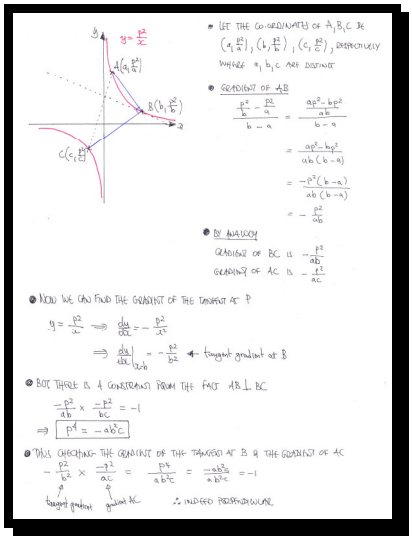
The distinct points A, B and C lie on the hyperbola with equation

$$xy = p^2,$$

where p is a positive constant.

Given that ABC is a right angle, show that the tangent to the hyperbola at B, is perpendicular to AC.

proof



Question 9 (**+)**

The general point $P\left(5t,\frac{5}{t}\right)$, $t\neq 0$, where t is a parameter, lies on the hyperbola, with Cartesian equation

$$xy=25.$$

a) Show that an equation of the normal to the hyperbola at the point P is

$$y=t^2x+\frac{5}{t}-5t^3$$

The normal to the hyperbola at P meets the hyperbola again at the point Q.

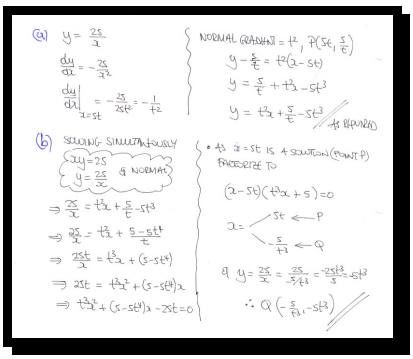
b) Show that the coordinates of Q are given by

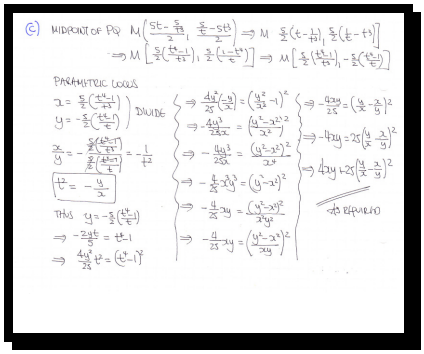
$$\left(-\frac{5}{t^3},-5t^3\right).$$

c) Show that the Cartesian form of the locus of the midpoint of PQ, as t varies, is given by

$$4xy+25\left(\frac{y}{x}-\frac{x}{y}\right)^2=0.$$

proof





Question 10 (**+)**

The general point $P\left(cp,\dfrac{c}{p}\right)$, $p\neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy=c^2,$$

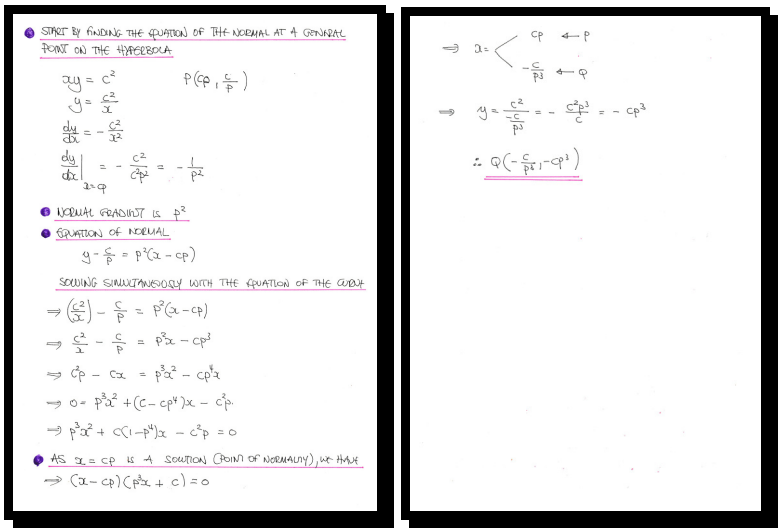
where c is a positive constant.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

Show that the coordinates of Q are

$$\left(-\frac{c}{p^3},-cp^3\right).$$

$\boxed{}$, proof



Question 11 (**+)**

The general point $P\left(\frac{p}{2},\frac{1}{2p}\right)$, $p\neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

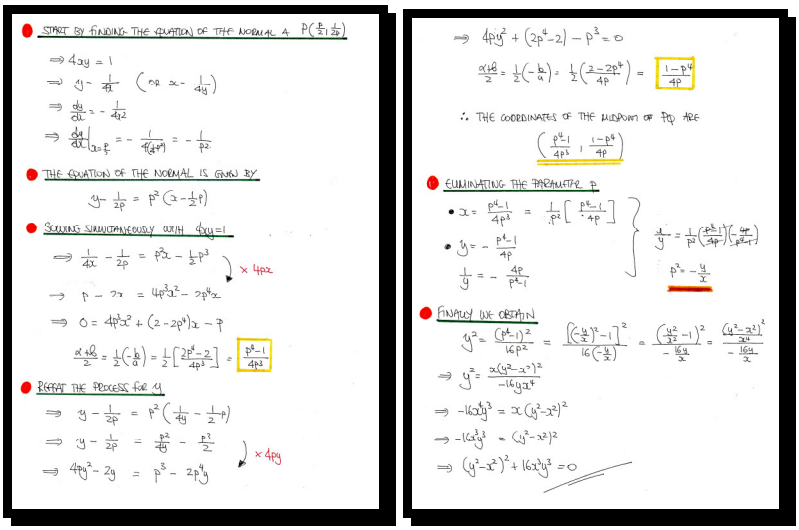
$$4xy=1.$$

The normal to the hyperbola at P meets the hyperbola again at the point Q.

Show that the Cartesian form of the locus of the midpoint of PQ, as p varies, is

$$\left(y^2-x^2\right)^2+16x^3y^3=0.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 12 (**+)**

Two distinct points $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$, lie on the hyperbola with Cartesian equation $xy=4$.

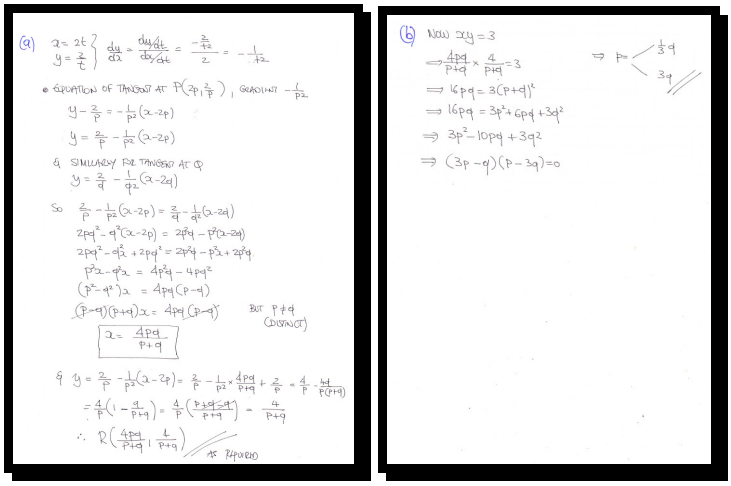
The tangents to the hyperbola at the points P and Q, meet at the point R.

a) Show that the coordinates of the point R are given by

$$x=\frac{4pq}{p+q}\ ,\quad y=\frac{4}{p+q}.$$

b) Given that the point R traces the rectangular hyperbola $xy=3$, find the two possible relationships between p and q, in the form $p=f(q)$

$$\boxed{p=3q}\ ,\quad \boxed{p=\frac{1}{3}q}$$



Question 13 (**+)**

The general point $P\left(2t,\frac{2}{t}\right)$, $t\neq 0$, where t is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy=4.$$

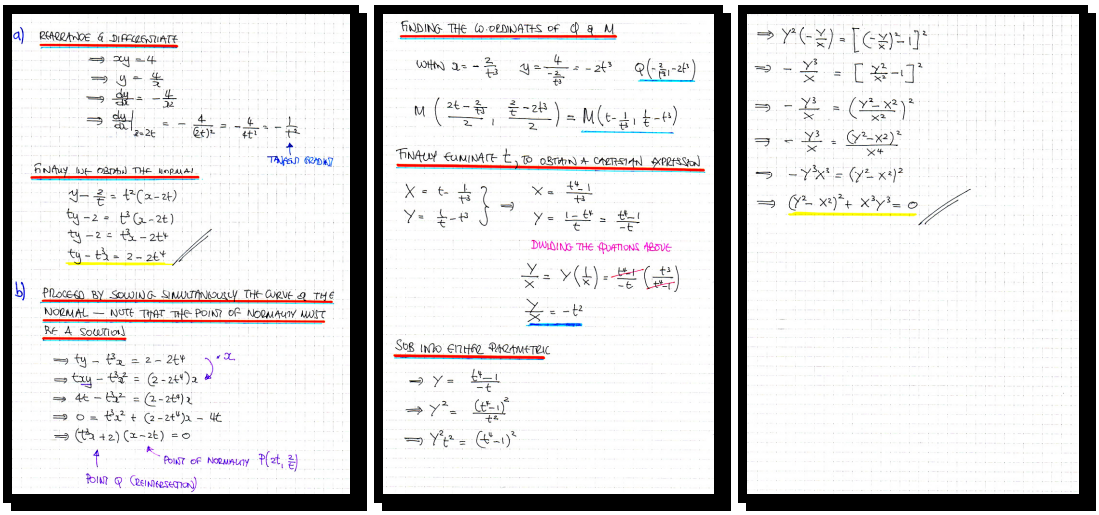
a) Find an equation of the normal to the hyperbola at the point P.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

The point M is the midpoint of PQ.

b) Find an equation of the locus of M, as t varies.
Give a simplified answer in the form $f(x,y)=0$.

$\square$, $ty-2=t^3x-2t^4$, $\left(y^2-x^2\right)^2+x^3y^3=0$



Question 14 (*****)

The point $P\left(p+\frac{1}{p},p-\frac{1}{p}\right)$, $p\neq 0$, lies on the rectangular hyperbola, with Cartesian equation

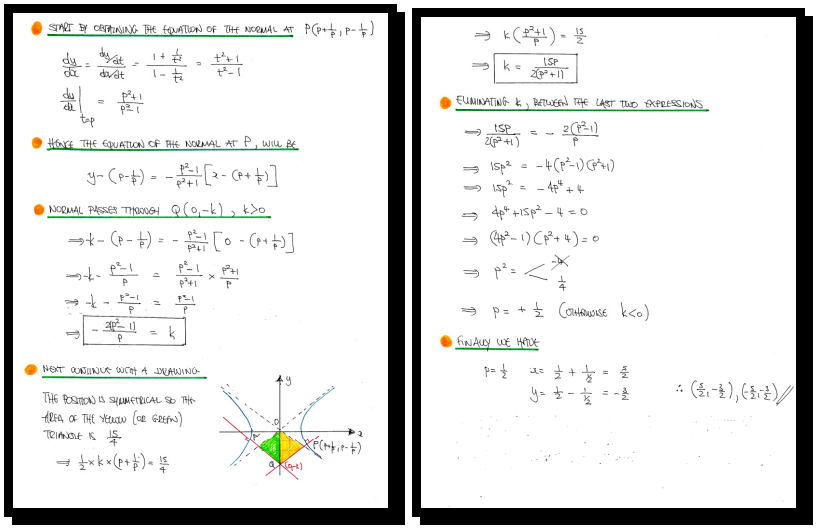
$$x^2-y^2=4\,.$$

The normal to the hyperbola at P meets the y axis at the point $Q(0,-k)$, $k>0$.

The area of the triangle OPQ, where O is the origin, is $\frac{15}{4}$.

Determine the two possible sets of coordinates for P.

$\square$, $\boxed{\left(\frac{5}{2},-\frac{3}{2}\right),\ \left(\frac{5}{2},-\frac{3}{2}\right)}$



Question 15 (*****)

The points P and Q are two distinct points which lie on the curve with equation

$$y=\frac{1}{x},\ x\in\mathbb{R},\ x\neq 0.$$

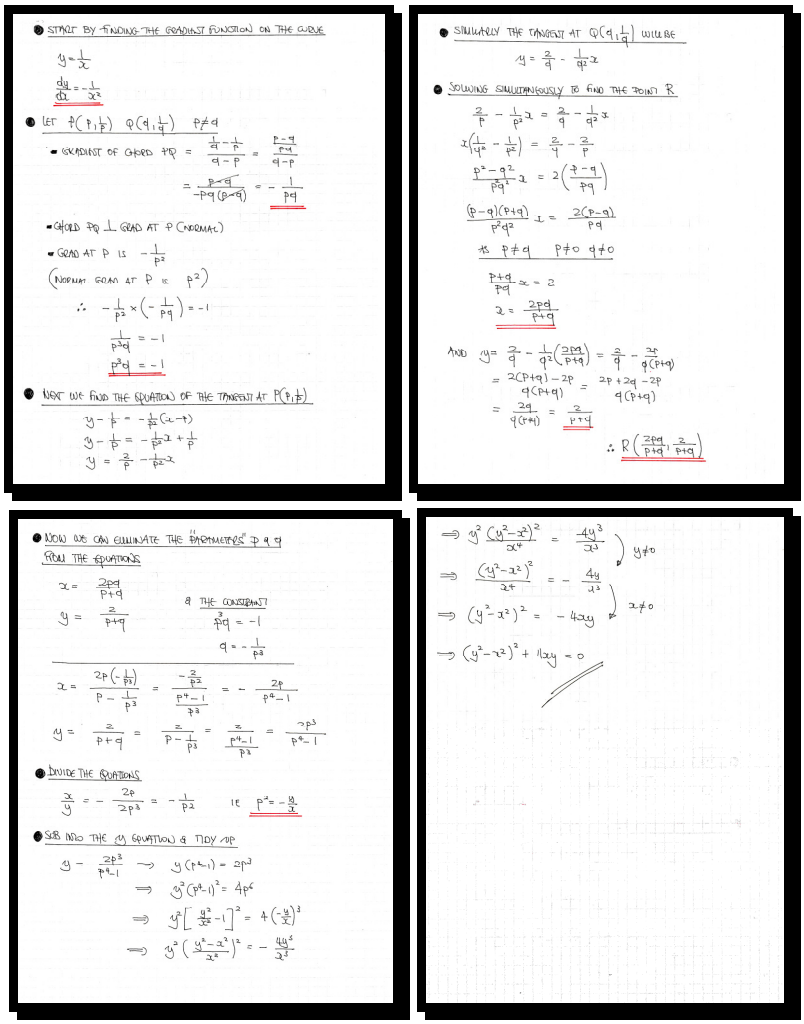
P and Q are free to move on the curve so that the straight line segment PQ is a normal to the curve at P.

The tangents to the curve at P and Q meet at the point R.

Show that R is moving on the curve with Cartesian equation

$$\left(y^2-x^2\right)^2+4xy=0.$$

$\boxed{}$, proof



Question 16 (*****)

The variable point P lies on the rectangular hyperbola, with Cartesian equation

$$xy = a^2,$$

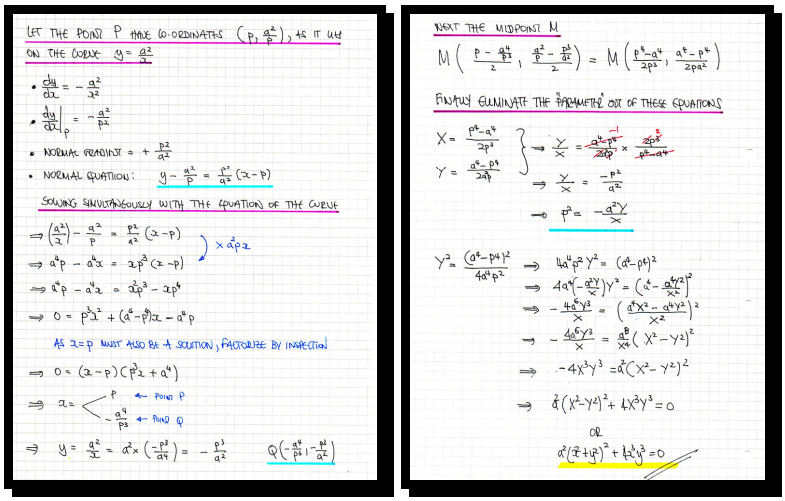
where a is a positive constant.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

The point M is the midpoint of PQ.

Determine, in the form $f(x, y) = 0$, an equation of the locus of M, for all the possible positions of P.

$\square$, $\boxed{a^2\left(y^2 - x^2\right)^2 + 4x^3y^3 = 0}$



HYPERBOLA

Question 1 (**)

A hyperbola H has foci at the points with coordinates $(-10,0)$ and $(10,0)$, and its Cartesian equation is given by

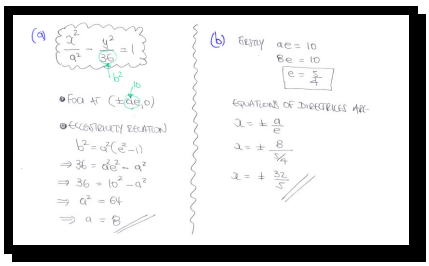
$$\frac{x^2}{a^2}-\frac{y^2}{36}=1,$$

where a is a positive constant.

a) Find the value of a.

b) Deduce the equations of the directrices of H.

$\boxed{a=8}$, $\boxed{x=\pm\frac{32}{5}}$

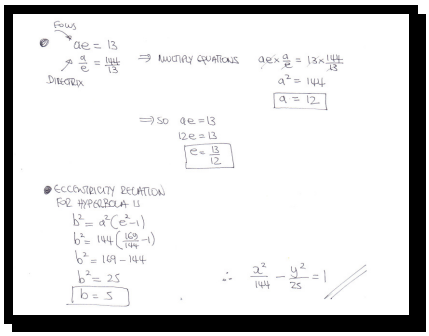


Question 2 (**)

A hyperbola H has foci at the points with coordinates $(\pm 13, 0)$ and the equations of its directrices are $x = \pm \frac{144}{13}$.

Determine a Cartesian equation for H.

$$\boxed{\frac{x^2}{144} - \frac{y^2}{25} = 1}$$



Question 3 **(***)**

A hyperbola is given parametrically by

$$x=\frac{3}{2}\left(t+\frac{1}{t}\right), \quad y=\frac{5}{2}\left(t-\frac{1}{t}\right), \quad t\neq 0.$$

a) Show that the Cartesian equation of the hyperbola can be written as

$$\frac{x^2}{9}-\frac{y^2}{25}=1.$$

b) Find …

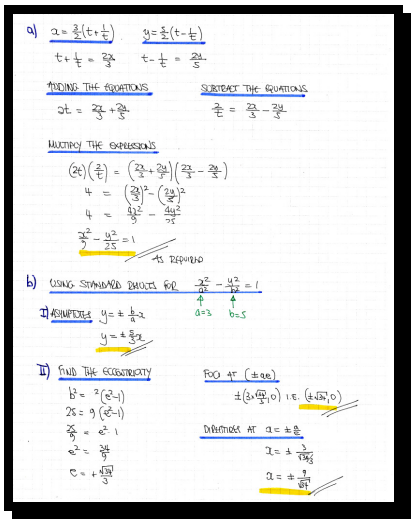
i. … the equations of its asymptotes.

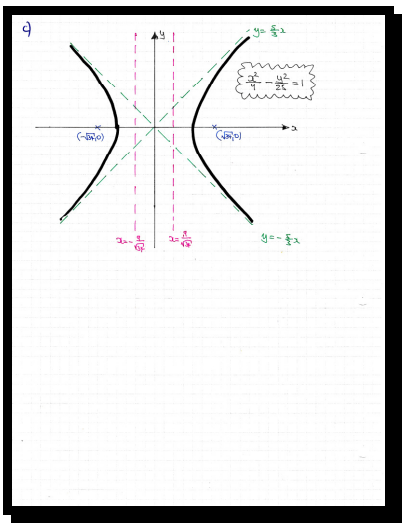
ii. … the coordinates of its foci.

iii. … the equations of its directrices.

c) Sketch the hyperbola indicating any intersections with the coordinate axes, as well as the information stated in part **(b)**.

$\boxed{}$, $\boxed{y=\pm\frac{5}{3}x}$ $\boxed{\left(\pm\sqrt{34},0\right)}$, $\boxed{x=\pm\frac{9}{\sqrt{34}}}$





Question 4 **(***)**

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola with equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

a) Show that an equation of the normal at P is given by

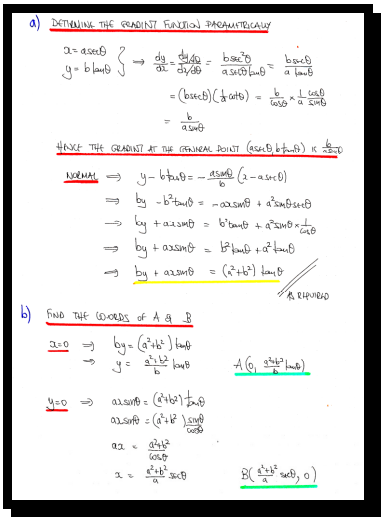
$$by+ax\sin\theta=\left(a^2+b^2\right)\tan\theta\ .$$

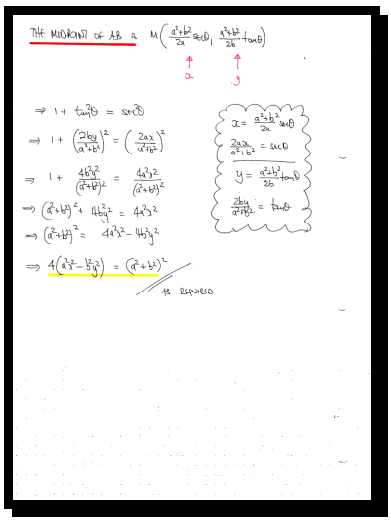
The normal to the hyperbola meets the coordinate axes at the points A and B.

b) Show that, as θ varies, the Cartesian locus of the midpoint of AB is given by

$$4\left(a^2x^2-b^2y^2\right)=\left(a^2+b^2\right)^2.$$

[ ], [proof]





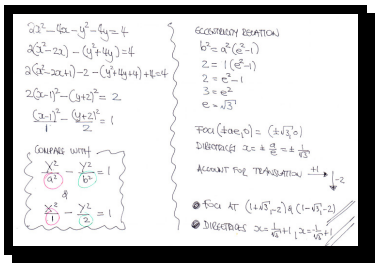
Question 5 (*+)**

A hyperbola has Cartesian equation

$$2x^2-4x-y^2-4y=4\,.$$

Find the coordinates of its foci and the equations of its directrices.

$$\boxed{\left(1+\sqrt{3},-2\right),\ \left(1-\sqrt{3},-2\right)},\quad \boxed{x=-\frac{1}{\sqrt{3}}+1,\quad x=\frac{1}{\sqrt{3}}+1}$$



Question 6 (****)

The general point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola with Cartesian equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

a) Show that an equation of the normal at P is given by

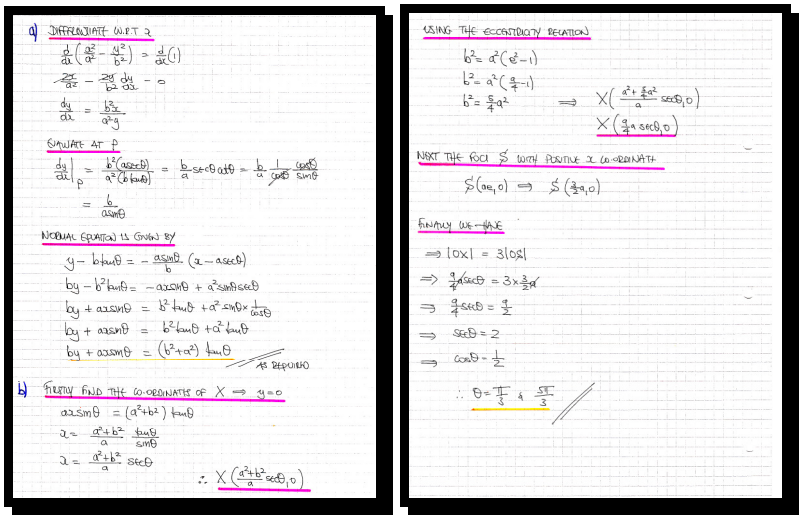
$$by+ax\sin\theta=\left(a^2+b^2\right)\tan\theta.$$

The normal to the hyperbola meets the x axis at the point X.

The eccentricity of the hyperbola is $\frac{3}{2}$ and its foci are denoted by S and S', where S has a positive x coordinate.

b) Given that $|OX|=3|OS|$, find the possible values of θ for $0\le\theta\le 2\pi$.

$\square$, $\boxed{\theta=\frac{\pi}{3},\frac{5\pi}{3}}$



Question 7 (****)

The equation of a hyperbola H is given in terms of a parameter t by

$$x=\sinh t\,,\quad y=\cosh t\,,\quad t\in\mathbb{R}\,.$$

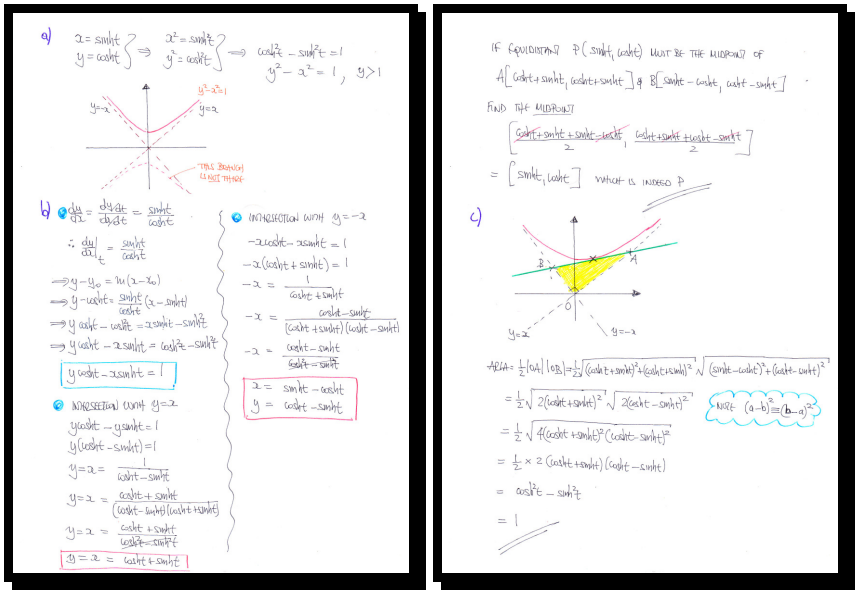
a) Sketch the graph of H, clearly marking the equation of each of its asymptotes.

The equation of the tangent to H at the point $P\left(\sinh t,\cosh t\right)$, meets each of the asymptotes at the points A and B.

b) Show that P is equidistant from A and B.

c) Show further that the area of the triangle OAB, where O is the origin, is exactly 1 square unit.

graph/proof



Question 8 **(****+)**

A hyperbola H and a line L have the following Cartesian equations

$$H:\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$L:\ y=mx+c\,,$$

where a, b, m and c are non zero constants.

a) Show that the x coordinates of the points of intersection between L and H satisfy the equation

$$\left(a^2m^2-b^2\right)x^2+\left(2a^2mc\right)x+a^2\left(b^2+c^2\right)=0\,.$$

b) Given the line is a tangent to the hyperbola show that

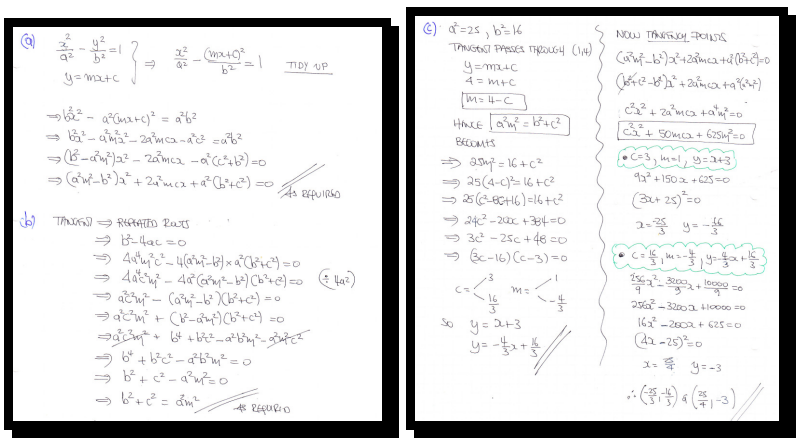
$$a^2m^2=b^2+c^2\,.$$

c) Find the equations of the two tangents to the hyperbola with Cartesian equation

$$\frac{x^2}{25}-\frac{y^2}{16}=1$$

that pass through the point $(1,4)$, and for each tangent the coordinates of their point of tangency.

$$\boxed{y=x+3,\ \left(-\tfrac{25}{3},-\tfrac{16}{3}\right)},\quad \boxed{y=-\tfrac{4}{3}x+\tfrac{16}{3},\ \left(\tfrac{25}{4},-3\right)}$$



Question 9 **(****+)**

A hyperbola H has Cartesian equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

The straight line T_1 is the tangent to H at the point $\left(a\cosh\theta, b\sinh\theta\right)$.

T_1 meets the x axis at the point P.

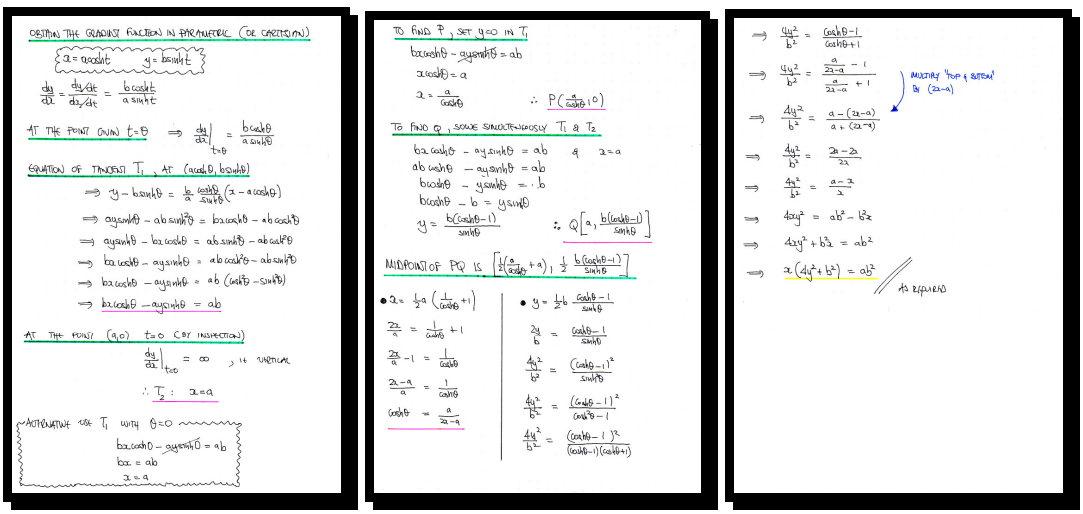
The straight line T_2 is a tangent to the hyperbola at the point $\left(a,0\right)$.

T_1 and T_2 meet each other at the point Q.

Given further that M is the midpoint of PQ, show that as θ varies, the locus of M traces the curve with equation

$$x\left(4y^2+b^2\right)=ab^2.$$

, proof



Question 10 (*****)

A hyperbola and an ellipse have respective equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{and} \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a>b>0$.

The tangent to the hyperbola, at a point whose both coordinates are positive, passes through the focus of the ellipse with positive x coordinate.

Show that the gradient of the above described tangent is 1.

V, [ ], proof

