COMPLEX NUMBERS PRACTICE (part 2)

ROOTS OF COMPLEX NUMBERS

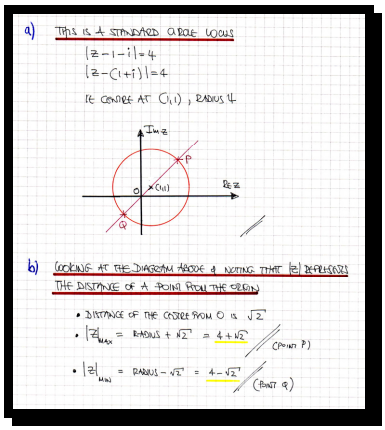
Question 1

$$z^4 = -16\,,\ z \in \mathbb{C}\,.$$

a) Solve the above equation, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \sqrt{2}\left(\pm 1 \pm \mathrm{i}\right)}$$



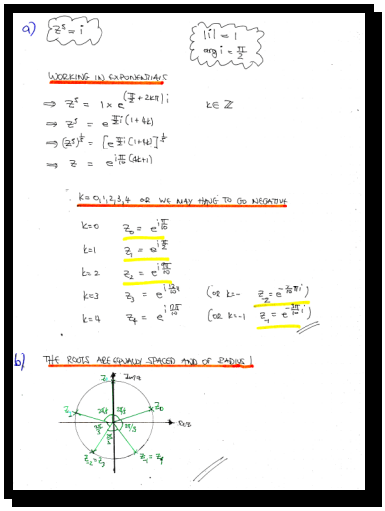
Question 2

$$z^5 = \mathrm{i}\,, \quad z \in \mathbb{C}\,.$$

a) Solve the equation, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{3\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{7\pi}{10}}}$$



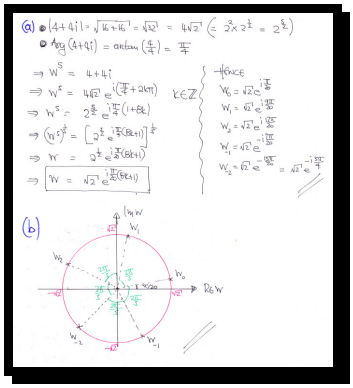
Question 3

$$z = 4 + 4\mathrm{i} \, .$$

a) Find the fifth roots of z.
Give the answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0, -\pi < \theta \leq \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{7\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}}$$



Question 4

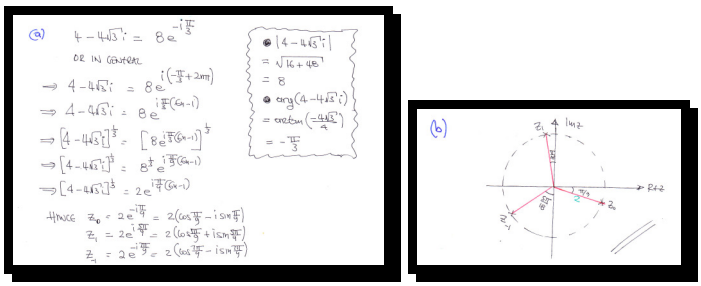
$$z = 4 - 4\sqrt{3}\mathrm{i}\,.$$

a) Find the cube roots of z.

Give the answers in polar form $r\left(\cos\theta + \mathrm{i}\sin\theta\right)$, $r > 0$, $-\pi < \theta \leq \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{z = 2\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right),\ z = 2\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right),\ z = 2\left(\cos\frac{7\pi}{9} - \mathrm{i}\sin\frac{7\pi}{9}\right)}$$



Question 5

The following complex number relationships are given

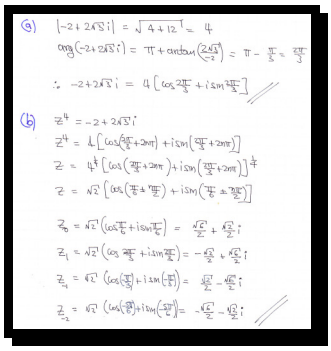
$$w=-2+2\sqrt{3}\mathrm{i}\,,\quad z^4=w\,.$$

a) Express w in the form $r\left(\cos\theta+\mathrm{i}\sin\theta\right)$, where $r>0$ and $-\pi<\theta\leq\pi$.

b) Find the possible values of z, giving the answers in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{w=2\left[\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]},$$

$$\boxed{z=\frac{1}{2}\left(\sqrt{6}+\mathrm{i}\sqrt{2}\right),\ z=\frac{1}{2}\left(-\sqrt{2}+\mathrm{i}\sqrt{6}\right),\ z=\frac{1}{2}\left(\sqrt{2}-\mathrm{i}\sqrt{6}\right),\ z=\frac{1}{2}\left(-\sqrt{6}-\mathrm{i}\sqrt{2}\right)}$$

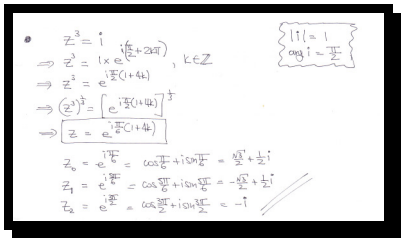


Question 6

Find the cube roots of the imaginary unit i, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

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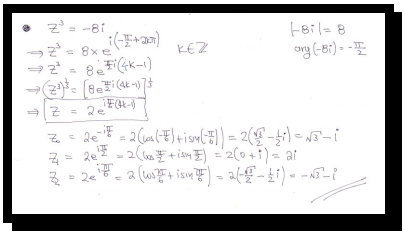
$$\boxed{z_1=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i},\ \ z_2=-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i},\ \ z_3=-\mathrm{i}}$$



Question 7

Find the cube roots of the complex number $-8\mathrm{i}$, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z_1=\sqrt{3}-\mathrm{i},\ \ z_2=-\sqrt{3}-\mathrm{i},\ \ z_3=2\mathrm{i}}$$

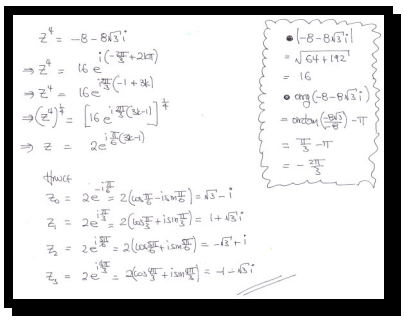


Question 8

$$z^4=-8-8\sqrt{3}\,\mathrm{i},\ z\in\mathbb{C}.$$

Solve the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z=\sqrt{3}-\mathrm{i}},\ \boxed{z=1+\sqrt{3}\,\mathrm{i}},\ \boxed{z=-\sqrt{3}+\mathrm{i}},\ \boxed{z=-1-\sqrt{3}\,\mathrm{i}}$$

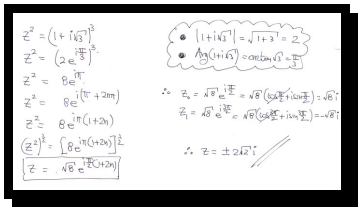


Question 9

$$z^2 = \left(1+\mathrm{i}\sqrt{3}\right)^3,\ z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z = \pm \mathrm{i} 2\sqrt{2}}$$



Question 10

$$z^3 = 32+32\sqrt{3}\,\mathrm{i},\ z \in \mathbb{C}.$$

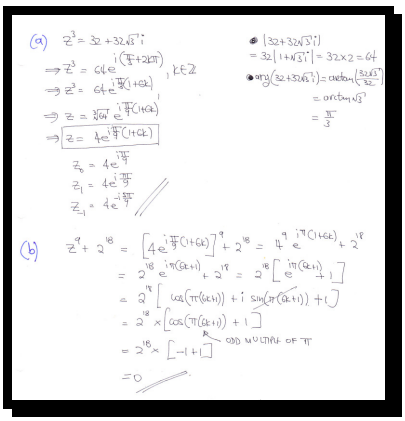
a) Solve the above equation.

Give the answers in exponential form $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r>0$, $-\pi < \theta \leq \pi$.

b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0.$$

$$\boxed{z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}}$$



Question 11

$$z^7 - 1 = 0, \ z \in \mathbb{C}.$$

One of the roots of the above equation is denoted by ω, where $0 < \arg \omega < \frac{\pi}{3}$.

a) Find ω in the form $\omega = r\mathrm{e}^{\mathrm{i}\theta}, \quad r > 0, \quad 0 < \theta \leq \frac{\pi}{3}$.

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$

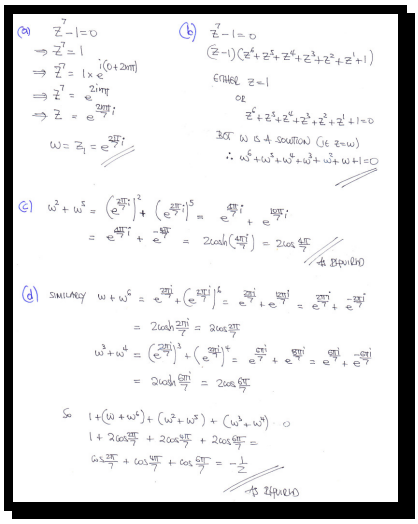
c) Show further that

$$\omega^2 + \omega^5 = 2\cos\left(\frac{4\pi}{7}\right).$$

d) Hence, using the results from the previous parts deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$

$$\boxed{\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{7}}}$$

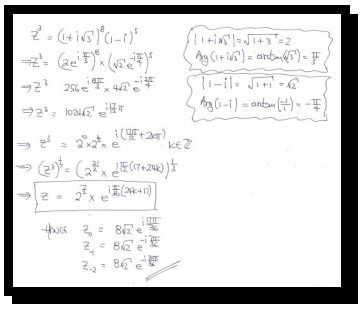


Question 12

$$z^3 = \left(1+\mathrm{i}\sqrt{3}\right)^8 \left(1-\mathrm{i}\right)^5 , \ z \in \mathbb{C} .$$

Find the three roots of the above equation, giving the answers in the form $k\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$, $k \in \mathbb{Z}$.

$$\boxed{z = 8\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}, \quad \theta = -\frac{31\pi}{36}, \ -\frac{7\pi}{36}, \ \frac{17\pi}{36}}$$



TRIGONOMETRIC IDENTITIES QUESTIONS

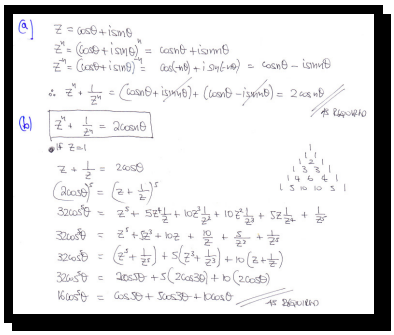
Question 1

If $z = \cos\theta + \mathrm{i}\sin\theta$, show clearly that …

a) … $z^n + \dfrac{1}{z^n} \equiv 2\cos n\theta$.

b) … $16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta$.

proof



Question 2

It is given that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

It is further given that

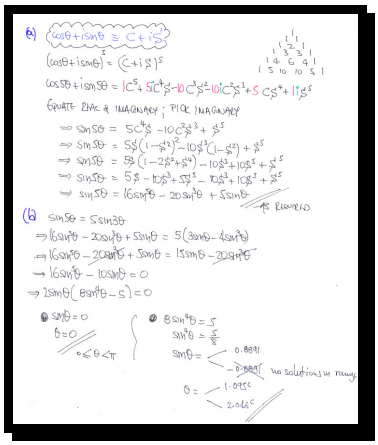
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

b) Solve the equation

$$\sin 5\theta = 5\sin 3\theta \quad \text{for} \quad 0 \le \theta < \pi ,$$

giving the solutions correct to 3 decimal places.

$$\boxed{\theta = 0,\ 1.095^{\text{c}},\ 2.046^{\text{c}}}$$



Question 3

The complex number z is given by

$$z = \mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \leq \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

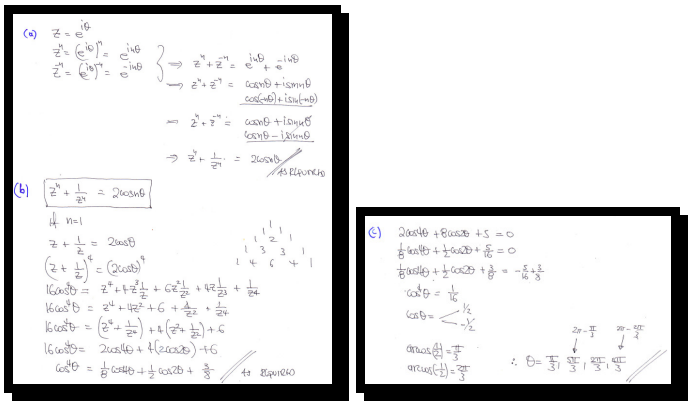
b) Hence show further that

$$\cos^4\theta \equiv \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} .$$

c) Solve the equation

$$2\cos 4\theta + 8\cos 2\theta + 5 = 0 , \ 0 \leq \theta < 2\pi .$$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$



Question 4

The complex number z is given by

$$z = \mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \leq \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

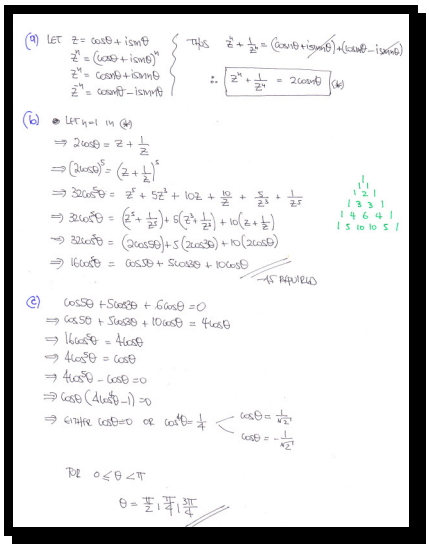
b) Hence show further that

$$16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta .$$

c) Use the results of part (a) and (b) to solve the equation

$$\cos 5\theta + 5\cos 3\theta + 6\cos\theta = 0 , \ 0 \leq \theta < \pi .$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}$$



Question 5

De Moivre's theorem asserts that

$$(\cos\theta+\mathrm{i}\sin\theta)^n \equiv \cos n\theta+\mathrm{i}\sin n\theta,\ \theta\in\mathbb{R},\ n\in\mathbb{Q}.$$

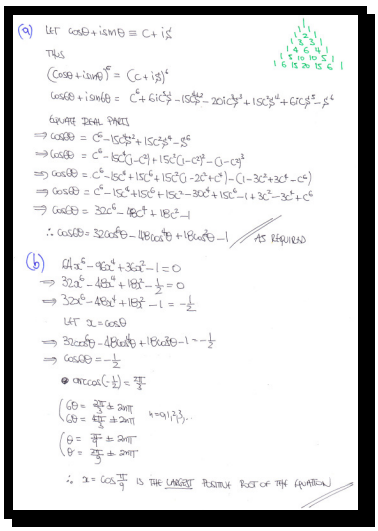
a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1.$$

b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6-96x^4+36x^2-1=0.$$

$$\boxed{x=\cos\left(\frac{\pi}{9}\right)}$$



Question 6

Euler's identity states

$$e^{i\theta} \equiv \cos\theta + i\sin\theta\,,\ \theta \in \mathbb{R}\,.$$

a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2\cos n\theta\,.$$

b) Hence show further that

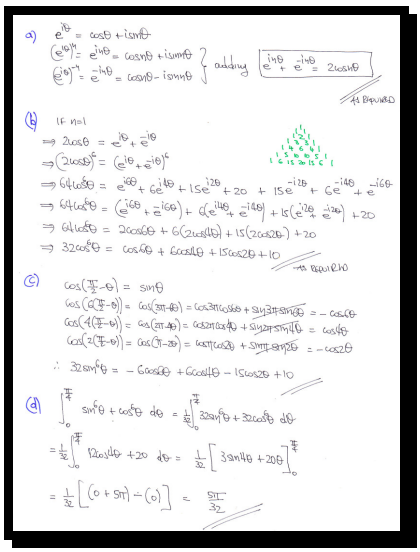
$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\,.$$

c) Use the fact that $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$ to find a similar expression for $32\sin^6\theta$.

d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta\,.$$

$$\boxed{32\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}\,,\ \boxed{\frac{5\pi}{32}}$$



Question 7

De Moivre's theorem asserts that

$$\left(\cos\theta+\mathrm{i}\sin\theta\right)^n\equiv\cos n\theta+\mathrm{i}\sin n\theta\,,\ \theta\in\mathbb{R}\,,\ n\in\mathbb{Q}\,.$$

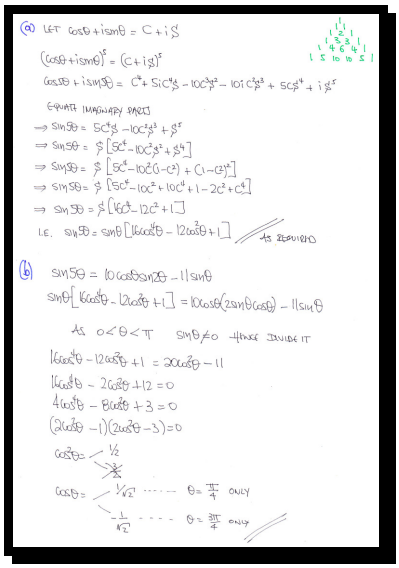
a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta=\sin\theta\left(16\cos^4\theta-12\cos^2\theta+1\right).$$

b) Hence, or otherwise, solve the equation

$$\sin 5\theta=10\cos\theta\sin 2\theta-11\sin\theta\,,\ 0<\theta<\pi\,.$$

$$\boxed{\theta=\frac{\pi}{4},\frac{3\pi}{4}}$$



Question 8

It is given that

$$\sin 5\theta \equiv \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

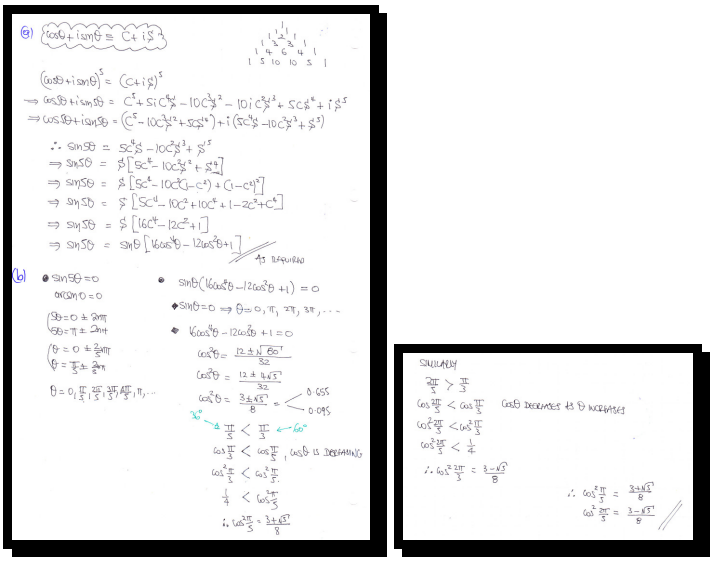
$$\sin 5\theta = 0.$$

b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{8}}, \quad \boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}}$$

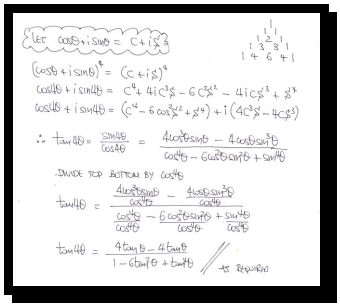


Question 9

By considering the binomial expansion of $(\cos\theta+\mathrm{i}\sin\theta)^4$ show that

$$\tan 4\theta \equiv \frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.$$

proof



Question 10

By using de Moivre's theorem followed by a suitable trigonometric identity, show clearly that …

a) … $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

b) … $\cos 6\theta \equiv \left(2\cos^2\theta - 1\right)\left(16\cos^4\theta - 16\cos^2\theta + 1\right)$

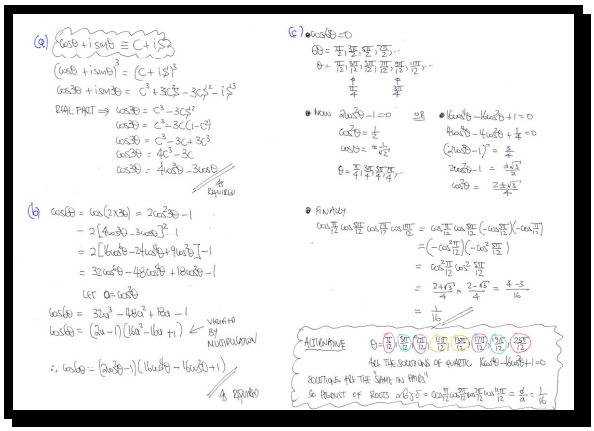
Consider the solutions of the equation.

$$\cos 6\theta = 0 \ , \ 0 \leq \theta \leq \pi \ .$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} \ .$$

$$\frac{1}{16}$$



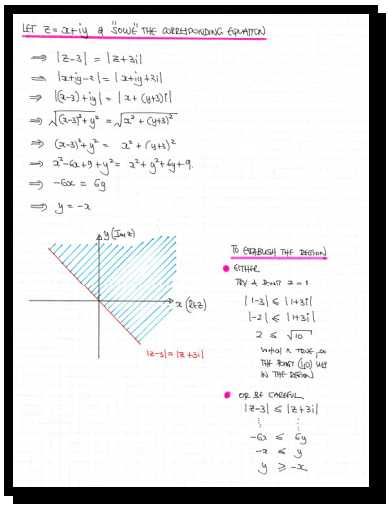
COMPLEX LOCI

Question 1

By finding a suitable Cartesian locus in the complex z plane, shade the region R that satisfies the inequality

$$|z-3| \leq |z+3\mathrm{i}|.$$

$$\boxed{x+y \geqslant 0}$$



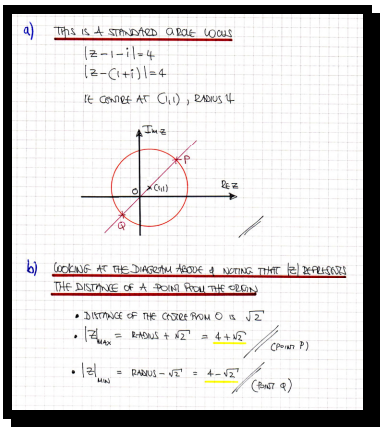
Question 2

$$|z-1-\mathrm{i}|=4\ ,\ z\in\mathbb{C}\ .$$

a) Sketch the locus of the points that satisfy the above equation in a standard Argand diagram.

b) Find the minimum and maximum values of $|z|$ for points that lie on this locus.

$$\boxed{z_{\min}=4-\sqrt{2}}\ ,\ \boxed{z_{\min}=4+\sqrt{2}}$$



Question 3

The complex number z represents the point $P(x, y)$ in the Argand diagram.

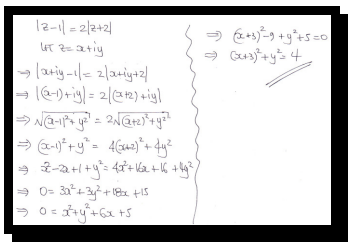
Given that

$$|z-1|=2|z+2|,$$

show that the locus of P is given by

$$(x+3)^2+y^2=4.$$

proof



Question 4

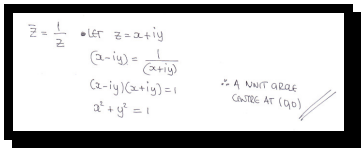
The complex number $z=x+\mathrm{i}y$ represents the point P in the complex plane.

Given that

$$\overline{z}=\frac{1}{z},\ z\neq 0$$

determine a Cartesian equation for the locus of P.

$x^2+y^2=1$



Question 5

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a) $|z-3+\mathrm{i}|=3$.

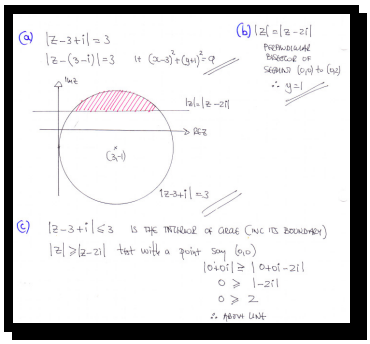
b) $|z|=|z-2\mathrm{i}|$.

Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-3+\mathrm{i}|\leq 3$$
$$|z|\geq|z-2\mathrm{i}|$$

$\boxed{(x-3)^2+(y+1)^2=9}$, $\boxed{y=1}$



Question 6

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

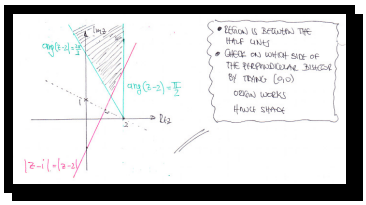
i. $|z-\mathrm{i}|=|z-2|$.

ii. $\arg(z-2)=\dfrac{\pi}{2}$.

b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-\mathrm{i}|\le|z-2| \qquad \text{and} \qquad \frac{\pi}{2}\le\arg(z-2)\le\frac{2\pi}{3}.$$

sketch



Question 7

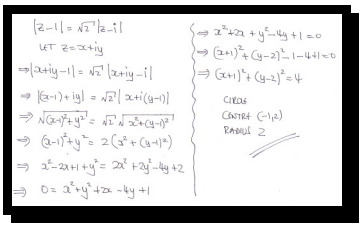
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z-1|=\sqrt{2}|z-\mathrm{i}|,$$

show that the locus of P is a circle, stating its centre and radius.

$$\boxed{(x+1)^2+(y-2)^2=4}\,,\quad \boxed{(-1,2),r=2}$$



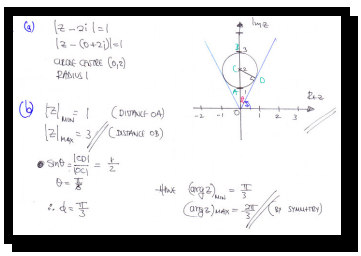
Question 8

$$|z-2\mathrm{i}|=1,\ z\in\mathbb{C}.$$

a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.

b) Find the minimum value and the maximum value of $|z|$, and the minimum value and the maximum of $\arg z$, for points that lie on this locus.

$$\boxed{|z|_{\min}=1}\,,\quad \boxed{|z|_{\max}=3}\,,\quad \boxed{\arg z_{\min}=\frac{\pi}{3}}\,,\quad \boxed{\arg z_{\max}=\frac{2\pi}{3}}$$



Question 9

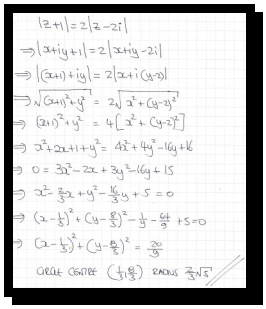
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z+1| = 2|z-2\mathrm{i}|,$$

show that the locus of P is a circle and state its radius and the coordinates of its centre.

$$\boxed{\left(\frac{1}{3}, \frac{8}{3}\right), \quad r = \frac{2}{3}\sqrt{5}}$$



Question 10

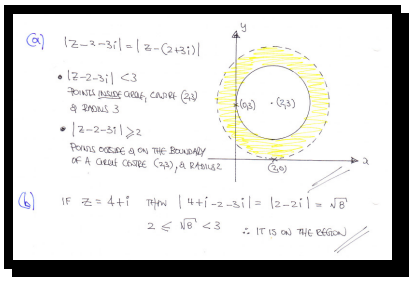
The complex number $z = x + \mathrm{i}y$ satisfies the relationship

$$2 \le |z - 2 - 3\mathrm{i}| < 3 .$$

a) Shade **accurately** in an Argand diagram the region represented by the above relationship.

b) Determine algebraically whether the point that represents the number $4 + \mathrm{i}$ lies inside or outside this region.

inside the region



Question 11

Two sets of loci in the Argand diagram are given by the following equations

$$|z| = |z+2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}.$$

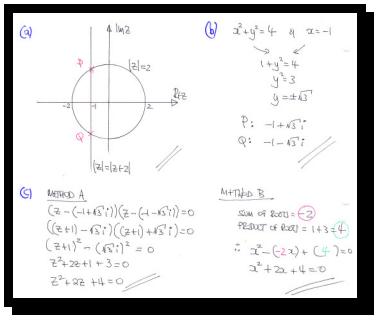
a) Sketch both these loci in the same Argand diagram.

The points P and Q in the Argand diagram satisfy both loci equations.

b) Write the complex numbers represented by P and Q, in the form $a + \mathrm{i}b$, where a and b are real numbers.

c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points P and Q.

$$\boxed{z = -1 \pm \sqrt{3}}, \quad \boxed{z^2 + 2z + 4 = 0}$$



Question 12

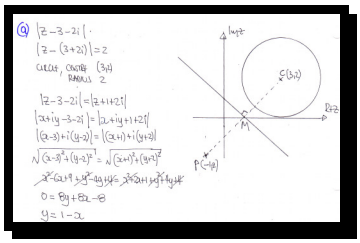
a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

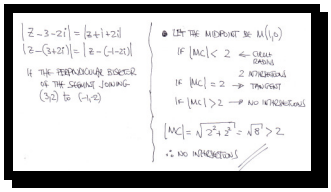
i. $|z-3-2\mathrm{i}|=2$.

ii. $|z-3-2\mathrm{i}|=|z+1+2\mathrm{i}|$.

b) Show by a **geometric** calculation that no points lie on both loci.

proof





Question 13

The point A represents the complex number on the z plane such that

$$|z-6\mathrm{i}|=2|z-3|,$$

and the point B represents the complex number on the z plane such that

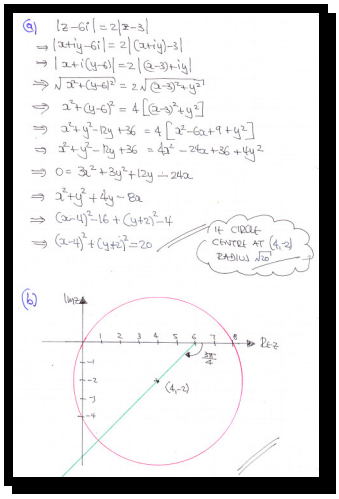
$$\arg(z-6)=-\frac{3\pi}{4}.$$

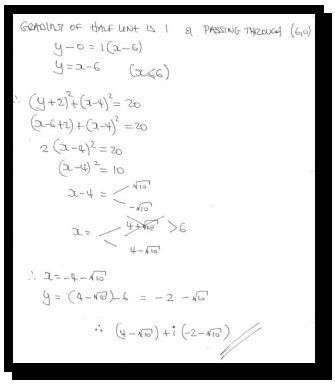
a) Show that the locus of A as z varies is a circle, stating its radius and the coordinates of its centre.

b) Sketch, on the same z plane, the locus of A and B as z varies.

c) Find the complex number z, so that the point A coincides with the point B.

$$\boxed{C(4,-2),\ r=\sqrt{20}},\quad \boxed{z=\left(4-\sqrt{10}\right)+\mathrm{i}\left(-2-\sqrt{10}\right)}$$





Question 14

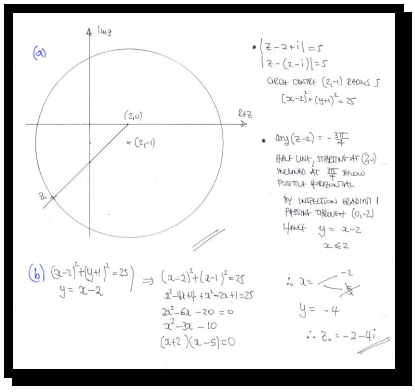
$$\left|z-2+\mathrm{i}\right|=5.$$

$$\arg\left(z-2\right)=-\frac{3\pi}{4}.$$

a) Sketch each of the above complex loci in the same Argand diagram.

b) Determine, in the form $x+\mathrm{i}y$, the complex number z_0 represented by the intersection of the two loci of part (a).

$$\boxed{z_0=-2-4\mathrm{i}}$$



Question 15

The locus of the point z in the Argand diagram, satisfy the equation

$$|z-2+\mathrm{i}|=\sqrt{3}\,.$$

a) Sketch the locus represented by the above equation.

The half line L with equation

$$y=mx-1,\quad x\geq 0,\quad m>0\,,$$

touches the locus described in part (a) at the point P.

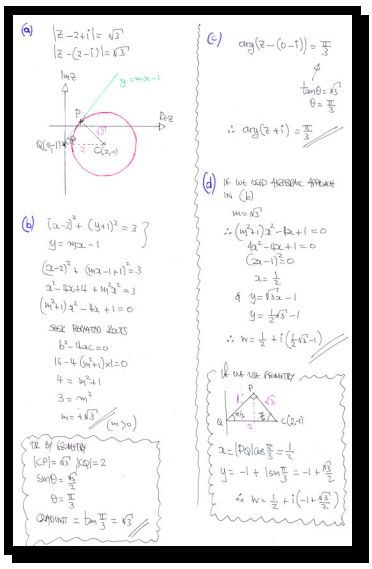
b) Find the value of m.

c) Write the equation of L, in the form

$$\arg\left(z-z_0\right)=\theta,\quad z_0\in\mathbb{C},\ -\pi<\theta\leq\pi\,.$$

d) Find the complex number w, represented by the point P.

$$\boxed{m=\sqrt{3}},\ \boxed{\arg\left(z+\mathrm{i}\right)=\frac{\pi}{3}},\ \boxed{w=\frac{1}{2}+\mathrm{i}\left(\frac{\sqrt{3}}{2}-1\right)}$$



Question 16

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \mathrm{i}\sqrt{3} \quad \text{and} \quad z_2 = \mathrm{i}z_1 .$$

a) Label accurately the points representing z_1 and z_2, in an Argand diagram.

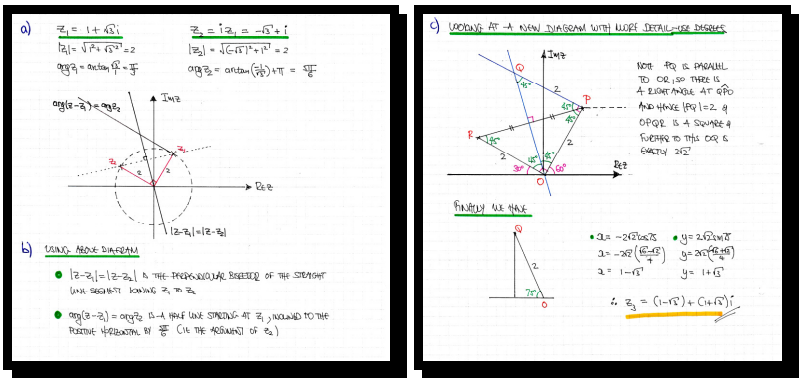
b) On the same Argand diagram, sketch the locus of the points z satisfying …

i. … $|z - z_1| = |z - z_2|$.

ii. … $\arg(z - z_1) = \arg z_2$.

c) Determine, in the form $x + \mathrm{i}y$, the complex number z_3 represented by the intersection of the two loci of part **(b)**.

$$\boxed{z_3 = \left(1 - \sqrt{3}\right) + \mathrm{i}\left(1 + \sqrt{3}\right)}$$



Question 17

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

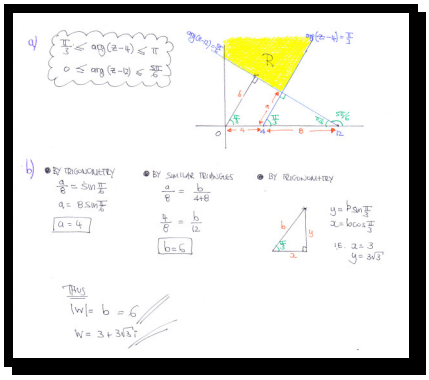
$$\frac{\pi}{3} \le \arg(z-4) \le \pi \quad \text{and} \quad 0 \le \arg(z-12) \le \frac{5\pi}{6}.$$

a) Sketch the region R, indicating clearly all the relevant details.

The complex number w lies in R, so that $|w|$ is minimum.

b) Find $|w|$, further giving w in the form $u+\mathrm{i}v$, where u and v are real numbers.

$$\boxed{|w|=3}, \quad \boxed{w=3+3\sqrt{3}\mathrm{i}}$$



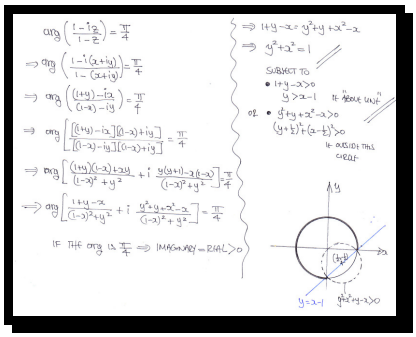
Question 18

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-\mathrm{i}z}{1-z}\right) = \frac{\pi}{4} \ , \ z \neq -\mathrm{i} \, .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{x^2 + y^2 = 1, \quad \text{such that} \quad y > x - 1}$$



Question 19

The complex number $x+iy$ in the z plane of an Argand diagram satisfies the inequality

$$x^2+y^2+x>0\,.$$

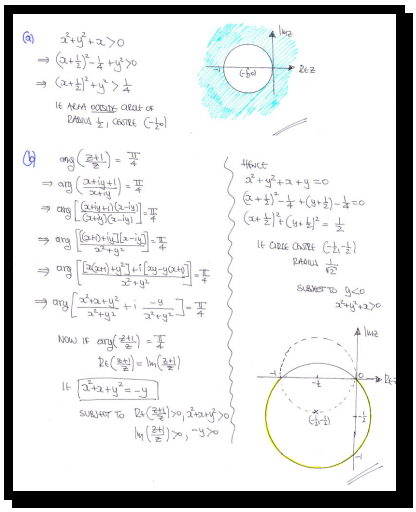
a) Sketch the region represented by this inequality.

A locus in the z plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}\,.$$

b) Sketch the locus represented by this equation.

sketch



Question 20

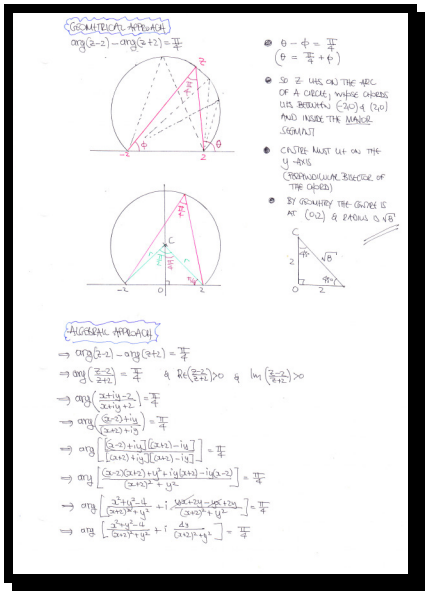
The complex number z satisfies the relationship

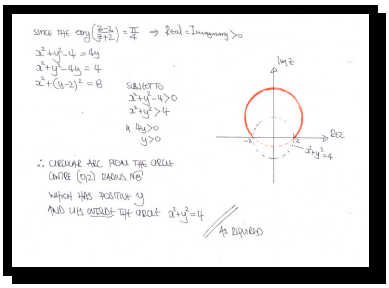
$$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}.$$

Show that the locus of z is a circular arc, stating …

- … the coordinates of its endpoints.
- … the coordinates of its centre.
- … the length of its radius.

$\boxed{(-2,0),(2,0)}$, $\boxed{(0,2)}$, $\boxed{r=2\sqrt{2}}$





COMPLEX FUNCTIONS

Question 1

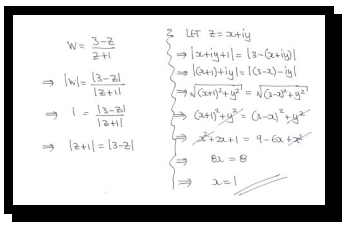
A transformation from the z plane to the w plane is defined by the complex function

$$w=\frac{3-z}{z+1},\ z\neq -1.$$

The locus of the points represented by the complex number $z=x+\mathrm{i}y$ is transformed to the circle with equation $|w|=1$ in the w plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number z.

$$\boxed{x=1}$$



Question 2

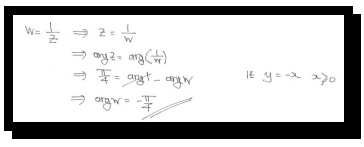
Find an equation of the locus of the points which lie on the half line with equation

$$\arg z=\frac{\pi}{4},\ z\neq 0$$

after it has been transformed by the complex function

$$w=\frac{1}{z}.$$

$$\boxed{\arg w=-\frac{\pi}{4}}$$



Question 3

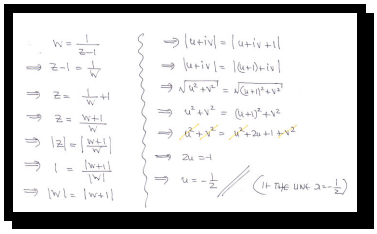
The complex function

$$w=\frac{1}{z-1},\ z\neq 1, z\in\mathbb{C},\ z\neq 1$$

transforms the point represented by $z=x+\mathrm{i}y$ in the z plane into the point represented by $w=u+\mathrm{i}v$ in the w plane.

Given that z satisfies the equation $|z|=1$, find a Cartesian locus for w.

$$\boxed{u=-\frac{1}{2}}$$



Question 4

The complex function $w = f(z)$ is given by

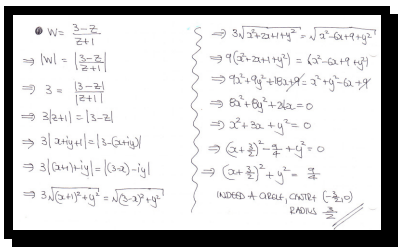
$$w = \frac{3-z}{z+1} \text{ where } z \in \mathbb{C}, \quad z \neq -1 .$$

A point P in the z plane gets mapped onto a point Q in the w plane.

The point Q traces the circle with equation $|w| = 3$.

Show that the locus of P in the z plane is also a circle, stating its centre and its radius.

$$\boxed{\text{centre } \left(-\frac{3}{2}, 0\right), \text{ radius } = \frac{3}{2}}$$



Question 5

The general point $P(x, y)$ which is represent by the complex number $z = x + \mathrm{i}y$ in the z plane, lies on the locus of

$$|z| = 1.$$

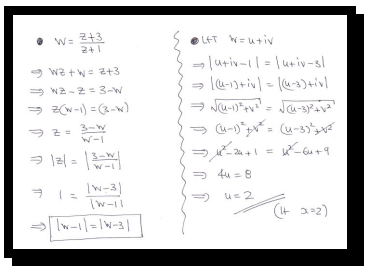
A transformation from the z plane to the w plane is defined by

$$w = \frac{z+3}{z+1}, \ z \neq -1,$$

and maps the point $P(x, y)$ onto the point $Q(u, v)$.

Find, in Cartesian form, the equation of the locus of the point Q in the w plane.

$$\boxed{u = 2}$$



Question 6

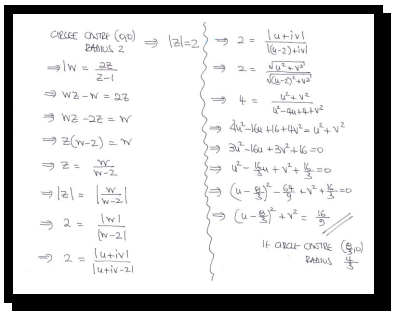
The point P represented by $z = x + \mathrm{i}y$ in the z plane is transformed into the point Q represented by $w = u + \mathrm{i}v$ in the w plane, by the complex transformation

$$w = \frac{2z}{z-1}, \ z \neq 1 .$$

The point P traces a circle of radius 2, centred at the origin O.

Find a Cartesian equation of the locus of the point Q.

$$\boxed{\left(u-\frac{8}{3}\right)^2 + v^2 = \frac{16}{9}}$$



Question 7

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

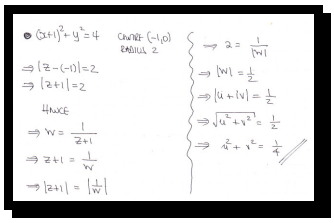
$$w = \frac{1}{z+1}, \ z \neq -1 .$$

If P is moving along the circle with equation

$$(x+1)^2 + y^2 = 4 ,$$

find in Cartesian form an equation of the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{4}}$$



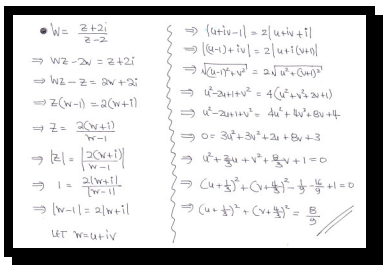
Question 8

A transformation from the z plane to the w plane is defined by the equation

$$w=\frac{z+2\mathrm{i}}{z-2},\ z\neq 2\,.$$

Find in the w plane, in Cartesian form, the equation of the image of the circle with equation $|z|=1,\ z\in\mathbb{C}$.

$$\boxed{\left(u+\frac{1}{3}\right)^2+\left(v+\frac{4}{3}\right)^2=\frac{8}{9}}$$



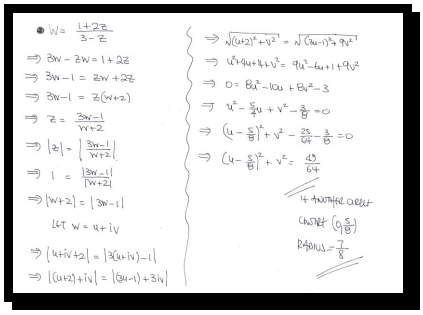
Question 9

A transformation from the z plane to the w plane is given by the equation

$$w=\frac{1+2z}{3-z},\ z\neq 3\,.$$

Show that the in the w plane, the image of the circle with equation $|z|=1,\ z\in\mathbb{C}$, is another circle, stating its centre and its radius .

$$\boxed{\left(u-\frac{5}{8}\right)^2+v^2=\frac{49}{64}},\ \boxed{\text{centre}\left(\frac{5}{8},0\right)},\ \boxed{r=\frac{7}{8}}$$



Question 10

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

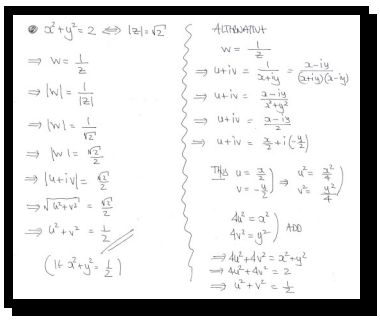
$$w = \frac{1}{z}, \ z \neq 0 .$$

If P is moving along the circle with equation

$$x^2 + y^2 = 2 ,$$

find in Cartesian form an equation for the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{2}}$$



Question 11

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

In the z plane, the point P is tracing the line with equation $y = x$.

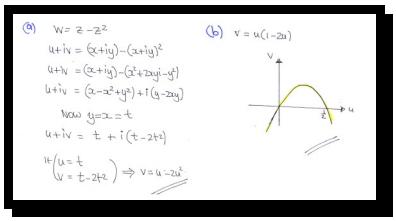
The complex numbers z and w are related by

$$w = z - z^2 .$$

a) Find, in Cartesian form, the equation of the locus of Q in the w plane.

b) Sketch the locus traced by Q.

$$\boxed{v = u - 2u^2 \text{ or } y = x - 2x^2}$$



Question 12

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

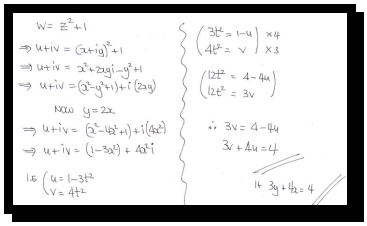
In the z plane, the point P is tracing the line with equation $y = 2x$.

Given that he complex numbers z and w are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of Q in the w plane.

$$\boxed{4u + 3v = 4 \text{ or } 4x + 3y = 4}$$



Question 13

A transformation of the z plane to the w plane is given by

$$w=\frac{1+3z}{1-z},\ z\in\mathbb{C},\ z\neq 1,$$

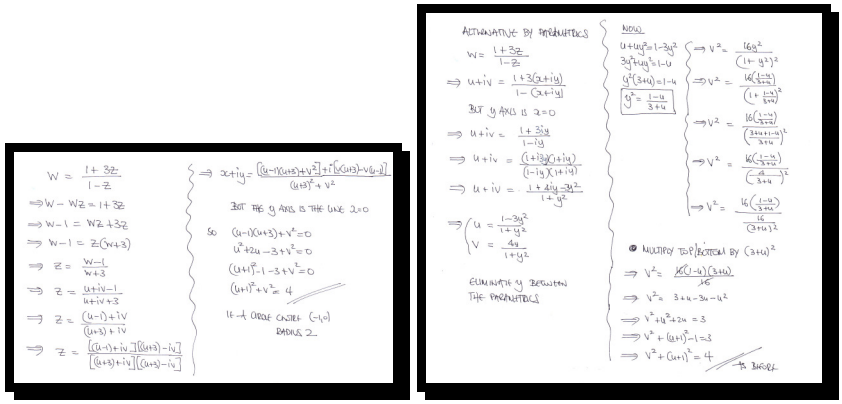
where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

The set of points that lie on the y axis of the z plane, are mapped in the w plane onto a curve C.

Show that a Cartesian equation of C is

$$(u+1)^2+v^2=4.$$

proof



Question 14

The complex function $w = f(z)$ is given by

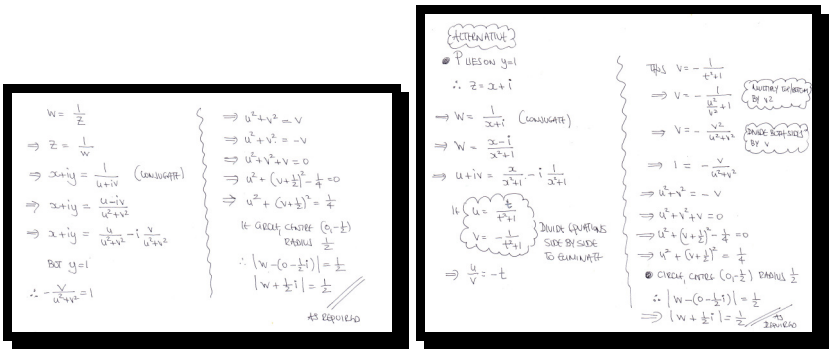
$$w = \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0 .$$

This function maps a general point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

Given that P lies on the line with Cartesian equation $y = 1$, show that the locus of Q is given by

$$\left| w + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2} .$$

proof



Question 15

A transformation of the z plane onto the w plane is given by

$$w=\frac{az+b}{z+c},\ z\in\mathbb{C},\ z\neq -c$$

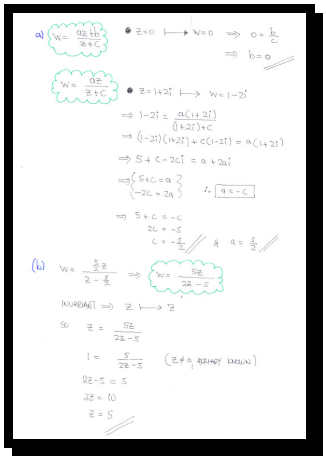
where a, b and c are real constants.

Under this transformation the point represented by the number $1+2\mathrm{i}$ gets mapped to its complex conjugate and the origin remains invariant.

a) Find the value of a, the value of b and the value of c.

b) Find the number, other than the number represented by the origin, which remains invariant under this transformation.

$$\boxed{a=\frac{5}{2}},\ \boxed{b=0},\ \boxed{c=-\frac{5}{2}},\ \boxed{z=5}$$



Question 16

A transformation of the z plane to the w plane is given by

$$w=\frac{1}{z-2},\ z\in\mathbb{C},\ z\neq 2$$

where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

The line with equation

$$2x+y=3$$

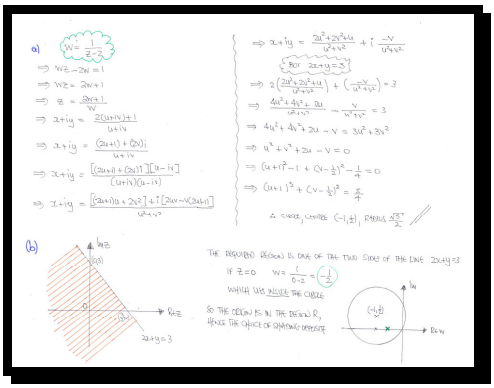
is mapped in the w plane onto a curve C.

a) Show that C represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region R in the z plane are mapped onto the points which lie inside C in the w plane.

b) Sketch and shade R in a suitable labelled Argand diagram, fully justifying the choice of region.

$$\boxed{\text{centre at } \left(-1,\frac{1}{2}\right),\ \text{radius}=\frac{\sqrt{5}}{2}}$$



Question 17

A transformation of the z plane to the w plane is given by

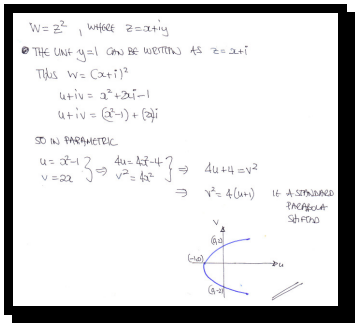
$$w = z^2, \ z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The line with equation $y = 1$ is mapped in the w plane onto a curve C.

Sketch the graph of C, marking clearly the coordinates of all points where the graph of C meets the coordinate axes.

sketch



Question 18

A transformation of points from the z plane onto points in the w plane is given by the complex relationship

$$w = z^2, \quad z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

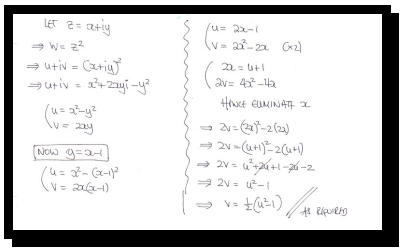
Show that if the point P in the z plane lies on the line with equation

$$y = x - 1,$$

the locus of this point in the w plane satisfies the equation

$$v = \frac{1}{2}\left(u^2 - 1\right).$$

proof



Question 19

A complex transformation from the z plane to the w plane is defined by

$$w=\frac{z+\mathrm{i}}{3+\mathrm{i}z},\ z\in\mathbb{C},\ z\neq 3\mathrm{i}.$$

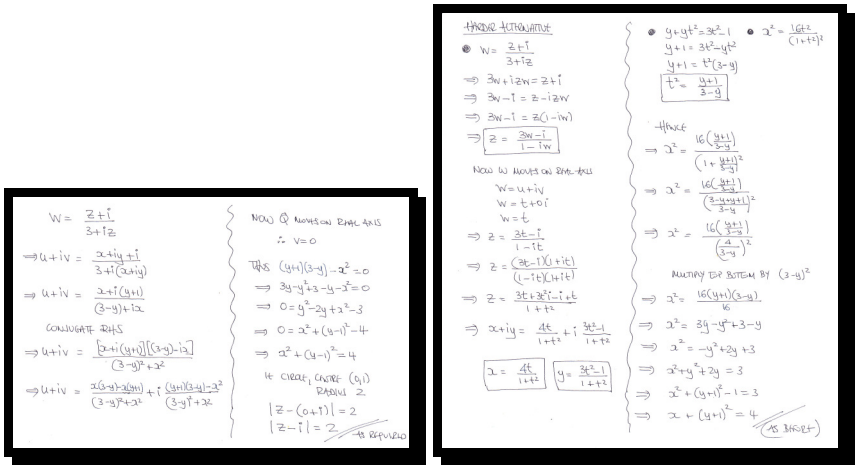
The point $P(x,y)$ is mapped by this transformation into the point $Q(u,v)$.

It is further given that Q lies on the real axis for all the possible positions of P.

Show that the P traces the curve with equation

$$|z-\mathrm{i}|=2.$$

proof



Question 20

A transformation of the z plane to the w plane is given by

$$w=\frac{2z+1}{z},\ z\in\mathbb{C},\ z\neq 0$$

where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

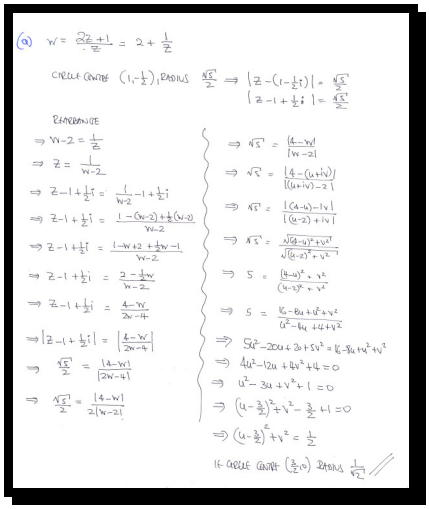
The circle C_1 with centre at $\left(1,-\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$ in the z plane is mapped in the w plane onto another curve C_2.

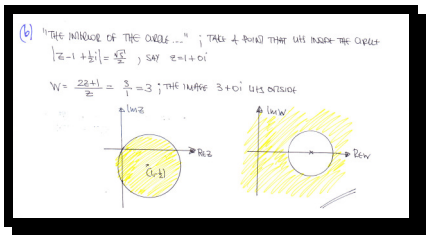
a) Show that C_2 is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside C_1 in the z plane are mapped onto points of a region R in the w plane.

b) Sketch and shade R in a suitably labelled Argand diagram, fully justifying the choice of the region.

$$\boxed{\text{centre at } \left(\frac{3}{2},0\right),\ \text{radius}=\frac{1}{\sqrt{2}}}$$





Question 21

A transformation of the z plane to the w plane is given by

$$w = z + \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0,$$

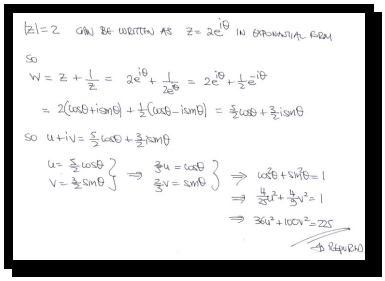
where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The locus of the points in the z plane that satisfy the equation $|z| = 2$ are mapped in the w plane onto a curve C.

By considering the equation of the locus $|z| = 2$ in exponential form, or otherwise, show that a Cartesian equation of C is

$$36u^2 + 100v^2 = 225.$$

proof



Question 22

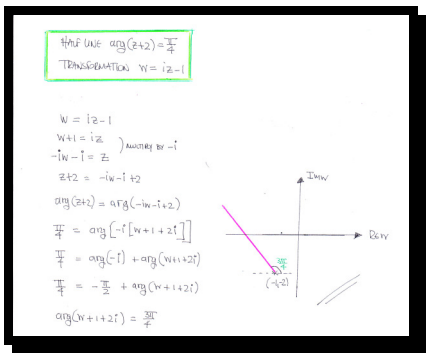
A transformation from the z plane to the w plane is defined by the equation

$$w = \mathrm{i}\, z - 1, \ z \in \mathbb{C}.$$

Sketch in the w plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg\left(z+2\right) = \frac{\pi}{4}, \ z \in \mathbb{C}.$$

graph



Question 23

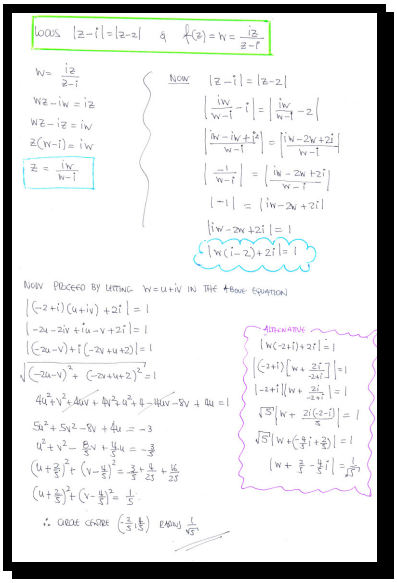
A transformation from the z plane to the w plane is defined by the equation

$$f(z)=\frac{\mathrm{i}z}{z-\mathrm{i}},\ z\in\mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-\mathrm{i}|=|z-2|,\ z\in\mathbb{C}.$$

$$\boxed{\left(u+\frac{2}{5}\right)^2+\left(v-\frac{4}{5}\right)^2=\frac{1}{5}}$$



Question 24

The complex function $w = f(z)$ is given by

$$w = \frac{1}{1-z} \ , \ z \neq 1 .$$

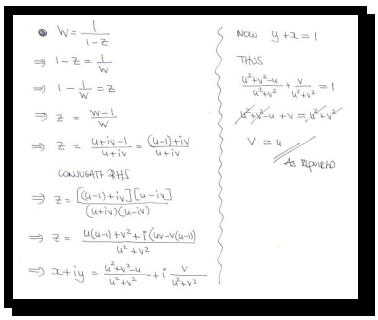
The point $P(x, y)$ in the z plane traces the line with Cartesian equation

$$y + x = 1 .$$

Show that the locus of the **image** of P in the w plane traces the line with equation

$$v = u .$$

proof



Question 25

The complex function $w = f(z)$ satisfies

$$w = \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0.$$

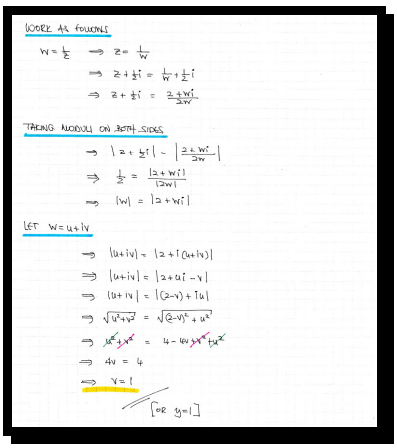
This function maps the point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

It is further given that P traces the curve with equation

$$\left| z + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of Q.

$$\boxed{v = 1}$$



Question 26

$$z=\cos\theta+\mathrm{i}\sin\theta\,,\ -\pi<\theta\leq\pi\,.$$

a) Show clearly that

$$\frac{2}{1+z}=1-\mathrm{i}\tan\frac{\theta}{2}\,.$$

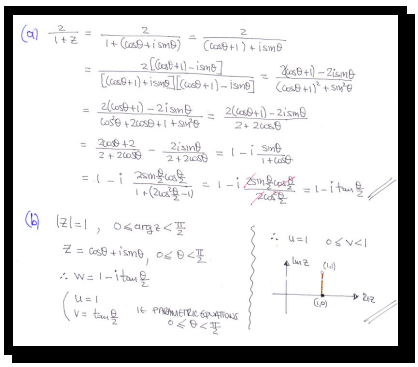
The complex function $w=f\left(z\right)$ is defined by

$$w=\frac{2}{1+z}\,,\ z\in\mathbb{C}\,,\ z\neq-1\,.$$

The circular arc $\left|z\right|=1$, for which $0\leq\arg z<\dfrac{\pi}{2}$, is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

proof/sketch



Question 27

The complex function with equation

$$f(z)=\frac{1}{z^2},\ z\in\mathbb{C},\ z\neq 0$$

maps the complex number $x+\mathrm{i}y$ from the z plane onto the complex number $u+\mathrm{i}v$ in the w plane.

The line with equation

$$y=mx,\ x\neq 0,$$

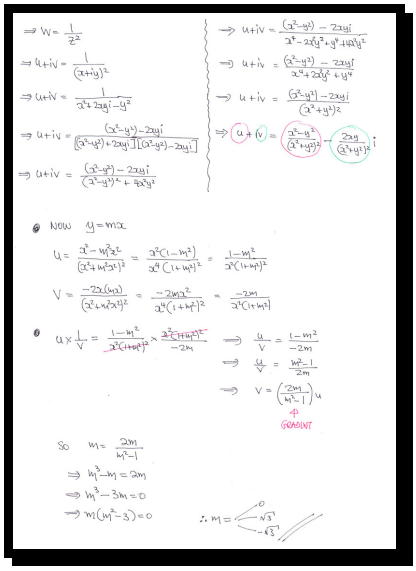
is mapped onto the line with equation

$$v=Mu,$$

where m and M are the respective gradients of the two lines.

Given that $m=M$, determine the three possible values of m.

$$\boxed{m=0,\pm\sqrt{3}}$$



Question 28

A complex transformation of points from the z plane onto points in the w plane is defined by the equation

$$w = z^2, \ z \in \mathbb{C} .$$

The point represented by $z = x + \mathrm{i}y$ is mapped onto the point represented by $w = u + \mathrm{i}v$.

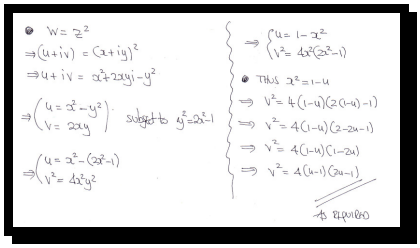
Show that if z traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of w satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

proof



Question 29

The complex function $w = f(z)$ is defined by

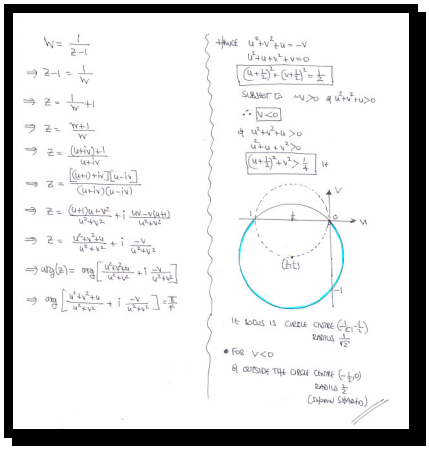
$$w = \frac{1}{z-1}, \ z \in \mathbb{C}, \ z \neq 1.$$

The half line with equation $\arg z = \frac{\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{\left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{2}, \ v<0, \ u^2+v^2+u>0}$$



Question 30

The complex function $w = f(z)$ is defined by

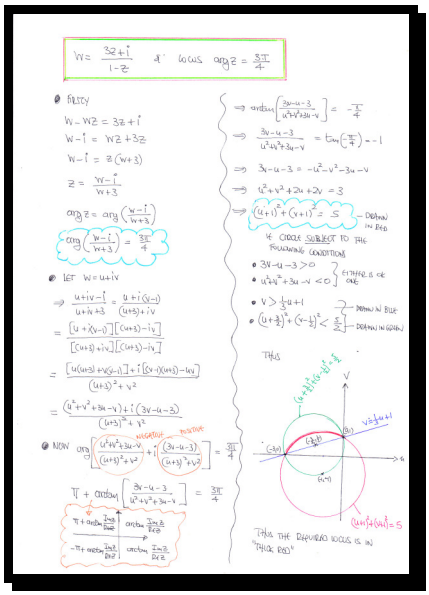
$$w = \frac{3z + \mathrm{i}}{1 - z}, \ z \in \mathbb{C}, \ z \neq 1.$$

The half line with equation $\arg z = \frac{3\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{(u+1)^2 + (v+1)^2 = 5, \quad v > \frac{1}{3}u + 1}$$



COMPLEX SERIES

Question 1

The following convergent series C and S are given by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta ...$$
$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta ...$$

a) Show clearly that

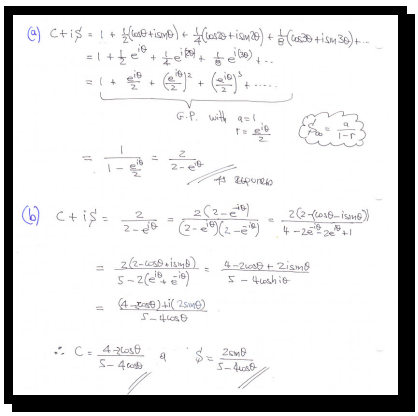
$$C + \mathrm{i}S = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}.$$

b) Hence show further that

$$C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta},$$

and find a similar expression for S.

$$\boxed{S = \frac{2\sin\theta}{5 - 4\cos\theta}}$$



Question 2

The following finite sums, C and S, are given by

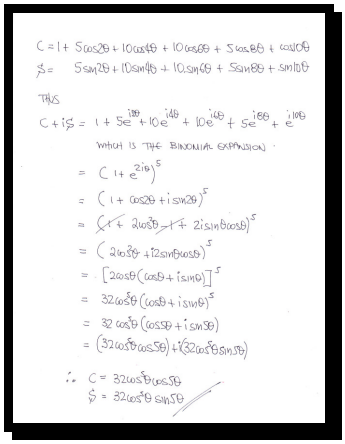
$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$
$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of $(1+A)^5$, show clearly that

$$C = 32\cos^5\theta\cos 5\theta,$$

and find a similar expression for S

$$\boxed{S = 32\cos^5\theta\sin 5\theta}$$



Question 3

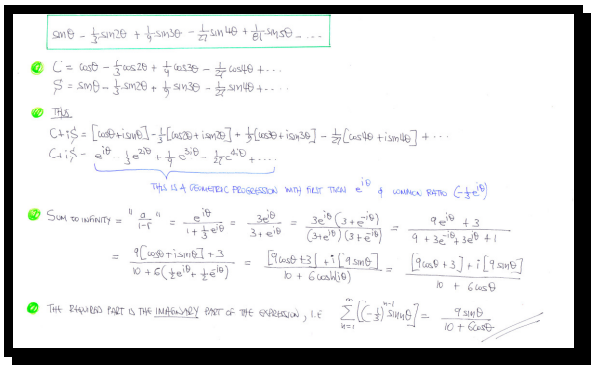
The following convergent series S is given below

$$S=\sin\theta-\frac{1}{3}\sin 2\theta+\frac{1}{9}\sin 3\theta-\frac{1}{27}\sin 4\theta\ ...$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S=\frac{9\sin\theta}{10+6\cos\theta}.$$

proof



Question 4

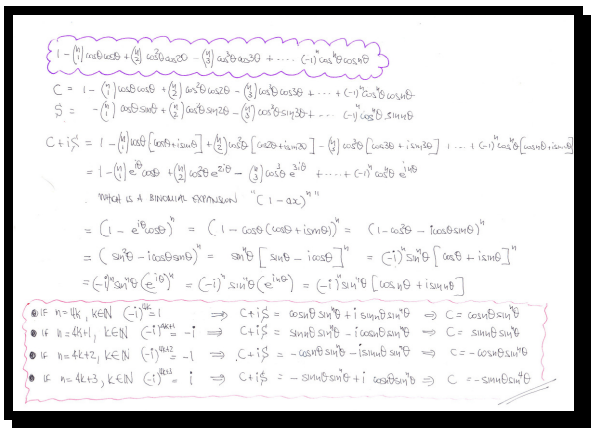
The sum C is given below

$$C=1-\binom{n}{1}\cos\theta\cos\theta+\binom{n}{2}\cos^2\theta\cos 2\theta-\binom{n}{1}\cos^3\theta\cos 3\theta+...+(-1)^n\cos^n\theta\cos n\theta$$

Given that $n\in\mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact simplified form.

$$\boxed{n=4k,\ k\in\mathbb{N}:C=\cos n\theta\sin^n\theta},\ \boxed{n=4k+1,\ k\in\mathbb{N}:C=\sin n\theta\sin^n\theta},$$
$$\boxed{n=4k+2,\ k\in\mathbb{N}:C=-\cos n\theta\sin^n\theta},\ \boxed{n=4k+3,\ k\in\mathbb{N}:C=-\sin n\theta\sin^n\theta}$$



Question 5

The following convergent series S is given below

$$S=\frac{\sin\theta}{1!}-\frac{\sin 2\theta}{2!}+\frac{\sin 3\theta}{3!}-\frac{\sin 4\theta}{4!}+\ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S=\mathrm{e}^{-\cos\theta}\sin\left(\sin\theta\right).$$

proof

