COMPLEX NUMBERS

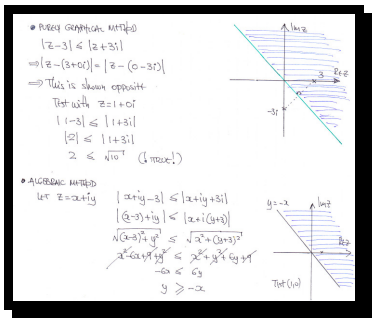
(Exam Questions II)

Question 1 (**)

By finding a suitable Cartesian locus for the complex z plane, shade the region R that satisfies the inequality

$$|z-3| \le |z+3\mathrm{i}| .$$

$$\boxed{x+y \ge 0}$$



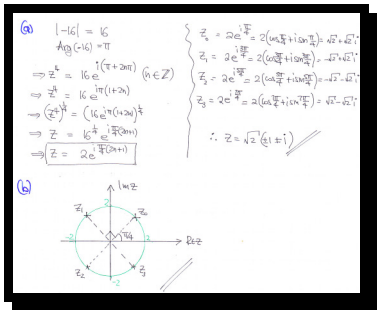
Question 2 (**)

$$z^4 = -16 , \ z \in \mathbb{C} .$$

a) Determine the solutions of the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \sqrt{2}\left(\pm 1 \pm \mathrm{i}\right)}$$



Question 3 (**)

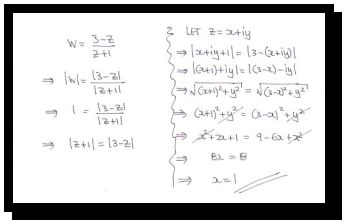
A transformation from the z plane to the w plane is defined by the complex function

$$w=\frac{3-z}{z+1},\ z\neq -1.$$

The locus of the points represented by the complex number $z=x+\mathrm{i}y$ is transformed to the circle with equation $|w|=1$ in the w plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number z.

$$\boxed{x=1}$$



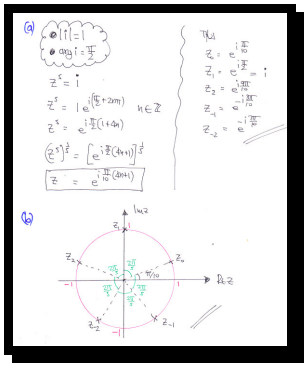
Question 4 (**)

$$z^5=\mathrm{i},\ z\in\mathbb{C}.$$

a) Solve the equation, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r>0$, $-\pi<\theta\leq\pi$.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z=\mathrm{e}^{\mathrm{i}\frac{\pi}{10}},\quad z=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\quad z=\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}},\quad z=\mathrm{e}^{-\mathrm{i}\frac{3\pi}{10}},\quad z=\mathrm{e}^{-\mathrm{i}\frac{7\pi}{10}}}$$



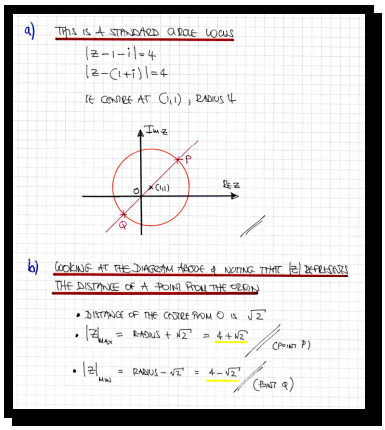
Question 5 (**)

$$|z-1-\mathrm{i}|=4\ ,\ z\in\mathbb{C}\ .$$

a) Sketch, in a standard Argand diagram, the locus of the points that satisfy the above equation.

b) Find the minimum and maximum value of $|z|$ for points that lie on this locus.

$\boxed{}$, $\boxed{z_{\min}=4-\sqrt{2}}$, $\boxed{z_{\min}=4+\sqrt{2}}$



Question 6 (**)

The complex number z represents the point $P(x, y)$ in the Argand diagram.

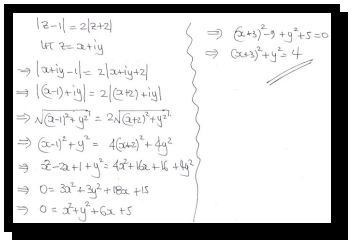
Given that

$$|z-1|=2|z+2|,$$

show that the locus of P is given by

$$(x+3)^2+y^2=4.$$

proof



Question 7 (**)

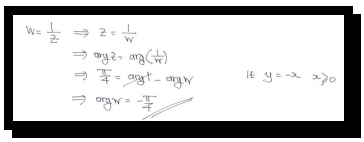
Find an equation of the locus of the points which lie on the half line with equation

$$\arg z=\frac{\pi}{4},\ z\neq 0$$

after it has been transformed by the complex function

$$w=\frac{1}{z}.$$

$$\arg w=-\frac{\pi}{4}$$



Question 8 (**)

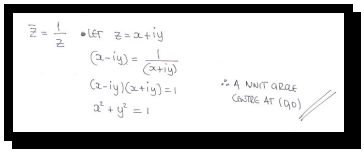
The complex number $z = x + \mathrm{i}y$ represents the point P in the complex plane.

Given that

$$\overline{z} = \frac{1}{z}, \ z \neq 0$$

determine a Cartesian equation for the locus of P.

$$\boxed{x^2 + y^2 = 1}$$



Question 9 (**)

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a) $|z-3+\mathrm{i}|=3$.

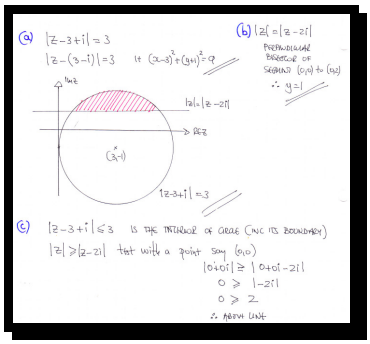
b) $|z|=|z-2\mathrm{i}|$.

Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-3+\mathrm{i}|\le 3$$
$$|z|\ge|z-2\mathrm{i}|$$

$\boxed{(x-3)^2+(y+1)^2=9}$, $\boxed{y=1}$



Question 10 (**)

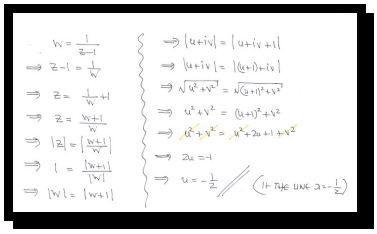
The complex function

$$w=\frac{1}{z-1},\ z\neq 1, z\in\mathbb{C},\ z\neq 1$$

transforms the point represented by $z=x+\mathrm{i}y$ in the z plane into the point represented by $w=u+\mathrm{i}v$ in the w plane.

Given that z satisfies the equation $|z|=1$, find a Cartesian locus for w.

$$\boxed{u=-\frac{1}{2}}$$



Question 11 (**)

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

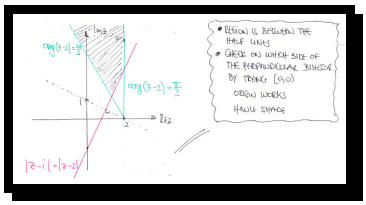
i. $|z-\mathrm{i}|=|z-2|$.

ii. $\arg(z-2)=\dfrac{\pi}{2}$.

b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-\mathrm{i}|\leq|z-2| \qquad \text{and} \qquad \frac{\pi}{2}\leq\arg(z-2)\leq\frac{2\pi}{3}.$$

sketch



Question 12 (**)

The complex function $w = f(z)$ is given by

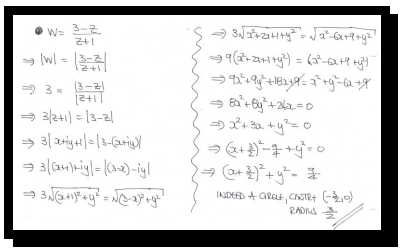
$$w = \frac{3-z}{z+1} \text{ where } z \in \mathbb{C}, \quad z \neq -1 .$$

A point P in the z plane gets mapped onto a point Q in the w plane.

The point Q traces the circle with equation $|w| = 3$.

Show that the locus of P in the z plane is also a circle, stating its centre and its radius.

$$\boxed{\text{centre } \left(-\frac{3}{2}, 0\right), \text{ radius } = \frac{3}{2}}$$



Question 13 (**)

The general point $P(x, y)$ which is represent by the complex number $z = x + \mathrm{i}y$ in the z plane, lies on the locus of

$$|z| = 1 .$$

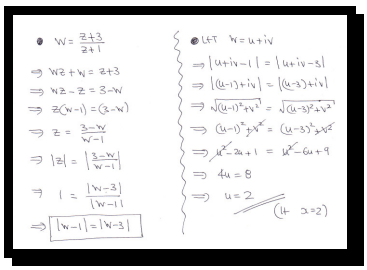
A transformation from the z plane to the w plane is defined by

$$w = \frac{z+3}{z+1}, \ z \neq -1,$$

and maps the point $P(x, y)$ onto the point $Q(u, v)$.

Find, in Cartesian form, the equation of the locus of the point Q in the w plane.

$\boxed{u = 2}$



Question 14 (**)

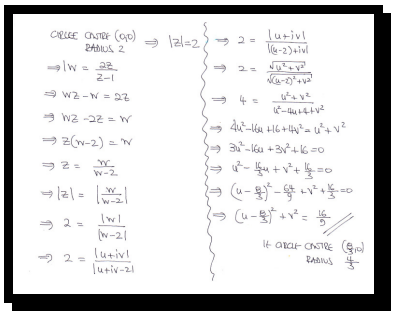
The point P represented by $z = x + \mathrm{i}y$ in the z plane is transformed into the point Q represented by $w = u + \mathrm{i}v$ in the w plane, by the complex transformation

$$w = \frac{2z}{z-1}, \ z \neq 1 .$$

The point P traces a circle of radius 2, centred at the origin O.

Find a Cartesian equation of the locus of the point Q.

$$\boxed{\left(u-\frac{8}{3}\right)^2 + v^2 = \frac{16}{9}}$$



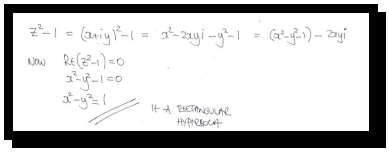
Question 15 (**)

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

It is further given that $z^2 - 1$ is purely imaginary for all values of z.

Find a Cartesian equation of the locus that P is tracing in the Argand diagram.

$$\boxed{x^2 - y^2 = 1}$$



Question 16 (+)**

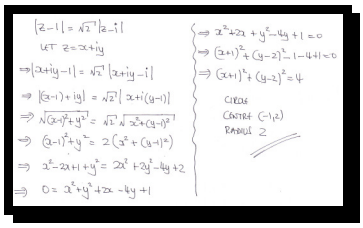
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z-1|=\sqrt{2}|z-\mathrm{i}|,$$

show that the locus of P is a circle, stating its centre and radius.

$$\boxed{(x+1)^2+(y-2)^2=4}\,,\ \boxed{(-1,2),r=2}$$



Question 17 (+)**

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

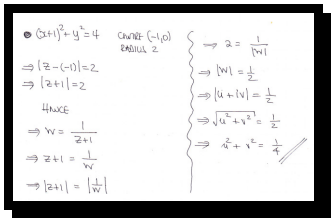
$$w = \frac{1}{z+1}, \ z \neq -1 .$$

If P is moving along the circle with equation

$$(x+1)^2 + y^2 = 4 ,$$

find in Cartesian form an equation of the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{4}}$$



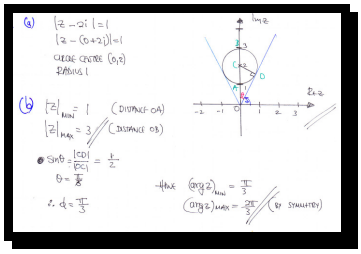
Question 18 **(**+)**

$$|z-2\mathrm{i}|=1,\ z\in\mathbb{C}.$$

a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.

b) Find the minimum value and the maximum value of $|z|$, and the minimum value and the maximum of $\arg z$, for points that lie on this locus.

$$\boxed{|z|_{\min}=1},\ \boxed{|z|_{\max}=3},\ \boxed{\arg z_{\min}=\frac{\pi}{3}},\ \boxed{\arg z_{\max}=\frac{2\pi}{3}}$$



Question 19 **(**+)**

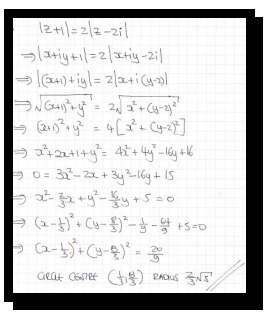
The complex number z represents the point $P(x,y)$ in the Argand diagram.

Given that

$$|z+1|=2|z-2\mathrm{i}|,$$

show that the locus of P is a circle and state its radius and the coordinates of its centre.

$$\boxed{\left(\frac{1}{3},\frac{8}{3}\right),\quad r=\frac{2}{3}\sqrt{5}}$$



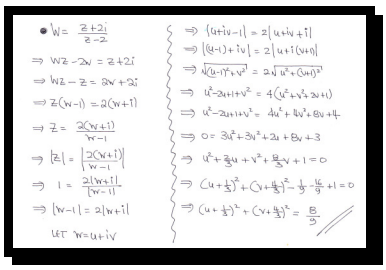
Question 20 (+)**

A transformation from the z plane to the w plane is defined by the equation

$$w=\frac{z+2\mathrm{i}}{z-2},\ z\neq 2\,.$$

Find in the w plane, in Cartesian form, the equation of the image of the circle with equation $|z|=1,\ z\in\mathbb{C}$.

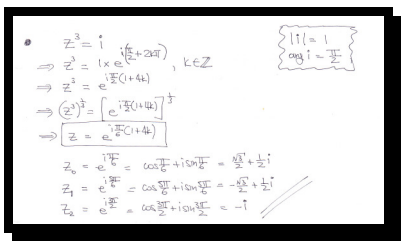
$$\boxed{\left(u+\frac{1}{3}\right)^2+\left(v+\frac{4}{3}\right)^2=\frac{8}{9}}$$



Question 21 (+)**

Find the cube roots of the imaginary unit i, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

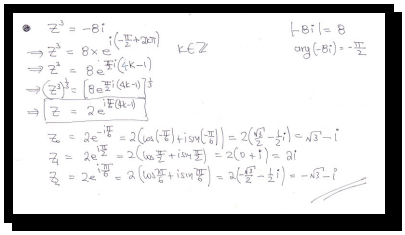
$$\boxed{z_1=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i},\ \ z_2=-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i},\ \ z_3=-\mathrm{i}}$$



Question 22 (+)**

Find the cube roots of the complex number $-8\mathrm{i}$, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z_1=\sqrt{3}-\mathrm{i},\ \ z_2=-\sqrt{3}-\mathrm{i},\ \ z_3=2\mathrm{i}}$$



Question 23 (+)**

The complex number z satisfies the relationship

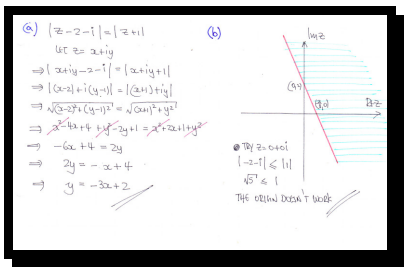
$$|z-2-\mathrm{i}|=|z+1|.$$

a) Find a Cartesian equation for the locus of z.

b) Shade in an Argand diagram the region that satisfy the inequality

$$|z-2-\mathrm{i}|\le|z+1|.$$

$$\boxed{y=2-3x}$$



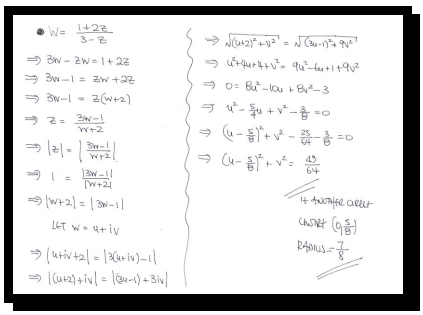
Question 24 (+)**

A transformation from the z plane to the w plane is given by the equation

$$w=\frac{1+2z}{3-z},\ z\neq 3.$$

Show that the in the w plane, the image of the circle with equation $|z|=1,\ z\in\mathbb{C}$, is another circle, stating its centre and its radius .

$$\boxed{\left(u-\frac{5}{8}\right)^2+v^2=\frac{49}{64}},\ \boxed{\text{centre}\left(\frac{5}{8},0\right)},\ \boxed{r=\frac{7}{8}}$$



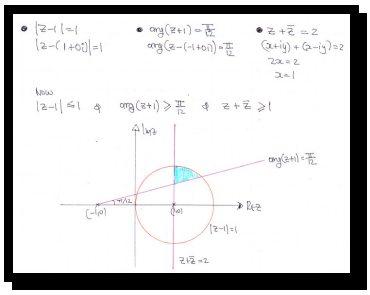
Question 25 (+)**

The complex number z satisfies all three relationships

$$|z-1|\leq 1,\quad \arg(z+1)\geq\frac{\pi}{12}\quad\text{and}\quad z+\overline{z}\geq 1.$$

Shade in an Argand diagram the region of the locus of z.

$$\boxed{\text{sketch}}$$



Question 26 (+)**

In separate Argand diagrams, the complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q, respectively.

The two numbers are related by the equation

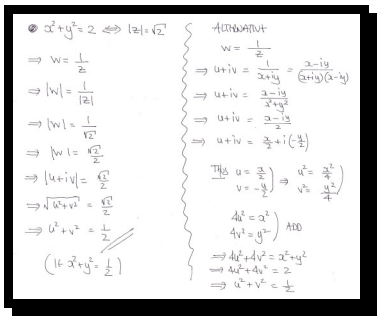
$$w = \frac{1}{z}, \ z \neq 0 .$$

If P is moving along the circle with equation

$$x^2 + y^2 = 2 ,$$

find in Cartesian form an equation for the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{2}}$$



Question 27 (+)**

The complex conjugate of z is denoted by $\overline{z}$.

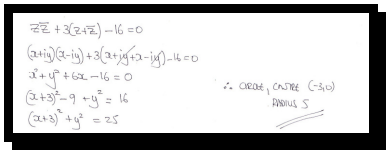
The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

Given further that

$$z\overline{z} + 3\left(z + \overline{z}\right) - 16 = 0$$

describe mathematically the locus of P.

circle, centre at $(-3,0)$, radius 5



Question 28 (***)

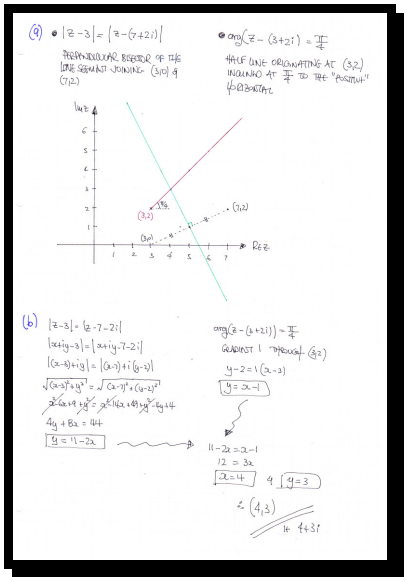
Two loci are defined in the complex plane by the relationships

$$|z-3|=|z-7-2\mathrm{i}| \quad \text{and} \quad \arg(z-3-2\mathrm{i})=\frac{\pi}{4}.$$

a) Sketch the two loci in the same Argand diagram.

b) Determine algebraically the complex number which lies on both loci.

$\boxed{4+3\mathrm{i}}$

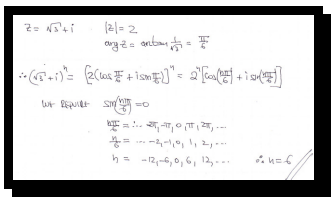


Question 29 (***)

Consider the expression $\left(\sqrt{3}+\mathrm{i}\right)^n$, where n is a positive integer.

Find the smallest positive value for n so that the expression is real.

$$\boxed{n=6}$$



Question 30 (***)

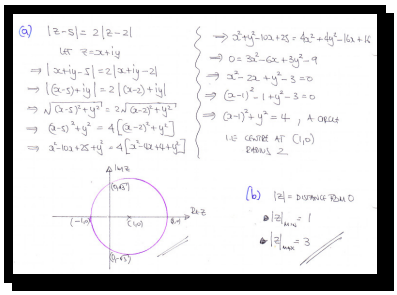
The complex number z satisfies the relationship

$$|z-5|=2|z-2|.$$

a) Sketch in an Argand diagram the locus of z.

b) State the minimum value of $|z|$ and maximum value of $|z|$, for points which lie on this locus.

$$\boxed{|z|_{\min}=1}\,,\quad \boxed{|z|_{\max}=3}$$



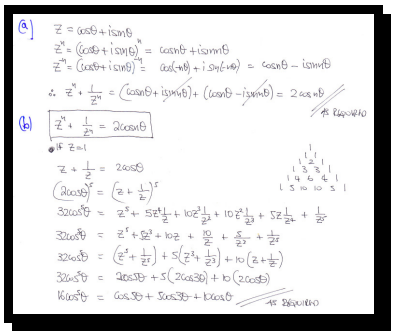
Question 31 **(***)**

If $z=\cos\theta+\mathrm{i}\sin\theta$, show clearly that …

a) … $z^n+\dfrac{1}{z^n}\equiv 2\cos n\theta$.

b) … $16\cos^5\theta\equiv\cos 5\theta+5\cos 3\theta+10\cos\theta$.

proof



Question 32 **(***)**

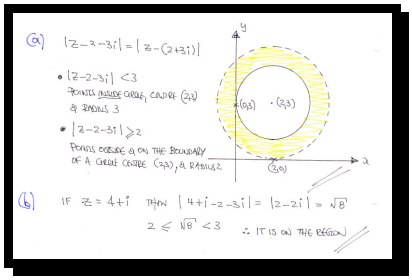
The complex number $z=x+\mathrm{i}y$ satisfies the relationship

$$2\le|z-2-3\mathrm{i}|<3.$$

a) Shade **accurately** in an Argand diagram the region represented by the above relationship.

b) Determine algebraically whether the point that represents the number $4+\mathrm{i}$ lies inside or outside this region.

inside the region



Question 33 (***)

The complex number is defined as $z=\cos\theta+\mathrm{i}\sin\theta$, $-\pi<\theta\leq\pi$.

a) Show clearly that …

i. … $z^n+\dfrac{1}{z^n}=2\cos\theta$.

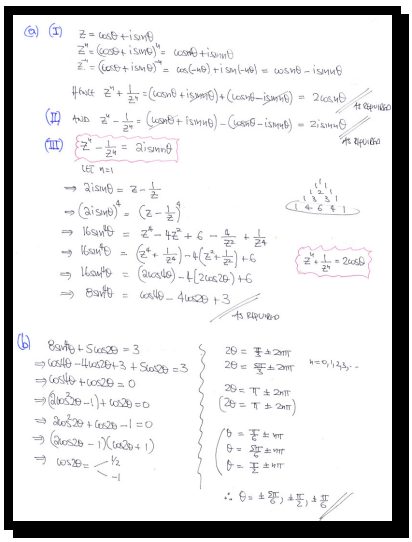
ii. … $z^n-\dfrac{1}{z^n}=2\mathrm{i}\sin\theta$.

iii. … $8\sin^4\theta=\cos 4\theta-4\cos 2\theta+3$.

b) Hence solve the equation

$$8\sin^4\theta+5\cos 2\theta=3\,,\quad -\pi<\theta\leq\pi\,.$$

$$\boxed{\theta=\pm\frac{5\pi}{6},\pm\frac{\pi}{2},\pm\frac{\pi}{6}}$$



Question 34 (***)

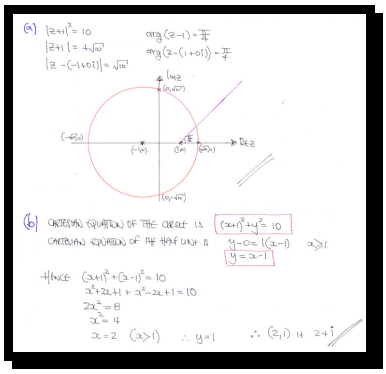
It is given that for $z \in \mathbb{C}$ the loci L_1 and L_2 have respective equations,

$$|z+1|^2 = 10 \qquad \text{and} \qquad \arg(z-1) = \frac{\pi}{4}.$$

a) Sketch L_1 and L_2 in the same Argand diagram.

b) Find the complex number that lies on both L_1 and L_2.

$\boxed{2+\mathrm{i}}$



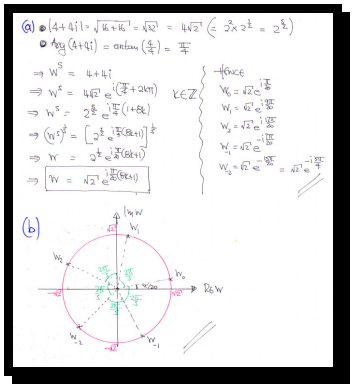
Question 35 (***)

$$z = 4 + 4\mathrm{i} \, .$$

a) Find the fifth roots of z.
Give the answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{7\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}}$$



Question 36 (***)

A straight line L and a circle C are to be drawn on a standard Argand diagram.

The equation of L is $\arg z = \dfrac{\pi}{3}$.

The centre of C lies on L and its radius is 3 units. The line with equation $\text{Im}\, z = 0$ is a tangent to C.

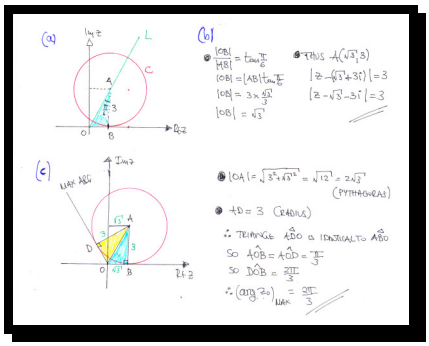
a) Sketch L and C on the same Argand diagram.

b) Determine an equation for C, giving the answer in the form $|z-\alpha| = k$, where α and k are constants.

The point that represents the complex number z_0 lies on C.

c) Determine the maximum value of $\arg z_0$, fully justifying the answer.

$$\boxed{|z-\sqrt{3}-3\text{i}| = 3}\,, \qquad \boxed{\arg z_0 = \frac{2\pi}{3}}$$



Question 37 (***)

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

In the z plane, the point P is tracing the line with equation $y = x$.

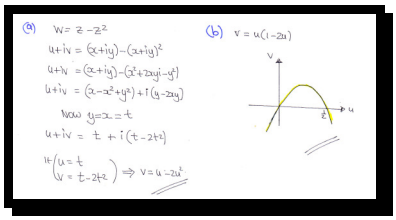
The complex numbers z and w are related by

$$w = z - z^2 .$$

a) Find, in Cartesian form, the equation of the locus of Q in the w plane.

b) Sketch the locus traced by Q.

$$\boxed{v = u - 2u^2 \text{ or } y = x - 2x^2}$$



Question 38 **(***)**

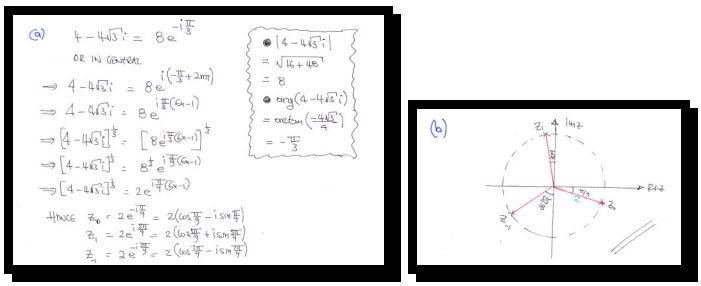
$$z = 4 - 4\sqrt{3}\mathrm{i}\,.$$

a) Find the cube roots of z.

Give the answers in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$, $-\pi < \theta \leq \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{z = 2\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right),\ z = 2\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right),\ z = 2\left(\cos\frac{7\pi}{9} - \mathrm{i}\sin\frac{7\pi}{9}\right)}$$



Question 39 (***)

The following complex number relationships are given

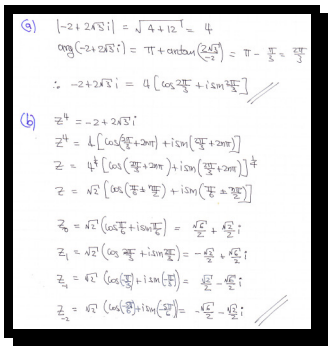
$$w=-2+2\sqrt{3}\mathrm{i}\,,\quad z^4=w\,.$$

a) Express w in the form $r\left(\cos\theta+\mathrm{i}\sin\theta\right)$, where $r>0$ and $-\pi<\theta\leq\pi$.

b) Find the possible values of z, giving the answers in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{w=2\left[\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]},$$

$$\boxed{z=\frac{1}{2}\left(\sqrt{6}+\mathrm{i}\sqrt{2}\right),\ z=\frac{1}{2}\left(-\sqrt{2}+\mathrm{i}\sqrt{6}\right),\ z=\frac{1}{2}\left(\sqrt{2}-\mathrm{i}\sqrt{6}\right),\ z=\frac{1}{2}\left(-\sqrt{6}-\mathrm{i}\sqrt{2}\right)}$$



Question 40 (***)

Two sets of loci in the Argand diagram are given by the following equations

$$|z|=|z+2| \quad \text{and} \quad |z|=2, \quad z\in\mathbb{C}.$$

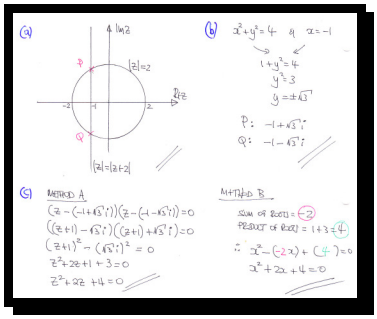
a) Sketch both these loci in the same Argand diagram.

The points P and Q in the Argand diagram satisfy both loci equations.

b) Write the complex numbers represented by P and Q, in the form $a+\mathrm{i}b$, where a and b are real numbers.

c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points P and Q.

$$\boxed{z=-1\pm\sqrt{3}}, \quad \boxed{z^2+2z+4=0}$$



Question 41 (***)

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

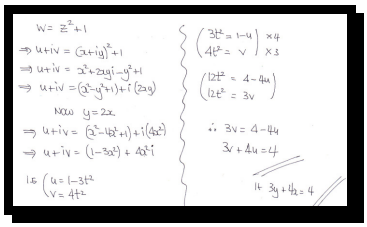
In the z plane, the point P is tracing the line with equation $y = 2x$.

Given that he complex numbers z and w are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of Q in the w plane.

$$\boxed{4u + 3v = 4 \quad \text{or} \quad 4x + 3y = 4}$$

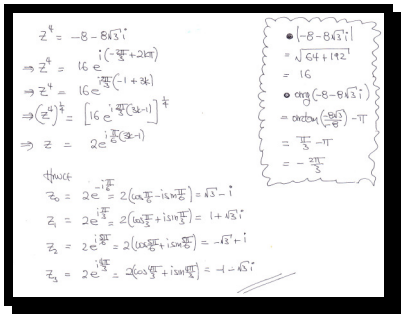


Question 42 (***)

$$z^4 = -8 - 8\sqrt{3}\,\mathrm{i}\,,\ z \in \mathbb{C}\,.$$

Solve the above equation, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z = \sqrt{3} - \mathrm{i}}\,,\ \boxed{z = 1 + \sqrt{3}\,\mathrm{i}}\,,\ \boxed{z = -\sqrt{3} + \mathrm{i}}\,,\ \boxed{z = -1 - \sqrt{3}\,\mathrm{i}}$$



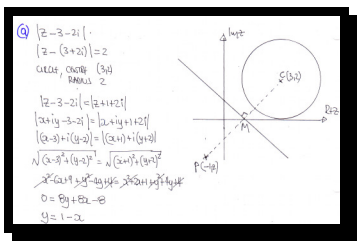
Question 43 (***)

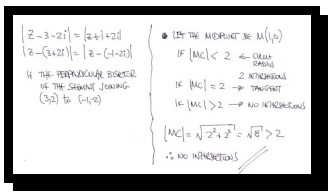
a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

i. $|z - 3 - 2\mathrm{i}| = 2\,.$

ii. $|z - 3 - 2\mathrm{i}| = |z + 1 + 2\mathrm{i}|\,.$

b) Show by a **geometric** calculation that no points lie on both loci.





Question 44 (***)

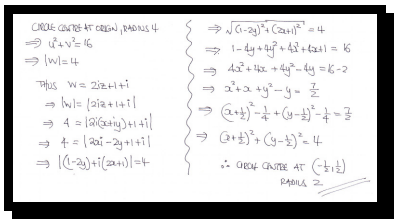
A circle C_1 in the z plane is mapped onto another circle C_2 in the w plane.

The mapping is defined by the relationship

$$w = 2\mathrm{i}z + 1 + \mathrm{i}\,.$$

Given C_2 has its centre at the origin and its radius is 4, find the coordinates of the centre of C_1 and the size of its radius.

$$\boxed{\left(-\frac{1}{2},\frac{1}{2}\right)}\,,\quad \boxed{r=2}$$



Question 45 (***)

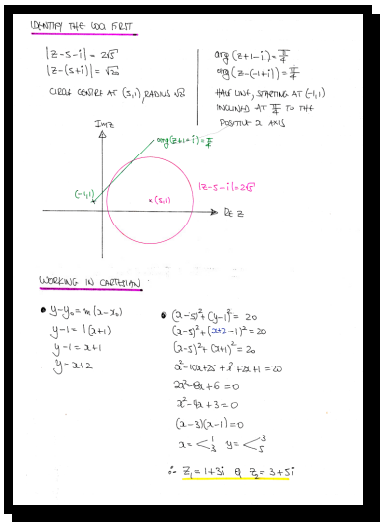
Sketch on a single Argand diagram the locus of the points z which satisfy

$$|z-5-\mathrm{i}|=2\sqrt{5} \quad \text{and} \quad \arg(z+1-\mathrm{i})=\frac{1}{4}\pi,$$

and hence find the complex numbers which lie on both of these loci.

No credit will be given to solutions based on a scale drawing.

$\boxed{}$, $\boxed{z_1=1+3\mathrm{i}}$, $\boxed{z_2=3+5\mathrm{i}}$



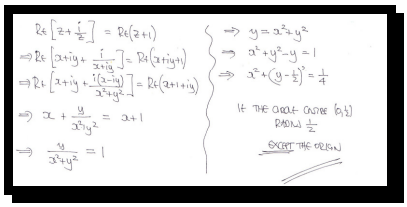
Question 46 (***)

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram and satisfies the relationship

$$\mathrm{Re}\left(z + \frac{\mathrm{i}}{z} \right) = \mathrm{Re}\left(z + 1 \right), \ z \neq 0 .$$

Describe mathematically the locus that P is tracing in the Argand diagram.

$$\boxed{\text{circle, centre at } \left(0, \frac{1}{2}\right), \text{ radius } \frac{1}{2}, \text{ except the origin}}$$



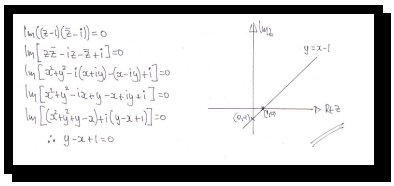
Question 47 (***)

The complex conjugate of z is denoted by $\overline{z}$.

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

Given that $\left(z - 1 \right)\left(\overline{z} - \mathrm{i} \right)$ is always real, sketch the locus of P.

$$\boxed{y = x - 1}$$



Question 48 (***)

The complex number z satisfies the equation

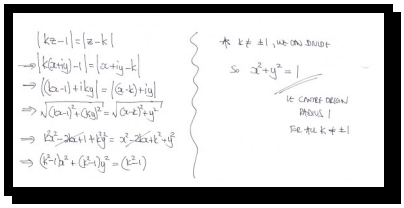
$$|kz-1|=|z-k|,$$

where k is a real constant such that $k \neq \pm 1$.

Show that for all the allowable values of the constant k, the point represented by z in an Argand diagram traces the circle with Cartesian equation

$$x^2+y^2=1.$$

proof



Question 49 **(***+)**

It is given that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

It is further given that

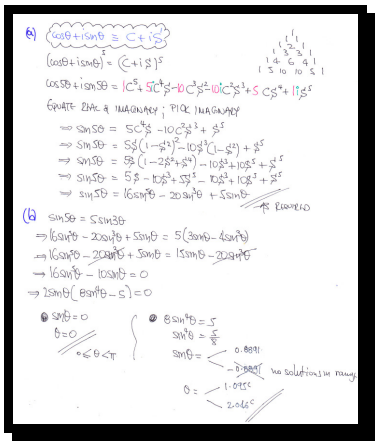
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

b) Solve the equation

$$\sin 5\theta = 5\sin 3\theta \quad \text{for} \quad 0 \le \theta < \pi ,$$

giving the solutions correct to 3 decimal places.

$$\boxed{\theta = 0,\ 1.095^{\text{c}},\ 2.046^{\text{c}}}$$



Question 50 (***)

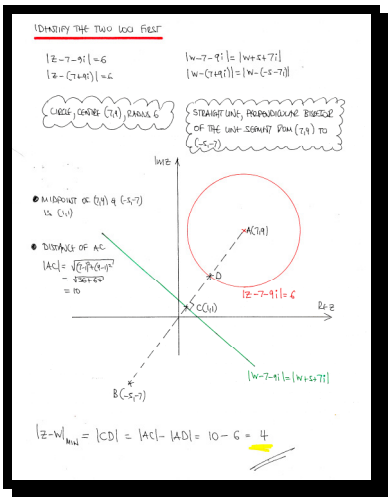
Sketch on a single Argand diagram the locus of the points z and w which satisfy

$$|z-7-9\mathrm{i}|=6 \quad \text{and} \quad |w-7-9\mathrm{i}|=|w+5+7\mathrm{i}|,$$

and hence find minimum value for $|z-w|$.

No credit will be given to solutions based on a scale drawing.

$\boxed{}$, $\boxed{|z-w|_{\min}=4}$



Question 51 (*+)**

The complex number z is defined as

$$z=\mathrm{e}^{\mathrm{i}\theta},\ -\pi<\theta\leq\pi.$$

a) Show clearly that …

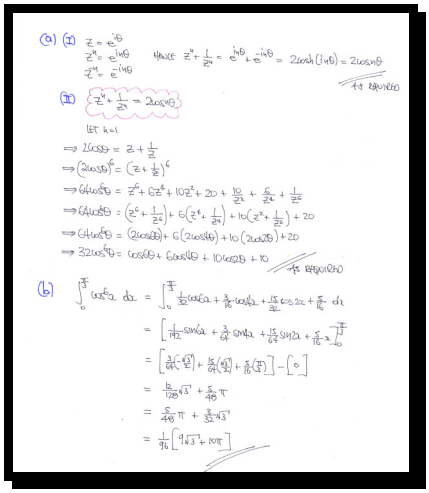
i. … $z^n+\dfrac{1}{z^n}=2\cos\theta$.

ii. … $32\cos^6\theta=\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10$.

b) Hence find an exact value for the integral

$$\int_0^{\frac{\pi}{3}}\cos^6 x\ \ dx.$$

$$\boxed{\frac{1}{96}\left(10\pi+9\sqrt{3}\right)}$$



Question 52 **(***+)**

$$z_1 = 2\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}.$$

a) Verify that z_1 is a solution of the equation

$$z^4 + 16 = 0,$$

and plot the four roots of the equation in an Argand diagram.

b) Find the values of the real constants a and b so that

$$(z - z_1)(z - \overline{z}_1) \equiv z^2 + az + b,$$

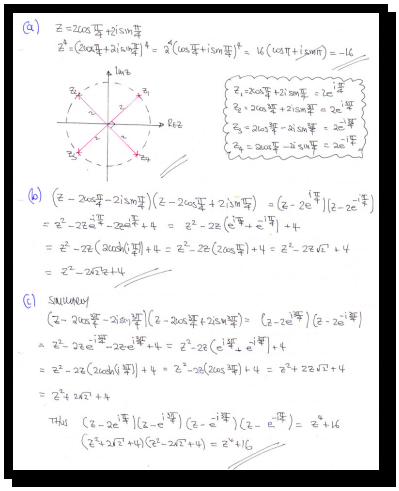
where $\overline{z}_1$ denotes the complex conjugate of z_1.

c) Hence show that

$$z^4 + 16 \equiv \left(z^2 + az + b\right)\left(z^2 + cz + d\right),$$

for some real constants c and d.

$$\boxed{a = -2\sqrt{2}}, \quad \boxed{b = 4}, \quad \boxed{c = 2\sqrt{2}}, \quad \boxed{d = 4},$$



Question 53 (*+)**

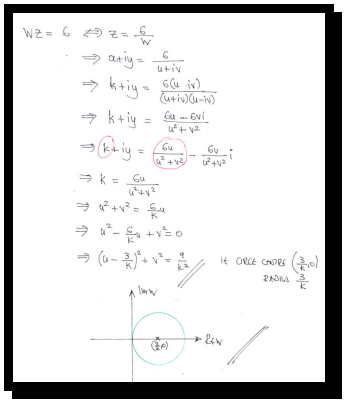
A transformation T maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $v = u + \mathrm{i}v$ in the w plane, and is defined as

$$zw = 6 \ , \ z \neq 0 \, .$$

The line with equation $x = k$, $k \in \mathbb{R}$, is mapped by T onto a circle C in the w plane.

Determine a Cartesian equation for C and sketch its graph in an Argand diagram.

$$\boxed{\left(u - \frac{3}{k}\right)^2 + v^2 = \frac{9}{k^2}}$$

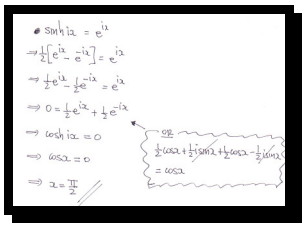


Question 54 (*+)**

Find a solution for the following equation

$$\sinh\left(\mathrm{i}x\right) = \mathrm{e}^{\mathrm{i}x} \ , \ x \in \mathbb{R} \, .$$

$$\boxed{x = \frac{\pi}{2}}$$



Question 55 (*+)**

Sketch on a standard Argand diagram the locus of the points z which satisfy

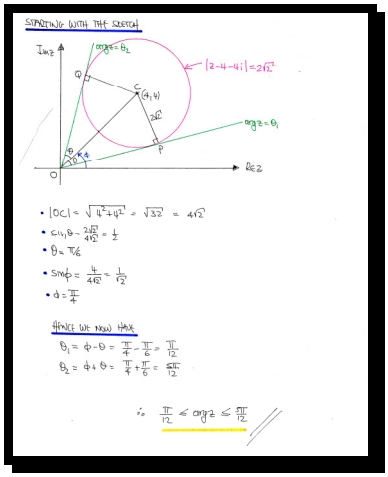
$$|z-4-4\mathrm{i}|=2\sqrt{2}\,,$$

and use it to prove that

$$\frac{1}{12}\pi \leq \arg z \leq \frac{5}{12}\pi\,.$$

No credit will be given to solutions based on a scale drawings.

$\boxed{}$, $\boxed{\text{proof}}$



Question 56 (*+)**

It is given that

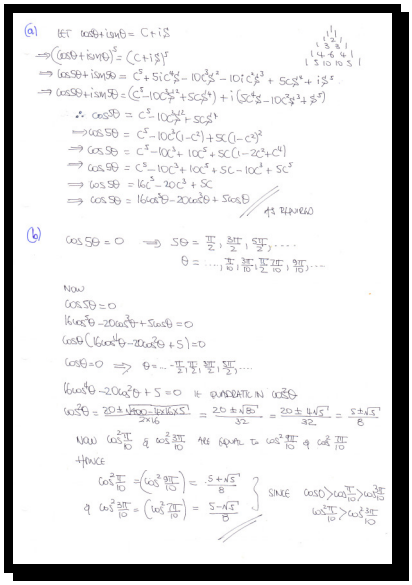
$$\cos 5\theta = 16\cos^5\theta - 20\cos^2\theta + 5\cos\theta .$$

a) Use de Moivre's theorem to prove the above trigonometric identity.

b) By considering the solution of the equation $\cos 5\theta = 0$, show that

$$\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8} .$$

proof

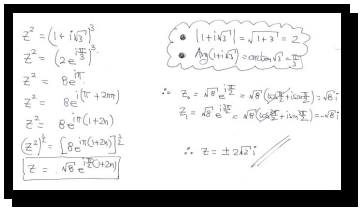


Question 57 (*+)**

$$z^2=\left(1+\mathrm{i}\sqrt{3}\right)^3,\ z\in\mathbb{C}.$$

Solve the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z=\pm\mathrm{i}2\sqrt{2}}$$



Question 58 (*+)**

A transformation of the z plane to the w plane is given by

$$w=\frac{1+3z}{1-z},\ z\in\mathbb{C},\ z\neq 1,$$

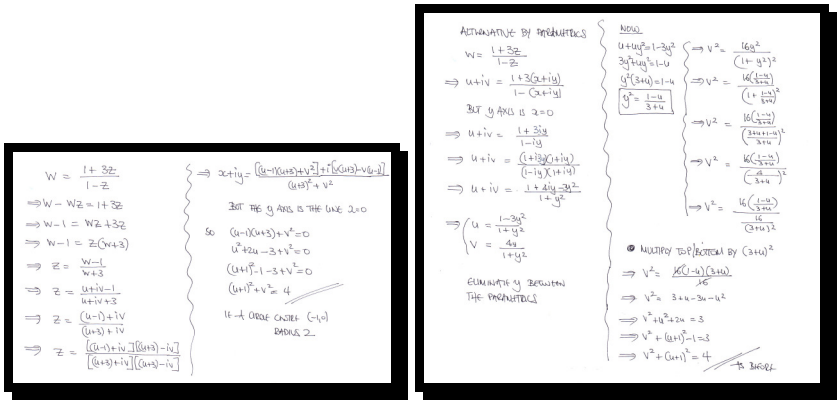
where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

The set of points that lie on the y axis of the z plane, are mapped in the w plane onto a curve C.

Show that a Cartesian equation of C is

$$(u+1)^2+v^2=4.$$

proof



Question 59 (*+)**

The point A represents the complex number on the z plane such that

$$|z-6\mathrm{i}|=2|z-3|,$$

and the point B represents the complex number on the z plane such that

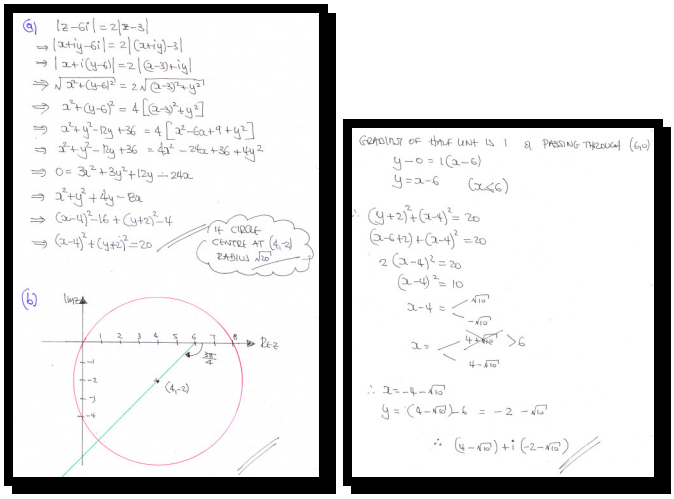
$$\arg(z-6)=-\frac{3\pi}{4}.$$

a) Show that the locus of A as z varies is a circle, stating its radius and the coordinates of its centre.

b) Sketch, on the same z plane, the locus of A and B as z varies.

c) Find the complex number z, so that the point A coincides with the point B.

$$\boxed{C(4,-2),\ r=\sqrt{20}},\quad \boxed{z=\left(4-\sqrt{10}\right)+\mathrm{i}\left(-2-\sqrt{10}\right)}$$



Question 60 (*+)**

The complex number z is given by

$$z=\mathrm{e}^{\mathrm{i}\theta},\ -\pi<\theta\leq\pi.$$

a) Show clearly that

$$z^n+\frac{1}{z^n}\equiv 2\cos n\theta.$$

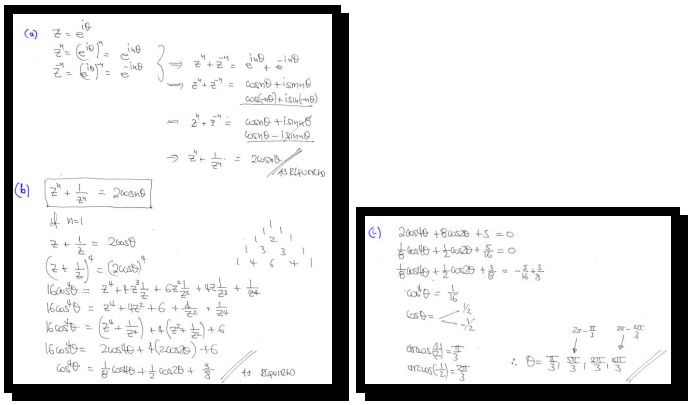
b) Hence show further that

$$\cos^4\theta\equiv\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}.$$

c) Solve the equation

$$2\cos 4\theta+8\cos 2\theta+5=0,\ 0\leq\theta<2\pi.$$

$$\boxed{\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}}$$



Question 61 (*+)**

The complex number $z=-9\mathrm{i}$ is given.

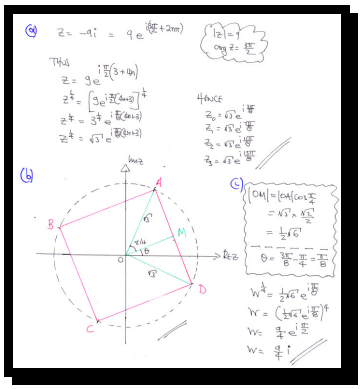
a) Determine the fourth roots of z, giving the answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta<2\pi$.

b) Plot the points represented by these roots in Argand diagram, and join them in order of increasing argument, labelled as A, B, C and D.

The midpoints of the sides of the quadrilateral $ABCD$ represent the 4^{th} roots of another complex number w.

c) Find w, giving the answer in the form $x+\mathrm{i}y$, where $x\in\mathbb{R}$, $y\in\mathbb{R}$.

$$\boxed{z=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\theta},\theta=\frac{3\pi}{8},\frac{7\pi}{8},\frac{11\pi}{8},\frac{15\pi}{8}},\quad\boxed{w=\frac{9}{4}\mathrm{i}}$$



Question 62 (*+)**

The complex numbers z and w, satisfy the relationship

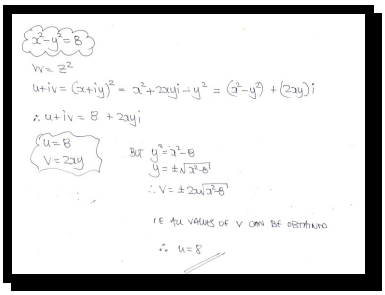
$$w = z^2 .$$

Given that in an Argand diagram, z is tracing the curve with equation

$$x^2 - y^2 = 8 ,$$

determine a Cartesian equation of the locus that w is tracing.

$$\boxed{u = 8 \text{ or } x = 8}$$



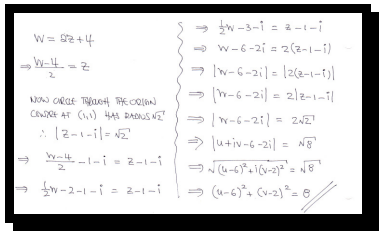
Question 63 (*+)**

The complex numbers z and w, satisfy the relationship

$$w = 2z + 4 , \ z \neq -2 .$$

Given that z is tracing a circle with centre at $(1,1)$ and radius $\sqrt{2}$ in an Argand diagram, determine a Cartesian equation of the locus that w is tracing.

$$\boxed{(u-6)^2 + (v-2)^2 = 8 \text{ or } (x-6)^2 + (y-2)^2 = 8}$$



Question 64 (*+)**

The complex number is defined as $z=\mathrm{e}^{\mathrm{i}\theta}$, $-\pi<\theta\leq\pi$.

a) Show that …

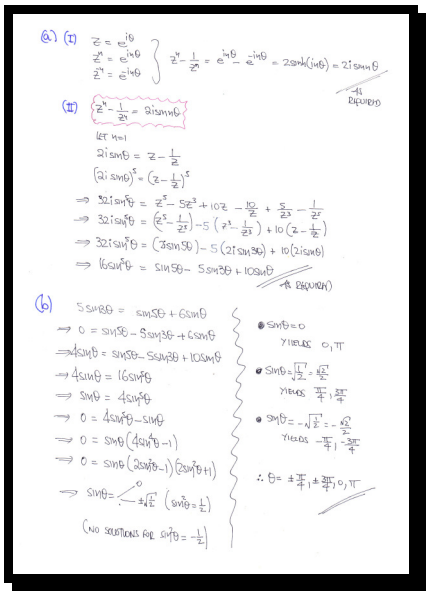
i. … $z^n-\dfrac{1}{z^n}=2\mathrm{i}\sin\theta$.

ii. … $16\sin^5\theta=\sin 5\theta-5\sin 3\theta+10\sin\theta$.

b) Hence solve the equation

$$5\sin 3\theta=\sin 5\theta+6\sin\theta,\ -\pi<\theta\leq\pi.$$

$$\boxed{\theta=0,\pm\frac{\pi}{4},\pm\frac{3\pi}{4},\pi}$$



Question 65 (*+)**

$$z^3 = 32 + 32\sqrt{3}\,\mathrm{i}\,,\ z \in \mathbb{C}\,.$$

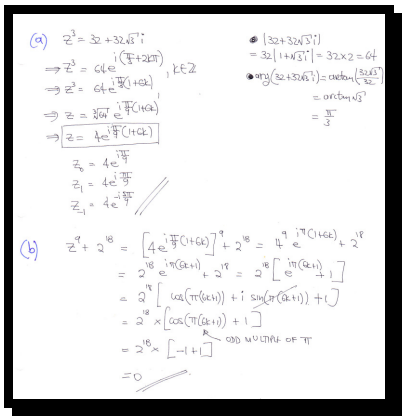
a) Solve the above equation.

Give the answers in exponential form $z = r\mathrm{e}^{\mathrm{i}\theta},\ r > 0,\ -\pi < \theta \leq \pi$.

b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0\,.$$

$$\boxed{z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}}$$



Question 66 (*+)**

The complex function $w = f(z)$ is given by

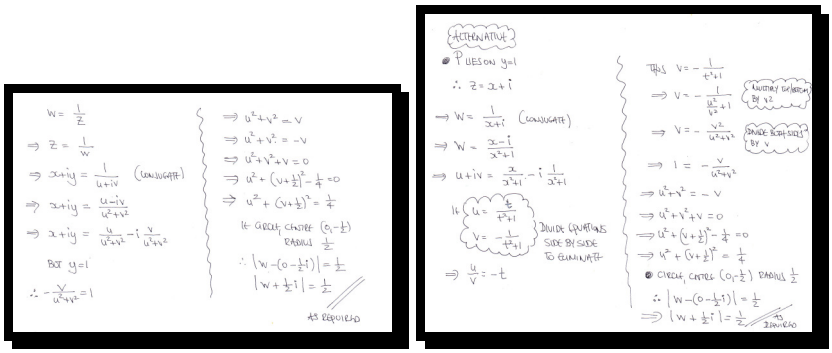
$$w = \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0 .$$

This function maps a general point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

Given that P lies on the line with Cartesian equation $y = 1$, show that the locus of Q is given by

$$\left| w + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2} .$$

proof



Question 67 **(***+)**

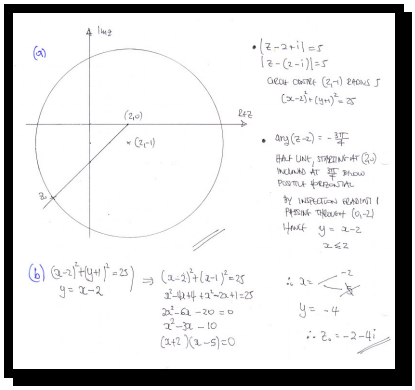
$$\left|z-2+\mathrm{i}\right|=5.$$

$$\arg\left(z-2\right)=-\frac{3\pi}{4}.$$

a) Sketch the above complex loci in the same Argand diagram.

b) Determine, in the form $x+\mathrm{i}y$, the complex number z_0 represented by the intersection of the two loci of part (a).

$$\boxed{z_0=-2-4\mathrm{i}}$$



Question 68 (*+)**

The complex number z is given in polar form as

$$\cos\left(\tfrac{2}{5}\pi\right)+\mathrm{i}\sin\left(\tfrac{2}{5}\pi\right).$$

a) Write z^2, z^3 and z^4 in polar form, each with argument θ, so that $0\le\theta<2\pi$.

In an Argand diagram the points A, B, C, D and E represent, in respective order, the complex numbers

$$1,\quad 1+z,\quad 1+z+z^2,\quad 1+z+z^2+z^3,\quad 1+z+z^2+z^3+z^4.$$

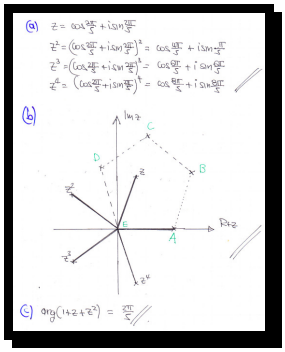
b) Sketch these points, in the sequential order given, in a standard Argand diagram.

c) State the exact argument of

$$1+z+z^2.$$

$\square$, $\boxed{z^2=\cos\frac{4\pi}{5}+\mathrm{i}\sin\frac{4\pi}{5}}$, $\boxed{z^3=\cos\frac{6\pi}{5}+\mathrm{i}\sin\frac{6\pi}{5}}$, $\boxed{z^4=\cos\frac{8\pi}{5}+\mathrm{i}\sin\frac{8\pi}{5}}$,

$\boxed{\arg\left(1+z+z^2\right)=\frac{2\pi}{5}}$



Question 69 (*+)**

The complex number z satisfies the following equation.

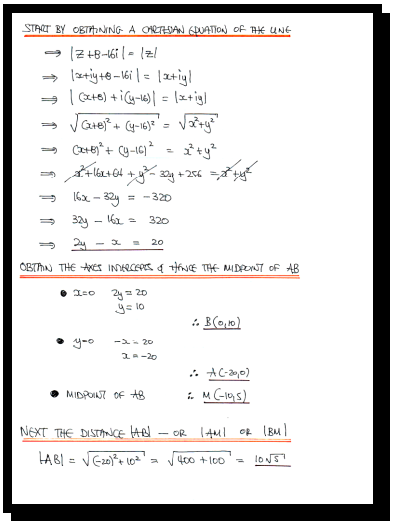
$$\left|z+8-16\mathrm{i}\right|=\left|z\right|.$$

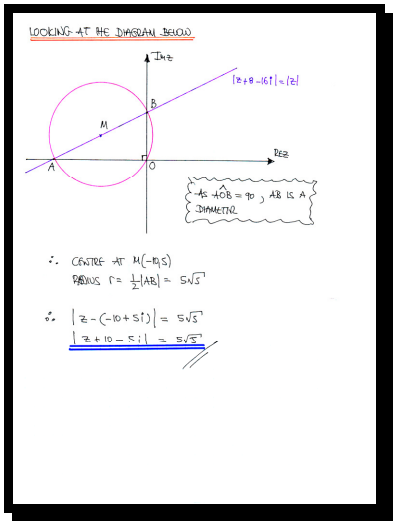
In a standard Argand diagram, the complex numbers represented by the points A and B lie on the real and imaginary axes, respectively.

Given further that A and B satisfy the above equation, determine an equation for the circle which passes though the points A, B and O, where O is the origin of the Argand diagram.

Give the answer in the form $\left|z-z_0\right|=r$, where $z_0\in\mathbb{C}$ and $r\in\mathbb{R}$.

$\boxed{}$, $\boxed{\left|z+10-5\mathrm{i}\right|=5\sqrt{5}}$





Question 70 (*+)**

The following convergent series C and S are given by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta ...$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta ...$$

a) Show clearly that

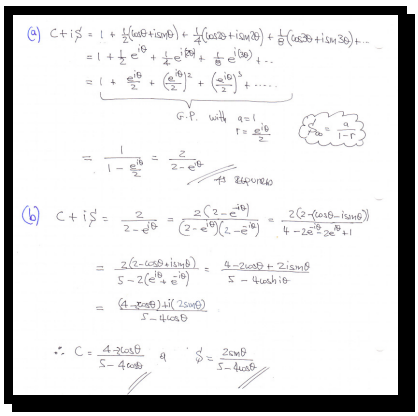
$$C + \mathrm{i}S = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}.$$

b) Hence show further that

$$C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta},$$

and find a similar expression for S.

$$\boxed{S = \frac{2\sin\theta}{5 - 4\cos\theta}}$$



Question 71 (*+)**

The complex number z is given by

$$z=\mathrm{e}^{\mathrm{i}\theta},\ -\pi<\theta\leq\pi.$$

a) Show clearly that

$$z^n+\frac{1}{z^n}\equiv 2\cos n\theta.$$

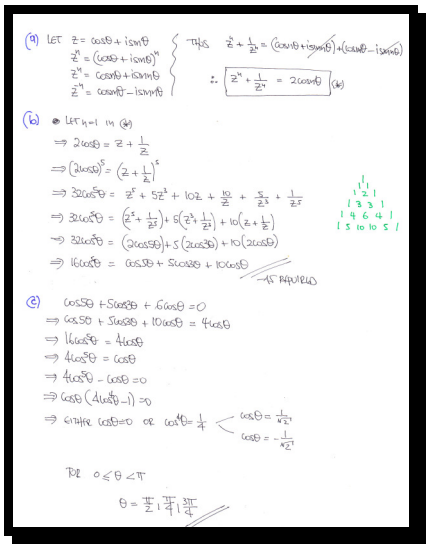
b) Hence show further that

$$16\cos^5\theta\equiv\cos 5\theta+5\cos 3\theta+10\cos\theta.$$

c) Use the results of parts **(a)** and **(b)** to solve the equation

$$\cos 5\theta+5\cos 3\theta+6\cos\theta=0,\ 0\leq\theta<\pi.$$

$$\boxed{\theta=\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4}}$$



Question 72 (*+)**

The complex number z lies on the curve with equation

$$|z+5-12\mathrm{i}|=6\,,\ z\in\mathbb{C}\,.$$

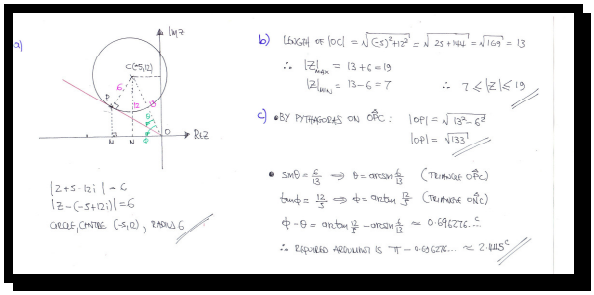
a) Sketch this curve in a standard Argand diagram.

b) Show that $a\le|z|\le b$, where a and b are integers.

The complex number z_0 lies on this curve so that its argument is the largest for all complex numbers which lie on this curve.

c) Determine the value of $|z_0|$ and the value of $\arg z_0$

$$\boxed{|z_0|=\sqrt{133}}\,,\ \boxed{\arg z_0\approx 2.445^{\text{c}}}$$



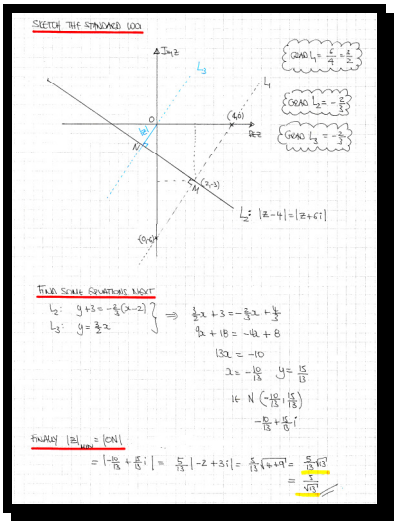
Question 73 **(***+)**

The complex number z satisfies

$$|z-4|=|z+6\mathrm{i}|.$$

Determine, as an exact simplified surd, the minimum value of $|z|$.

$\square$, $\boxed{|z|_{\min}=\frac{5}{\sqrt{13}}}$



Question 74 (****)

A transformation of the z plane onto the w plane is given by

$$w=\frac{az+b}{z+c},\ z\in\mathbb{C},\ z\neq -c$$

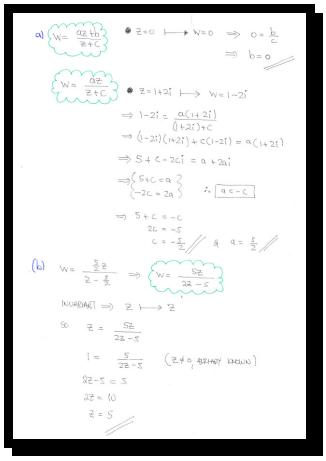
where a, b and c are real constants.

Under this transformation the point represented by the number $1+2\mathrm{i}$ gets mapped to its complex conjugate and the origin remains invariant.

a) Find the value of a, the value of b and the value of c.

b) Find the number, other than the number represented by the origin, which remains invariant under this transformation.

$$\boxed{a=\frac{5}{2}},\ \boxed{b=0},\ \boxed{c=-\frac{5}{2}},\ \boxed{z=5}$$



Question 75 (****)

$$z^7-1=0,\ z\in\mathbb{C}.$$

One of the roots of the above equation is denoted by ω, where $0<\arg\omega<\frac{\pi}{3}$.

a) Find ω in the form $\omega=re^{i\theta}$, $r>0$, $0<\theta\leq\frac{\pi}{3}$.

b) Show clearly that

$$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0.$$

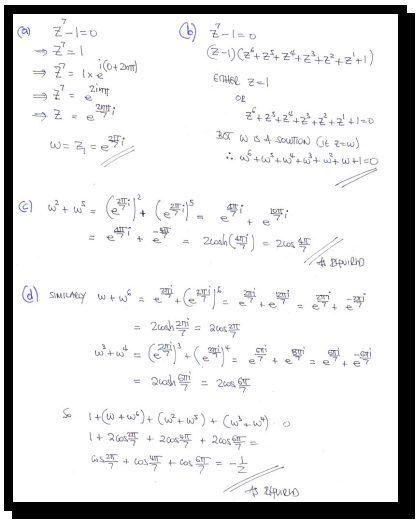
c) Show further that

$$\omega^2+\omega^5=2\cos\left(\frac{4\pi}{7}\right).$$

d) Hence, using the results from the previous parts, deduce that

$$\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\frac{1}{2}.$$

$$\boxed{\omega=e^{i\frac{2\pi}{7}}}$$



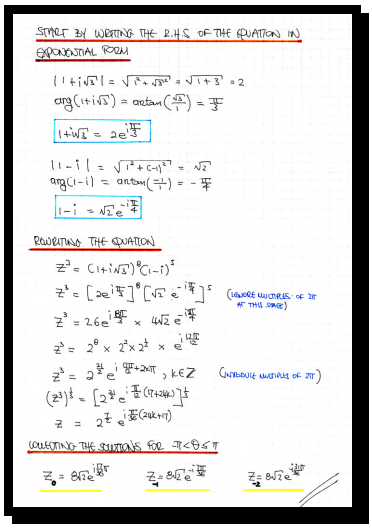
Question 76 (****)

$$z^3=\left(1+\mathrm{i}\sqrt{3}\right)^8\left(1-\mathrm{i}\right)^5,\ z\in\mathbb{C}.$$

Determine the three roots of the above equation.

Give the answers in the form $k\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi<\theta\leq\pi$, $k\in\mathbb{Z}$.

$\square$, $\boxed{z=8\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta},\quad \theta=-\frac{31\pi}{36},\ -\frac{7\pi}{36},\ \frac{17\pi}{36}}$



Question 77 (****)

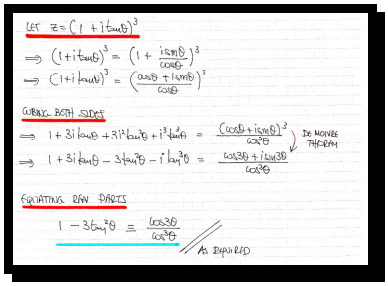
The complex number is defined as

$$z=\left(1+\mathrm{i}\tan\theta\right)^3,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

By considering the real part of z, or otherwise, prove the validity of the following trigonometric identity

$$1-3\tan^2\theta\equiv\frac{\cos 3\theta}{\cos^3\theta}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 78 (****)

Consider the following expression

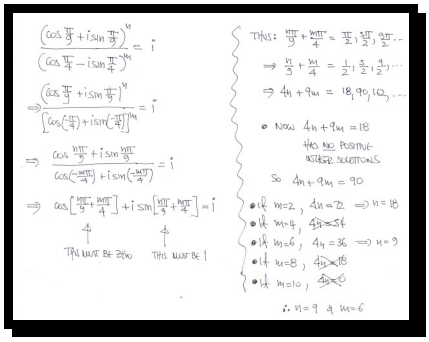
$$\frac{\left(\cos\frac{\pi}{9}+\mathrm{i}\sin\frac{\pi}{9}\right)^{n}}{\left(\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}\right)^{m}}=\mathrm{i}$$

The values of n and m are such so that

$$\{m\in\mathbb{N}:1\leq m\leq 9\} \text{ and } \{m\in\mathbb{N}:1\leq m\leq 9\}.$$

Determine, by a full mathematical method, the value of n and the value of m.

$$\boxed{m=6}\,,\ \boxed{n=9}$$



Question 79 (****)

A transformation T maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $v = u + \mathrm{i}v$ in the w plane, and is defined as

$$w = \frac{2}{\overline{z} - 1}, \quad z \in \mathbb{C}, \quad z \neq 1,$$

where $\overline{z}$ is the complex conjugate of z.

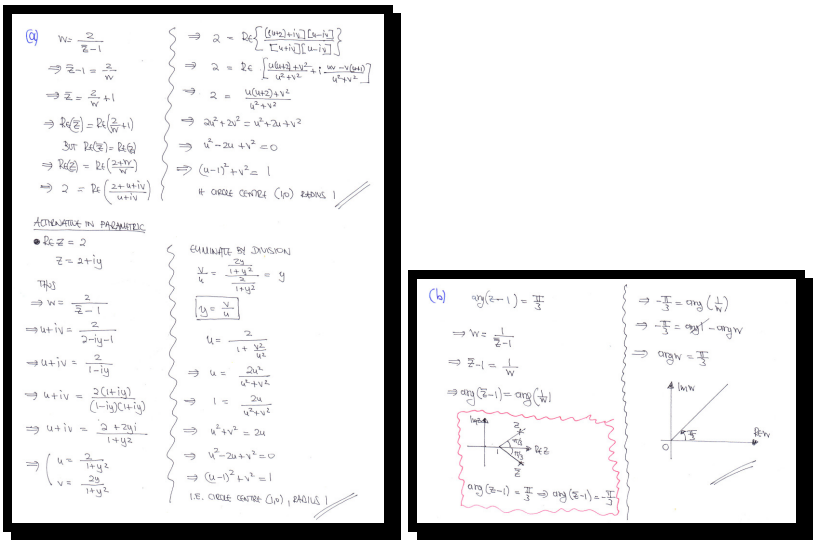
The line with equation $\mathrm{Re}\, z = 2$ is mapped by T onto a circle C in the w plane.

a) Determine the coordinates of the centre of C and the length of its radius.

b) Find an equation of the image in the w plane of the half line with equation

$$\arg(z - 1) = \frac{\pi}{3}.$$

$$\boxed{(1,0), \quad r = 1}, \quad \boxed{\arg w = \frac{\pi}{3}}$$



Question 80 (****)

A complex function $w = f(z)$ is defined as

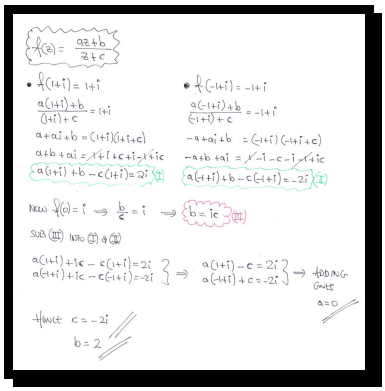
$$w = \frac{az+b}{z+c}, \ z \in \mathbb{C}, \ z \neq -c.$$

The constants a, b and c are complex.

Under the function f the points $1+\mathrm{i}$ and $-1+\mathrm{i}$ are invariant, while the origin is mapped onto i.

Determine the values of the constants a, b and c.

$$\boxed{a=0}, \ \boxed{b=2}, \ \boxed{c=-2\mathrm{i}}$$



Question 81 (****)

De Moivre's theorem asserts that

$$(\cos\theta+\mathrm{i}\sin\theta)^n \equiv \cos n\theta+\mathrm{i}\sin n\theta,\ \theta\in\mathbb{R},\ n\in\mathbb{Q}.$$

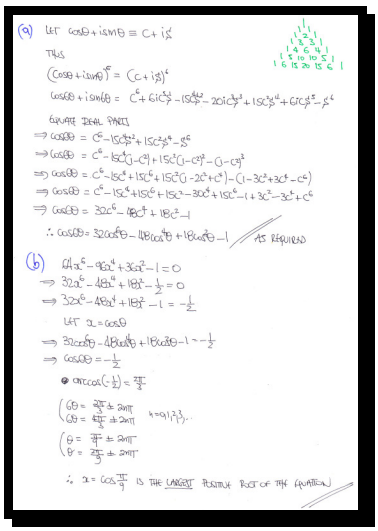
a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1.$$

b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6-96x^4+36x^2-1=0.$$

$$\boxed{x=\cos\left(\frac{\pi}{9}\right)}$$



Question 82 (****)

A transformation of the z plane to the w plane is given by

$$w=\frac{1}{z-2},\ z\in\mathbb{C},\ z\neq 2$$

where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

The line with equation

$$2x+y=3$$

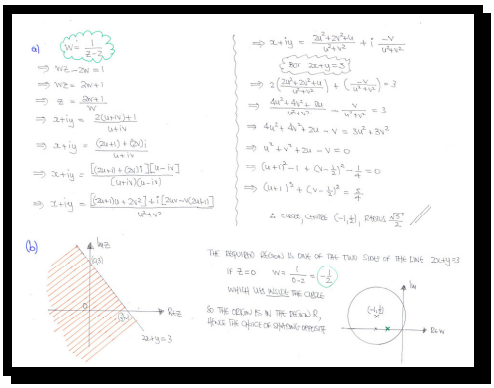
is mapped in the w plane onto a curve C.

a) Show that C represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region R in the z plane are mapped onto the points which lie inside C in the w plane.

b) Sketch and shade R in a suitable labelled Argand diagram, fully justifying the choice of region.

$$\boxed{\text{centre at } \left(-1,\frac{1}{2}\right),\ \text{radius}=\frac{\sqrt{5}}{2}}$$



Question 83 **(****)**

The locus of the point z in the Argand diagram, satisfy the equation

$$|z-2+\mathrm{i}|=\sqrt{3}\,.$$

a) Sketch the locus represented by the above equation.

The half line L with equation

$$y=mx-1, \quad x\geq 0, \quad m>0,$$

touches the locus described in part (a) at the point P.

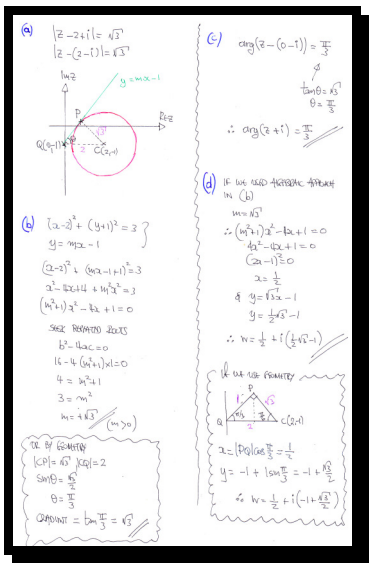
b) Find the value of m.

c) Write the equation of L, in the form

$$\arg\left(z-z_0\right)=\theta, \quad z_0\in\mathbb{C}, \quad -\pi<\theta\leq\pi\,.$$

d) Find the complex number w, represented by the point P.

$$\boxed{m=\sqrt{3}}, \quad \boxed{\arg\left(z+\mathrm{i}\right)=\frac{\pi}{3}}, \quad \boxed{w=\frac{1}{2}+\mathrm{i}\left(\frac{\sqrt{3}}{2}-1\right)}$$

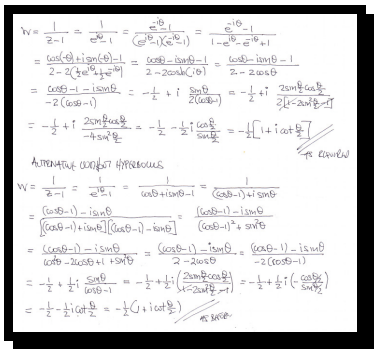


Question 84 **(****)**

If $z=\mathrm{e}^{\mathrm{i}\theta}$, $-\pi<\theta\leq\pi$, and $w=\dfrac{1}{z-1}$ show clearly that

$$w=-\frac{1}{2}\left[1+\mathrm{i}\cot\left(\frac{\theta}{2}\right)\right].$$

proof

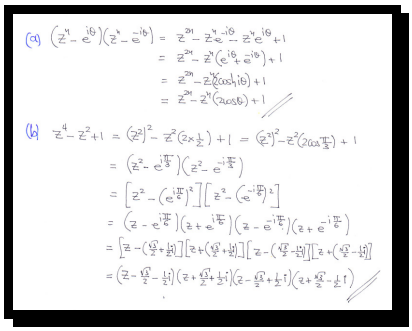


Question 85 **(****)**

a) Simplify fully $\left(z^n-\mathrm{e}^{\mathrm{i}\theta}\right)\left(z^n-\mathrm{e}^{-\mathrm{i}\theta}\right)$.

b) Hence factorize z^4-z^2+1 into 4 linear complex factors.

$$\boxed{z^{2n}-z^n\left(2\cos\theta\right)+1},\quad \boxed{\left(z+\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)\left(z+\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)\left(z-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)}$$



Question 86 (****)

Let $z=\cos\theta+\mathrm{i}\sin\theta=C+\mathrm{i}S$, $-\pi<\theta\leq\pi$.

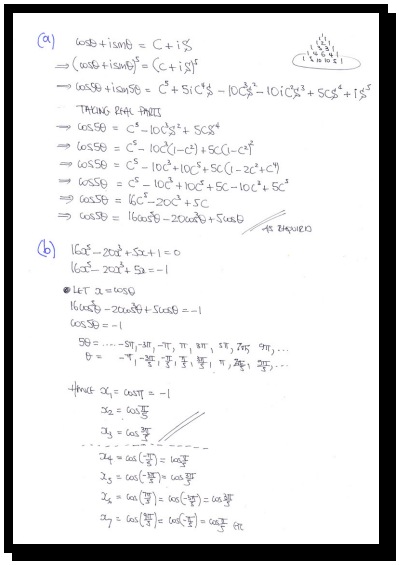
a) Use De Moivre's theorem to show that

$$\cos 5\theta\equiv 16\cos^5\theta-20\cos^3\theta+5\cos\theta.$$

b) Hence or otherwise find, in exact form where appropriate, 3 distinct solutions of the quintic equation

$$16x^5-20x^3+5x+1=0.$$

$$\boxed{x=-1,\ \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5}}$$



Question 87 (****)

Euler's identity states

$$e^{i\theta} \equiv \cos\theta + i\sin\theta\,,\ \theta \in \mathbb{R}\,.$$

a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2\cos n\theta\,.$$

b) Hence show further that

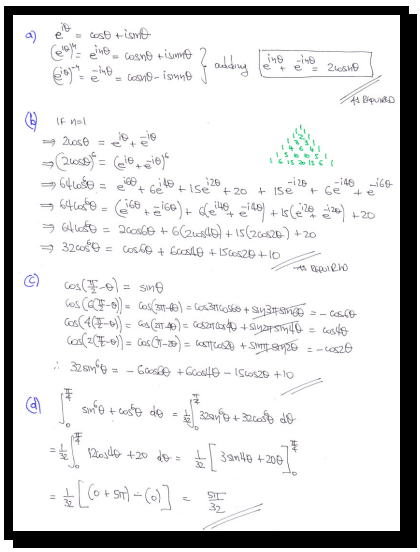
$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\,.$$

c) Use the fact that $\cos\left(\frac{\pi}{2}-\theta\right) \equiv \sin\theta$ to find a similar expression for $32\sin^6\theta$.

d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta\,.$$

$$\boxed{32\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}\,,\ \boxed{\frac{5\pi}{32}}$$



Question 88 (****)

A transformation of the z plane to the w plane is given by

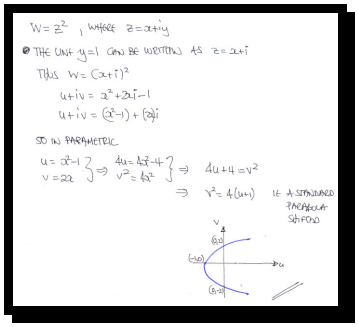
$$w = z^2, \ z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The straight line with equation $y = 1$ is mapped in the w plane onto a curve C.

Sketch the graph of C, marking clearly the coordinates of all points where the graph of C meets the coordinate axes.

sketch



Question 89 (****)

De Moivre's theorem asserts that

$$(\cos\theta+\mathrm{i}\sin\theta)^n\equiv\cos n\theta+\mathrm{i}\sin n\theta,\ \theta\in\mathbb{R},\ n\in\mathbb{Q}.$$

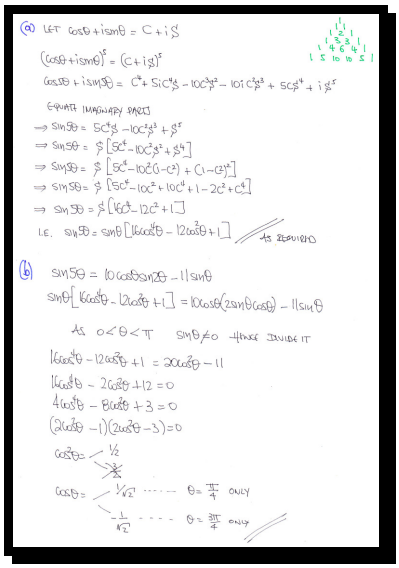
a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta=\sin\theta\left(16\cos^4\theta-12\cos^2\theta+1\right).$$

b) Hence, or otherwise, solve the equation

$$\sin 5\theta=10\cos\theta\sin 2\theta-11\sin\theta,\ 0<\theta<\pi.$$

$$\boxed{\theta=\frac{\pi}{4},\frac{3\pi}{4}}$$



Question 90 (****)

A transformation of points from the z plane onto points in the w plane is given by the complex relationship

$$w=z^2,\quad z\in\mathbb{C},$$

where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

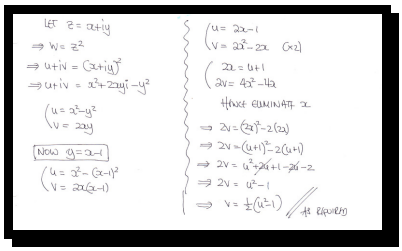
Show that if the point P in the z plane lies on the line with equation

$$y=x-1,$$

the locus of this point in the w plane satisfies the equation

$$v=\frac{1}{2}\left(u^2-1\right).$$

proof



Question 91 (****)

It is given that

$$\sin 5\theta \equiv \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

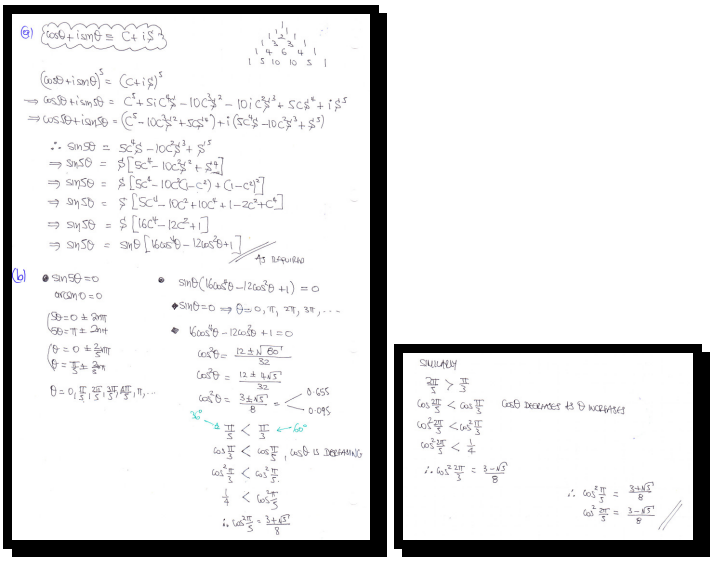
$$\sin 5\theta = 0.$$

b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{8}}, \quad \boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}}$$



Question 92 (****)

The complex number z is given by

$$z=\cos\theta+\mathrm{i}\sin\theta\,,\ -\pi<\theta\leq\pi\,.$$

a) Show clearly that

$$z^n+\frac{1}{z^n}\equiv 2\cos n\theta\,.$$

b) Hence show further that if $z=\cos\theta+\mathrm{i}\sin\theta$, the equation

$$3z^4-5z^3+8z^2-5z+3=0$$

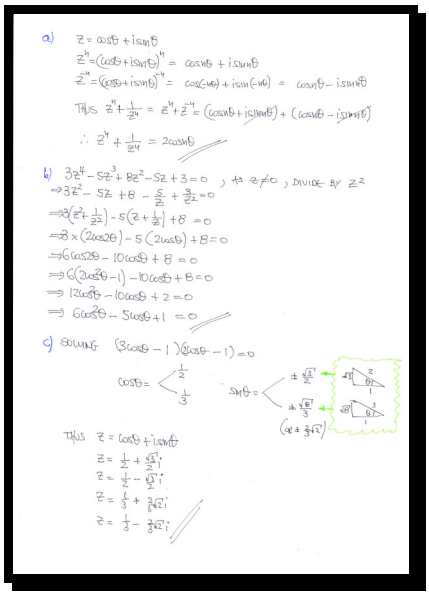
transforms into the equation

$$6\cos^2\theta-5\cos\theta+1=0\,.$$

c) Hence find in exact surd form the four roots of the equation

$$3z^4-5z^3+8z^2-5z+3=0\,.$$

$$\boxed{z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i},\ z=\frac{1}{3}\pm\frac{2}{3}\sqrt{2}\mathrm{i},}$$



Question 93 (****)

A complex transformation from the z plane to the w plane is defined by

$$w=\frac{z+\mathrm{i}}{3+\mathrm{i}z},\ z\in\mathbb{C},\ z\neq 3\mathrm{i}.$$

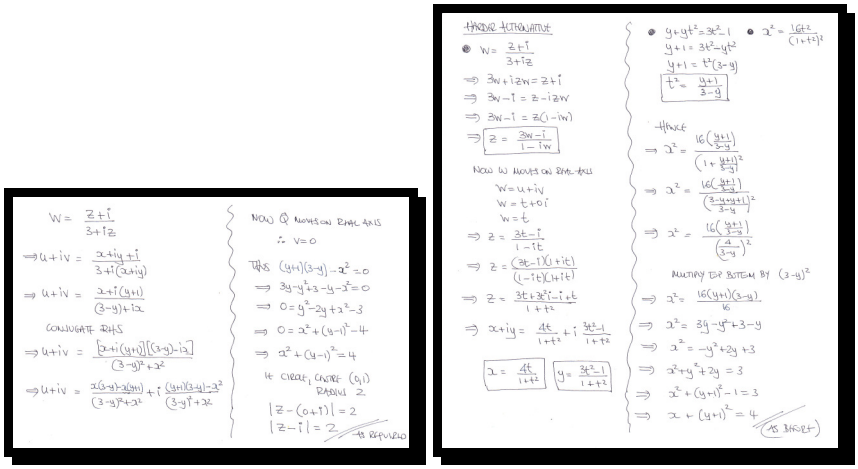
The point $P(x,y)$ is mapped by this transformation into the point $Q(u,v)$.

It is further given that Q lies on the real axis for all the possible positions of P.

Show that the P traces the curve with equation

$$|z-\mathrm{i}|=2.$$

proof



Question 94 (****)

The complex number z is given by $z=\mathrm{e}^{\mathrm{i}\theta}$, $-\pi<\theta\leq\pi$

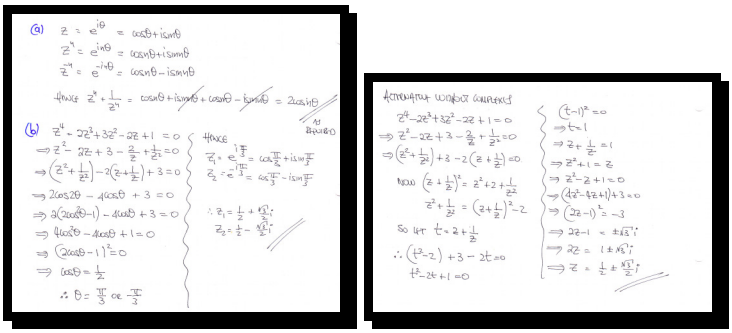
a) Show clearly that

$$z^n+\frac{1}{z^n}=2\cos n\theta\,.$$

b) Hence solve the equation

$$z^4-2z^3+3z^2-2z+1=0\,.$$

$$\boxed{z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}}$$



Question 95 (****)

A transformation of the z plane to the w plane is given by

$$w=\frac{2z+1}{z},\ z\in\mathbb{C},\ z\neq 0$$

where $z=x+\mathrm{i}y$ and $w=u+\mathrm{i}v$.

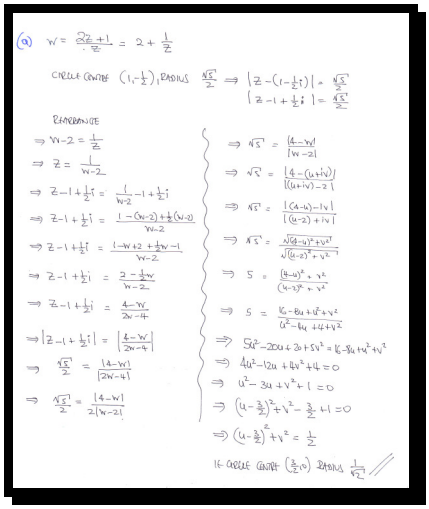
The circle C_1 with centre at $\left(1,-\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$ in the z plane is mapped in the w plane onto another curve C_2.

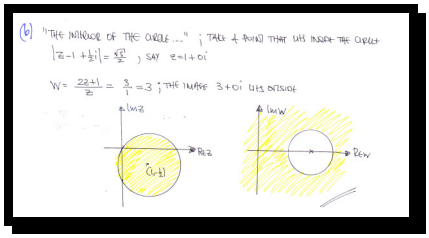
a) Show that C_2 is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside C_1 in the z plane are mapped onto points of a region R in the w plane.

b) Sketch and shade R in a suitably labelled Argand diagram, fully justifying the choice of the region.

$$\boxed{\text{centre at } \left(\frac{3}{2},0\right),\ \text{radius}=\frac{1}{\sqrt{2}}}$$





Question 96 (****)

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \mathrm{i}\sqrt{3} \qquad \text{and} \qquad z_2 = \mathrm{i}z_1 .$$

a) Label accurately the points representing z_1 and z_2, in an Argand diagram.

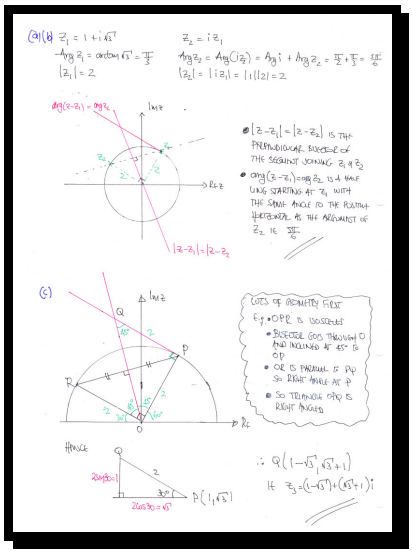
b) On the same Argand diagram, sketch the locus of the points z satisfying …

i. … $|z - z_1| = |z - z_2|$.

ii. … $\arg(z - z_1) = \arg z_2$.

c) Determine, in the form $x + \mathrm{i}y$, the complex number z_3 represented by the intersection of the two loci of part **(b)**.

$\square$, $\boxed{z_3 = \left(1 - \sqrt{3}\right) + \mathrm{i}\left(1 + \sqrt{3}\right)}$



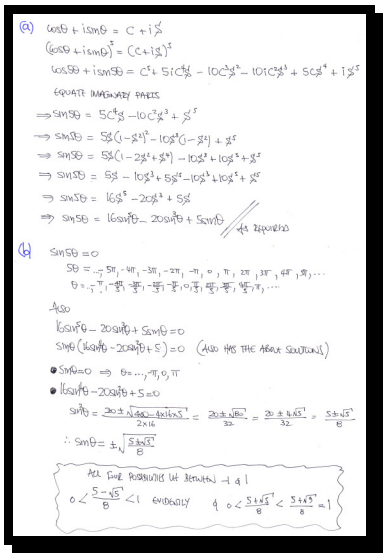
Question 97 (****)

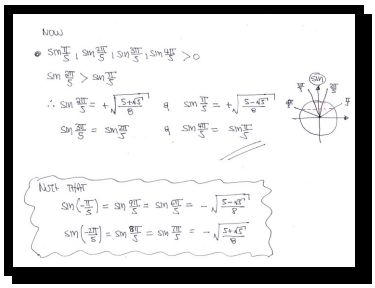
a) Use De Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

b) By considering the solutions of the equation $\sin 5\theta = 0$, find in exact surd form the values of $\sin\left(\frac{n\pi}{5}\right)$, for $n = 1,2,3,4$.

$$\boxed{\sin\frac{\pi}{5} = \sin\frac{4\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}} \quad , \quad \boxed{\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}}$$





Question 98 (****)

A transformation of the z plane to the w plane is given by

$$w = z + \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0,$$

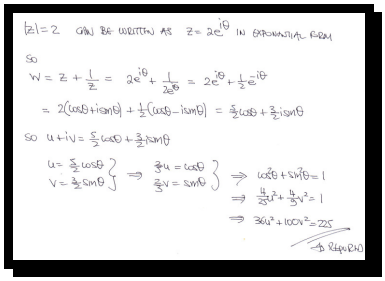
where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The locus of the points in the z plane that satisfy the equation $|z| = 2$ are mapped in the w plane onto a curve C.

By considering the equation of the locus $|z| = 2$ in exponential form, or otherwise, show that a Cartesian equation of C is

$$36u^2 + 100v^2 = 225.$$

proof



Question 99 (****)

a) Use De Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

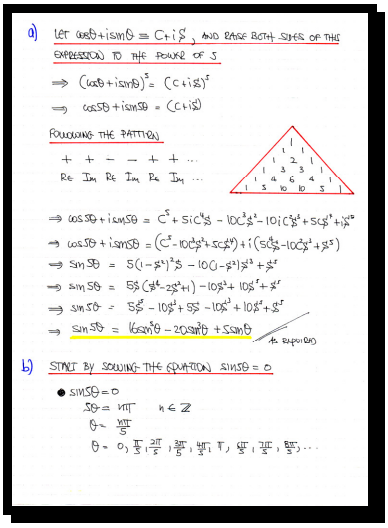
b) By considering the solutions of the equation $\sin 5\theta = 0$, find in trigonometric form the four solutions of the equation

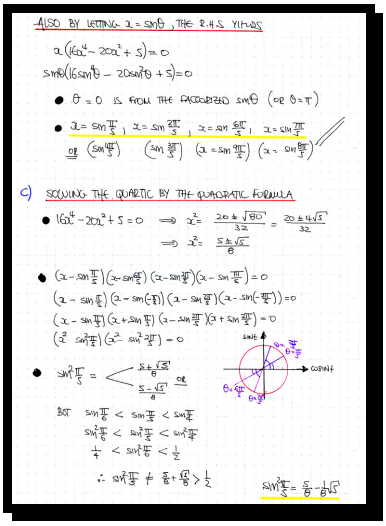
$$16x^4 - 20x^2 + 5 = 0 .$$

c) Hence show, with full justification, that

$$\sin^2\left(\frac{\pi}{5}\right) = \frac{5-\sqrt{5}}{8} .$$

$\boxed{}$, $\boxed{x = \sin\left(\frac{1}{5}k\pi\right), \quad k = 1,2,6,7}$





Question 100 (****)

The complex function $w = f(z)$ is given by

$$w = \frac{1}{1-z} \ , \ z \neq 1 .$$

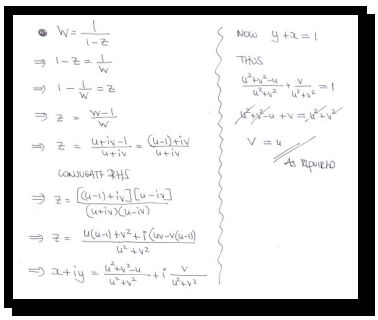
The point $P(x, y)$ in the z plane traces the line with Cartesian equation

$$y + x = 1 .$$

Show that the locus of the **image** of P in the w plane traces the line with equation

$$v = u .$$

proof

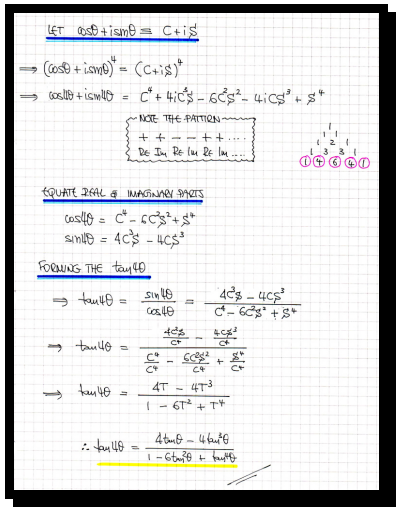


Question 101 (****)

By considering the binomial expansion of $\left(\cos\theta+\mathrm{i}\sin\theta\right)^4$ show that

$$\tan 4\theta \equiv \frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.$$

[ ], [proof]



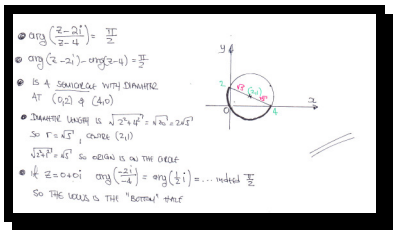
Question 102 (****)

In an Argand diagram which represents the z plane, the complex number $z=x+\mathrm{i}y$ satisfies the relationship

$$\arg\left(\frac{z-2\mathrm{i}}{z-4}\right)=\frac{\pi}{2}.$$

Sketch the curve that the locus of z traces.

[sketch]



Question 103 (****)

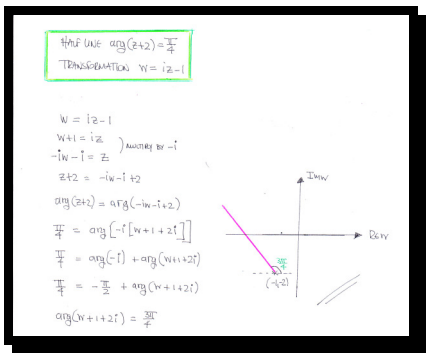
A transformation from the z plane to the w plane is defined by the equation

$$w=\mathrm{i}\,z-1,\ z\in\mathbb{C}\,.$$

Sketch in the w plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg\left(z+2\right)=\frac{\pi}{4},\ z\in\mathbb{C}\,.$$

graph



Question 104 (****)

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

It is given that

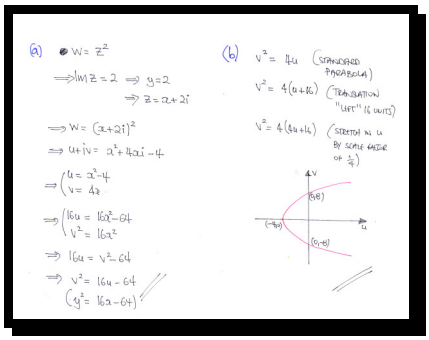
$$f(z)=z^2,\ z\in\mathbb{C}.$$

The line with equation $\operatorname{Im} z=2$ in the z plane is mapped onto the curve C in the w plane.

a) Find a Cartesian equation for C.

b) Sketch the graph of C.

$$\boxed{v^2=16u-64}$$



Question 105 (****)

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

It is given that

$$f(z)=\frac{4}{z},\ z\in\mathbb{C},\ z\neq 0.$$

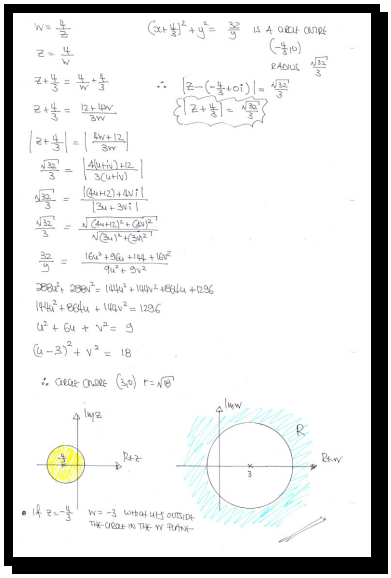
The points from the z plane, except the origin, which lie inside and on the boundary of the circle with equation

$$\left(x+\frac{4}{3}\right)^2+y^2=\frac{32}{9},$$

are mapped onto the region R in the w plane.

Shade the region R in a clearly labelled Argand diagram.

sketch



Question 106 **(****)**

$$z = \mathrm{e}^{\mathrm{i}\theta}\,,\ -\pi < \theta \le \pi\,.$$

a) Show that …

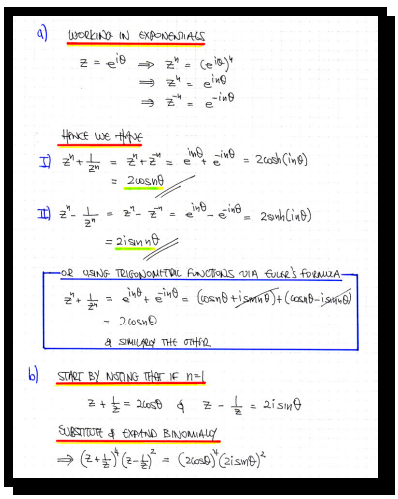
i. … $z^n + \dfrac{1}{z^n} = 2\cos n\theta\,.$

ii. … $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta\,.$

b) Hence show further that

$$\cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta\,.$$

[ ], [proof]


a) WORKING IN EXPONENTIALS

$z = e^{i\theta} \Rightarrow z^n = (e^{i\theta})^n$

$\Rightarrow z^n = e^{in\theta}$

$\Rightarrow z^{-n} = e^{-in\theta}$

HENCE WE HAVE

I) $z^n + \frac{1}{z^n} = z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = 2\cos(in\theta)$

$= 2\cos n\theta$

II) $z^n - \frac{1}{z^n} = z^n - z^{-n} = e^{in\theta} - e^{-in\theta} = 2\sin(in\theta)$

$= 2i\sin n\theta$

OR USING TRIGONOMETRIC FUNCTIONS VIA EULER'S FORMULA

$z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$= 2\cos n\theta$

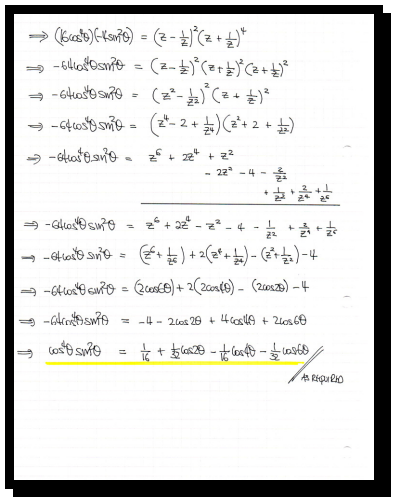
& SIMILARLY THE OTHER.

b) START BY NOTING THAT IF $n=1$

$z + \frac{1}{z} = 2\cos\theta$ & $z - \frac{1}{z} = 2i\sin\theta$

SUBSTITUTE & EXPAND BINOMIALLY

$\Rightarrow \left(z + \frac{1}{z}\right)^4 \left(z - \frac{1}{z}\right)^2 = (2\cos\theta)^4 (2i\sin\theta)^2$


$\Rightarrow (16\cos^4\theta)(-4\sin^2\theta) = \left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^2 - \frac{1}{z^2}\right)^2 \left(z + \frac{1}{z}\right)^2$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^4 - 2 + \frac{1}{z^4}\right)\left(z^2 + 2 + \frac{1}{z^2}\right)$

$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 + z^2 - 2z^2 - 4 - \frac{2}{z^2} + \frac{1}{z^4} + \frac{2}{z^4} + \frac{1}{z^6}$

$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 - z^2 - 4 - \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^6 + \frac{1}{z^6}\right) + 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) - 4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = 2\cos(6\theta) + 2(2\cos(4\theta)) - (2\cos(2\theta)) - 4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = -4 - 2\cos 2\theta + 4\cos 4\theta + 2\cos 6\theta$

$\Rightarrow \cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$

AS REQUIRED

Question 107 (****)

The locus of a point, represented by the complex number z, satisfies the relationship

$$\left|z+1+\mathrm{i}\right|=\left|z-1+2\mathrm{i}\right|.$$

When this locus is transformed by the complex function

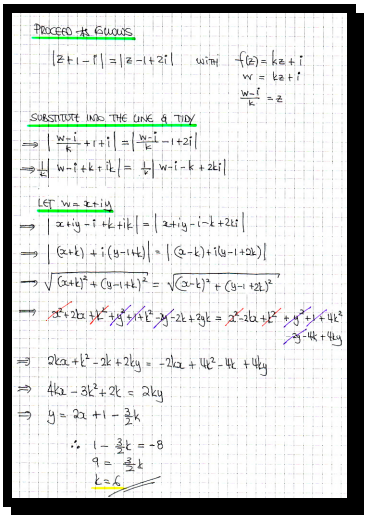
$$f\left(z\right)=kz+\mathrm{i}\,,\ k\in\mathbb{R}\,,$$

the image of the locus traces the straight line with Cartesian equation

$$y=2x-8\,.$$

Determine the value of k.

$\boxed{}$, $\boxed{k=6}$



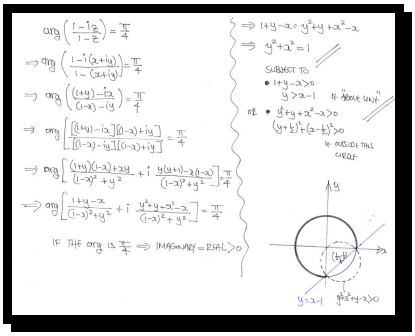
Question 108 (****)

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-\mathrm{i}z}{1-z}\right) = \frac{\pi}{4} \ , \ z \neq -\mathrm{i} \, .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{u^2 + v^2 = 1, \quad \text{such that} \quad v > u - 1}$$



Question 109 (****)

The complex function $w = f(z)$ satisfies

$$w = \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0.$$

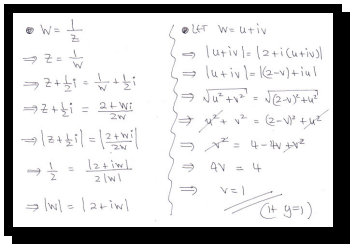
This function maps the point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

It is further given that P traces the curve with equation

$$\left| z + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of Q.

$\boxed{v = 1}$

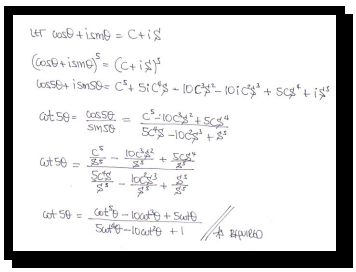


Question 110 (****)

Use De Moivre's theorem to show that

$$\cot 5\theta \equiv \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}.$$

proof



Question 111 (****)

A transformation T from the z plane to the w plane is defined by

$$w=\frac{z-\mathrm{i}}{z+1},\ z\in\mathbb{C},\ z\neq -1.$$

T transforms the circle with equation $|z|=1$ in the z plane, into the straight line L in the w plane.

a) Find a Cartesian equation for L.

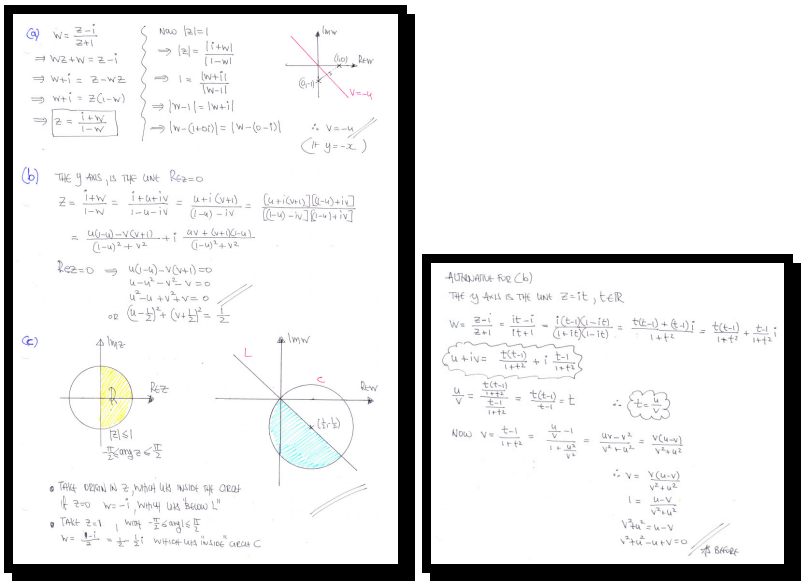
T transforms the y axis in the z plane, into the curve C in the w plane.

b) Find a Cartesian equation for C.

The region R in the z plane, satisfies $|z|\leq 1$ such that $-\frac{\pi}{2}\leq \arg z\leq \frac{\pi}{2}$.

c) Shade the image of R under T in the w plane.

$\boxed{y=-x \text{ or } v=-u}$, $\boxed{u^2+v^2-u+v=0}$



Question 112 (****)

A transformation T maps the point $x+\mathrm{i}y$ from the z plane to the point $u+\mathrm{i}v$ in the w plane, and is defined by

$$w=\frac{z+\mathrm{i}}{z},\ z\in\mathbb{C},\ z\neq 0.$$

T transforms the line with equation $y=x$ in the z plane, except the origin, into the straight line L_1 in the w plane.

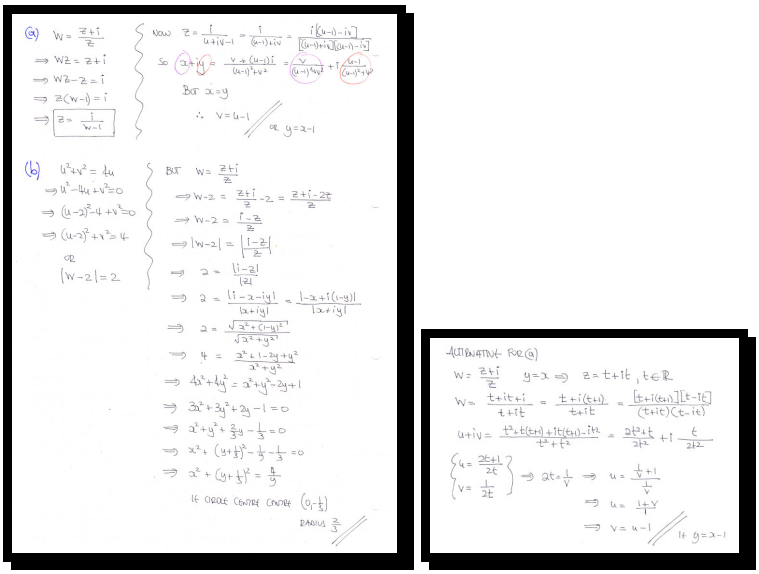
a) Find a Cartesian equation for L_1.

T transforms the circle C_1 in the z plane, into the circle C_2 in the w plane.

b) Find the coordinates of the centre of C_1 and the length of its radius, given the Cartesian equation of C_2 is

$$u^2+v^2=4u.$$

$$\boxed{y=x-1 \text{ or } v=u-1},\ \boxed{\left(0,-\frac{1}{3}\right),\ r=\frac{2}{3}}$$



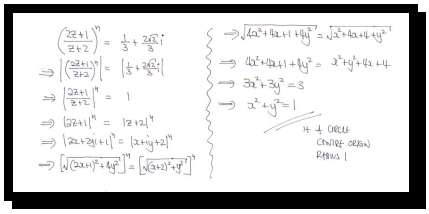
Question 113 (****)

The complex number z satisfies the relationship

$$\left(\frac{2z+1}{z+2}\right)^n=\frac{1}{3}+\frac{2\sqrt{2}}{3}\mathrm{i},\quad z\neq -2,\quad n\in\mathbb{N}.$$

Show that the point represented by z in an Argand diagram represents a circle, stating the coordinates of its centre and the size of its radius.

$$\boxed{(0,0),\ r=1}$$



Question 114 (****)

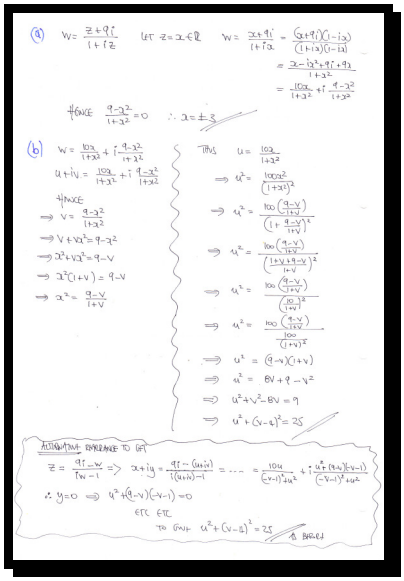
The numbers z and w satisfy the relationship

$$w=\frac{z+9\mathrm{i}}{1+\mathrm{i}z},\ z\neq\mathrm{i}\,.$$

a) Given that $w\in\mathbb{R}$, find the possible values of z.

b) Given instead that $z\in\mathbb{R}$, find a Cartesian equation of the locus of the point represented by w, in an Argand diagram.

$$\boxed{z=\pm 3,\text{ or } x=\pm 3},\ \boxed{u^2+(v-4)^2=25}$$



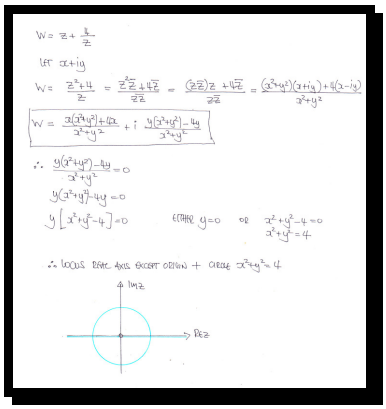
Question 115 (****)

The numbers z and w satisfy the relationship

$$w = z + \frac{4}{z}, \quad z \in \mathbb{C}, \ z \neq 0 .$$

Given that w is always real sketch in a suitably labelled Argand diagram the locus of the possible positions of z.

graph



Question 116 (****)

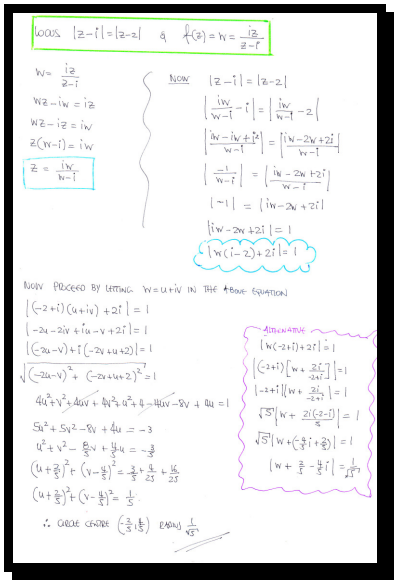
A transformation from the z plane to the w plane is defined by the equation

$$f(z)=\frac{\mathrm{i}z}{z-\mathrm{i}},\ z\in\mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-\mathrm{i}|=|z-2|,\ z\in\mathbb{C}.$$

$$\boxed{\left(u+\frac{2}{5}\right)^2+\left(v-\frac{4}{5}\right)^2=\frac{1}{5}}$$



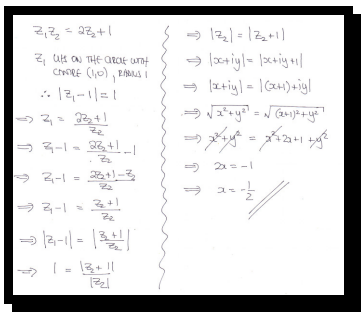
Question 117 (****)

The complex numbers z_1 and z_2, satisfy the relationship

$$z_1 z_2 = 2z_2 + 1, \ z_2 \neq 0.$$

Given that z_1 is tracing a circle with centre at $(1,0)$ and radius 1 in an Argand diagram, determine a Cartesian equation of the locus that z_2 is tracing.

$$\boxed{x = -\frac{1}{2}}$$



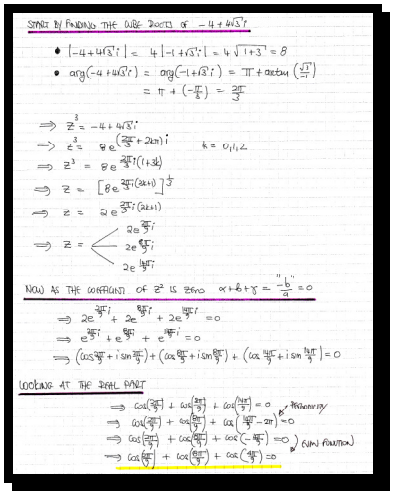
Question 118 (****)

$$z^3+4=4\sqrt{3}\,\mathrm{i}\,.$$

By considering the sum of the three roots of the above cubic equation show clearly that

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0\,.$$

[ ], [proof]

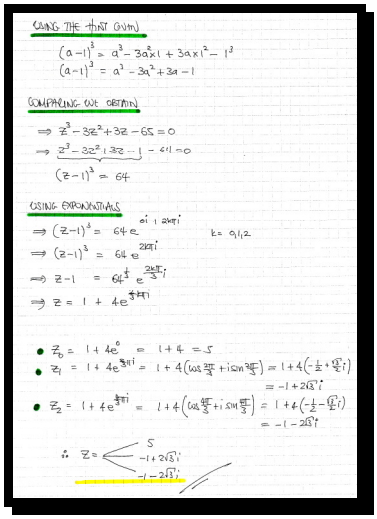


Question 119 (****)

$$z^3-3z^2+3z-65=0,\ z\in\mathbb{C}.$$

By considering the binomial expansion of $(a-1)^3$, or otherwise, find in exact form where appropriate the three solutions of the above equation.

$\square$, $\boxed{z=5,\ -1\pm i\frac{\sqrt{3}}{2}}$



Question 120 **(****+)**

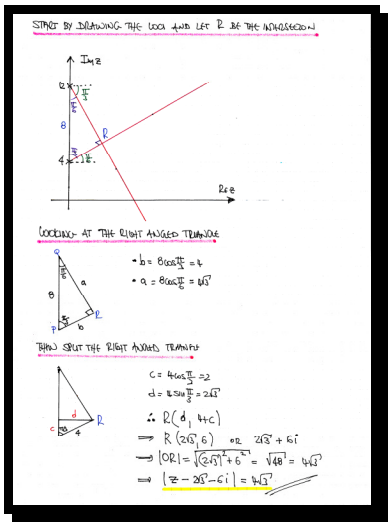
The complex number w is the point of intersection of the following two loci in a standard Argand diagram

$$\arg(z-4\mathrm{i})=\frac{\pi}{6} \quad \text{and} \quad \arg(z-12\mathrm{i})=-\frac{\pi}{3}.$$

Determine the equation of the circle which passes though w and the origin of the Argand diagram.

Give the answer in the form $|z-w|=r$, where w and r must be stated.

$\boxed{}$, $\boxed{|z-2\sqrt{3}-6\mathrm{i}|=4\sqrt{3}}$



Question 121 (**+)**

The complex number $17+k\text{i}$, where k is a real constant, satisfies the locus

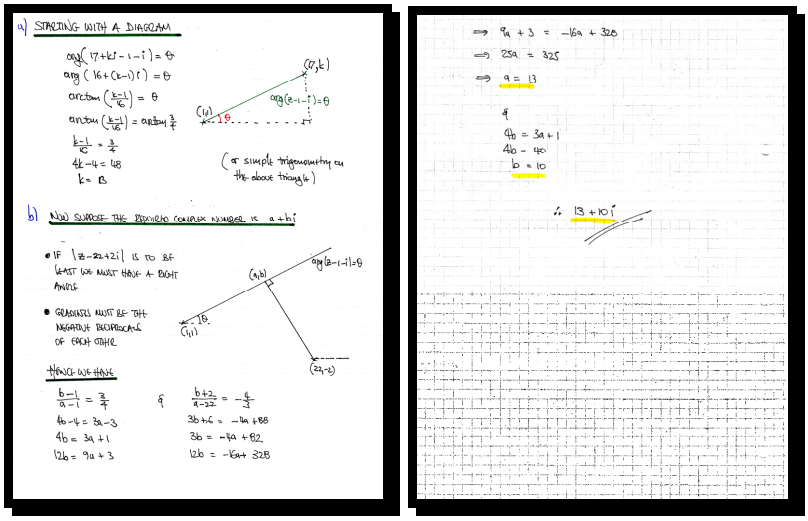
$$\arg\left(z-1-\text{i}\right)=\theta,$$

where $\theta=\arctan\dfrac{3}{4}$.

a) Determine the value of k.

b) Find the complex number z which satisfies the locus $\arg\left(z-1-\text{i}\right)=\theta$ so that $\left|z-22+2\text{i}\right|$ is least.

$\boxed{}$, $\boxed{k=13}$, $\boxed{13+10\text{i}}$



Question 122 (**+)**

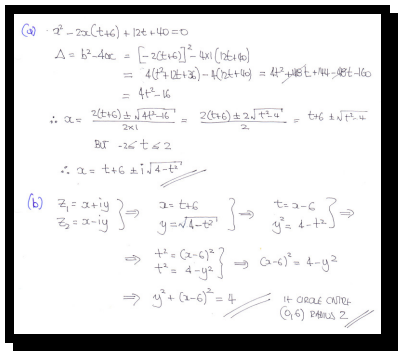
The quadratic equation

$$x^2-2x\left(t+6\right)+12t+40=0,$$

where t is a parameter such that $-2\le t\le 2$, has complex roots.

Show that for all t such that $-2\le t\le 2$, the roots of this quadratic equation lie on a circle in an Argand diagram.

$\boxed{}$, $\boxed{x=t+6\pm \mathrm{i}\sqrt{4-t^2}}$



Question 123 (**+)**

The complex function $w = f(z)$ is defined by

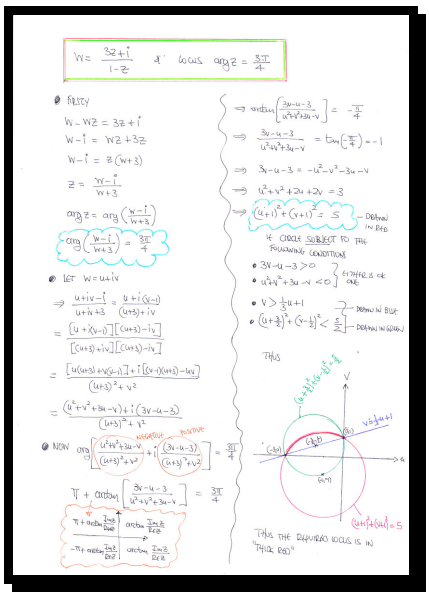
$$w = \frac{3z+\mathrm{i}}{1-z}, \ z \in \mathbb{C}, \ z \neq 1.$$

The half line with equation $\arg z = \frac{3\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{(u+1)^2+(v+1)^2=5, \quad v>\frac{1}{3}u+1}$$



Question 124 (**+)**

It is given that

$$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

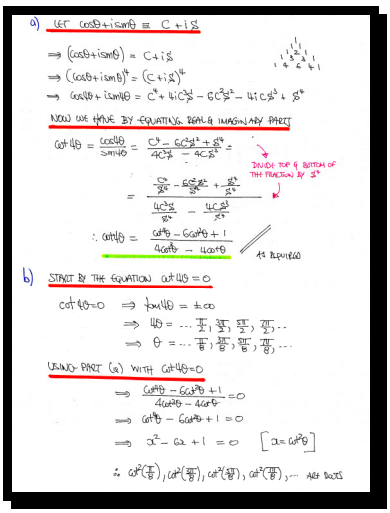
b) Deduce that $x = \cot^2\left(\frac{\pi}{8}\right)$ is one of the two solutions of the equation

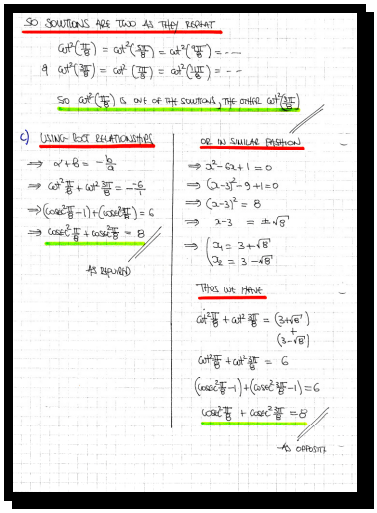
$$x^2 - 6x + 1 = 0.$$

c) Show further that

$$\operatorname{cosec}^2\left(\frac{\pi}{8}\right) + \operatorname{cosec}^2\left(\frac{3\pi}{8}\right) = 8.$$

[ ], [proof]





Question 125 (**+)**

In an Argand diagram which represents the z plane, the complex number $z=x+\mathrm{i}y$ satisfies the relationship

$$\arg\left(\frac{z-2\mathrm{i}}{z-4}\right)=\frac{\pi}{2}.$$

a) Sketch the curve that the locus of z traces.

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

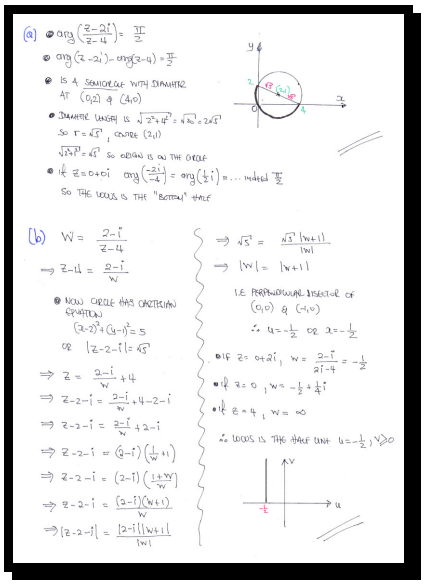
It is given that

$$f(z)=\frac{2-\mathrm{i}}{z-4},\ z\in\mathbb{C},\ z\neq 4.$$

The points in the z plane which lie on the locus described in part (a) are mapped onto a line in the w plane.

b) Sketch this line in an Argand diagram representing the w plane.

sketch



Question 126 (**+)**

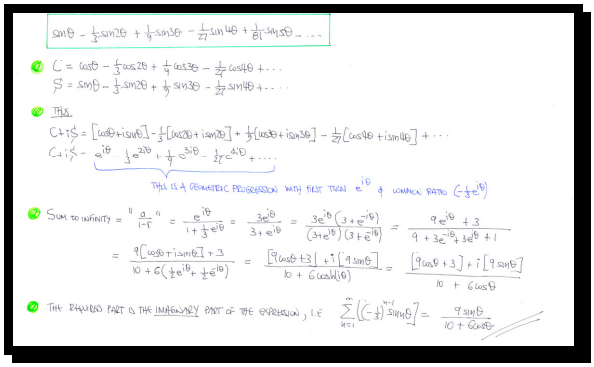
The following convergent series S is given below

$$S=\sin\theta-\frac{1}{3}\sin 2\theta+\frac{1}{9}\sin 3\theta-\frac{1}{27}\sin 4\theta\ ...$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S=\frac{9\sin\theta}{10+6\cos\theta}.$$

proof



Question 127 **(****+)**

$$f(z)=z^6+8z^3+64,\ z\in\mathbb{C}.$$

a) Given that $f(z)=0$, show that

$$z^3=-4\pm4\sqrt{3}\mathrm{i}.$$

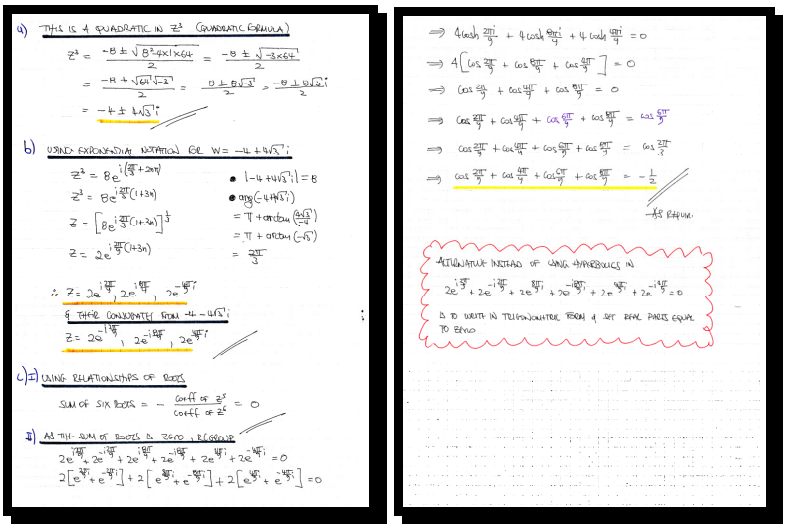
b) Find the six solutions of the equation $f(z)=0$, giving the answers in the form $z=re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$.

c) Show further that …

i. … the sum of the six roots is zero.

ii. … $\cos\dfrac{2\pi}{9}+\cos\dfrac{4\pi}{9}+\cos\dfrac{6\pi}{9}+\cos\dfrac{8\pi}{9}=-\dfrac{1}{2}$.

$\square$, $\boxed{z=2e^{\mathrm{i}\varphi},\varphi=\pm\dfrac{2\pi}{9},\pm\dfrac{4\pi}{9},\pm\dfrac{8\pi}{9}}$



Question 128 **(****+)**

$$z=\cos\theta+\mathrm{i}\sin\theta\,,\ -\pi<\theta\leq\pi\,.$$

a) Show clearly that

$$\frac{2}{1+z}=1-\mathrm{i}\tan\frac{\theta}{2}\,.$$

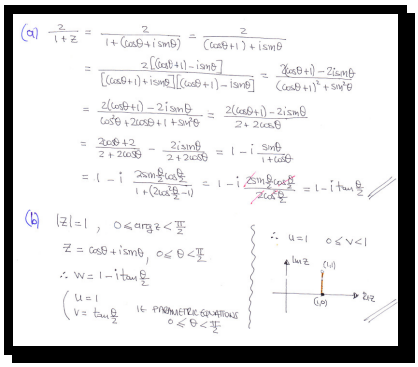
The complex function $w=f\left(z\right)$ is defined by

$$w=\frac{2}{1+z}\,,\ z\in\mathbb{C}\,,\ z\neq-1\,.$$

The circular arc $\left|z\right|=1$, for which $0\leq\arg z<\dfrac{\pi}{2}$, is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

proof/sketch



Question 129 **(****+)**

De Moivre's theorem states that

$$(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta,\ n\in\mathbb{Q}.$$

a) Use De Moivre's theorem to show that

$$\tan 5\theta\equiv\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}.$$

b) Use part **(a)** to find the solutions of the equation

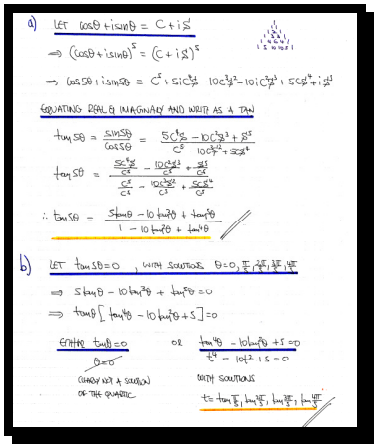
$$t^4-10t^2+5=0,$$

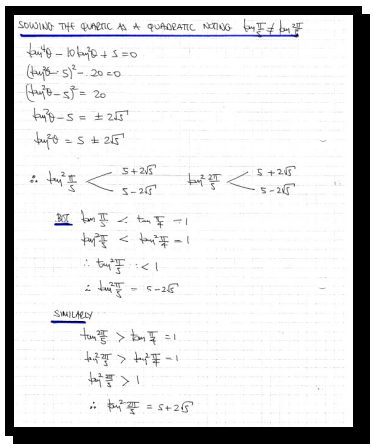
giving the answers in the form $t=\tan\varphi,\ 0<\varphi<\pi$.

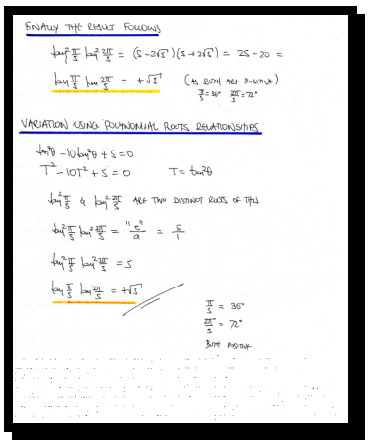
c) Show further that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5}.$$

$\square$, $\boxed{t=\tan\frac{\pi}{5},\tan\frac{2\pi}{5},\tan\frac{3\pi}{5},\tan\frac{4\pi}{5}}$







Question 130 (**+)**

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

It is given that

$$f(z)=\frac{z}{z-\mathrm{i}},\ z\in\mathbb{C},\ z\neq\mathrm{i}.$$

A circle C_1 with centre at $z=\mathrm{i}$ and radius 1 is mapped onto the circle C_2 in the w plane.

a) Find the coordinates of the centre of C_2, and the length of its radius.

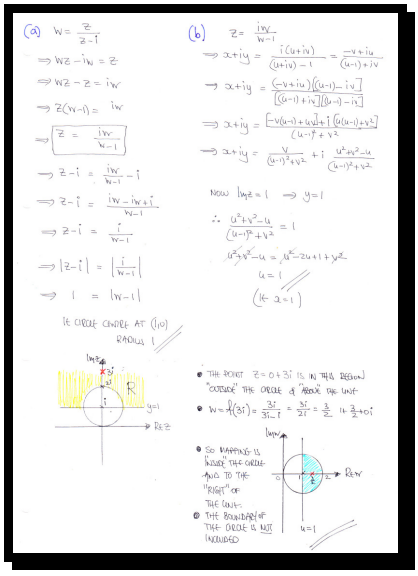
The straight line $z=\mathrm{i}$ is mapped onto another line L in the w plane.

b) Find an equation for this line.

The region R in the z plane lies outside C_1 such that $\mathrm{Im}\,z\geq 1$.

c) Shade in a clearly labelled diagram the image of R in the w plane.

$\boxed{(1,0),\ r=1}$, $\boxed{u=1\ \text{ or }\ x=1}$



Question 131 (**+)**

$$z^5-1=0\,,\ z\in\mathbb{C}\,,\ -\pi<\arg z\le\pi\,.$$

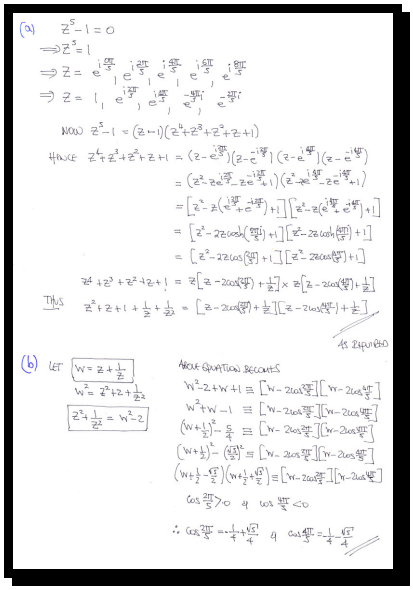
a) By considering the four complex roots of the above equation show clearly that

$$z^2+z+1+\frac{1}{z}+\frac{1}{z^2}=\left[z+\frac{1}{z}-2\cos\left(\frac{2\pi}{5}\right)\right]\left[z+\frac{1}{z}-2\cos\left(\frac{4\pi}{5}\right)\right].$$

b) Use the substitution $w=z+\dfrac{1}{z}$ in the above equation, to find in exact surd form the values of

$$\cos\left(\frac{2\pi}{5}\right)\ \text{and}\ \cos\left(\frac{4\pi}{5}\right).$$

$$\boxed{\cos\left(\frac{2\pi}{5}\right)=\frac{-1+\sqrt{5}}{4}},\ \boxed{\cos\left(\frac{4\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}}$$



Question 132 (**+)**

The complex number $x+iy$ in the z plane of an Argand diagram satisfies the inequality

$$x^2+y^2+x>0\,.$$

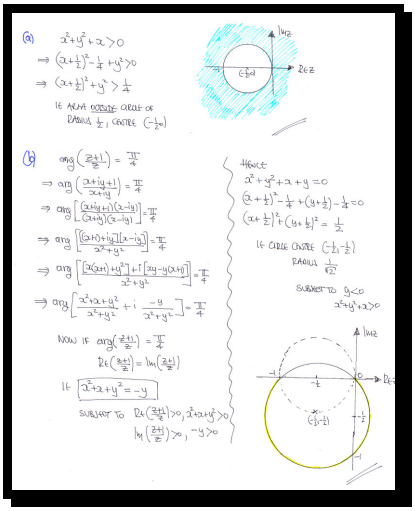
a) Sketch the region represented by this inequality.

A locus in the z plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}\,.$$

b) Sketch the locus represented by this equation.

sketch



Question 133 (**+)**

The following finite sums, C and S, are given by

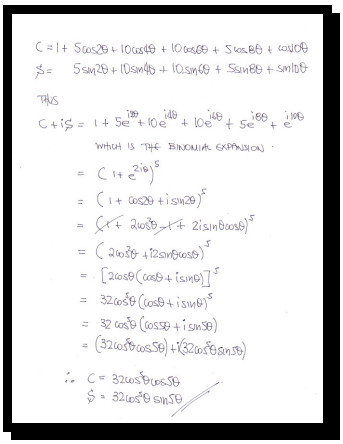
$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$
$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of $(1+A)^5$, show clearly that

$$C = 32\cos^5\theta\cos 5\theta,$$

and find a similar expression for S

$$\boxed{S = 32\cos^5\theta\sin 5\theta}$$



Question 134 (**+)**

The complex function with equation

$$f(z)=\frac{1}{z^2},\ z\in\mathbb{C},\ z\neq 0$$

maps the complex number $x+\mathrm{i}y$ from the z plane onto the complex number $u+\mathrm{i}v$ in the w plane.

The line with equation

$$y=mx,\ x\neq 0,$$

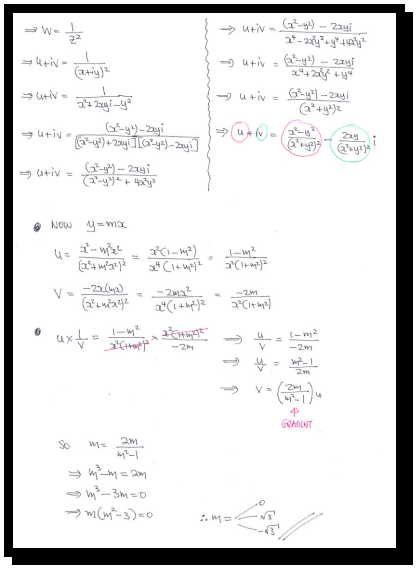
is mapped onto the line with equation

$$v=Mu,$$

where m and M are the respective gradients of the two lines.

Given that $m=M$, determine the three possible values of m.

$$\boxed{m=0,\pm\sqrt{3}}$$



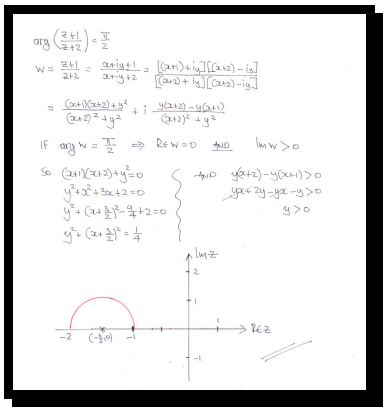
Question 135 (**+)**

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{z+1}{z+2}\right) = \frac{\pi}{2} \ , \ z \neq -2 \, .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\left(x+\frac{3}{2}\right)^2 + y^2 = \frac{1}{4}, \quad \text{such that} \quad y > 0$$



Question 136 (**+)**

$$z^n = 1,\ z \in \mathbb{C},\ n \in \mathbb{N}.$$

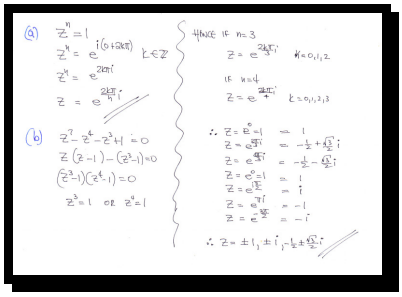
a) Solve the above equation, giving the general solution in terms of n and any suitably defined parameters.

b) Hence solve the equation

$$z^7 - z^4 - z^3 + 1 = 0,\ z \in \mathbb{C},$$

giving the answers in the form $x + \mathrm{i}y$, $x, y \in \mathbb{R}$, where appropriate.

$$\boxed{z = \mathrm{e}^{\frac{2k\pi}{n}\mathrm{i}}, k \in \mathbb{Z}},\ \boxed{z = \pm 1,\ \pm \mathrm{i},\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}}$$



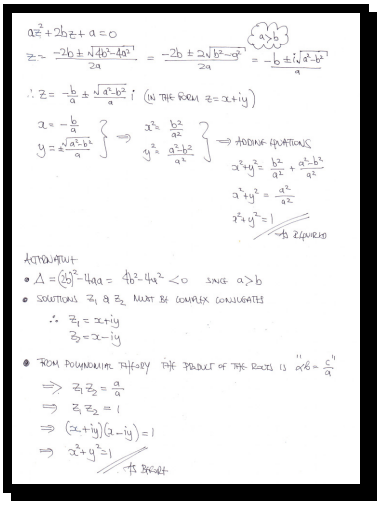
Question 137 (**+)**

Given that $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a > b > 0$, show that in an Argand diagram, the roots of the quadratic equation

$$az^2 + 2bz + a = 0,$$

lie on the circle with equation $x^2 + y^2 = 1$.

proof



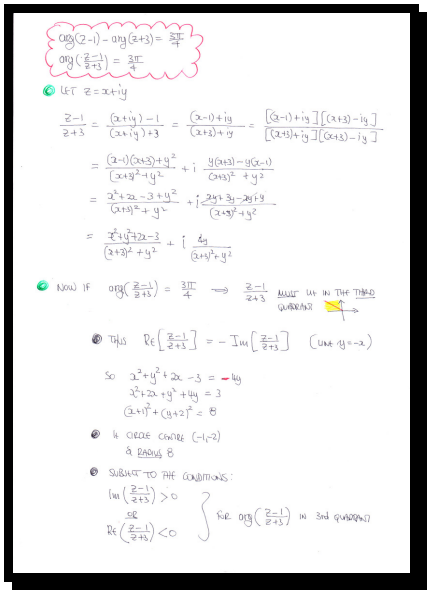
Question 138 (**+)**

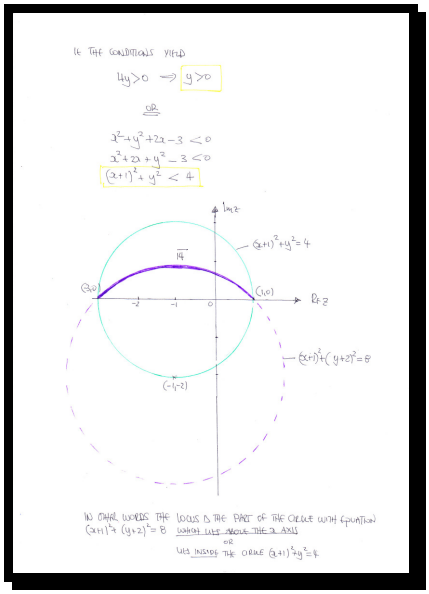
The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg(z-1) - \arg(z+3) = \frac{3\pi}{4} \ , \ z \neq -3 \, .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{(x+1)^2 + (y+2)^2 = 8, \quad \text{such that} \quad y > 0}$$



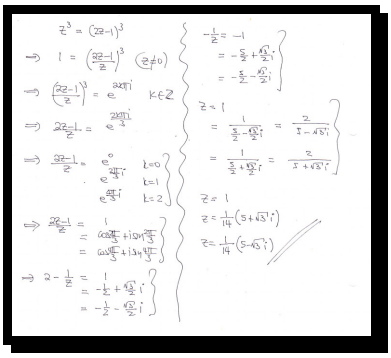


Question 139 (**+)**

$$z^3=(2z-1)^3,\ z\in\mathbb{C}.$$

Find in the form $x+\mathrm{i}y$ the exact solutions of the above equation.

$$\boxed{z=1,\ \frac{1}{14}\left(5\pm\mathrm{i}\sqrt{3}\right)}$$



Question 140 (**+)**

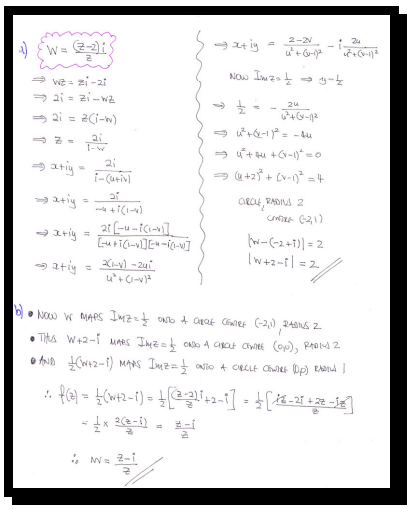
$$f(z)\equiv\frac{(z-2)\mathrm{i}}{z},\ z=x+\mathrm{i}y,\ x\in\mathbb{R},\ y\in\mathbb{R}.$$

The complex function f maps complex numbers onto complex numbers, which can be graphed in two separate Argand diagrams.

a) Given that $\mathrm{Im}\,z=\frac{1}{2}$, determine an equation of the locus of the image of the points under f.

b) Hence determine a complex function $g(z)$, which maps $\mathrm{Im}\,z=\frac{1}{2}$ onto a unit circle, centre at the origin O.

$$\boxed{|w+2-\mathrm{i}|=2},\quad \boxed{g(z)=w=\frac{z-\mathrm{i}}{z}}$$



Question 141 **(****+)**

$$f(z)=(z-4)^3,\ z\in\mathbb{C}.$$

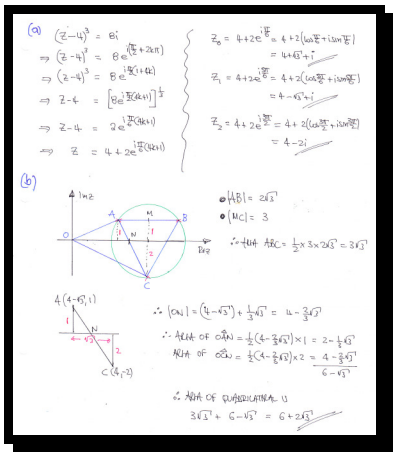
a) Solve the equation $f(z)=8\mathrm{i}$, giving the answers in the form $x+\mathrm{i}y$.

The points A, B and C represent in an Argand diagram the roots of the equation $f(z)=8\mathrm{i}$. The points A and B represent the roots whose imaginary parts are positive and the point A represents the root with the smaller real part.

b) Show that the area of the quadrilateral $OABC$, where O is the origin, is

$$6+2\sqrt{3}.$$

$$\boxed{z=4+\sqrt{3}+\mathrm{i},\ \ z=4-\sqrt{3}+\mathrm{i},\ \ z=4-2\mathrm{i}}$$



Question 142 (**+)**

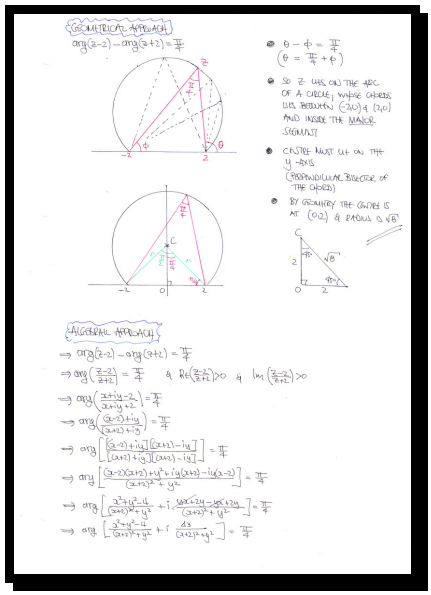
The complex number z satisfies the relationship

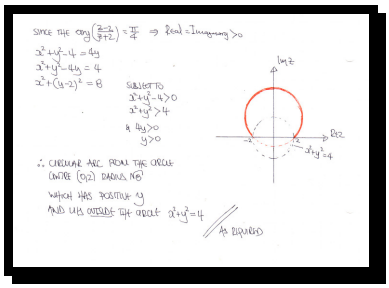
$$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}.$$

Show that the locus of z is a circular arc, stating …

- … the coordinates of its endpoints.
- … the coordinates of its centre.
- … the length of its radius.

$$\boxed{(-2,0),(2,0)},\ \boxed{(0,2)},\ \boxed{r=2\sqrt{2}}$$





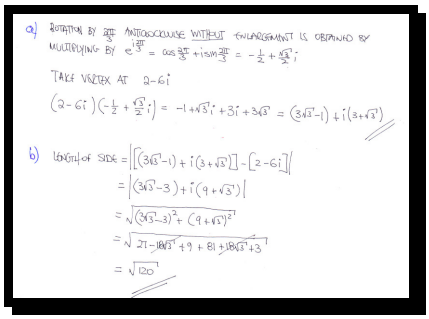
Question 143 (**+)**

An equilateral triangle T is drawn in a standard Argand diagram. The origin O is located at the centre of T. One of the vertices of T is represented by the complex number $2-6\mathrm{i}$.

a) Find, in exact simplified form the complex number represented by another vertex of T.

b) Calculate, in exact surd form, the area of T.

$$\boxed{\left(3\sqrt{3}-1\right)+\mathrm{i}\left(3+\sqrt{3}\right)}\,,\quad \boxed{\text{area}=\sqrt{120}}$$



Question 144 (**+)**

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

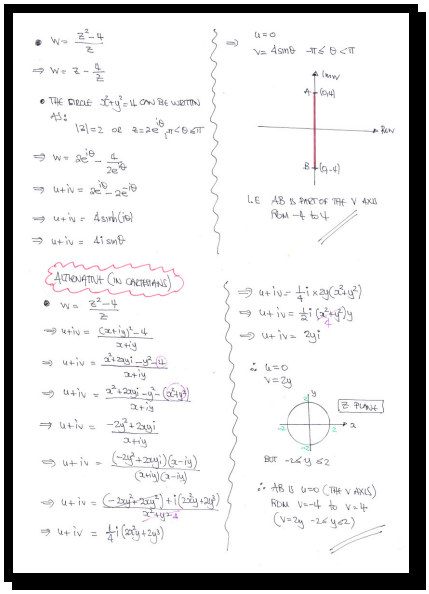
It is given that

$$f(z)=\frac{z^2-4}{z},\ z\in\mathbb{C},\ z\neq 0.$$

The circle C with equation $x^2+y^2=4$ in the z plane is mapped onto a line segment AB in the w plane.

Find a Cartesian equation for AB, stating the coordinates of its endpoints.

$\boxed{(1,0),\ r=1}$, $\boxed{u=1 \text{ or } x=1}$



Question 145 **(****+)**

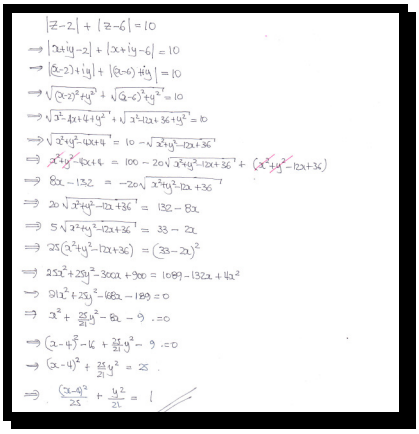
The complex number z satisfies the relationship

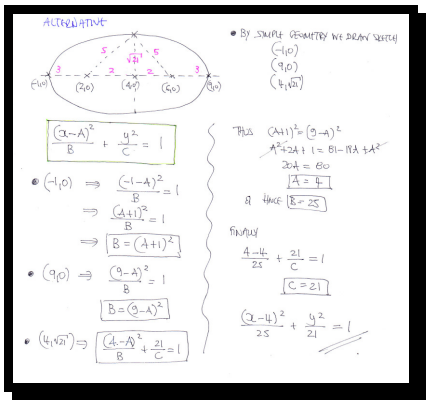
$$|z-2|+|z-6|=10.$$

Determine a simplified Cartesian equation for the locus of z, giving the final answer in the form

$$f(x,y)=1.$$

$$\boxed{\frac{(x-4)^2}{25}+\frac{y^2}{21}=1}$$





Question 146 (**+)**

$$f(z)\equiv(z+2\mathrm{i})^2,\ z\in\mathbb{C}.$$

The complex function f maps points, of the form $x+\mathrm{i}y$, from the z plane onto points, of the form $u+\mathrm{i}v$, in the w plane.

The straight line L lies in the z plane and has Cartesian equation

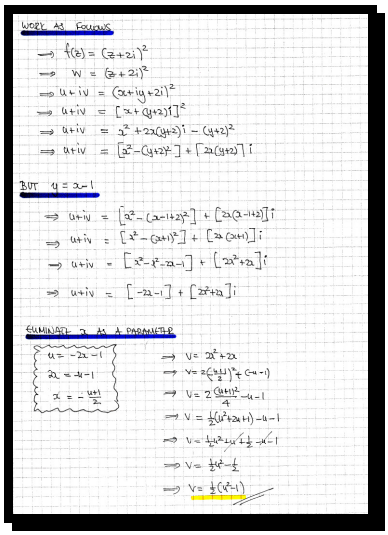
$$y=x-1.$$

Find an equation of the image of L in the w plane, giving the answer in the form

$$v=g(u),$$

where g, is a real function to be found.

$\square$, $\boxed{v=\frac{1}{2}\left(u^2-1\right)}$



Question 147 (**+)**

Use de Moivre's theorem followed by a suitable trigonometric identity, to show that …

a) … $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

b) … $\cos 6\theta \equiv \left(2\cos^2\theta - 1\right)\left(16\cos^4\theta - 16\cos^2\theta + 1\right)$

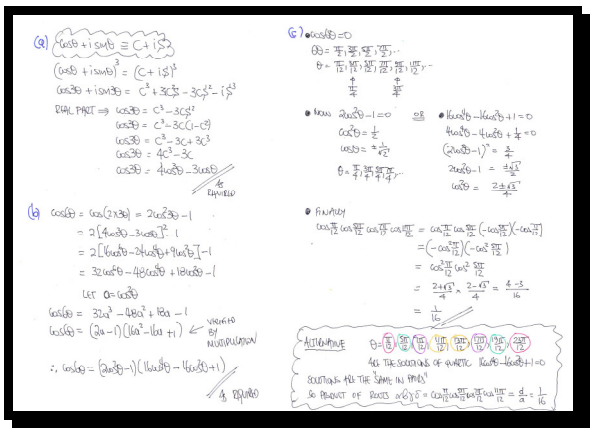
Consider the solutions of the equation.

$$\cos 6\theta = 0\,,\ 0 \le \theta \le \pi\,.$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12}\,.$$

$$\boxed{\frac{1}{16}}$$



Question 148 (**+)**

A transformation T maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $v = u + \mathrm{i}v$ in the w plane, and is defined as

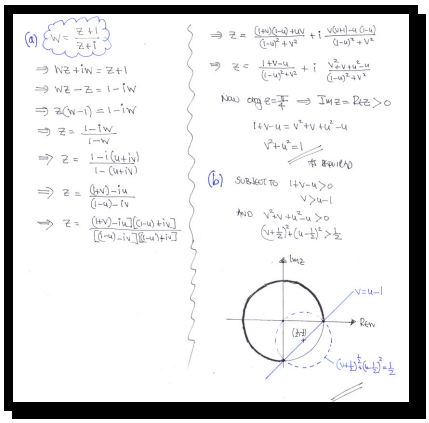
$$w = \frac{z+1}{z+\mathrm{i}} \ , \ z \neq -\mathrm{i} \, .$$

The points that lie on the half line with equation $\arg z = \dfrac{\pi}{4}$ are mapped by T onto points which lie on a circle.

a) Determine a Cartesian equation for this circle.

b) Show that the image of the half line with equation $\arg z = \dfrac{\pi}{4}$ is not the entire circle found in part (b).

$$\boxed{u^2 + v^2 = 1}$$

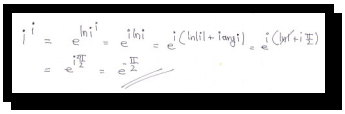


Question 149 (**+)**

Show that if $z = \mathrm{i}$

$$\mathrm{z}^{\mathrm{z}} = \mathrm{e}^{-\frac{\pi}{2}} \, .$$

proof



Question 150 (**+)**

The complex function $w=f(z)$ maps points of the form $z=x+\mathrm{i}y$ from the z plane onto points of the form $w=u+\mathrm{i}v$ in the w plane.

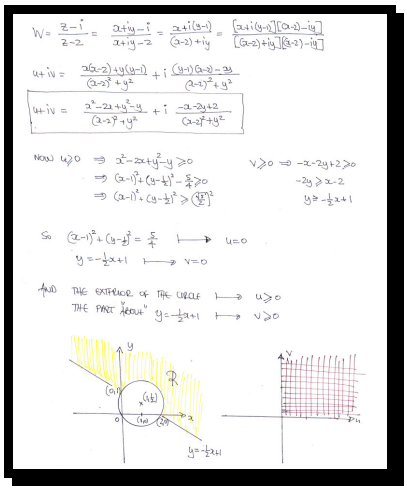
It is given that

$$f(z)=\frac{z-\mathrm{i}}{z-2},\ z\in\mathbb{C},\ z\neq 2.$$

The points of a region R in the z plane are mapped onto points of a region R' in the w plane. The region R' consists of points such that $u\geq 0$ and $v\geq 0$.

Shade, with justification, in an accurate Argand diagram the region R.

sketch



Question 151 **(****+)**

$$f(\theta)=(\cos\theta+\mathrm{i}\sin\theta)^4+(\cos\theta-\mathrm{i}\sin\theta)^4.$$

a) By considering a simplified expression of $f(\theta)$, show that

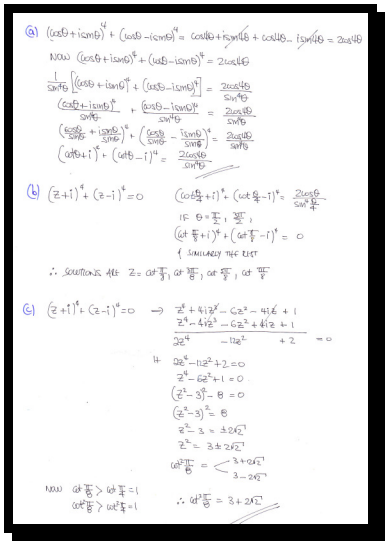
$$(\cot\theta+\mathrm{i})^4+(\cot\theta-\mathrm{i})^4=\frac{2\cos 4\theta}{\sin^4\theta}.$$

b) Find in the form $z=\cot\left(\frac{k\pi}{8}\right)$, the four solution of the equation

$$(z+\mathrm{i})^4+(z-\mathrm{i})^4=0.$$

c) Hence, show clearly that $\cot^2\left(\frac{\pi}{8}\right)=3+2\sqrt{2}$.

$$\boxed{x=\cot\left(\frac{k\pi}{8}\right),\ k=1,3,5,7}$$



Question 152 (**+)**

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

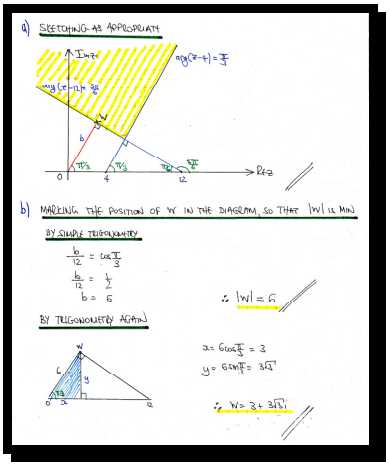
$$\frac{\pi}{3}\le \arg(z-4)\le \pi \quad \text{and} \quad 0\le \arg(z-12)\le \frac{5\pi}{6}.$$

a) Sketch the region R, indicating clearly all the relevant details.

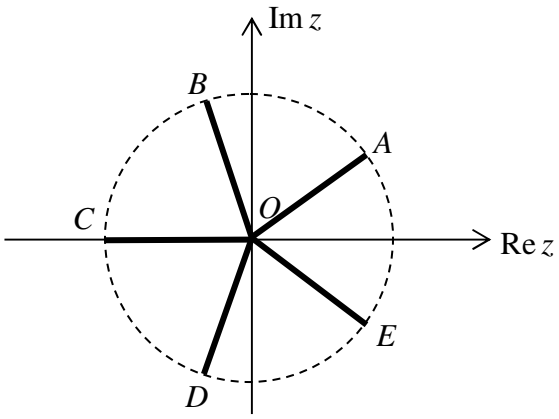
The complex number w lies in R, so that $|w|$ is minimum.

b) Find $|w|$, further giving w in the form $u+\mathrm{i}v$, where u and v are real numbers.

$\square$, $\boxed{|w|=6}$, $\boxed{w=3+3\sqrt{3}\mathrm{i}}$



Question 153 (**+)**



The figure above shows in a standard Argand diagram, the five roots of the equation $z^5+32=0$, indicated by the points A to E on a circle of radius r.

a) State the value of r.

b) State the five roots of the equation

$$z^5+32=0,$$

giving the answers in the form $z=r\left(\cos\theta+\mathrm{i}\sin\theta\right)$, $-\pi<\theta\leq\pi$.

c) Show that a quadratic equation satisfied by the roots indicated by B and D is

$$z^2+4z\cos\left(\frac{2\pi}{5}\right)+4=0.$$

d) Find a similar quadratic satisfied by the roots indicated by A and E.

[continues overleaf]

[continued from overleaf]

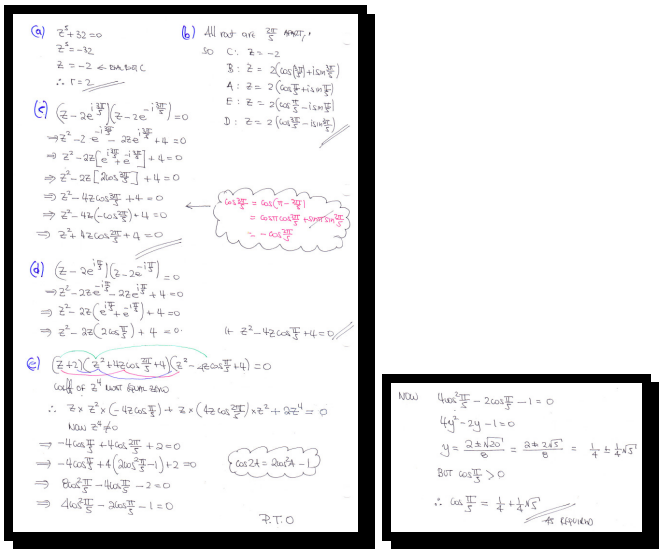
Consider the coefficients of z^4 in the following equations

$$z^5+32=0 \quad \text{and} \quad (z-z_C)\left[(z-z_B)(z-z_D)\right]\left[(z-z_A)(z-z_E)\right]=0\,.$$

e) Show that $\cos\left(\dfrac{\pi}{5}\right)=\dfrac{1}{4}+\dfrac{1}{4}\sqrt{5}$.

(you may find the cosine double angle formula useful)

$$\boxed{r=2}\,,\quad \boxed{z=2\left(\cos n\theta+\mathrm{i}\sin n\theta\right), n=-2,-1,0,1,2}\,,\quad \boxed{z^2-4z\cos\left(\frac{\pi}{5}\right)+4=0}$$



Question 154 (**+)**

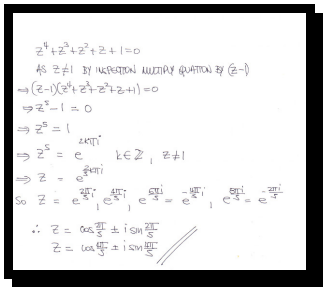
$$z^4+z^3+z^2+z+1=0,\ z\in\mathbb{C}.$$

By using the identity

$$a^n-1\equiv(a-1)\left(a^n+a^{n-1}+a^{n-2}+...+a^2+a+1\right),$$

or otherwise, find in exact trigonometric form the four solutions of the above equation.

$$\boxed{z=\cos\frac{2\pi}{5}\pm\mathrm{i}\sin\frac{2\pi}{5},\ \cos\frac{4\pi}{5}\pm\mathrm{i}\sin\frac{4\pi}{5}}$$



Question 155 **(****+)**

$$f(z) \equiv z^2,\ z \in \mathbb{C}.$$

The complex function f maps points, of the form $x+\mathrm{i}y$, from the z plane onto points, of the form $u+\mathrm{i}v$, in the w plane.

The curve C lies in the z plane and has Cartesian equation

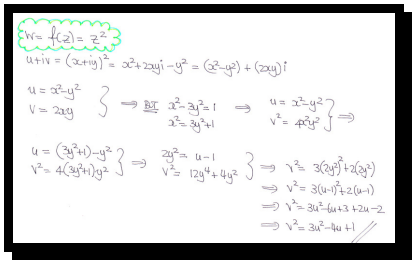
$$x^2 - 3y^2 = 1.$$

Find an equation of the image of C in the w plane, giving the answer in the form

$$v^2 = Au^2 + Bu + C,$$

where A, B and C are real constants to be found.

$$\boxed{v^2 = 3u^2 - 4u + 1}$$



Question 156 **(****+)**

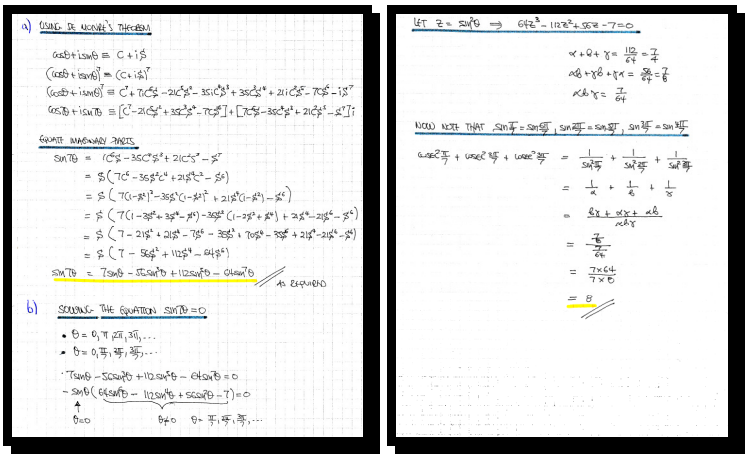
a) Show that

$$\sin 7\theta \equiv 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

b) By considering a suitable polynomial equation based on the result of part **(a)** show further

$$\operatorname{cosec}^2\left(\tfrac{1}{7}\pi\right) + \operatorname{cosec}^2\left(\tfrac{2}{7}\pi\right) + \operatorname{cosec}^2\left(\tfrac{3}{7}\pi\right) = 8$$

[ ], [proof]



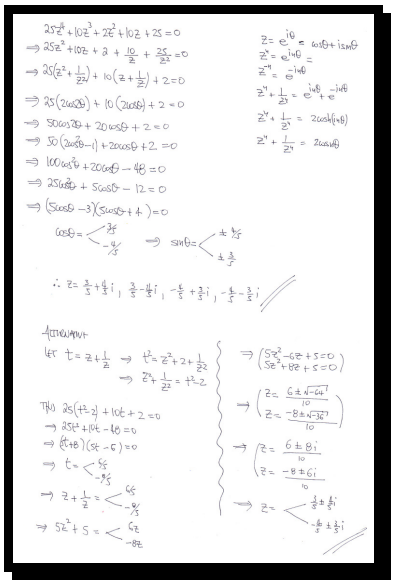
Question 157 (**+)**

The following equation has no real solutions

$$25z^4+10z^3+2z^2+10z+25=0\,.$$

Find the four complex solution of the above equation, giving the answer in the form $a+b\mathrm{i}$, where $a\in\mathbb{C}$ and $b\in\mathbb{C}$.

$$\boxed{z=\frac{3}{5}+\frac{4}{5}\mathrm{i},\quad z=\frac{3}{5}-\frac{4}{5}\mathrm{i},\quad z=-\frac{4}{5}+\frac{3}{5}\mathrm{i},\quad z=-\frac{4}{5}-\frac{3}{5}\mathrm{i}}$$



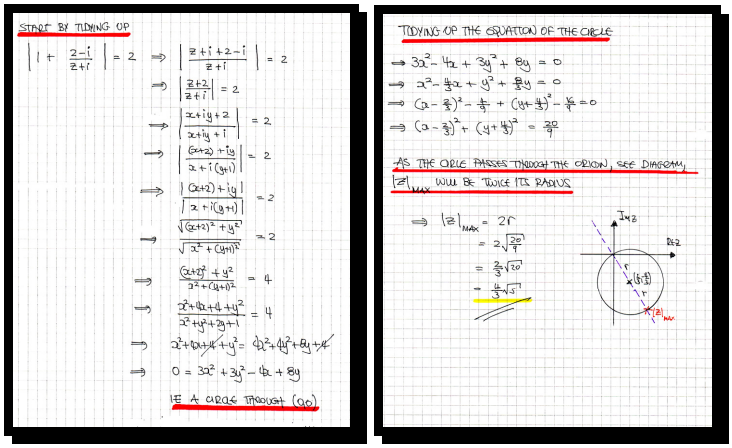
Question 158 (**+)**

$$f(z)=\frac{2-\mathrm{i}}{z+\mathrm{i}},\ z\in\mathbb{C},\ z\neq-\mathrm{i}.$$

Find the greatest value of the modulus of z, given further that

$$\left|1+f(z)\right|=2.$$

$\square$, $\boxed{|z|_{\max}=\frac{4}{3}\sqrt{5}}$



Question 159 (**+)**

The complex function $w = f(z)$ is defined by

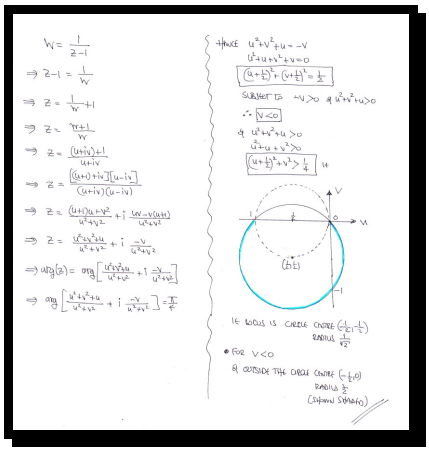
$$w = \frac{1}{z-1}, \ z \in \mathbb{C}, \ z \neq 1.$$

The half line with equation $\arg z = \frac{\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{2}, \ v<0, \ u^2+v^2+u>0$$



Question 160 (**+)**

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

a) Use De Moivre's theorem to prove the validity of the above trigonometric identity.

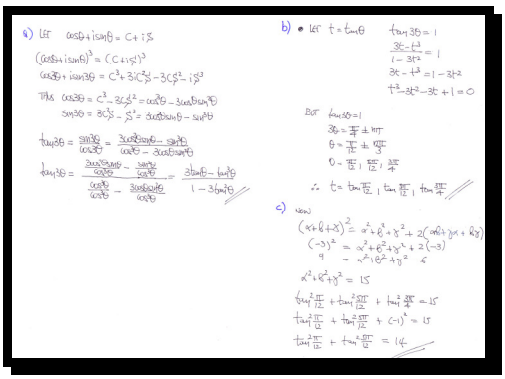
b) Hence find in exact trigonometric form the solutions of the equation

$$t^3 - 3t^2 - 3t + 1 = 0.$$

c) Use the answer of part **(b)** to show further that

$$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} = 14.$$

$$\boxed{t = \tan\frac{\pi}{12},\ \tan\frac{5\pi}{12},\ \tan\frac{3\pi}{4}}$$



Question 161 (**+)**

The locus L_1 of a point in an Argand diagram satisfies

$$\arg(z-2)-\arg(z-2\mathrm{i})=\frac{3\pi}{4},\ z\in\mathbb{C}.$$

a) Find a Cartesian equation for L_1.

b) Show that all the points which lie on L_1 satisfy

$$\left|\frac{z-4}{z-1}\right|=k,$$

where k is an integer to be found.

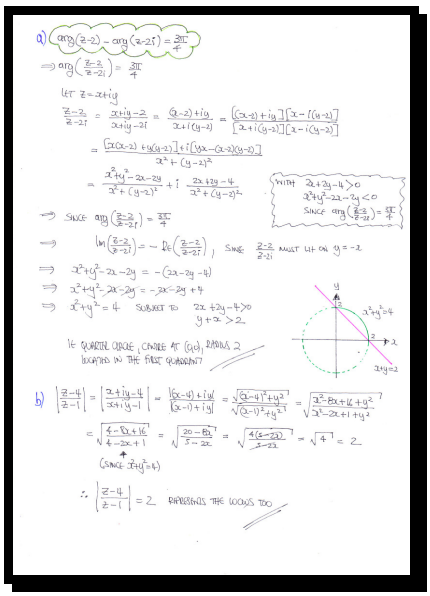
The locus L_2 of a different point in the same Argand diagram satisfies

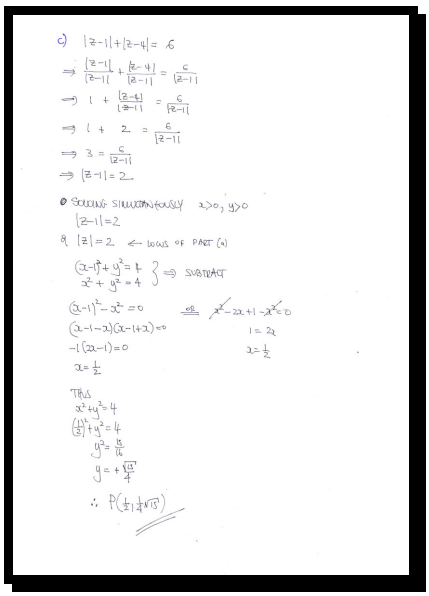
$$|z-1|+|z-4|=6,\ z\in\mathbb{C}.$$

The point P lies on L_1 and L_1.

c) Find the complex number represented by P.

$$\boxed{L_1: x^2+y^2=4,\ x>0,\ y>0},\ \boxed{k=2},\ \boxed{P:\frac{1}{2}+\frac{1}{4}\mathrm{i}\sqrt{15}}$$





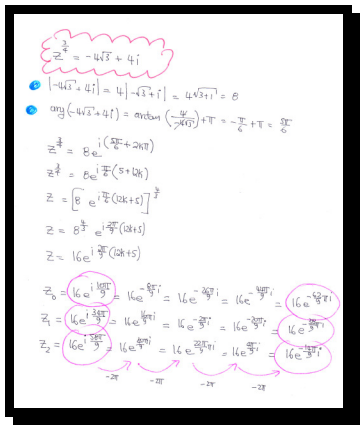
Question 162 (**+)**

Solve the equation

$$z^{\frac{3}{4}} = -4\sqrt{3} + 4\mathrm{i}\,,\ z \in \mathbb{C}\,.$$

Give each of the roots in exponential form.

$$\boxed{z = 16\mathrm{e}^{\frac{8}{9}\pi\mathrm{i}} = 16\mathrm{e}^{-\frac{62}{9}\pi\mathrm{i}}}\,,\ \boxed{z = 16\mathrm{e}^{\frac{34}{9}\pi\mathrm{i}} = 16\mathrm{e}^{-\frac{38}{9}\pi\mathrm{i}}}\,,\ \boxed{z = 16\mathrm{e}^{\frac{58}{9}\pi\mathrm{i}} = 16\mathrm{e}^{-\frac{14}{9}\pi\mathrm{i}}}$$



Question 163 (**+)**

The complex number w is defined as $w=\mathrm{e}^{\frac{2}{5}\pi\mathrm{i}}$.

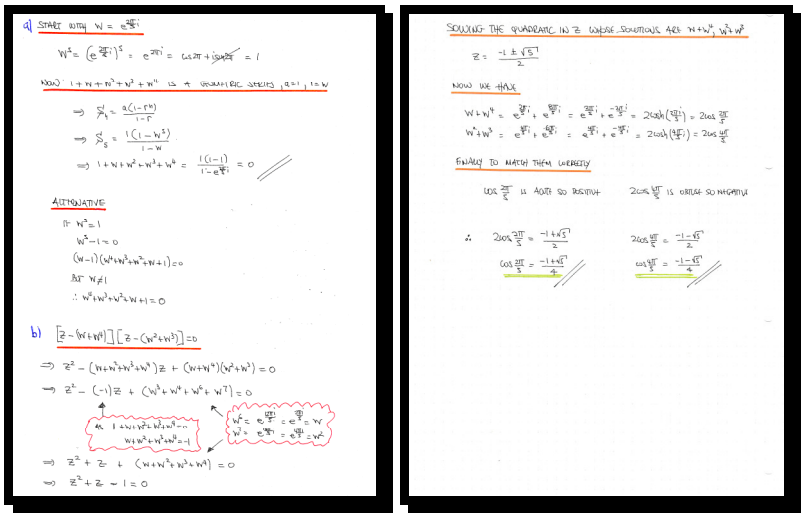
a) Prove that

$$1+w+w^2+w^3+w^4=0\,.$$

b) Derive a quadratic equation with integer coefficients whose roots are $\left(w+w^4\right)$ and $\left(w^2+w^3\right)$, and hence show with full justification that

$$\cos\left(\frac{2}{5}\pi\right)=\frac{-1+\sqrt{5}}{4}\quad\text{and}\quad\cos\left(\frac{4}{5}\pi\right)=\frac{-1-\sqrt{5}}{4}\,.$$

[ ], [proof]



Question 164 (**+)**

A complex transformation of points from the z plane onto points in the w plane is defined by the equation

$$w = z^2,\ z \in \mathbb{C}.$$

The point represented by $z = x + \mathrm{i}y$ is mapped onto the point represented by $w = u + \mathrm{i}v$.

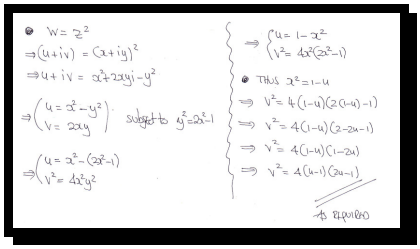
Show that if z traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of w satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

proof

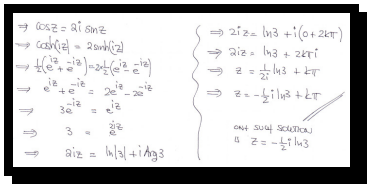


Question 165 (*****)

Find a solution of the equation

$$\cos z = 2\mathrm{i}\sin z,\ z \in \mathbb{C}.$$

$$z = k\pi - \tfrac{1}{2}\mathrm{i}\ln 3,\ k \in \mathbb{Z}$$



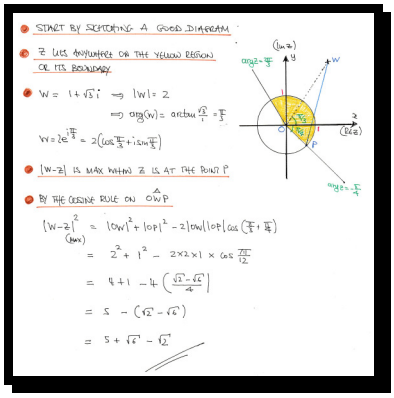
Question 166 (**+)**

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

$$-\frac{1}{4}\pi \le \arg z \le \frac{2}{3}\pi \quad \text{and} \quad |z| \le 1 .$$

Determine, in exact surd form, the maximum value of $|w-z|^2$, where $w = 1+\mathrm{i}\sqrt{3}$.

$\boxed{}$, $\boxed{|w-z|^2 = 5+\sqrt{6}-\sqrt{2}}$



Question 167 (**+)**

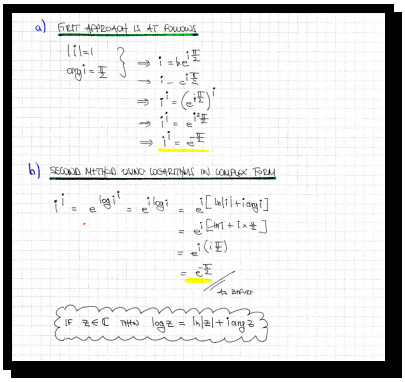
It is required to find the principal value of i^{i}, in exact simplified form, where i is the imaginary unit.

a) Show, with detailed workings, that

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{-\frac{1}{2}\pi}.$$

b) Use a different method to that used in part **(a)**, to verify the exact answer given in part **(a)**.

[ ], [proof]



Question 168 (*****)

The finite sum C is given below.

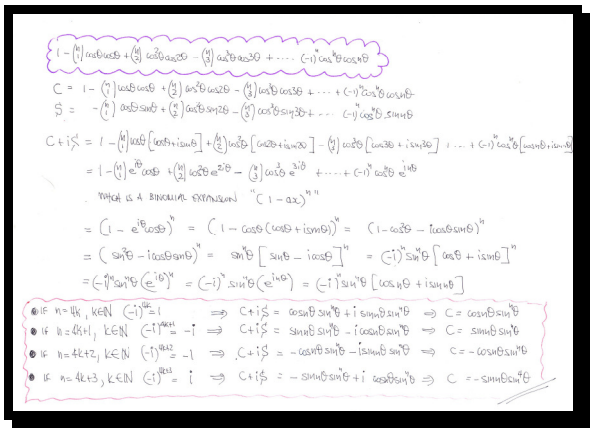
$$C=\sum_{r=0}^{n}\left[\binom{n}{r}(-1)^n\cos^n\theta\cos n\theta\right].$$

Given that $n\in\mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact fully simplified form.

$\boxed{}$, $\boxed{n=4k,\ k\in\mathbb{N}:C=\cos n\theta\sin^n\theta}$, $\boxed{n=4k+1,\ k\in\mathbb{N}:C=\sin n\theta\sin^n\theta}$,

$\boxed{n=4k+2,\ k\in\mathbb{N}:C=-\cos n\theta\sin^n\theta}$, $\boxed{n=4k+3,\ k\in\mathbb{N}:C=-\sin n\theta\sin^n\theta}$



Question 169 (*****)

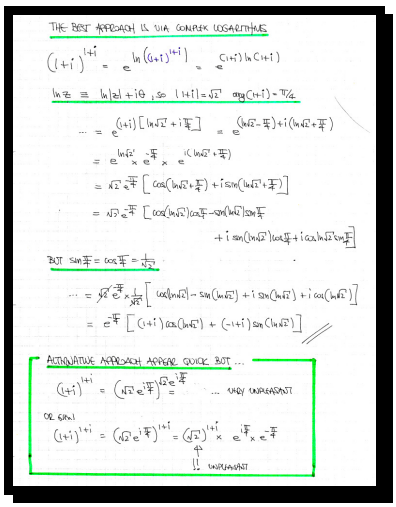
The complex number w is defined as $w=z^z$, where $z=1+\mathrm{i}$.

Show, with details workings, that

$$w=\mathrm{e}^{-\frac{1}{4}\pi}\left[(1+\mathrm{i})\cos(\ln k)+(-1+\mathrm{i})\sin(\ln k)\right],$$

where $(1+\mathrm{i})\cos(\ln k)+$ is an exact real constant to be found.

$\square$, $\boxed{k=\sqrt{2}}$



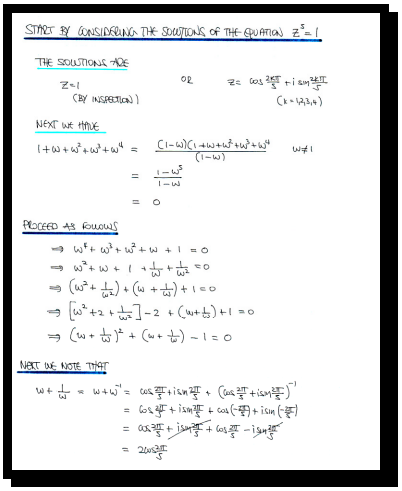
Question 170 (*****)

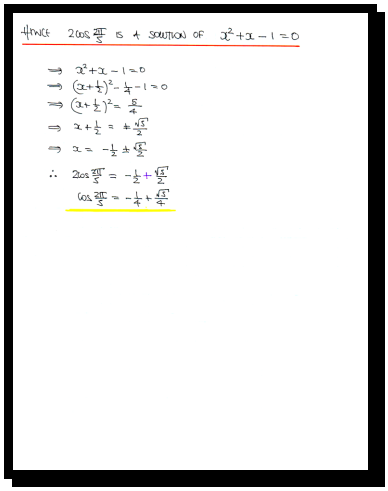
Use complex numbers to prove that

$$\cos\left(\frac{2}{5}\pi\right)=-\frac{1}{4}+\frac{1}{4}\sqrt{5}$$

A detailed method must support this proof.

[ ], [proof]



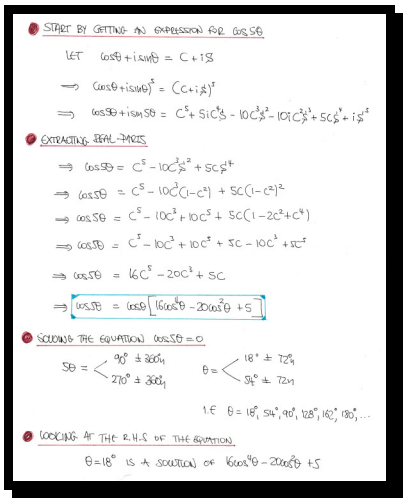


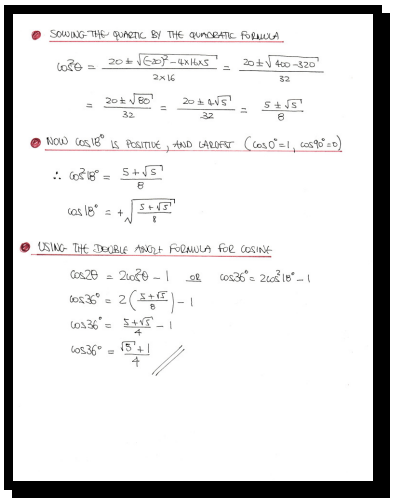
Question 171 (*****)

Use De Moivre's theorem to find a multiple angle cosine expression and use this expression to show that

$$\cos 36^\circ = \frac{1}{4}\left(1+\sqrt{5}\right).$$

[ ], [proof]





Question 172 (*****)

$$w=\frac{2-\mathrm{i}z}{z},\ z\in\mathbb{C},\ z\neq 0.$$

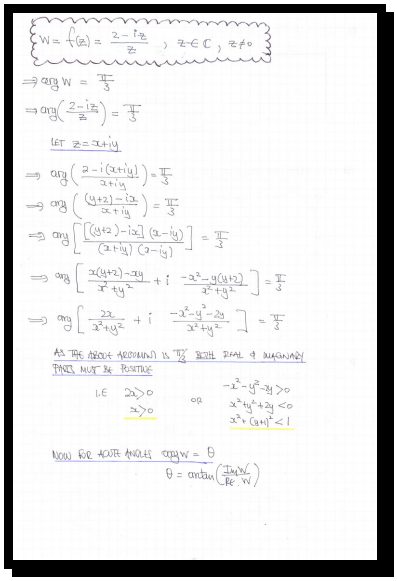
The complex function $w=f(z)$, maps the point $P(x,y)$ from the z complex plane onto the point $Q(u,v)$ on the w complex plane.

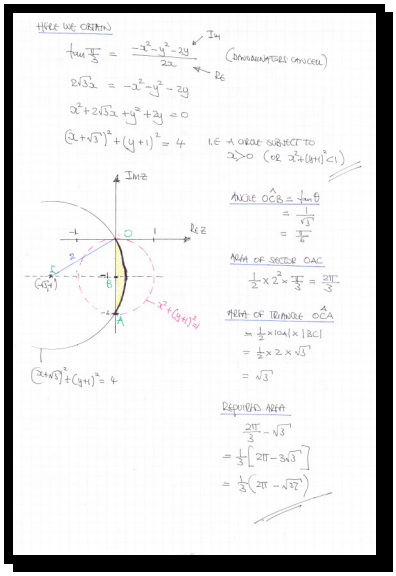
The curve C in the z complex plane is mapped in the w complex plane onto the curve with equation

$$\arg w=\frac{1}{3}\pi .$$

Determine a Cartesian equation of C, and hence find an exact simplified value for the area of the finite region bounded by C, and the y axis.

$$\boxed{},\ \boxed{\left(x+\sqrt{3}\right)^2+\left(y+1\right)^2=4\ \cup\ x>0},\ \boxed{\frac{2}{3}\pi-\sqrt{3}}$$





Question 173 (*****)

a) Show that

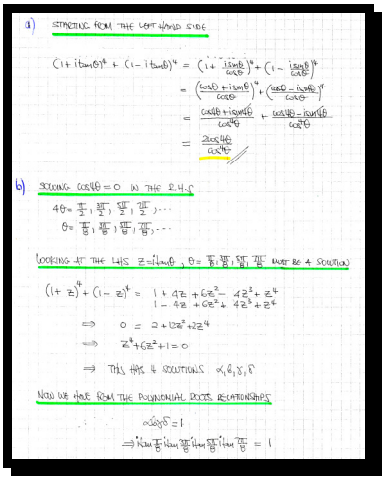
$$(1+\mathrm{i}\tan\theta)^4+(1-\mathrm{i}\tan\theta)^4\equiv\frac{2\cos 4\theta}{\cos^4\theta}$$

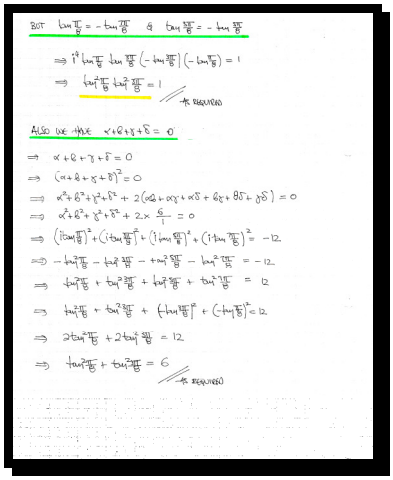
b) By considering a suitable polynomial equation based on the result of part **(a)** show further

i. $\tan^2\left(\frac{1}{8}\pi\right)\tan^2\left(\frac{3}{8}\pi\right)=1$

ii. $\tan^2\left(\frac{1}{8}\pi\right)+\tan^2\left(\frac{3}{8}\pi\right)=6$

[ ], [proof]





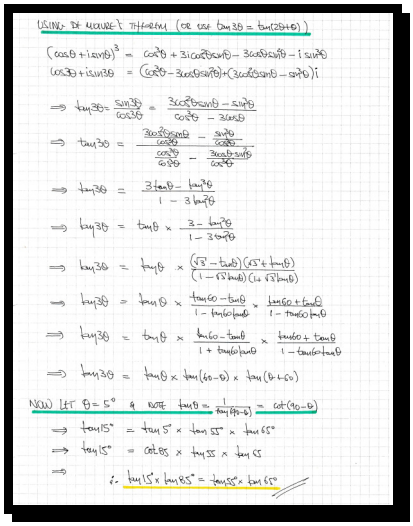
Question 174 **(*****)**

$$\tan(3\theta^\circ) \equiv \tan(\theta^\circ)\times\tan(60^\circ-\theta^\circ)\times\tan(60^\circ+\theta^\circ)$$

Prove the validity of the above trigonometric identity and hence show that

$$\tan 15^\circ \times \tan 85^\circ \;=\; \tan 55^\circ \times \tan 65^\circ .$$

[ ], [proof]



Question 175 (*****)

$$I = \int \cos\left(\ln x\right) dx \quad \text{and} \quad J = \int \sin\left(\ln x\right) dx$$

a) Use an appropriate method to find expressions for I and J.

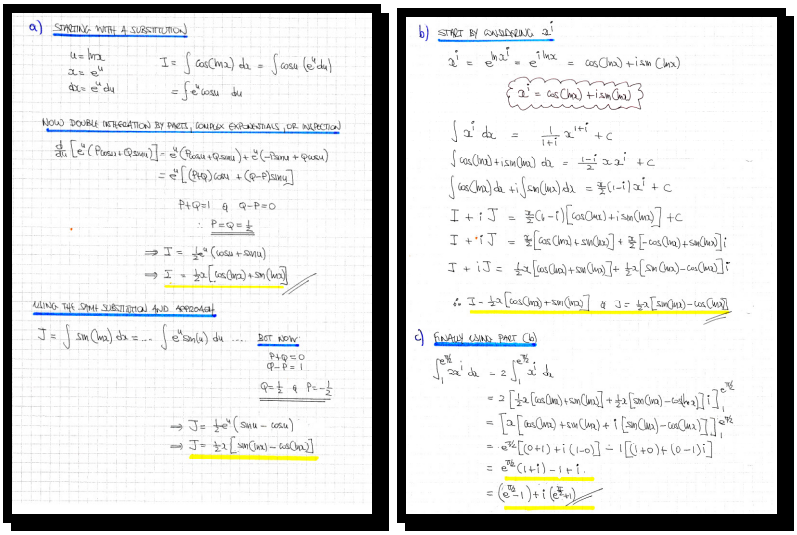
b) Use the integral $\int x^{\mathrm{i}}\, dx$, where i is the imaginary unit, to verify the answers given in part **(a)**.

c) Find an exact simplified value for

$$\int_1^{\mathrm{e}^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\ dx\,.$$

$\square$, $\boxed{I=\frac{1}{2}x\left[\sin\left(\ln x\right)+\cos\left(\ln x\right)\right]}$, $\boxed{J=\frac{1}{2}x\left[\sin\left(\ln x\right)-\cos\left(\ln x\right)\right]}$,

$$\boxed{\int_1^{\mathrm{e}^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\ dx = \left(\mathrm{e}^{\frac{1}{2}\pi}-1\right)+\left(\mathrm{e}^{\frac{1}{2}\pi}+1\right)\mathrm{i}}$$



Question 176 (*****)

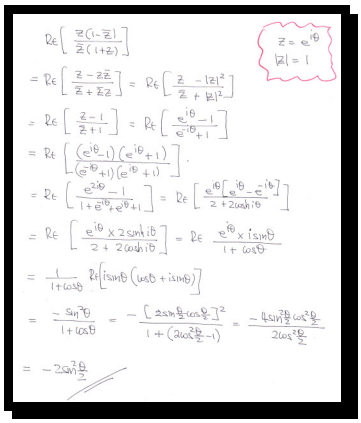
The complex number z has unit modulus and $\arg z = \theta$, $-\pi < \theta \leq \pi$.

The complex conjugate of z is denoted by $\overline{z}$.

Using a detailed method, show that

$$\text{Re}\left[\frac{z(1-\overline{z})}{\overline{z}(1+z)}\right] = -2\sin\left(\frac{1}{2}\theta\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 177 (*****)

The complex number $z = z_1 + z_2$ where

$$z_1 = 3 + 4\mathrm{i} \quad \text{and} \quad z_2 = 4\mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \leq \pi$$

a) Sketch in an Argand diagram the locus of z.

The complex number z_3 lies on the locus of z such that the argument of z_3 takes its maximum value.

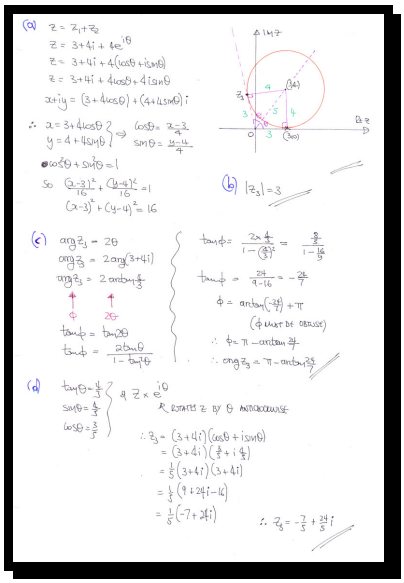
b) State the value of $|z_3|$.

c) Show clearly that

$$\arg z_3 = \pi - \arctan \frac{24}{7}.$$

d) Find z_3 in the form $x + \mathrm{i}y$.

$\square$, $\boxed{|z_3| = 3}$, $\boxed{|z|_{\max} = 3}$, $\boxed{z_3 = -\frac{7}{5} + \frac{24}{5}\mathrm{i}}$



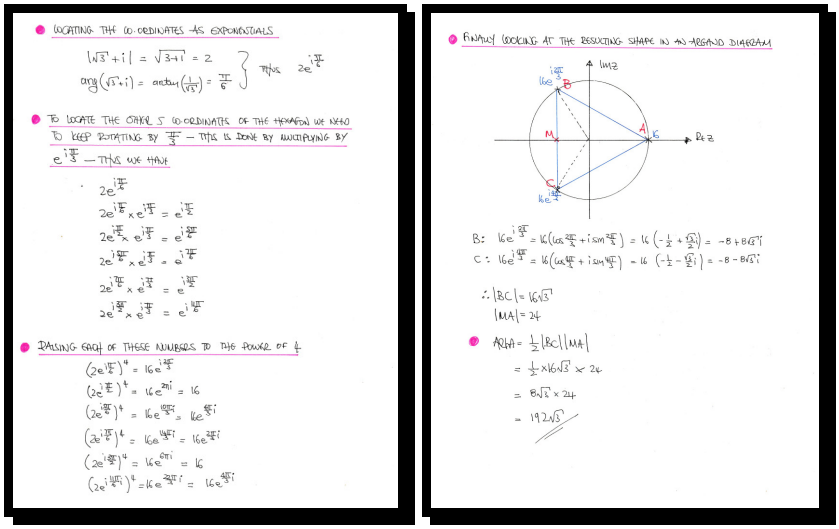
Question 178 (*****)

In a standard Argand diagram the complex number $\sqrt{3}+\mathrm{i}$, represents one of the vertices of a regular hexagon, with centre at the origin O.

The complex numbers that represent these 6 vertices are all raised to the power of 4, creating a closed shape S, whose sides are straight line segments.

Determine the area of S.

$\boxed{}$, $\boxed{\text{proof}}$



Question 179 (*****)

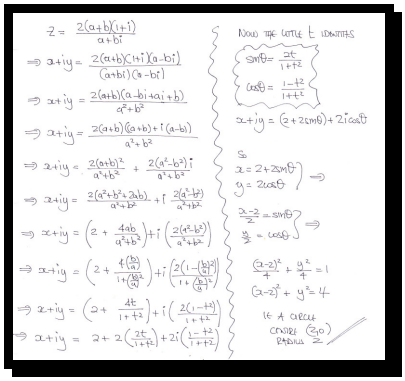
The complex number z is given by

$$z=\frac{2(a+b)(1+\mathrm{i})}{a+b\mathrm{i}},\ a+b\neq 0,$$

where a and b are real parameters.

Show, that for all allowable values of a and b, the point represented by z is tracing a circle, determining the coordinates of its centre and the size of its radius.

$\boxed{}$, $\boxed{(2,0)}$, $\boxed{r=2}$



Question 180 (*****)

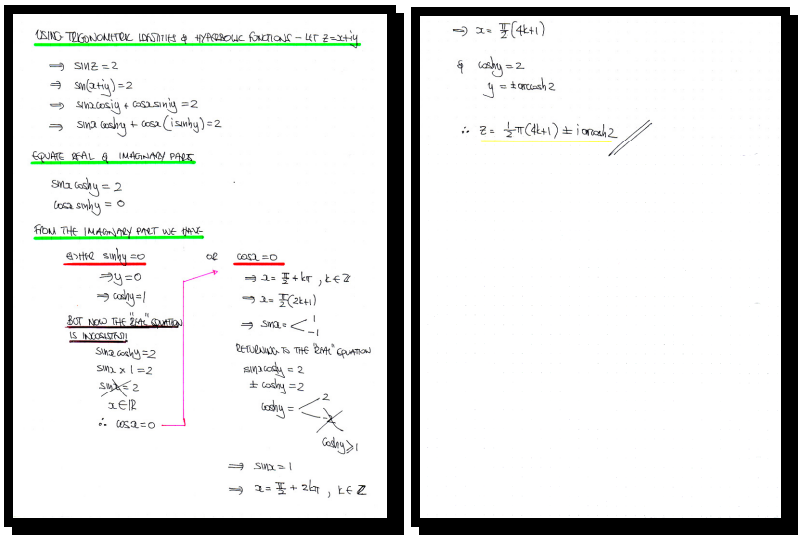
Show clearly that the general solution of the equation

$$\sin z = 2\,,\ z \in \mathbb{C}\,,$$

can be written in the form

$$z = \frac{\pi}{2}(4k+1) \pm \mathrm{i}\,\mathrm{arcosh}\,2\,,\ k \in \mathbb{Z}\,.$$

[ ], [proof]



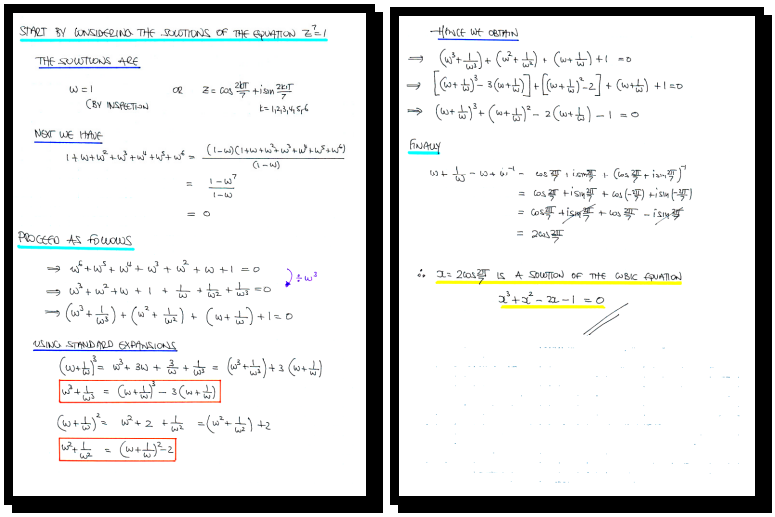
Question 181 (*****)

Use complex numbers to prove that $\cos\left(\frac{2}{7}\pi\right)$ is a solution of the cubic equation

$$x^3+x^2-2x-1=0\,.$$

You may **not** use verification in this proof.

[ ], [proof]



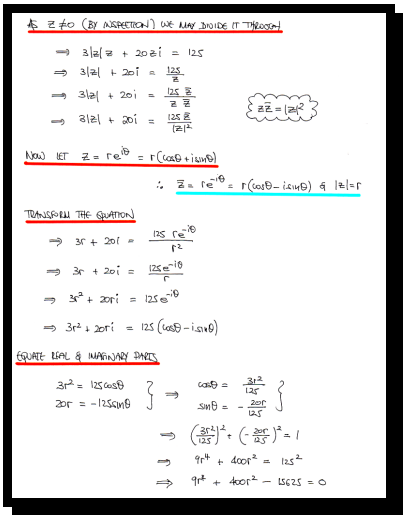
Question 182 (*****)

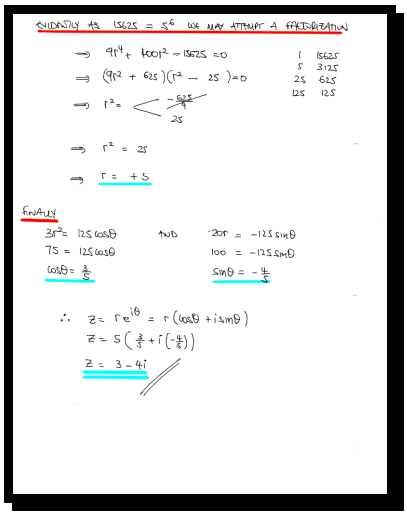
Solve the following equation

$$3|z|z + 20zi = 125, \quad z \in \mathbb{C}.$$

Give the answer in the form $x+\mathrm{i}\,y$, where x and y are real.

$\boxed{}$, $\boxed{z=3-4\mathrm{i}}$





Question 183 (*****)

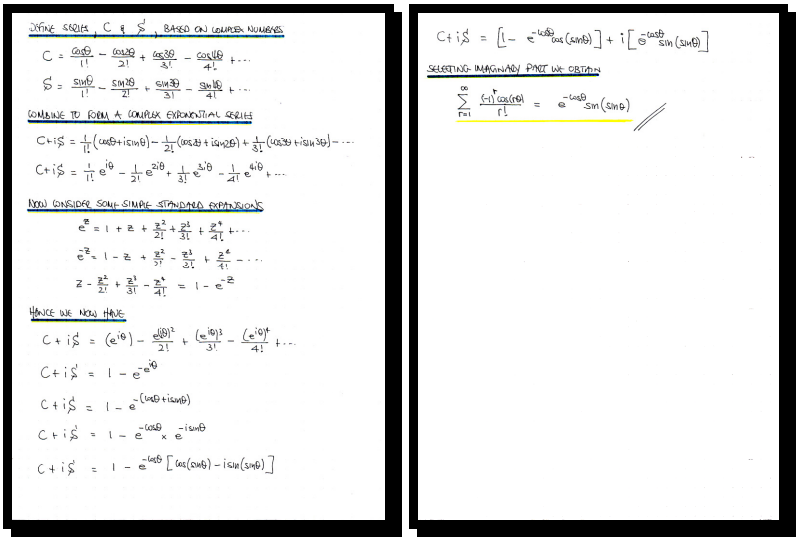
The following convergent series S is given below

$$S=\frac{\sin\theta}{1!}-\frac{\sin 2\theta}{2!}+\frac{\sin 3\theta}{3!}-\frac{\sin 4\theta}{4!}+\ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S=\mathrm{e}^{-\cos\theta}\sin\left(\sin\theta\right).$$

, proof



Question 184 (*****)

The point P in an Argand diagram represents the complex number z, which satisfies

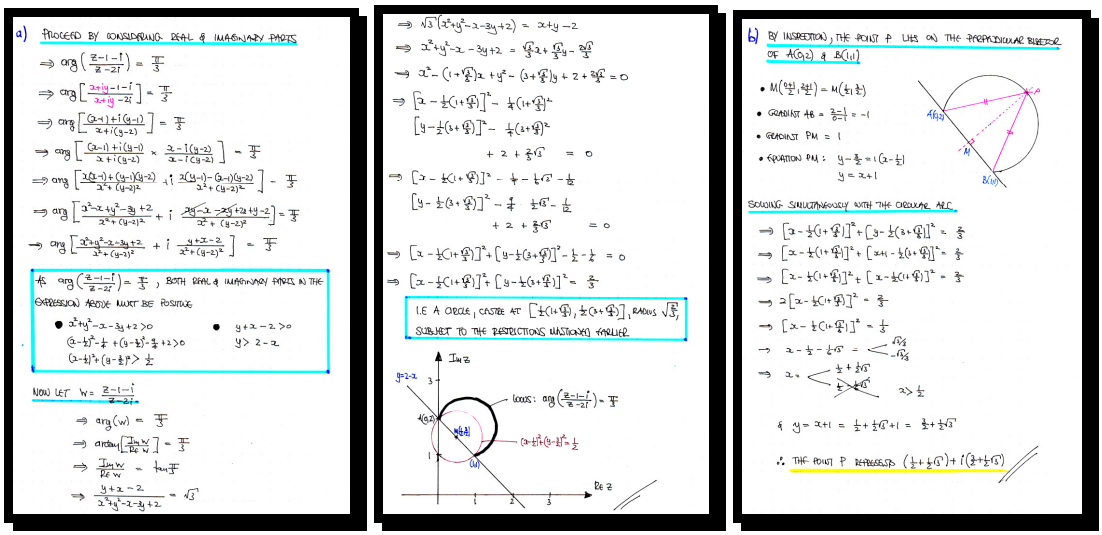
$$\arg\left[\frac{z-1-\mathrm{i}}{z-2\mathrm{i}}\right]=\frac{\pi}{3},\ z\neq 2\mathrm{i}\,.$$

It further given that P lies on the arc AB of a circle centred at C and of radius r.

a) Sketch in an Argand diagram the circular arc AB, stating the coordinates of C and the value of r.

b) Given further that $|PA|=|PB|$, find the complex number represented by P.

$$\boxed{},\ \boxed{C\left[\frac{1}{2}\left(1+\frac{1}{3}\sqrt{3}\right),\frac{1}{2}\left(3+\frac{1}{3}\sqrt{3}\right)\right]},\ \boxed{r=\sqrt{\frac{2}{3}}},\ \boxed{\frac{1}{2}\left(1+\sqrt{3}\right)+\frac{1}{2}\left(3+\sqrt{3}\right)\mathrm{i}}$$

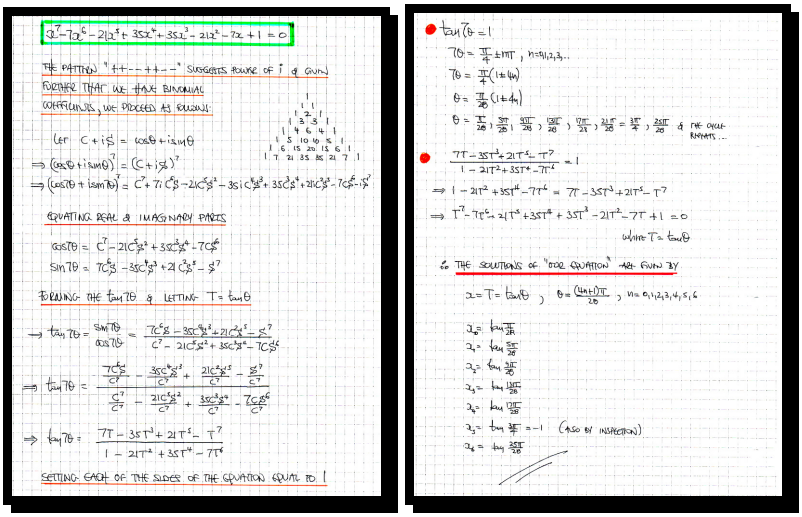


Question 185 (*****)

Find, in exact trigonometric form where appropriate, the real solutions of the following polynomial equation

$$x^7-7x^6-21x^5+35x^4+35x^3-21x^2-7x+1=0\,.$$

$\square$, $x=\tan\left(\frac{\pi}{28}\right)$, $x=\tan\left(\frac{5\pi}{28}\right)$, $x=\tan\left(\frac{9\pi}{28}\right)$, $x=\tan\left(\frac{13\pi}{28}\right)$, $x=\tan\left(\frac{17\pi}{28}\right)$, $x=\tan\left(\frac{3\pi}{4}\right)=-1$, $x=\tan\left(\frac{25\pi}{28}\right)$



Question 186 (*****)

By showing a detailed method involving complex numbers, sum the following series.

$$\sum_{n=0}^{\infty}\left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right].$$

$\square$, $\boxed{\frac{3}{2}}$

