COMPLEX NUMBERS

(part 1)

BASIC COMPLEX ALGEBRA

Question 1

Simplify the following complex number expressions, giving the final answer in the form $a+b\mathrm{i}$, where $a\in\mathbb{R}$, $b\in\mathbb{R}$.

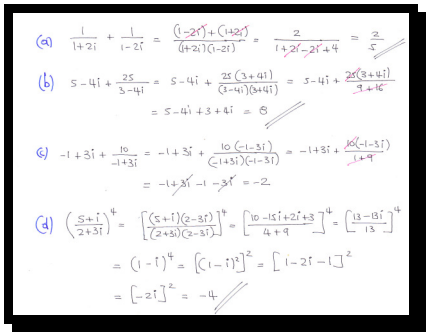
a) $\dfrac{1}{1+2\mathrm{i}}+\dfrac{1}{1-2\mathrm{i}}$

b) $5-4\mathrm{i}+\dfrac{25}{3-4\mathrm{i}}$

c) $-1+3\mathrm{i}+\dfrac{10}{-1+3\mathrm{i}}$

d) $\left(\dfrac{5+\mathrm{i}}{2+3\mathrm{i}}\right)^4$

$\boxed{\frac{2}{5}}$, $\boxed{8}$, $\boxed{-2}$, $\boxed{-4}$

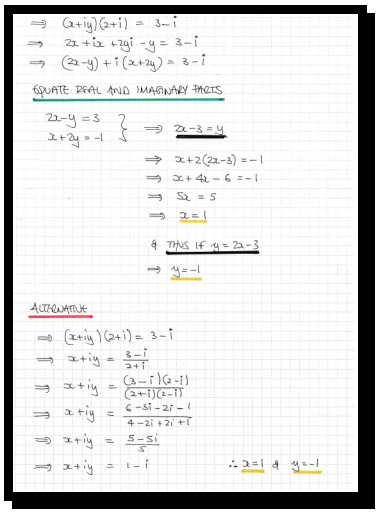


Question 2

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x+\mathrm{i}y)(2+\mathrm{i}) = 3-\mathrm{i}\,.$$

$$\boxed{(x, y) = (1, -1)}$$

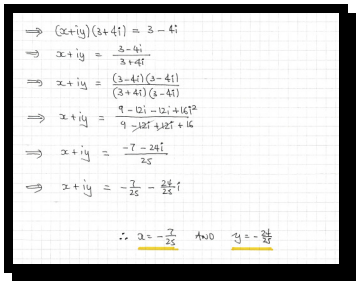


Question 3

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x+\mathrm{i}y)(3+4\mathrm{i}) = 3-4\mathrm{i}\,.$$

$$\boxed{(x, y) = \left(-\frac{7}{25}, -\frac{24}{25}\right)}$$

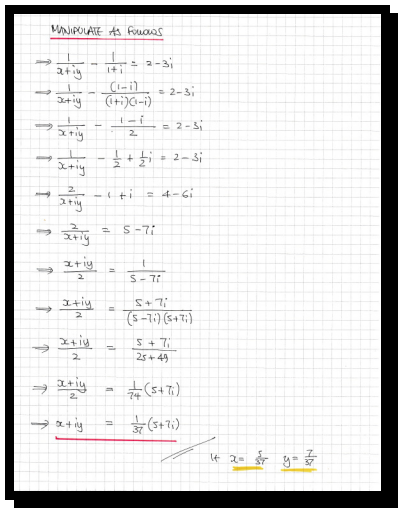


Question 4

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}-\frac{1}{1+\mathrm{i}}=2-3\mathrm{i}\,.$$

$$\boxed{(x,y)=\left(\frac{5}{37},\frac{7}{37}\right)}$$

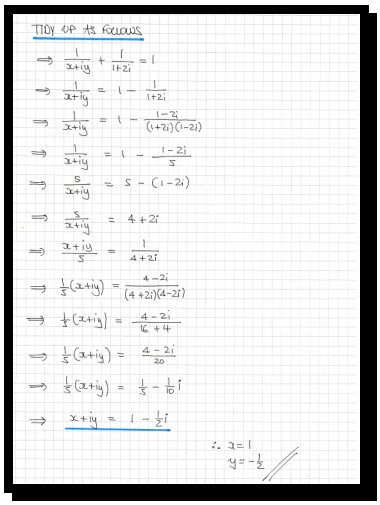


Question 5

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}+\frac{1}{1+2\mathrm{i}}=1.$$

$$\boxed{(x,y)=\left(1,-\frac{1}{2}\right)}$$

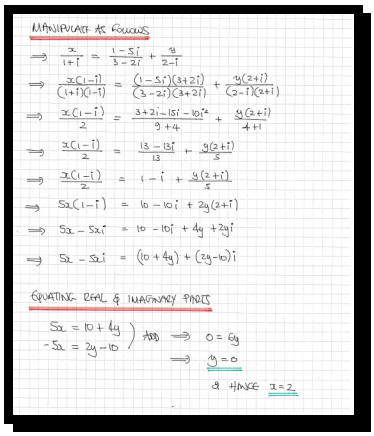


Question 6

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{x}{1+\mathrm{i}} = \frac{1-5\mathrm{i}}{3-2\mathrm{i}} + \frac{y}{2-\mathrm{i}}.$$

$$\boxed{(x, y) = (2, 0)}$$



Question 7

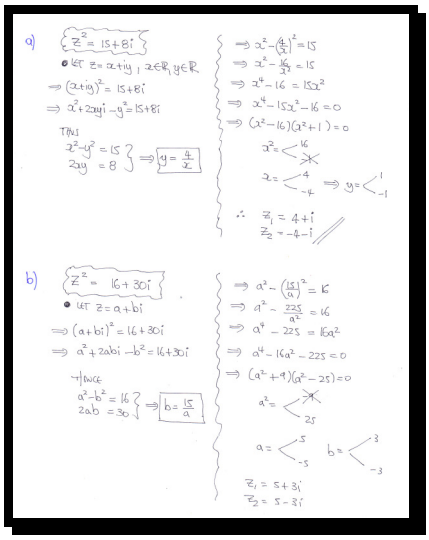
Find the square roots of the following complex numbers.

a) $15+8\mathrm{i}$

b) $16+30\mathrm{i}$

Give the answers in the form $a+b\mathrm{i}$, where $a\in\mathbb{R}$ and $b\in\mathbb{R}$.

$\boxed{\pm(4+\mathrm{i})}$, $\boxed{\pm(5+3\mathrm{i})}$



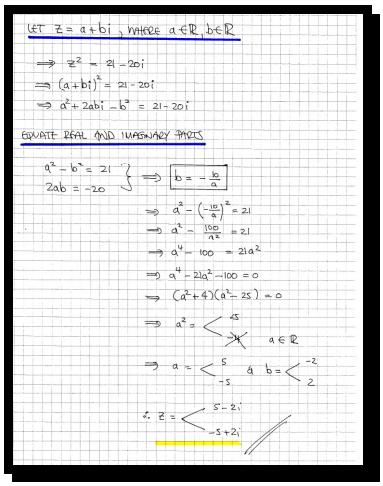
Question 8

Solve the following equation.

$$z^2 = 21 - 20\mathrm{i}, \quad z \in \mathbb{C}.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\boxed{z = \pm(5 - 2\mathrm{i})}$$



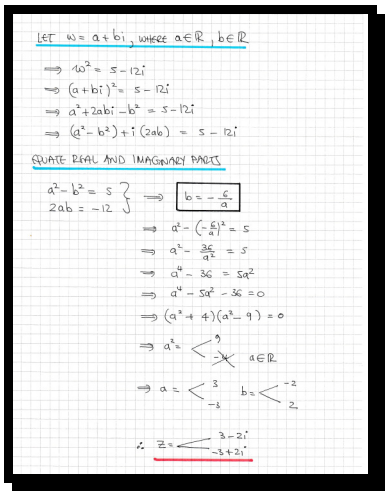
Question 9

Solve the following equation.

$$w^2 = 5 - 12\mathrm{i}\,, \quad w \in \mathbb{C}\,.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\boxed{w = \pm(3 - 2\mathrm{i})}$$

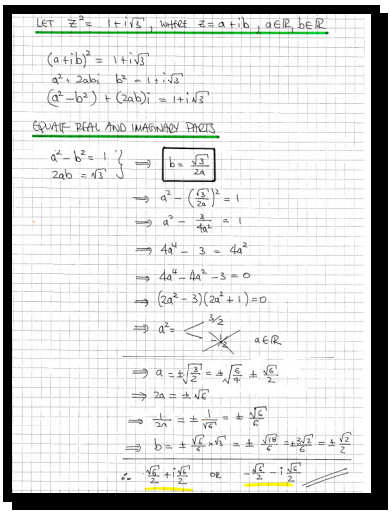


Question 10

Find the square roots of $1+\mathrm{i}\sqrt{3}$.

Give the answers in the form $a+b\mathrm{i}$, where $a\in\mathbb{R}$ and $b\in\mathbb{R}$.

$$\boxed{\pm\frac{1}{2}\left(\sqrt{6}+\mathrm{i}\sqrt{2}\right)}$$

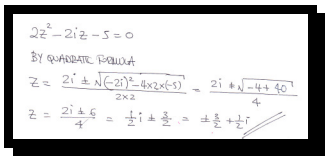


Question 11

Solve the equation

$$2z^2 - 2\mathrm{i}z - 5 = 0,\ z \in \mathbb{C}.$$

$$\boxed{z = \pm\frac{3}{2} + \frac{1}{2}\mathrm{i}}$$



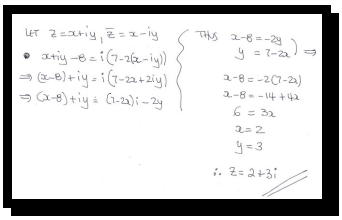
Question 12

$$z - 8 = \mathrm{i}(7 - 2\overline{z}),\ z \in \mathbb{C}.$$

The complex conjugate of z is denoted by $\overline{z}$.

Determine the value of z in the above equation, giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$$\boxed{z = 2 + 3\mathrm{i}}$$



Question 13

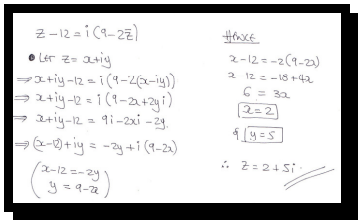
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z-12=\mathrm{i}\left(9-2\overline{z}\right),$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+5\mathrm{i}}$$



Question 14

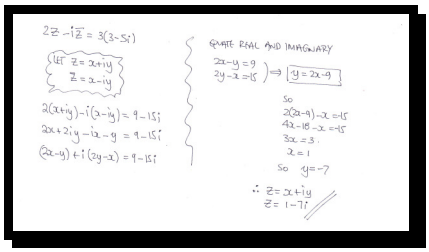
The complex number z satisfies the equation

$$2z-\mathrm{i}\overline{z}=3\left(3-5\mathrm{i}\right),$$

where $\overline{z}$ denotes the complex conjugate of z.

Determine the value of z, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=1-7\mathrm{i}}$$



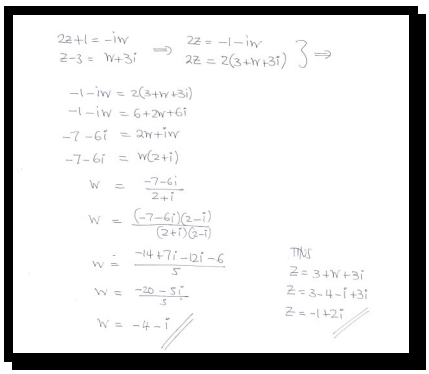
Question 15

Find the value of z and the value of w in the following simultaneous equations

$$2z+1=-\mathrm{i}w$$

$$z-3=w+3\mathrm{i}\,.$$

$$\boxed{z=-1+2\mathrm{i},\ w=-4-\mathrm{i}}$$



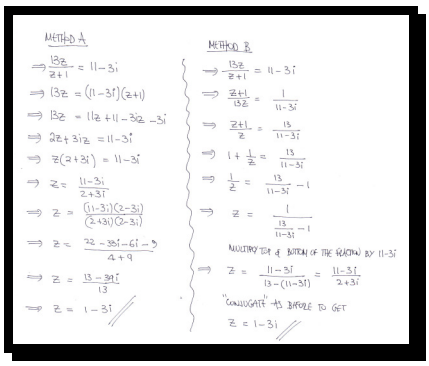
Question 16

Solve the equation

$$\frac{13z}{z+1}=11-3\mathrm{i}\,,\quad z\in\mathbb{C}\,,$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=1-3\mathrm{i}}$$



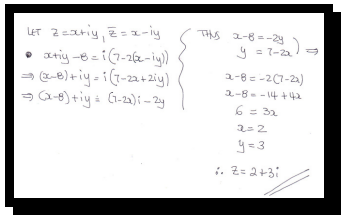
Question 17

$$z-8=\mathrm{i}\left(7-2\overline{z}\right),\ z\in\mathbb{C}.$$

The complex conjugate of z is denoted by $\overline{z}$.

Determine the value of z in the above equation, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+3\mathrm{i}}$$



Question 18

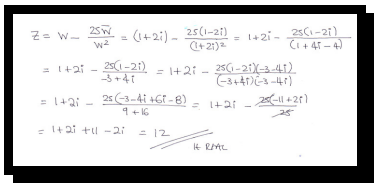
The complex conjugate of w is denoted by $\overline{w}$.

Given further that

$$w=1+2\mathrm{i}\ \text{ and }\ z=w-\frac{25\overline{w}}{w^2},$$

show clearly that z is a real number, stating its value.

$$\boxed{12}$$



Question 19

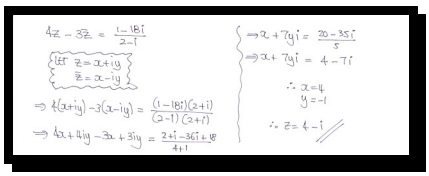
The complex number z satisfies the equation

$$4z-3\overline{z}=\frac{1-18\mathrm{i}}{2-\mathrm{i}},$$

where $\overline{z}$ denotes the complex conjugate of z.

Solve the equation, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=4-\mathrm{i}}$$



Question 20

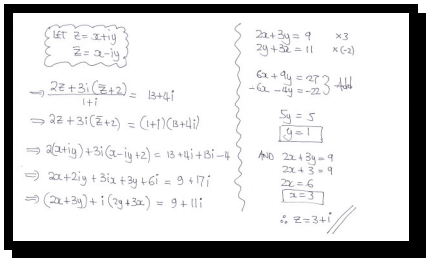
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2z+3\mathrm{i}\left(\overline{z}+2\right)}{1+\mathrm{i}}=13+4\mathrm{i},$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=3+\mathrm{i}}$$



Question 21

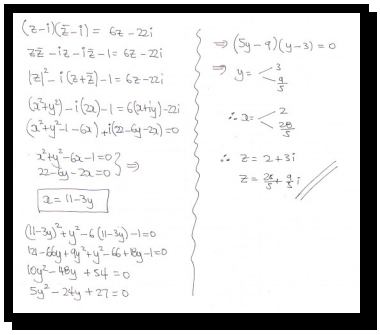
The complex conjugate of z is denoted by $\overline{z}$.

Find the two solutions of the equation

$$(z-\mathrm{i})(\overline{z}-\mathrm{i})=6z-22\mathrm{i}\,,\ z\in\mathbb{C}\,,$$

giving the answers in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z_1=2+3\mathrm{i}}\,,\ \boxed{z_2=\frac{28}{5}+\frac{9}{5}\mathrm{i}}$$



Question 22

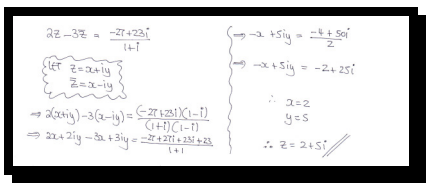
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$2z-3\overline{z}=\frac{-27+23\mathrm{i}}{1+\mathrm{i}}\,,$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+5\mathrm{i}}$$



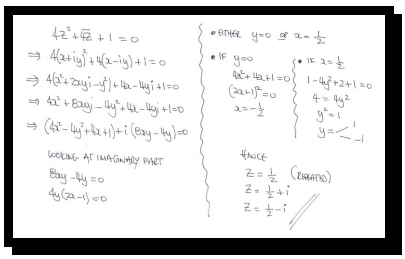
Question 23

Find the three solutions of the equation

$$4z^2+4\overline{z}+1=0\,,\ z\in\mathbb{C}\,,$$

where $\overline{z}$ denotes the complex conjugate of z.

$$\boxed{z=\frac{1}{2},\ \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i}}$$



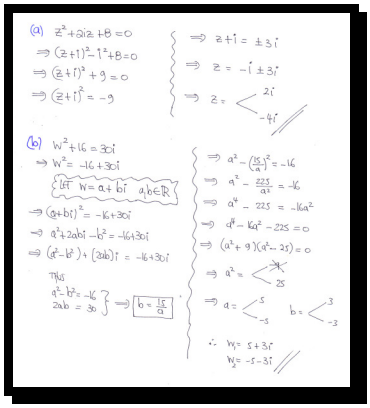
Question 24

Solve the following equations.

a) $z^2+2\mathrm{i}z+8=0\,,\ z\in\mathbb{C}\,.$

b) $w^2+16=30\mathrm{i}\,,\ w\in\mathbb{C}\,.$

$$\boxed{z_1=2\mathrm{i},\ z_2=-4\mathrm{i}}\,,\quad \boxed{w=\pm(3+5\mathrm{i})}$$



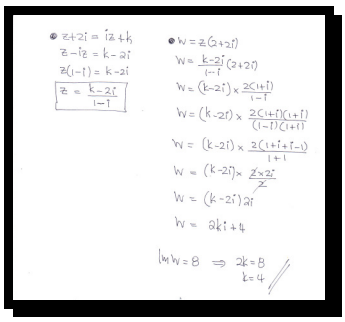
Question 25

It is given that

$$z+2\mathrm{i}=\mathrm{i}z+k\ ,\ k\in\mathbb{R} \qquad \text{and} \qquad \frac{w}{z}=2+2\mathrm{i}\ ,\ \operatorname{Im}w=8\ .$$

Determine the value of k.

$$\boxed{k=4}$$



Question 26

The complex number z satisfies the equation

$$z^2 = 3 + 4\mathrm{i}\,.$$

a) Find the possible values of …

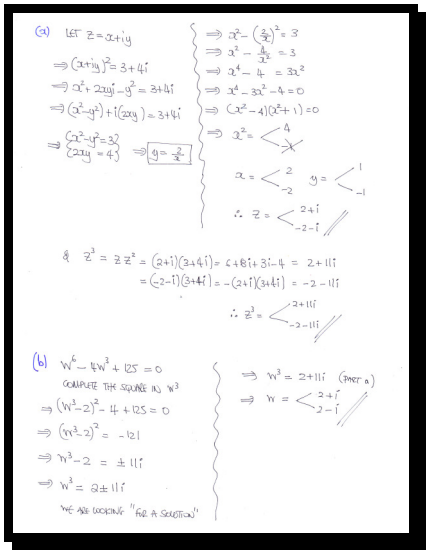
a.… z.

b.… z^3.

b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0,\ w \in \mathbb{C},$$

$$\boxed{z = \pm(2+\mathrm{i})},\ \boxed{z^3 = 2 \pm 11\mathrm{i}},\ \boxed{w = \pm(2+\mathrm{i})}$$



Question 27

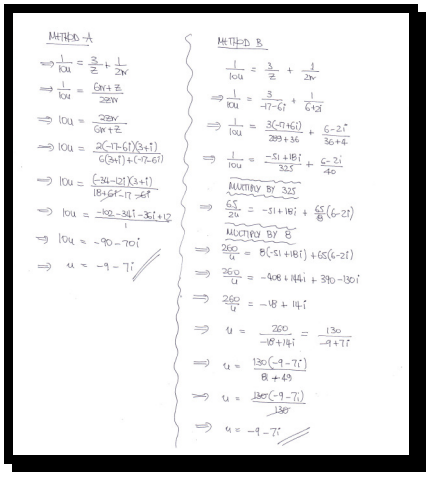
It is given that

$$z=-17-6\mathrm{i} \text{ and } w=3+\mathrm{i}.$$

Find the value of u given further that

$$\frac{1}{10u}=\frac{3}{z}+\frac{1}{2w}.$$

$$\boxed{u=-9-7\mathrm{i}}$$



Question 28

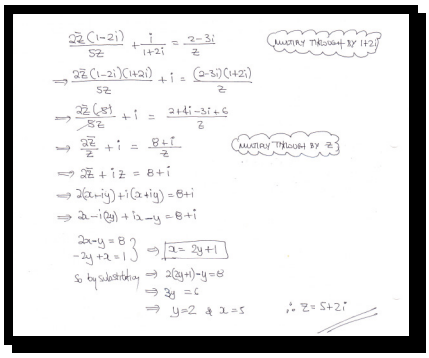
The complex conjugate of the complex number z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2\overline{z}\left(1-2\mathrm{i}\right)}{5z}+\frac{\mathrm{i}}{1+2\mathrm{i}}=\frac{2-3\mathrm{i}}{z},$$

giving the answer in the form $x+\mathrm{i}y$.

$$\boxed{z=5+2\mathrm{i}}$$



Question 29

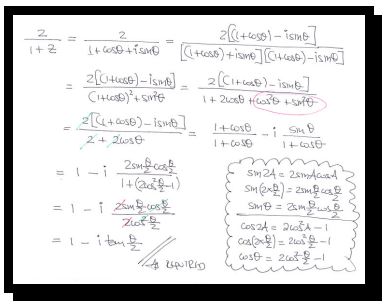
It is given that

$$z = \cos\theta + \mathrm{i}\sin\theta,\ 0 \le z < 2\pi .$$

Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\left(\frac{\theta}{2}\right).$$

proof



Question 30

By considering the solutions of the equation

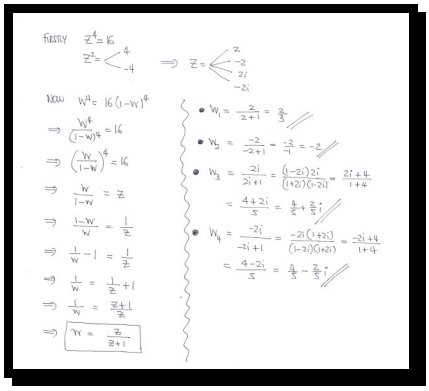
$$z^4 = 16,$$

find the solutions of the equation

$$w^4 = 16(1-w)^4,$$

giving the answers in the form $x+\mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\boxed{z_1 = 2, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{4}{5}+\mathrm{i}\frac{2}{5}, \quad z_4 = \frac{4}{5}-\mathrm{i}\frac{2}{5}}$$



Question 31

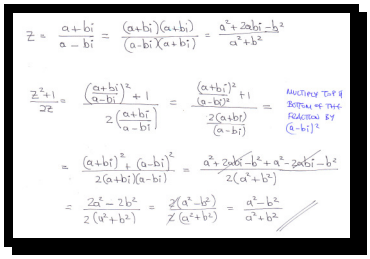
The complex number z is given by

$$z=\frac{a+b\mathrm{i}}{a-b\mathrm{i}},\ a\in\mathbb{R},\ b\in\mathbb{R}.$$

Show clearly that

$$\frac{z^2+1}{2z}=\frac{a^2-b^2}{a^2+b^2}.$$

proof



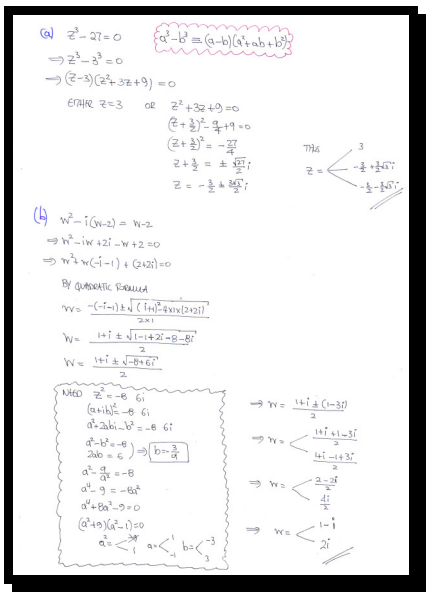
Question 32

Solve the following equations.

a) $z^3-27=0$.

b) $w^2-\mathrm{i}(w-2)=(w-2)$.

$$\boxed{z_1=3,\quad z_2=\frac{3}{2}\left(-1\pm\sqrt{3}\right)},\ \boxed{w_1=2\mathrm{i},\quad w_2=1-\mathrm{i}}$$

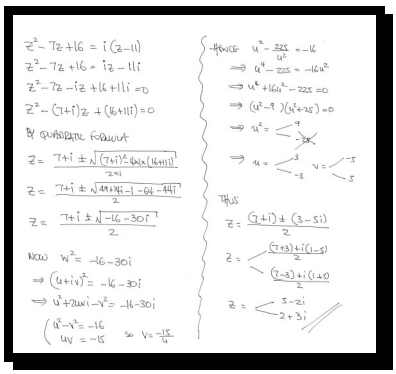


Question 33

Solve the quadratic equation

$$z^2-7z+16=\mathrm{i}(z-11),\ z\in\mathbb{C}.$$

$$\boxed{z=2+3\mathrm{i},\ z=5-2\mathrm{i}}$$



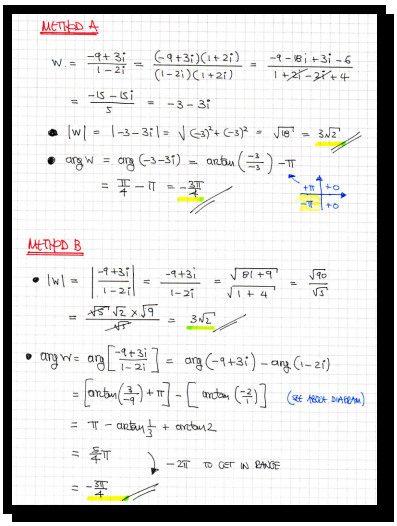
MODULUS AND ARGUMENT

Question 1

$$w=\frac{-9+3\mathrm{i}}{1-2\mathrm{i}}.$$

Find the modulus and the argument of the complex number w.

$$\boxed{|w|=3\sqrt{2}},\quad \boxed{\arg w=-\frac{3\pi}{4}}$$



Question 2

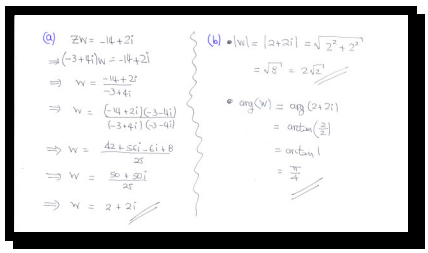
$$z=-3+4\text{i} \qquad \text{and} \qquad zw=-14+2\text{i}\,.$$

By showing clear workings, find …

a) … w in the form $a+b\text{i}$, where a and b are real numbers.

b) … the modulus and the argument of w.

$$\boxed{w=2+2\text{i}}\,,\quad \boxed{|w|=2\sqrt{2}}\,,\quad \boxed{\arg w=\frac{\pi}{4}}$$



Question 3

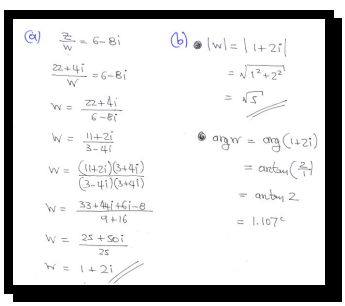
$$z=22+4\text{i} \qquad \text{and} \qquad \frac{z}{w}=6-8\text{i}\,.$$

By showing clear workings, find …

a) … w in the form $a+b\text{i}$, where a and b are real numbers .

b) … the modulus and the argument of w.

$$\boxed{w=1+2\text{i}}\,,\quad \boxed{|w|=\sqrt{5}}\,,\quad \boxed{\arg w\approx 1.11^{\text{c}}}$$

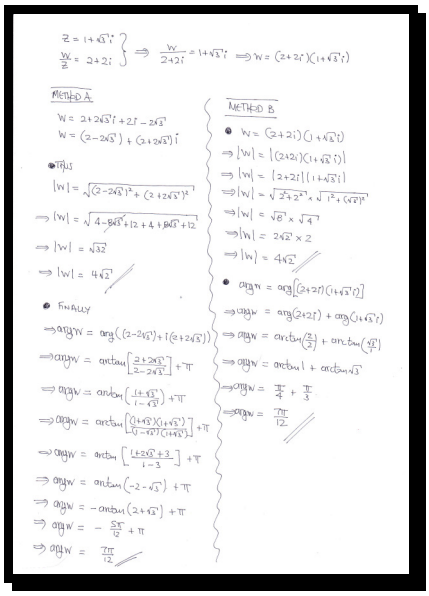


Question 4

$$z = 1 + \sqrt{3}\mathrm{i} \qquad \text{and} \qquad \frac{w}{z} = 2 + 2\mathrm{i}\,.$$

Find the exact value of the modulus of w and the exact value of the argument of w.

$$\boxed{|w| = 4\sqrt{2}}\,, \quad \boxed{\arg w = \frac{7\pi}{12}}$$



Question 5

The following complex numbers are given.

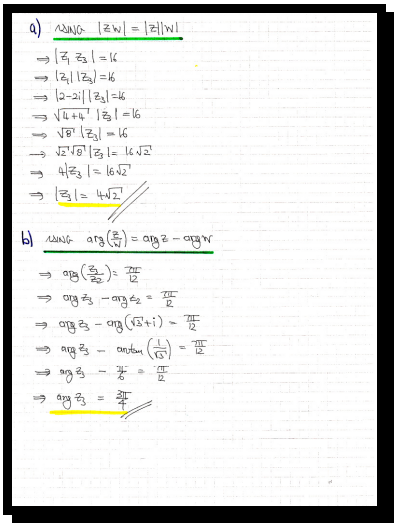
$$z_1 = 2-2\mathrm{i}\,, \quad z_2 = \sqrt{3}+\mathrm{i} \quad \text{and} \quad z_3 = a+b\mathrm{i} \quad \text{where} \quad a\in\mathbb{R}\,,\ b\in\mathbb{R}\,.$$

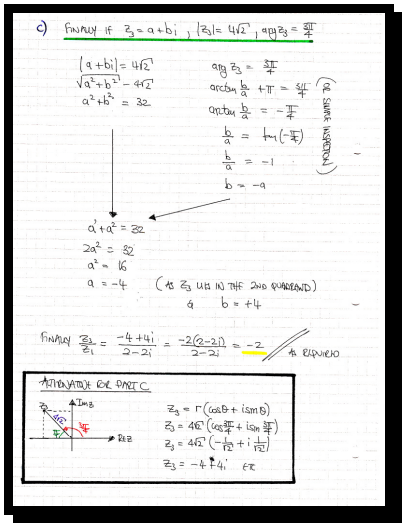
a) If $|z_1 z_3| = 16$, find the modulus z_3.

b) Given further that $\arg\left(\dfrac{z_3}{z_2}\right) = \dfrac{7\pi}{12}$, determine the argument of z_3.

c) Find the values of a and b, and hence show $\dfrac{z_3}{z_1} = -2$.

$$\boxed{|z_3| = 4\sqrt{2}}\,, \quad \boxed{\arg z_3 = \frac{3\pi}{4}}\,, \quad \boxed{a=-4}\,, \quad \boxed{b=4}$$





Question 6

$$z=\sqrt{3}+\mathrm{i} \quad \text{and} \quad w=3\mathrm{i}\,.$$

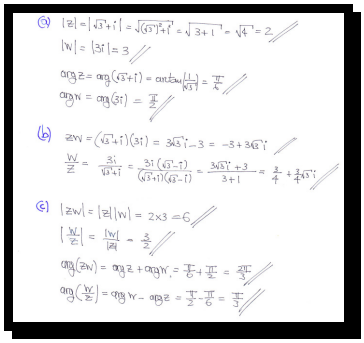
a) Find, in exact form where appropriate, the modulus and argument of z and the modulus and argument of w.

b) Determine simplified expressions for zw and $\dfrac{w}{z}$, giving the answers in the form $x+\mathrm{i}y$, where $x\in\mathbb{R}$, $y\in\mathbb{R}$.

c) Find, in exact form where appropriate, the modulus and argument of zw and the modulus and argument of $\dfrac{w}{z}$.

$$\boxed{|z|=2,\ \ |w|=3}\,,\ \boxed{\arg z=\frac{\pi}{6},\ \arg w=\frac{\pi}{2}}\,,\ \boxed{zw=-3+3\sqrt{3}\mathrm{i}}\,,\ \boxed{\frac{w}{z}=\frac{3}{4}+\frac{3}{4}\sqrt{3}\mathrm{i}}\,,$$

$$\boxed{|zw|=6,\ \ \left|\frac{w}{z}\right|=\frac{3}{2}}\,,\ \boxed{\arg(zw)=\frac{2\pi}{3},\arg\left(\frac{w}{z}\right)=\frac{\pi}{3}}$$



Question 7

The following complex numbers are given

$$z=\frac{1+\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad w=\frac{\sqrt{2}}{1-\mathrm{i}}.$$

a) Calculate the modulus of z and the modulus of w.

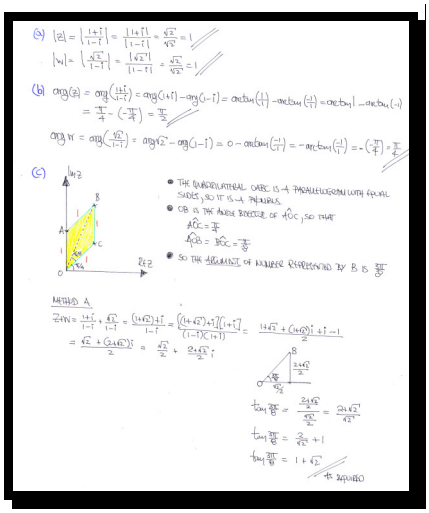
b) Find the argument of z and the argument of w.

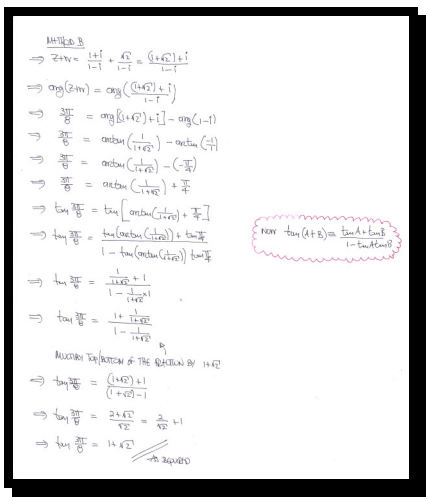
In a standard Argand diagram, the points A, B and C represent the numbers z, $z+w$ and w respectively. The origin of the Argand diagram is denoted by O.

c) By considering the quadrilateral $OABC$ and the argument of $z+w$, show that

$$\tan\left(\frac{3\pi}{8}\right)=1+\sqrt{2}.$$

$\boxed{|z|=1}$, $\boxed{|w|=1}$ $\boxed{\arg z=\frac{\pi}{2}}$, $\boxed{\arg w=\frac{\pi}{4}}$





Question 8

$$\frac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+3\mathrm{i}} = q(1+\mathrm{i}) \ , \quad q \in \mathbb{R} \ .$$

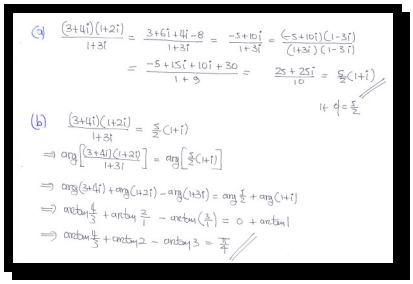
a) Find the value of q.

b) Hence simplify

$$\arctan\frac{4}{3} + \arctan 2 - \arctan 3 \, ,$$

giving the answer in terms of π.

$$\boxed{q = \frac{5}{2}} \ , \quad \boxed{\frac{1}{4}\pi}$$



Question 9

It is given that

$$z=\frac{1+8\mathrm{i}}{1-2\mathrm{i}}\ .$$

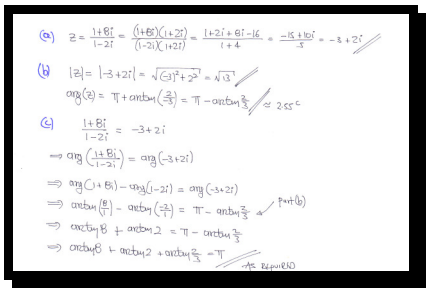
a) Express z in the form $x+\mathrm{i}y$.

b) Find the modulus and argument of z.

c) Show clearly that

$$\arctan 8+\arctan 2+\arctan\frac{2}{3}=\pi\ .$$

$\boxed{z=-3+2\mathrm{i}}$, $\boxed{|z|=\sqrt{13}}$, $\boxed{\arg z\approx 2.55^{\mathrm{c}}}$



COMPLEX POLYNOMIAL QUESTIONS

Question 1

The cubic equation

$$2z^3-5z^2+cz-5=0,\ c\in\mathbb{R},$$

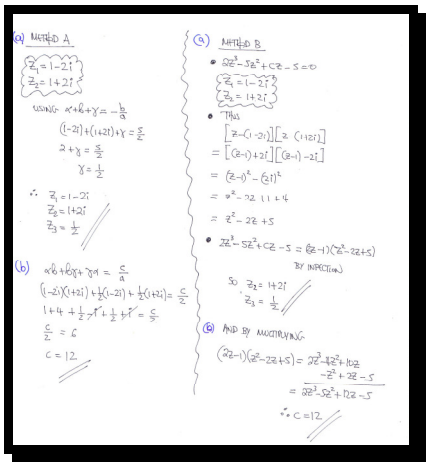
has a solution $z=1-2\mathrm{i}$.

Find in any order …

a) … the other two solutions of the equations.

b) … the value of c.

$$\boxed{z_2=1+2\mathrm{i}},\ \boxed{z_3=\frac{1}{2}},\ \boxed{c=12}$$



Question 2

The following cubic equation is given

$$z^3 + az^2 + bz - 5 = 0,$$

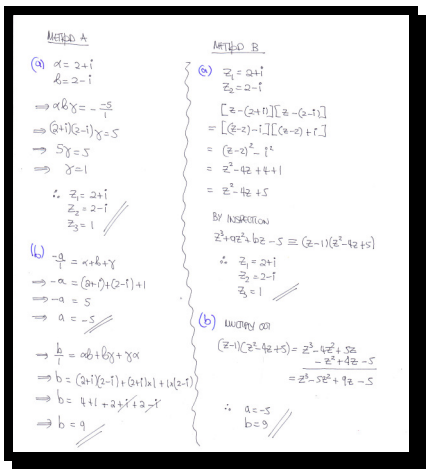
where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

One of the roots of the above cubic equation is $2+\mathrm{i}$.

a) Find the other two roots.

b) Determine the value of a and the value of b.

$\boxed{z_2 = 2-\mathrm{i}}$, $\boxed{z_3 = 1}$, $\boxed{a = -5}$, $\boxed{b = 9}$



Question 3

The following cubic equation is given

$$z^3 + pz^2 + 6z + q = 0,$$

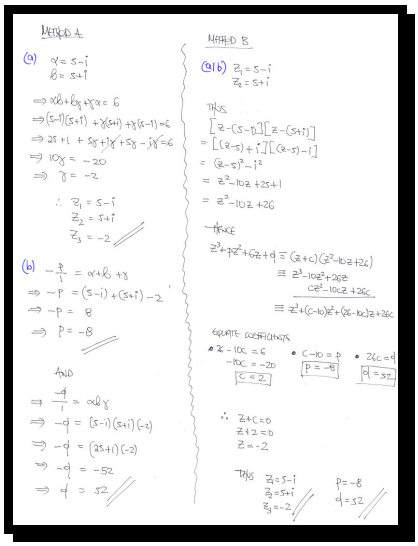
where $p \in \mathbb{R}$, $q \in \mathbb{R}$.

One of the three solutions of the above cubic equation is $5 - \mathrm{i}$.

a) Find the other two solutions of the equation.

b) Determine the value of p and the value of q.

$$\boxed{z_2 = 5 + \mathrm{i},\ z_3 = 2}, \quad \boxed{p = -8}, \quad \boxed{q = 52}$$



Question 4

The following cubic equation is given

$$z^3+2z^2+az+b=0\,,$$

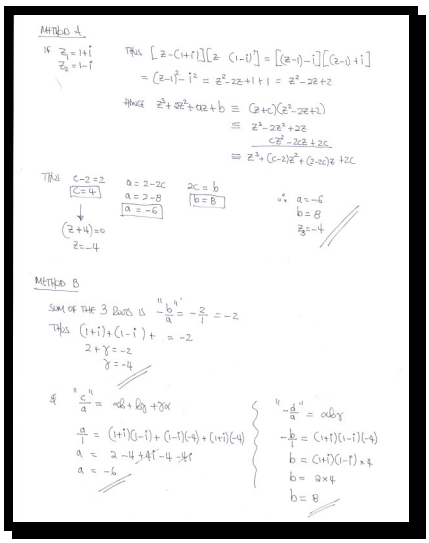
where $a\in\mathbb{R}$, $b\in\mathbb{R}$.

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Find the value of a and the value of b.

$\boxed{z=-4}$, $\boxed{a=-6}$, $\boxed{b=8}$



Question 5

The following cubic equation is given

$$z^3 + Az^2 + Bz + 26 = 0,$$

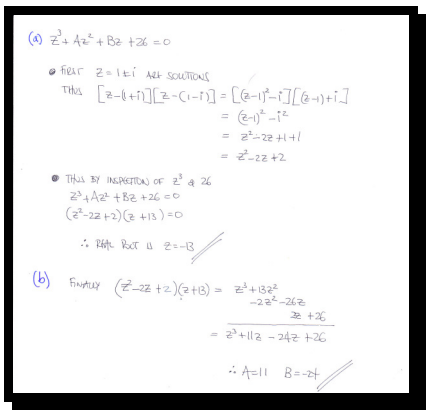
where $A \in \mathbb{R}$, $B \in \mathbb{R}$

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Determine the value of A and the value of B.

$\boxed{z = -13}$, $\boxed{A = 11}$, $\boxed{B = -24}$



Question 6

The cubic equation

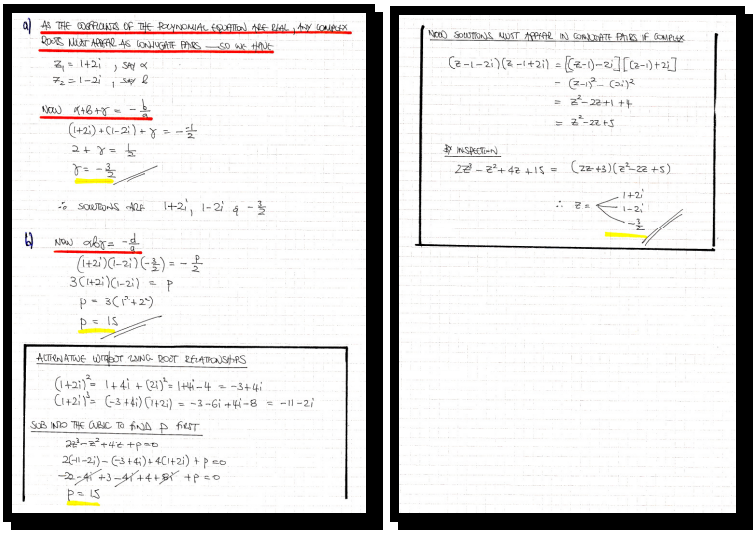
$$2z^3 - z^2 + 4z + p = 0, \ p \in \mathbb{R},$$

is satisfied by $z = 1 + 2\mathrm{i}$.

a) Find the other two roots of the equation.

b) Determine the value of p.

$$\boxed{1 - 2\mathrm{i}, -\frac{3}{2}}, \quad \boxed{p = 15}$$



Question 7

Consider the cubic equation

$$z^3+z+10=0,\ z\in\mathbb{C}.$$

a) Verify that $1+2\mathrm{i}$ is a root of this equation.

b) Find the other two roots.

$$\boxed{z_1=1-2\mathrm{i},\quad z_2=-2}$$



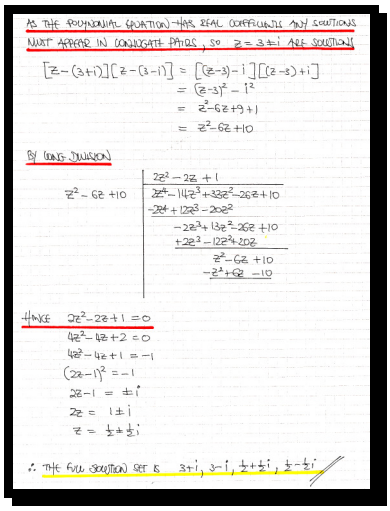
Question 8

Solve the equation

$$2z^4-14z^3+33z^2-26z+10=0,\ z\in\mathbb{C}$$

given that one of its roots is $3+\mathrm{i}$.

$$\boxed{z=3+\mathrm{i},\quad z=3-\mathrm{i},\quad z=\frac{1}{2}+\frac{1}{2}\mathrm{i},\quad z=\frac{1}{2}-\frac{1}{2}\mathrm{i}}$$



Question 9

$$2z^3+pz^2+qz+16=0,\ p\in\mathbb{R},\ q\in\mathbb{R}.$$

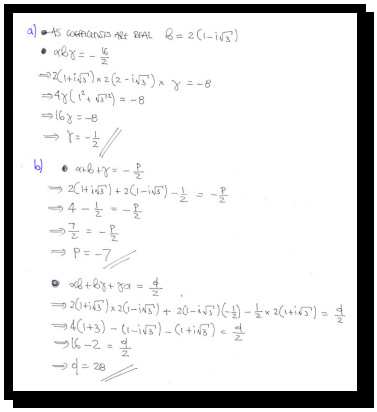
The above cubic equation has roots α, β and γ, where γ is real.

It is given that $\alpha=2\left(1+\mathrm{i}\sqrt{3}\right)$.

a) Find the other two roots, β and γ.

b) Determine the values of p and q.

$$\boxed{\beta=2\left(1-\mathrm{i}\sqrt{3}\right)},\ \boxed{\gamma=-\frac{1}{2}},\ \boxed{p=-7},\ \boxed{q=28}$$



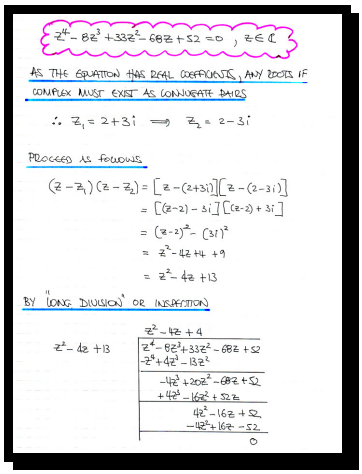
Question 10

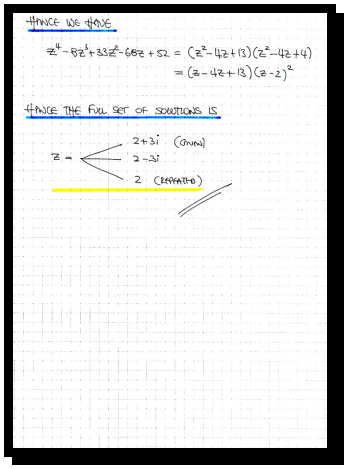
$$z^4-8z^3+33z^2-68z+52=0,\ z\in\mathbb{C}.$$

One of the roots of the above quartic equation, is $2+3\mathrm{i}$.

Find the other roots of the equation.

$$\boxed{z=2-3\mathrm{i},\ z=2}$$





Question 11

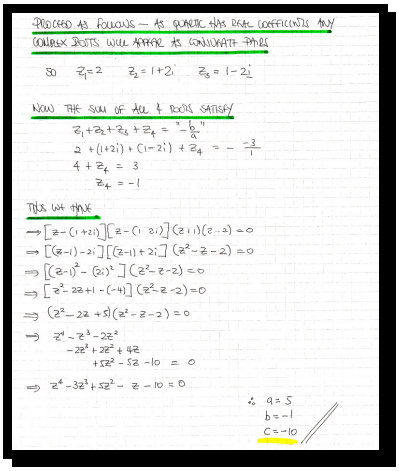
It is given that $z=2$ and $z=1+2\mathrm{i}$ are solutions of the equation

$$z^4-3z^3+az^2+bz+c=0\,.$$

where a, b and c are real constants.

Determine the values of a, b and c.

$\boxed{a=5}$, $\boxed{b=-1}$, $\boxed{c=-10}$



Question 12

If $1-2\mathrm{i}$ is a root of the quartic equation

$$z^4-6z^3+18z^2-30z+25=0$$

find the other three roots.

$$\boxed{z_2=1+2\mathrm{i},\quad z_3=2-\mathrm{i},\quad z_4=2+\mathrm{i}}$$

