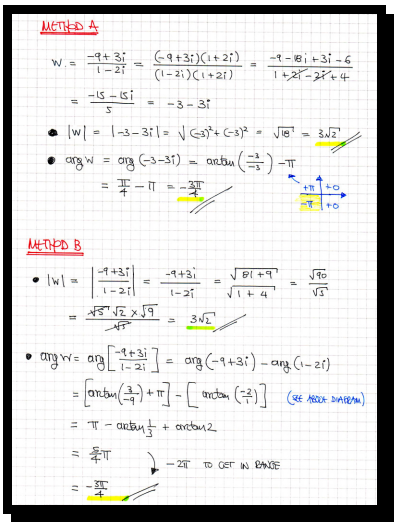
COMPLEX NUMBERS
(Exam Questions I)

Question 1 (**)

$$w=\frac{-9+3\mathrm{i}}{1-2\mathrm{i}}.$$

Find the modulus and the argument of the complex number w.

$\square$, $\boxed{|w|=3\sqrt{2}}$, $\boxed{\arg w=-\frac{3\pi}{4}}$

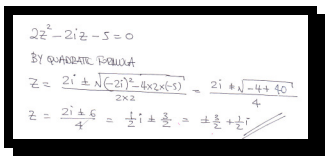


Question 2 (**)

Solve the equation

$$2z^2-2\mathrm{i}z-5=0,\ z\in\mathbb{C}.$$

$$\boxed{z=\pm\frac{3}{2}+\frac{1}{2}\mathrm{i}}$$

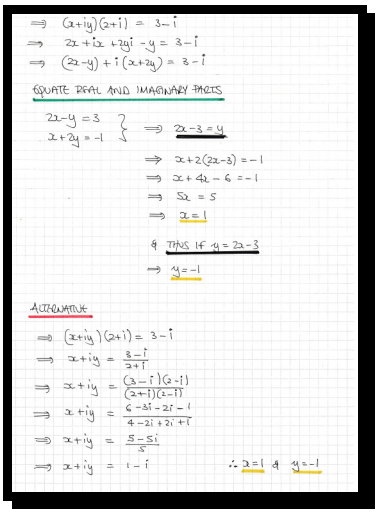


Question 3 (**)

Find the value of x and the value of y in the following equation, given further that $x\in\mathbb{R}$, $y\in\mathbb{R}$.

$$(x+\mathrm{i}y)(2+\mathrm{i})=3-\mathrm{i}\,.$$

$\square$, $\boxed{(x,y)=(1,-1)}$

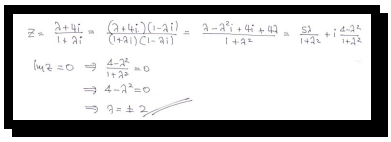


Question 4 (**)

$$z=\frac{\lambda+4\mathrm{i}}{1+\lambda\mathrm{i}}\,,\ \lambda\in\mathbb{R}\,.$$

Given that z is a real number, find the possible values of λ.

$\boxed{\lambda=\pm 2}$

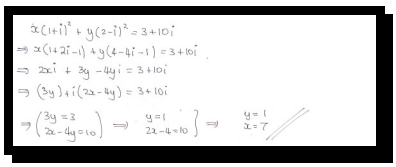


Question 5 (**)

Find the values of x and y in the equation

$$x(1+\mathrm{i})^2+y(2-\mathrm{i})^2=3+10\mathrm{i}\,,\ x\in\mathbb{R}\,,\ y\in\mathbb{R}\,.$$

$\boxed{x=7}$, $\boxed{y=1}$

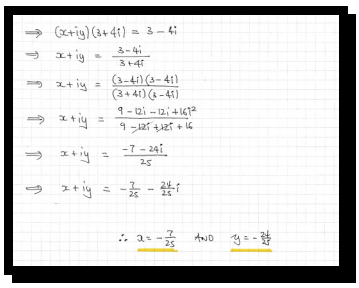


Question 6 (**)

Find the value of x and the value of y in the following equation, given further that $x\in\mathbb{R}$, $y\in\mathbb{R}$.

$$(x+\mathrm{i}y)(3+4\mathrm{i})=3-4\mathrm{i}\,.$$

$\boxed{}$, $\boxed{(x,y)=\left(-\frac{7}{25},-\frac{24}{25}\right)}$



Question 7 (**)

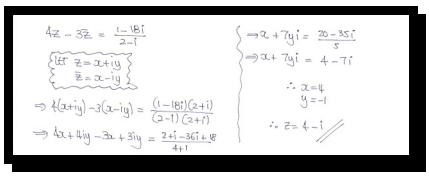
The complex number z satisfies the equation

$$4z-3\overline{z}=\frac{1-18\mathrm{i}}{2-\mathrm{i}},$$

where $\overline{z}$ denotes the complex conjugate of z.

Solve the equation, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=4-\mathrm{i}}$$



Question 8 (**)

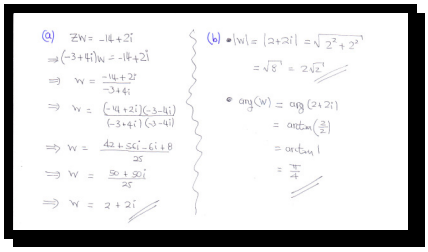
$$z=-3+4\mathrm{i} \qquad \text{and} \qquad zw=-14+2\mathrm{i}.$$

By showing clear workings, find …

a) … w in the form $a+b\mathrm{i}$, where a and b are real numbers.

b) … the modulus and the argument of w.

$$\boxed{w=2+2\mathrm{i}}, \quad \boxed{|w|=2\sqrt{2}}, \quad \boxed{\arg w=\frac{\pi}{4}}$$



Question 9 **(**)**

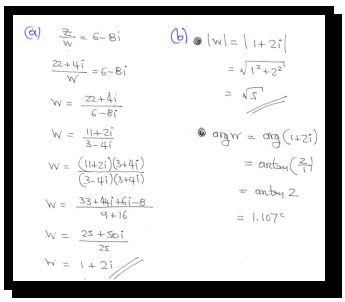
$$z = 22 + 4\mathrm{i} \quad \text{and} \quad \frac{z}{w} = 6 - 8\mathrm{i} .$$

By showing clear workings, find …

a) … w in the form $a + b\mathrm{i}$, where a and b are real numbers .

b) … the modulus and the argument of w.

$$\boxed{w = 1 + 2\mathrm{i}} , \quad \boxed{|w| = \sqrt{5}} , \quad \boxed{\arg w \approx 1.11^{\mathrm{c}}}$$

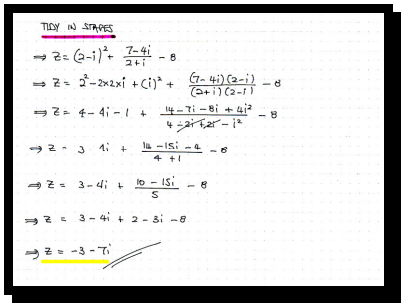


Question 10 **(**)**

$$z = (2 - \mathrm{i})^2 + \frac{7 - 4\mathrm{i}}{2 + \mathrm{i}} - 8 .$$

Express z in the form $x + \mathrm{i}y$, where x and y are real numbers.

$$\boxed{} , \quad \boxed{-3 - 7\mathrm{i}}$$



Question 11 (**)

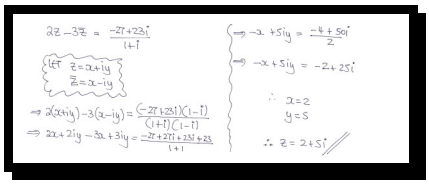
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$2z-3\overline{z}=\frac{-27+23\mathrm{i}}{1+\mathrm{i}},$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+5\mathrm{i}}$$



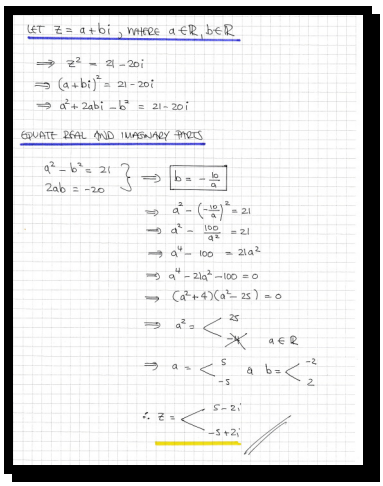
Question 12 (+)**

Solve the following equation.

$$z^2=21-20\mathrm{i}\,,\quad z\in\mathbb{C}.$$

Give the answers in the form $a+b\mathrm{i}$, where $a\in\mathbb{R}$ and $b\in\mathbb{R}$.

$$\boxed{\phantom{z=\pm(5-2\mathrm{i})}},\ \boxed{z=\pm(5-2\mathrm{i})}$$



Question 13 (+)**

The cubic equation

$$2z^3-5z^2+cz-5=0,\ c\in\mathbb{R},$$

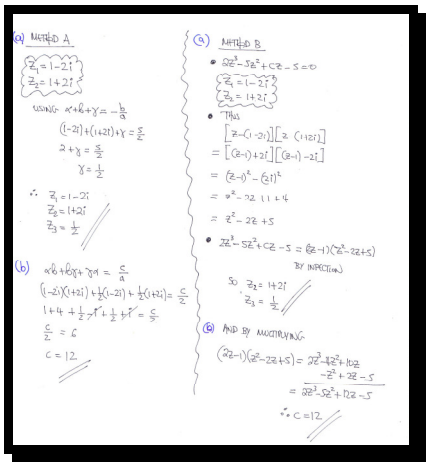
has a solution $z=1-2\mathrm{i}$.

Find in any order …

a) … the other two solutions of the equations.

b) … the value of c.

$$\boxed{z_2=1+2\mathrm{i}},\ \boxed{z_3=\frac{1}{2}},\ \boxed{c=12}$$



Question 14 (+)**

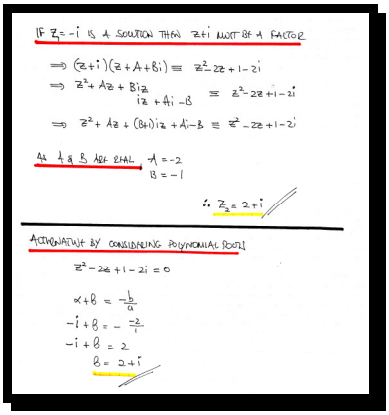
The quadratic equation

$$z^2-2z+1-2\mathrm{i}=0,\ c\in\mathbb{R},$$

has a solution $z=-\mathrm{i}$.

Find the other solution.

$\boxed{}$, $\boxed{z_2=2+\mathrm{i}}$



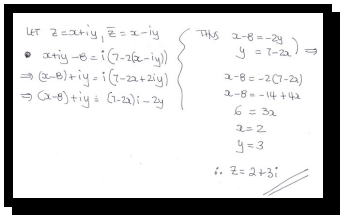
Question 15 **(**+)**

$$z-8=\mathrm{i}\left(7-2\overline{z}\right),\ z\in\mathbb{C}.$$

The complex conjugate of z is denoted by $\overline{z}$.

Determine the value of z in the above equation, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+3\mathrm{i}}$$



Question 16 **(**+)**

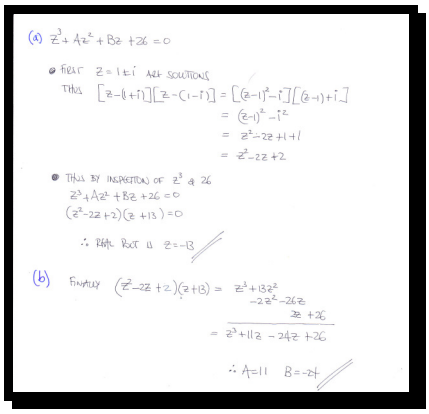
$$z^3+Az^2+Bz+26=0,\text{ where } A\in\mathbb{R},\ B\in\mathbb{R}$$

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Determine the values of A and B.

$$\boxed{z=-13},\ \boxed{A=11},\ \boxed{B=-24}$$



Question 17 (+)**

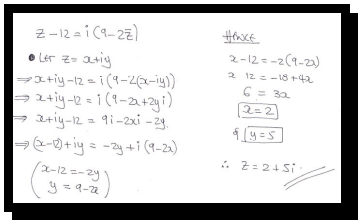
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z-12=\mathrm{i}\left(9-2\overline{z}\right),$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=2+5\mathrm{i}}$$



Question 18 (+)**

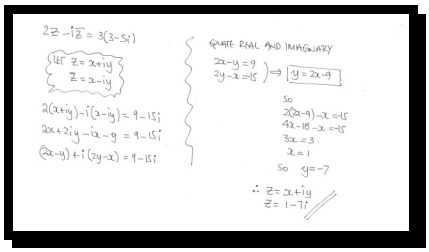
The complex number z satisfies the equation

$$2z-\mathrm{i}\overline{z}=3\left(3-5\mathrm{i}\right),$$

where $\overline{z}$ denotes the complex conjugate of z.

Determine the value of z, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=1-7\mathrm{i}}$$



Question 19 (+)**

The cubic equation

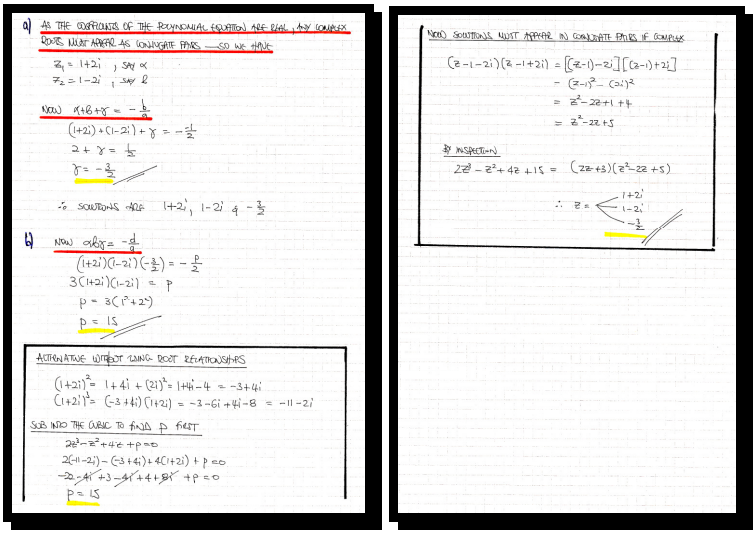
$$2z^3 - z^2 + 4z + p = 0, \ p \in \mathbb{R},$$

is satisfied by $z = 1 + 2\mathrm{i}$.

a) Find the other two roots of the equation.

b) Determine the value of p.

$\boxed{}$, $\boxed{1-2\mathrm{i}, -\frac{3}{2}}$, $\boxed{p = 15}$



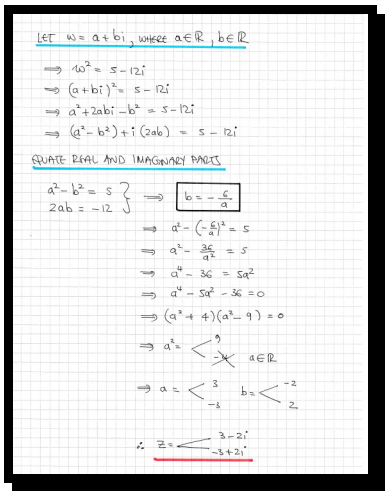
Question 20 (+)**

Solve the following equation.

$$w^2 = 5 - 12\mathrm{i}\,, \quad w \in \mathbb{C}\,.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$\boxed{}$, $\boxed{w = \pm(3 - 2\mathrm{i})}$

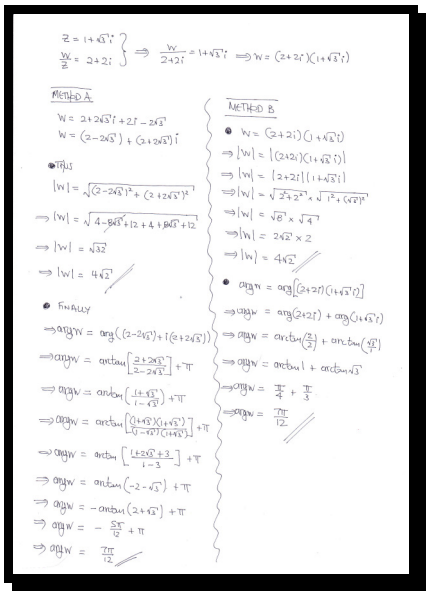


Question 21 (+)**

$$z = 1 + \sqrt{3}\mathrm{i} \qquad \text{and} \qquad \frac{w}{z} = 2 + 2\mathrm{i} \, .$$

Find the exact value of the modulus of w and the exact value of the argument of w.

$$\boxed{|w| = 4\sqrt{2}} \, , \quad \boxed{\arg w = \frac{7\pi}{12}}$$



Question 22 (+)**

The following cubic equation is given

$$z^3+az^2+bz-5=0,$$

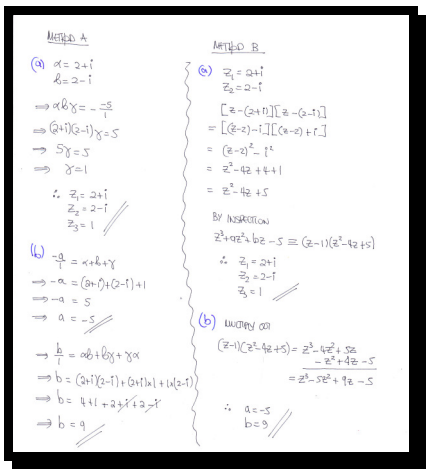
where $a\in\mathbb{R}$, $b\in\mathbb{R}$.

One of the roots of the above cubic equation is $2+\mathrm{i}$.

a) Find the other two roots.

b) Determine the value of a and the value of b.

$$\boxed{z_2=2-\mathrm{i}}\,,\ \boxed{z_3=1}\,,\ \boxed{a=-5}\,,\ \boxed{b=9}$$



Question 23 (+)**

The following cubic equation is given

$$z^3 + pz^2 + 6z + q = 0,$$

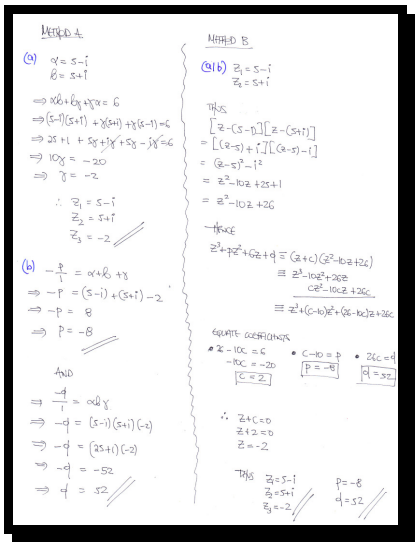
where $p \in \mathbb{R}$, $q \in \mathbb{R}$.

One of the three solutions of the above cubic equation is $5 - \mathrm{i}$.

a) Find the other two solutions of the equation.

b) Determine the value of p and the value of q.

$$\boxed{z_2 = 5 + \mathrm{i},\ z_3 = 2}, \quad \boxed{p = -8}, \quad \boxed{q = 52}$$



Question 24 (+)**

The complex number z is defined as

$$z=\mathrm{i}(1+\mathrm{i})(1-2\mathrm{i})^2.$$

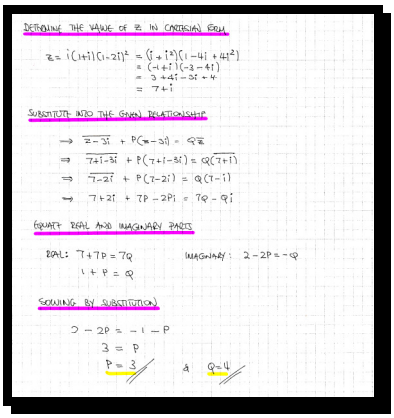
It is further given that

$$\overline{z-3\mathrm{i}}\ +\ P(z-3\mathrm{i})\ =\ Q\,\overline{z}$$

where P and Q are **real** constants.

Find the value of P and the value of Q.

$\boxed{}$, $\boxed{P=3}$, $\boxed{Q=4}$



Question 25 (***)

$$z=\sqrt{3}+\mathrm{i} \quad \text{and} \quad w=3\mathrm{i}\,.$$

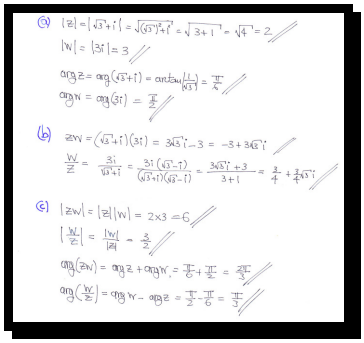
a) Find, in exact form where appropriate, the modulus and argument of z and the modulus and argument of w.

b) Determine simplified expressions for zw and $\dfrac{w}{z}$, giving the answers in the form $x+\mathrm{i}y$, where $x\in\mathbb{R}$, $y\in\mathbb{R}$.

c) Find, in exact form where appropriate, the modulus and argument of zw and the modulus and argument of $\dfrac{w}{z}$.

$$\boxed{|z|=2,\ |w|=3},\ \boxed{\arg z=\frac{\pi}{6},\ \arg w=\frac{\pi}{2}},\ \boxed{zw=-3+3\sqrt{3}\mathrm{i}},\ \boxed{\frac{w}{z}=\frac{3}{4}+\frac{3}{4}\sqrt{3}\mathrm{i}},$$

$$\boxed{|zw|=6,\ \left|\frac{w}{z}\right|=\frac{3}{2}},\ \boxed{\arg(zw)=\frac{2\pi}{3},\arg\left(\frac{w}{z}\right)=\frac{\pi}{3}}$$

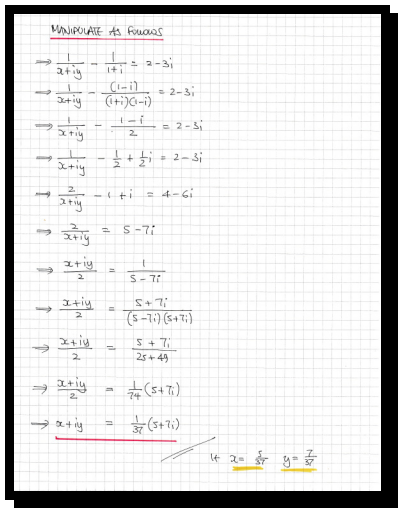


Question 26 (***)

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}-\frac{1}{1+\mathrm{i}}=2-3\mathrm{i}\,.$$

$\square$, $\boxed{(x,y)=\left(\frac{5}{37},\frac{7}{37}\right)}$

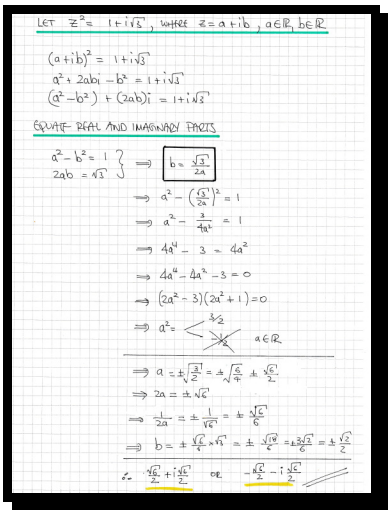


Question 27 (***)

Find the square roots of $1+\mathrm{i}\sqrt{3}$.

Give the answers in the form $a+b\mathrm{i}$, where $a\in\mathbb{R}$ and $b\in\mathbb{R}$.

$\square$, $\boxed{\pm\frac{1}{2}\left(\sqrt{6}+\mathrm{i}\sqrt{2}\right)}$



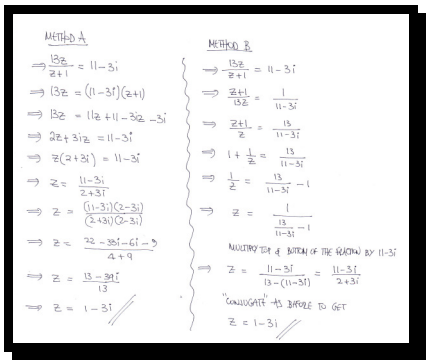
Question 28 **(***)**

Solve the equation

$$\frac{13z}{z+1}=11-3\mathrm{i}\,,\quad z\in\mathbb{C}\,,$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$\boxed{z=1-3\mathrm{i}}$



Question 29 **(***)**

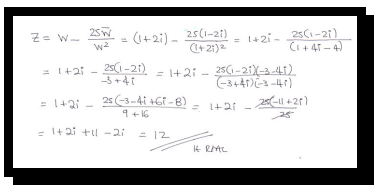
The complex conjugate of w is denoted by $\overline{w}$.

Given further that

$$w=1+2\mathrm{i}\ \text{ and }\ z=w-\frac{25\overline{w}}{w^2}\,,$$

show clearly that z is a real number, stating its value.

$\boxed{12}$



Question 30 (***)

The following cubic equation is given

$$z^3+2z^2+az+b=0\,,$$

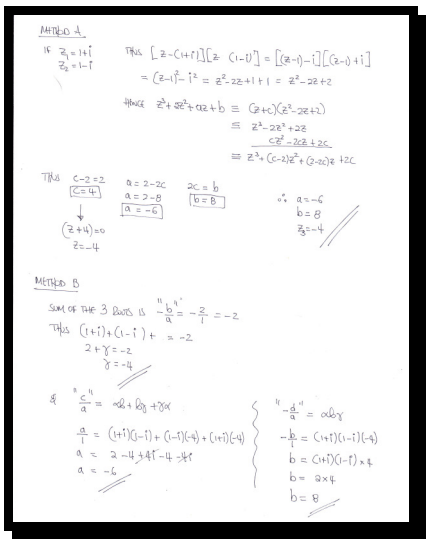
where $a\in\mathbb{R}$, $b\in\mathbb{R}$.

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Find the value of a and the value of b.

$$\boxed{z=-4}\,,\quad \boxed{a=-6}\,,\quad \boxed{b=8}$$



Question 31 (***)

The following complex numbers are given.

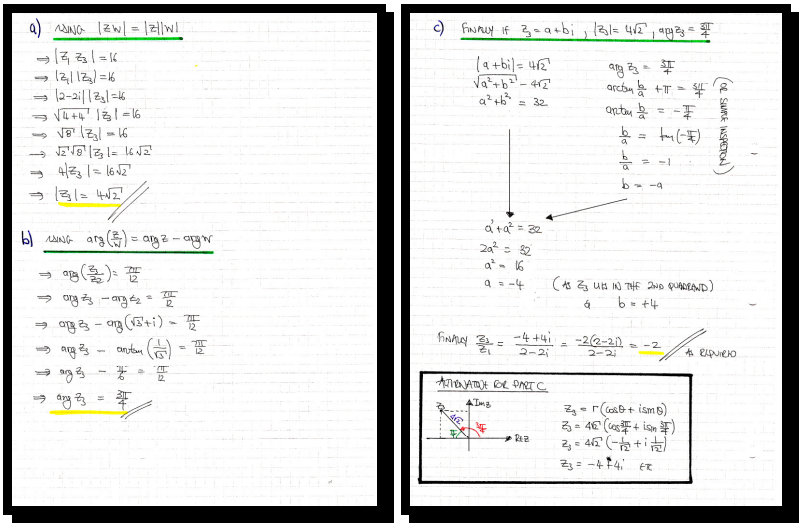
$$z_1 = 2-2\mathrm{i}\,, \quad z_2 = \sqrt{3}+\mathrm{i} \quad \text{and} \quad z_3 = a+b\mathrm{i} \quad \text{where} \quad a\in\mathbb{R}\,,\ b\in\mathbb{R}\,.$$

a) If $|z_1 z_3| = 16$, find the modulus z_3.

b) Given further that $\arg\left(\dfrac{z_3}{z_2}\right) = \dfrac{7\pi}{12}$, determine the argument of z_3.

c) Find the values of a and b, and hence show $\dfrac{z_3}{z_1} = -2$.

$\square$, $\boxed{|z_3| = 4\sqrt{2}}$, $\boxed{\arg z_3 = \dfrac{3\pi}{4}}$, $\boxed{a=-4}$, $\boxed{b=4}$



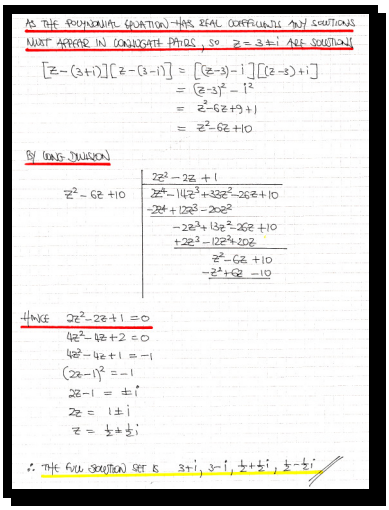
Question 32 (***)

Solve the equation

$$2z^4-14z^3+33z^2-26z+10=0,\ z\in\mathbb{C}$$

given that one of its roots is $3+\mathrm{i}$.

$\boxed{}$, $\boxed{z=3+\mathrm{i},\quad z=3-\mathrm{i},\quad z=\frac{1}{2}+\frac{1}{2}\mathrm{i},\quad z=\frac{1}{2}-\frac{1}{2}\mathrm{i}}$



Question 33 (***)

$$2z^3+pz^2+qz+16=0,\ p\in\mathbb{R},\ q\in\mathbb{R}.$$

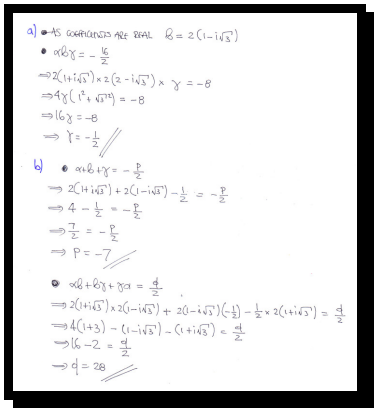
The above cubic equation has roots α, β and γ, where γ is real.

It is given that $\alpha=2\left(1+\mathrm{i}\sqrt{3}\right)$.

a) Find the other two roots, β and γ.

b) Determine the values of p and q.

$$\boxed{\beta=2\left(1-\mathrm{i}\sqrt{3}\right)},\ \boxed{\gamma=-\frac{1}{2}},\ \boxed{p=-7},\ \boxed{q=28}$$

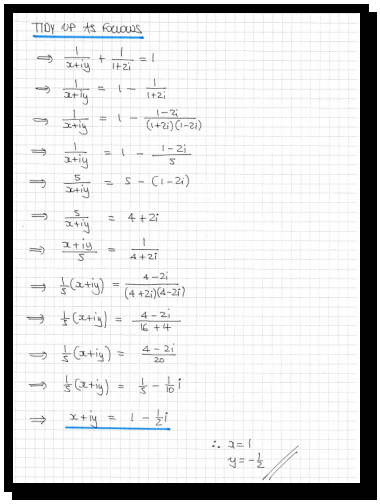


Question 34 (***)

Find the value of x and the value of y in the following equation, given that $x, y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}+\frac{1}{1+2\mathrm{i}}=1.$$

$\square$, $\boxed{(x,y)=\left(1,-\frac{1}{2}\right)}$



Question 35 (***)

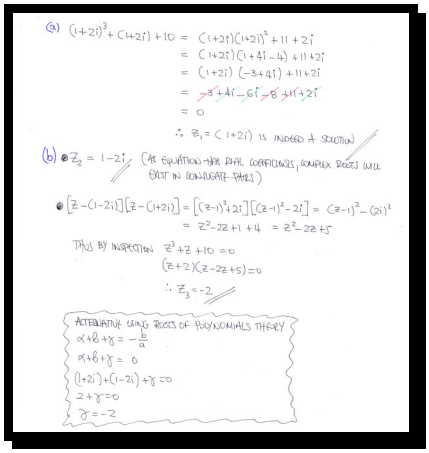
Consider the cubic equation

$$z^3+z+10=0, \quad z\in\mathbb{C}.$$

a) Verify that $1+2\mathrm{i}$ is a root of this equation.

b) Find the other two roots.

$$\boxed{z_1=1-2\mathrm{i}, \quad z_2=-2}$$



Question 36 (***)

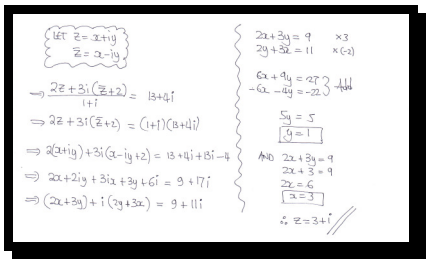
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2z+3\mathrm{i}\left(\overline{z}+2\right)}{1+\mathrm{i}}=13+4\mathrm{i},$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=3+\mathrm{i}}$$



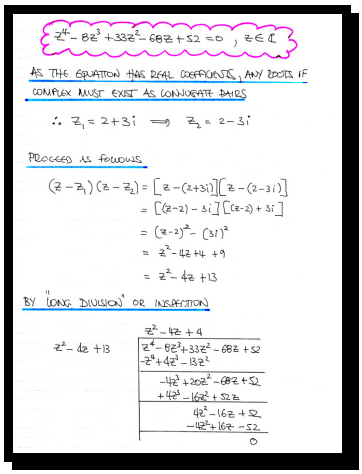
Question 37 **(***)**

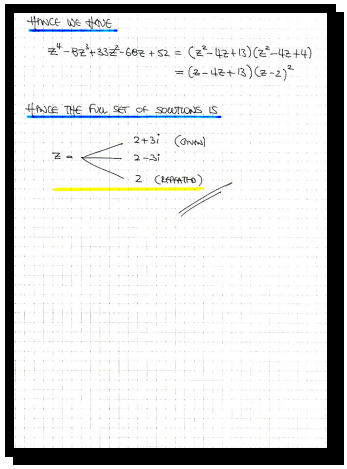
$$z^4-8z^3+33z^2-68z+52=0,\ z\in\mathbb{C}.$$

One of the roots of the above quartic equation is $2+3\mathrm{i}$.

Find the other roots of the equation.

$\boxed{\phantom{z=2-3\mathrm{i},\ z=2}}$, $\boxed{z=2-3\mathrm{i},\ z=2}$



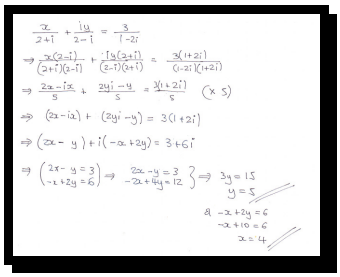


Question 38 **(***)**

Find the values of x and y in the equation

$$\frac{x}{2+\mathrm{i}}+\frac{\mathrm{i}y}{2-\mathrm{i}}=\frac{3}{1-2\mathrm{i}},\ x\in\mathbb{R},\ y\in\mathbb{R}.$$

$\boxed{x=4}$, $\boxed{y=5}$



Question 39 (***)

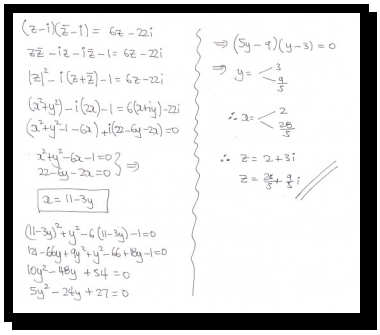
The complex conjugate of z is denoted by $\overline{z}$.

Find the two solutions of the equation

$$(z-\mathrm{i})(\overline{z}-\mathrm{i})=6z-22\mathrm{i}\,,\ z\in\mathbb{C}\,,$$

giving the answers in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z_1=2+3\mathrm{i}}\,,\ \boxed{z_2=\frac{28}{5}+\frac{9}{5}\mathrm{i}}$$

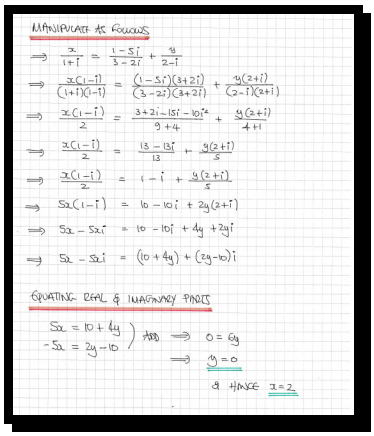


Question 40 (***)

Find the value of x and the value of y in the following equation, given further that $x\in\mathbb{R}$, $y\in\mathbb{R}$.

$$\frac{x}{1+\mathrm{i}}=\frac{1-5\mathrm{i}}{3-2\mathrm{i}}+\frac{y}{2-\mathrm{i}}.$$

$\square$, $\boxed{(x,y)=(2,0)}$



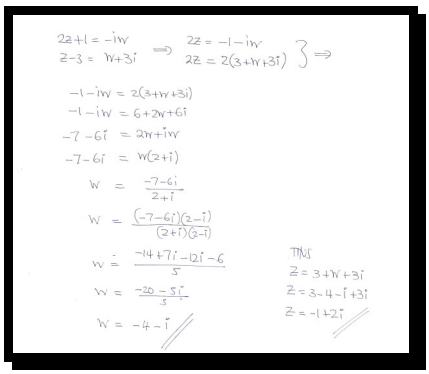
Question 41 (***)

Find the value of z and the value of w in the following simultaneous equations

$$2z+1=-\mathrm{i}w$$

$$z-3=w+3\mathrm{i}\,.$$

$$\boxed{z=-1+2\mathrm{i},\ w=-4-\mathrm{i}}$$



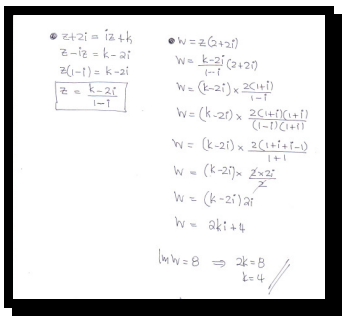
Question 42 (***)

It is given that

$$z+2\mathrm{i}=\mathrm{i}z+k\,,\ k\in\mathbb{R}\qquad\text{and}\qquad\frac{w}{z}=2+2\mathrm{i}\,,\ \operatorname{Im}w=8\,.$$

Determine the value of k.

$$\boxed{k=4}$$



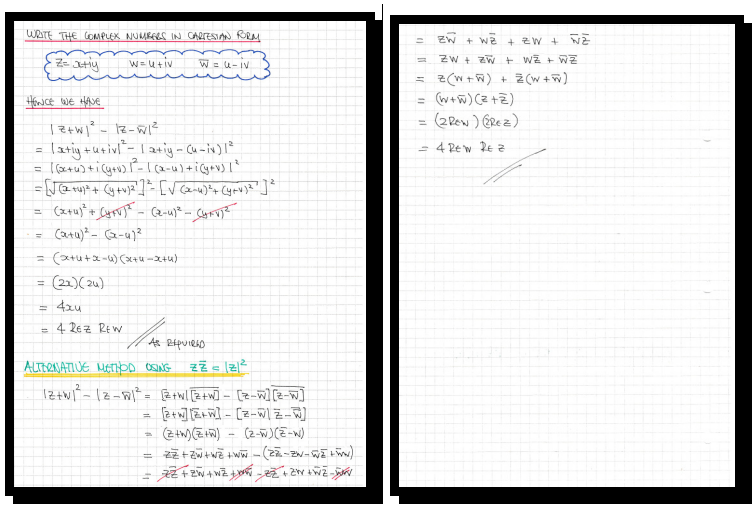
Question 43 (*+)**

Given that z and w are complex numbers prove that

$$|z+w|^2-|z-\overline{w}|^2=4\,\mathrm{Re}\,z\,\mathrm{Re}\,w\,,$$

where $\overline{w}$ denotes the complex conjugate of w.

[ ], [proof]



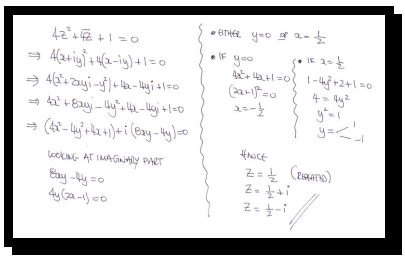
Question 44 (*+)**

Find the three solutions of the equation

$$4z^2+4\overline{z}+1=0\,,\ z\in\mathbb{C}\,,$$

where $\overline{z}$ denotes the complex conjugate of z.

$$\boxed{z=\frac{1}{2},\ \frac{1}{2}+\mathrm{i},\ \frac{1}{2}-\mathrm{i}}$$



Question 45 (*+)**

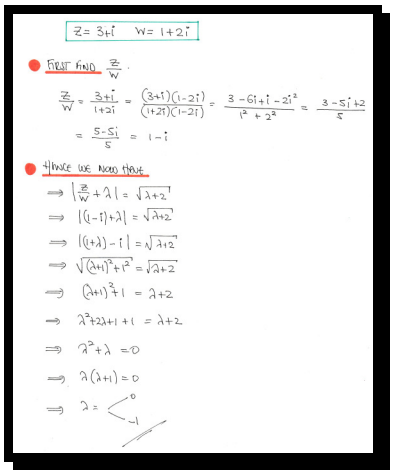
The complex numbers z and w are defined as

$$z=3+\mathrm{i}\quad\text{and}\quad w=1+2\mathrm{i}\,.$$

Determine the possible values of the real constant λ if

$$\left|\frac{z}{w}+\lambda\right|=\sqrt{\lambda+2}\,.$$

$$\boxed{},\ \boxed{\lambda=0,\ \ \lambda=-1}$$



Question 46 (*+)**

The complex number z satisfies the equation

$$z^2 = 3 + 4\mathrm{i}\,.$$

a) Find the possible values of …

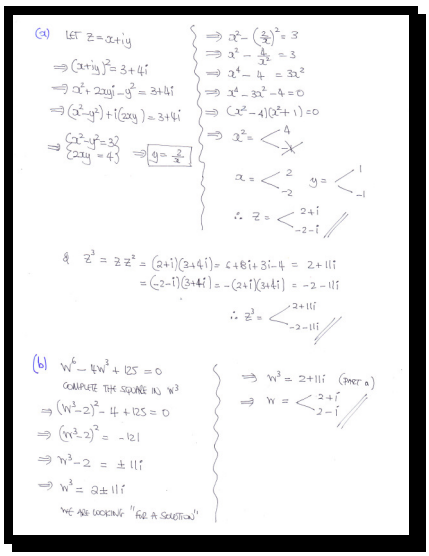
i. … z.

ii. … z^3.

b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0\,,\ w \in \mathbb{C}\,,$$

$$\boxed{z = \pm(2+\mathrm{i})}\,,\ \boxed{z^3 = 2 \pm 11\mathrm{i}}\,,\ \boxed{w = \pm(2+\mathrm{i})}$$



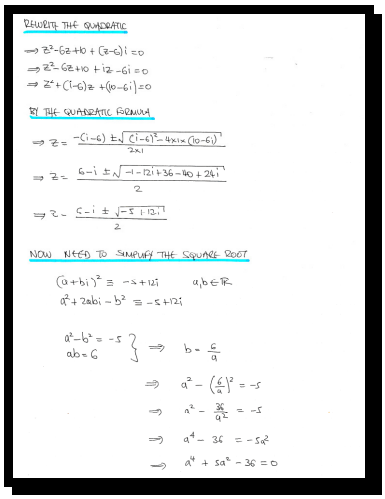
Question 47 **(***+)**

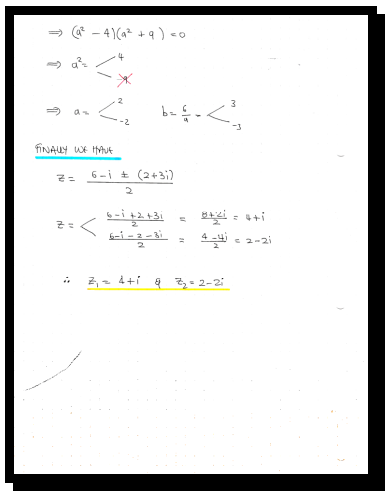
Solve the following quadratic equation

$$z^2-6z+10+(z-6)\mathrm{i}=0, \quad z\in\mathbb{C}.$$

Give the answers in the form $a+b\mathrm{i}$, $a\in\mathbb{R}$, $b\in\mathbb{R}$.

$\boxed{}$, $\boxed{z_1=4+\mathrm{i}}$, $\boxed{z_2=2-2\mathrm{i}}$





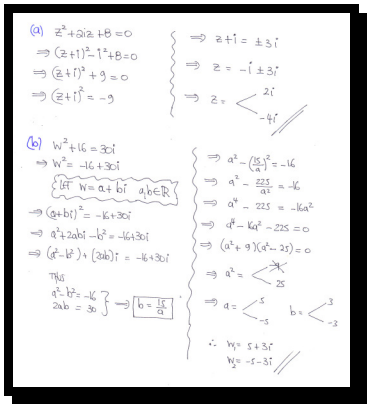
Question 48 (*+)**

Solve teach of the following equations.

a) $z^2+2\mathrm{i}z+8=0$, $z\in\mathbb{C}$.

b) $w^2+16=30\mathrm{i}$, $w\in\mathbb{C}$.

$\boxed{z_1=2\mathrm{i},\ z_2=-4\mathrm{i}}$, $\boxed{w=\pm(3+5\mathrm{i})}$



Question 49 (*+)**

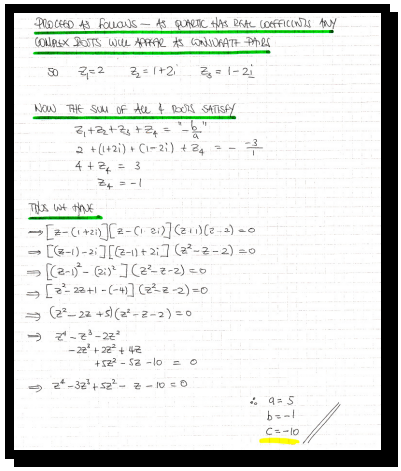
It is given that $z=2$ and $z=1+2\mathrm{i}$ are solutions of the equation

$$z^4-3z^3+az^2+bz+c=0\,.$$

where a, b and c are real constants.

Determine the values of a, b and c.

$\boxed{}$, $\boxed{a=5}$, $\boxed{b=-1}$, $\boxed{c=-10}$



Question 50 (*+)**

The following complex numbers are given

$$z=\frac{1+\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad w=\frac{\sqrt{2}}{1-\mathrm{i}}.$$

a) Calculate the modulus of z and the modulus of w.

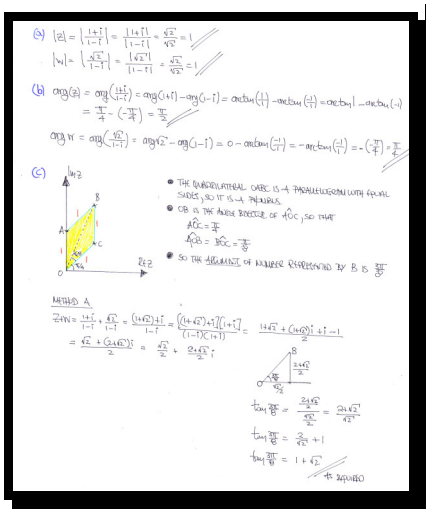
b) Find the argument of z and the argument of w.

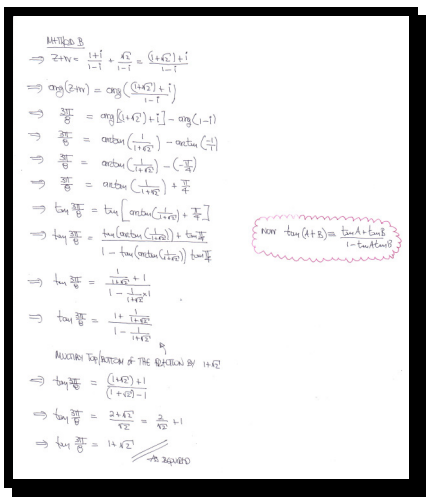
In a standard Argand diagram, the points A, B and C represent the numbers z, $z+w$ and w respectively. The origin of the Argand diagram is denoted by O.

c) By considering the quadrilateral $OABC$ and the argument of $z+w$, show that

$$\tan\left(\frac{3\pi}{8}\right)=1+\sqrt{2}\,.$$

$\boxed{|z|=1}$, $\boxed{|w|=1}$ $\boxed{\arg z=\frac{\pi}{2}}$, $\boxed{\arg w=\frac{\pi}{4}}$





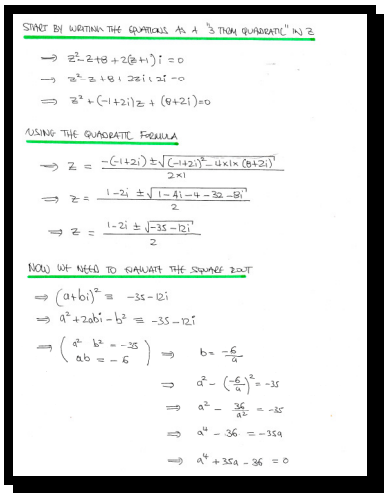
Question 51 (*+)**

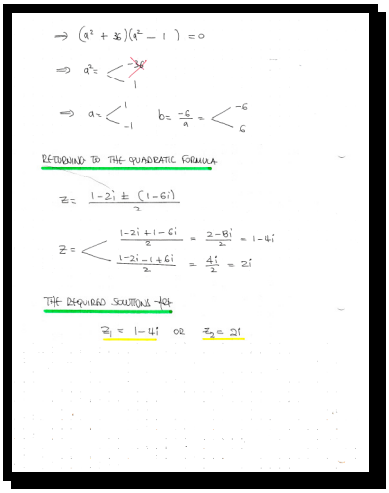
Solve the following quadratic equation

$$z^2 - z + 8 + 2(z+1)\mathrm{i} = 0, \quad z \in \mathbb{C}.$$

Give the answers in the form $a+b\mathrm{i}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$.

$\boxed{}$, $\boxed{z_1 = 2\mathrm{i}}$, $\boxed{z_2 = 1-4\mathrm{i}}$





Question 52 (*+)**

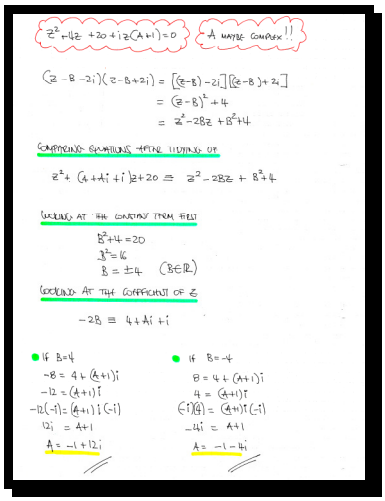
The quadratic equation

$$z^2+4z+20+\mathrm{i}\,z(A+1)=0\ ,$$

where A is a constant, has complex conjugate roots.

If one of the roots of this quadratic is $z=B+2\mathrm{i}$, where B is a **real** constant, find the possible values of A.

$\boxed{}$, $\boxed{A=-1+12\mathrm{i}\ \cup\ A=-1-4\mathrm{i}}$



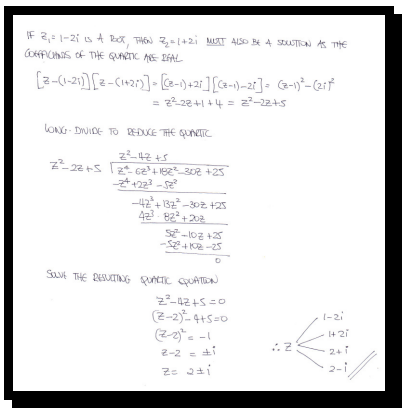
Question 53 (*+)**

If $1-2\mathrm{i}$ is a root of the quartic equation

$$z^4-6z^3+18z^2-30z+25=0$$

find the other three roots.

$$\boxed{z_2=1+2\mathrm{i},\quad z_3=2-\mathrm{i},\quad z_4=2+\mathrm{i}}$$



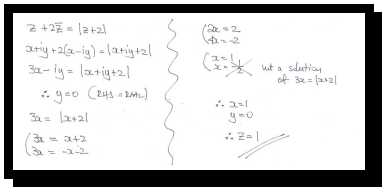
Question 54 (****)

The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z+2\overline{z}=|z+2|,\ z\in\mathbb{C}.$$

$$\boxed{z=1}$$



Question 55 (****)

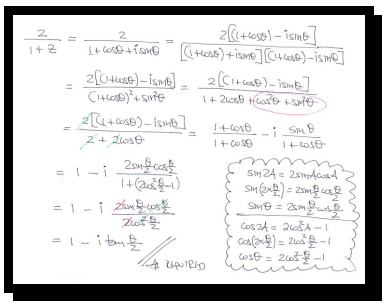
It is given that

$$z=\cos\theta+\mathrm{i}\sin\theta,\ 0\le z<2\pi .$$

Show clearly that

$$\frac{2}{1+z}=1-\mathrm{i}\tan\left(\frac{\theta}{2}\right).$$

proof



Question 56 **(****)**

$$\frac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+3\mathrm{i}}=q(1+\mathrm{i})\ ,\quad q\in\mathbb{R}\ .$$

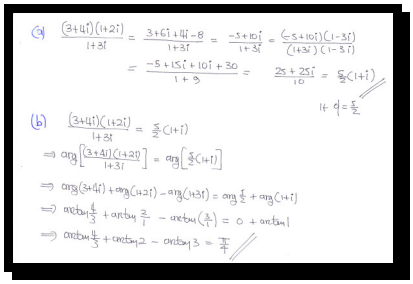
a) Find the value of q.

b) Hence simplify

$$\arctan\frac{4}{3}+\arctan 2-\arctan 3\,,$$

giving the answer in terms of π.

$$\boxed{q=\frac{5}{2}}\ ,\quad \boxed{\frac{1}{4}\pi}$$



Question 57 (****)

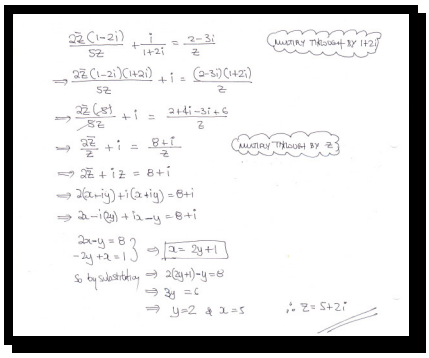
The complex conjugate of the complex number z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2\overline{z}\left(1-2\mathrm{i}\right)}{5\mathrm{z}}+\frac{\mathrm{i}}{1+2\mathrm{i}}=\frac{2-3\mathrm{i}}{\mathrm{z}},$$

giving the answer in the form $x+\mathrm{i}y$.

$$\boxed{z=5+2\mathrm{i}}$$



Question 58 (****)

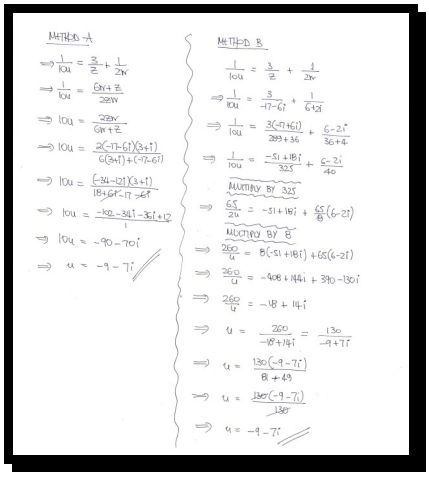
It is given that

$$z=-17-6\mathrm{i} \qquad \text{and} \qquad w=3+\mathrm{i}\,.$$

Find the value of u given further that

$$\frac{1}{10u}=\frac{3}{z}+\frac{1}{2w}\,.$$

$$\boxed{u=-9-7\mathrm{i}}$$



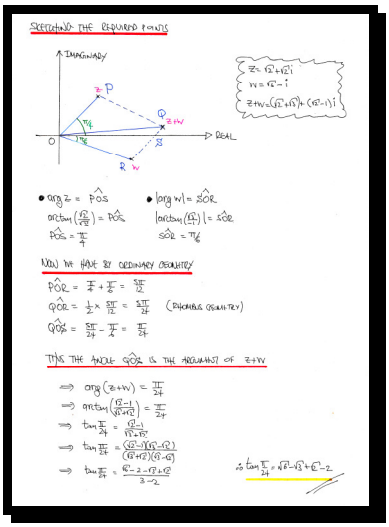
Question 59 (****)

Sketch on a standard Argand diagram the locus of the points $z=\sqrt{2}\left(1+\mathrm{i}\right)$, $w=\sqrt{3}-\mathrm{i}$ and $z+w$, and use geometry to prove that

$$\tan\left(\frac{\pi}{24}\right)=\sqrt{6}-\sqrt{3}+\sqrt{2}-2\,.$$

You must justify all the steps in this proof.

$\boxed{}$, $\boxed{\text{proof}}$



Question 60 (****)

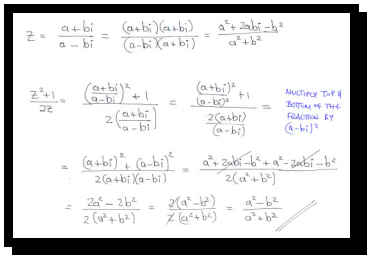
The complex number z is given by

$$z=\frac{a+b\mathrm{i}}{a-b\mathrm{i}},\ a\in\mathbb{R},\ b\in\mathbb{R}.$$

Show clearly that

$$\frac{z^2+1}{2z}=\frac{a^2-b^2}{a^2+b^2}.$$

proof



Question 61 (****)

It is given that

$$z=\frac{1+8\mathrm{i}}{1-2\mathrm{i}} \ .$$

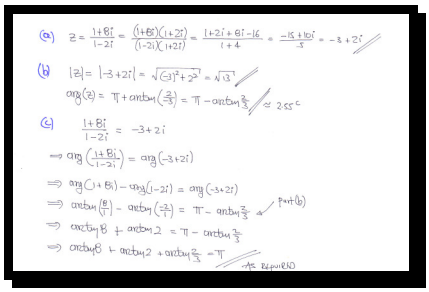
a) Express z in the form $x+\mathrm{i}y$, where x and y are real numbers.

b) Find the modulus and argument of z.

c) Show clearly that

$$\arctan 8+\arctan 2+\arctan\frac{2}{3}=\pi \ .$$

$\boxed{z=-3+2\mathrm{i}}$, $\boxed{|z|=\sqrt{13}}$, $\boxed{\arg z\approx 2.55^{\mathrm{c}}}$



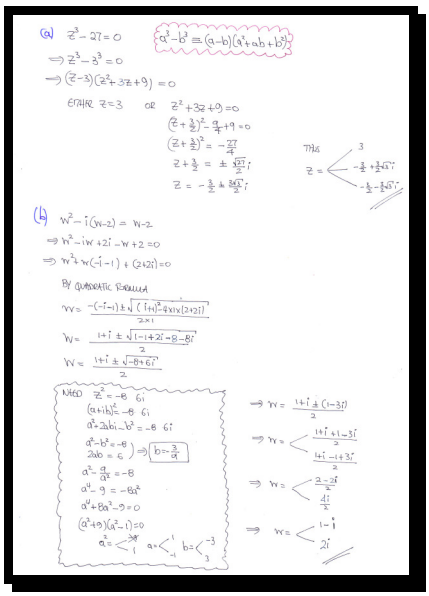
Question 62 (**+)**

Solve each of the following equations.

a) $z^3-27=0$.

b) $w^2-\mathrm{i}(w-2)=(w-2)$.

$$\boxed{z_1=3,\quad z_2=\frac{3}{2}\left(-1\pm\sqrt{3}\right)},\ \boxed{w_1=2\mathrm{i},\quad w_2=1-\mathrm{i}}$$

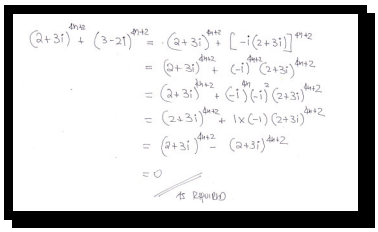


Question 63 **(****+)**

$$z=(2+3\mathrm{i})^{4n+2}+(3-2\mathrm{i})^{4n+2},\ n\in\mathbb{N}.$$

Show clearly that $z=0$ for all $n\in\mathbb{N}$.

proof



Question 64 **(****+)**

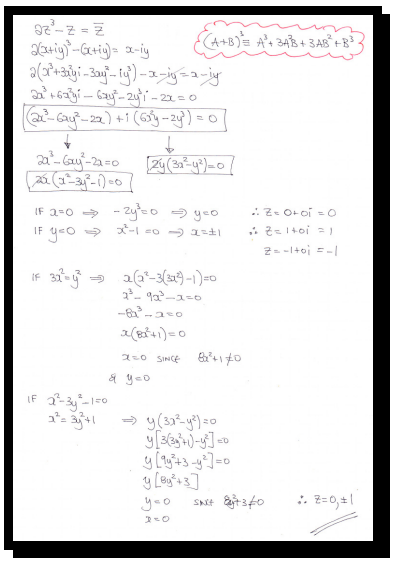
The complex conjugate of z is denoted by $\overline{z}$.

Show clearly that the equation

$$2z^3-z=\overline{z},$$

is satisfied either by $z=0$ or $z=\pm 1$.

proof

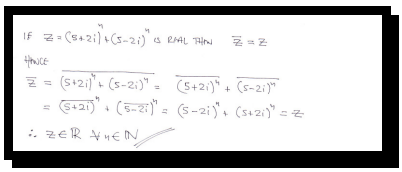


Question 65 (**+)**

$$z=(5+2\mathrm{i})^n+(5-2\mathrm{i})^n,\ n\in\mathbb{N}.$$

Show clearly that z is a real number.

proof



Question 66 (**+)**

The complex number z satisfies the relationship

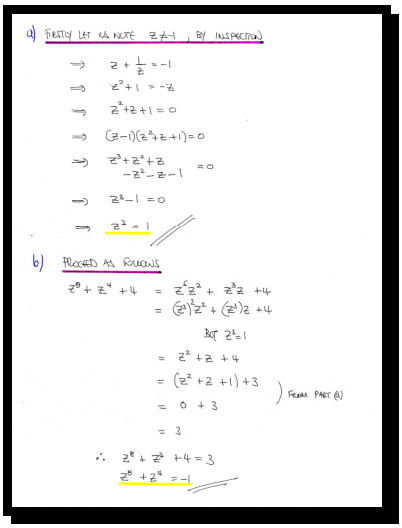
$$z+\frac{1}{z}=-1\,,\ z\neq 0\,.$$

Show clearly that ….

a) … $z^3=1$.

b) … $z^8+z^4=-1$.

[ ] , proof

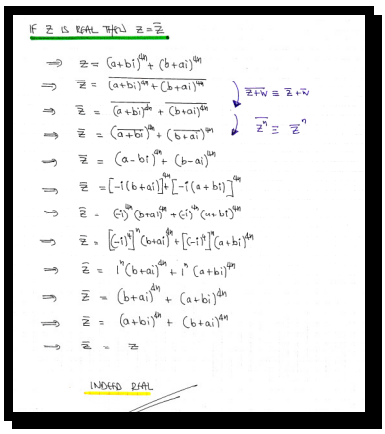


Question 67 (**+)**

$$z=(a+bi)^{4n}+(b+ai)^{4n},\ a\in\mathbb{R},\ b\in\mathbb{R},\ n\in\mathbb{N}.$$

Show that z is a real number.

[ ], [proof]

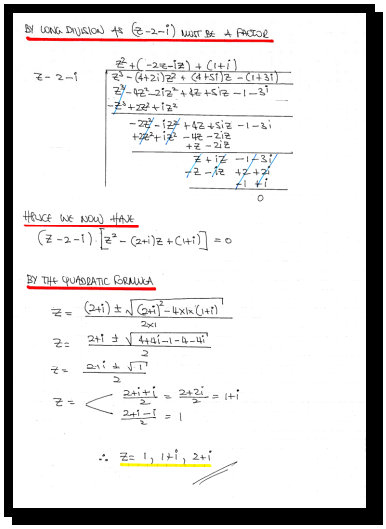


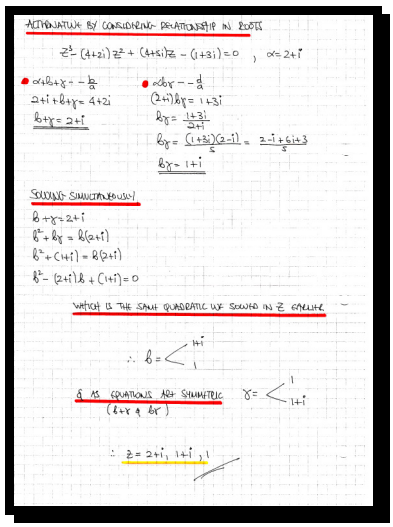
Question 68 (**+)**

$$z^3-(4+2\mathrm{i})z^2+(4+5\mathrm{i})z-(1+3\mathrm{i})=0\,,\ z\in\mathbb{C}\,.$$

Given that one of the solutions of the above cubic equation is $z=2+\mathrm{i}$, find the other two solutions.

$\boxed{}$, $\boxed{z=1}$, $\boxed{z=1+\mathrm{i}}$





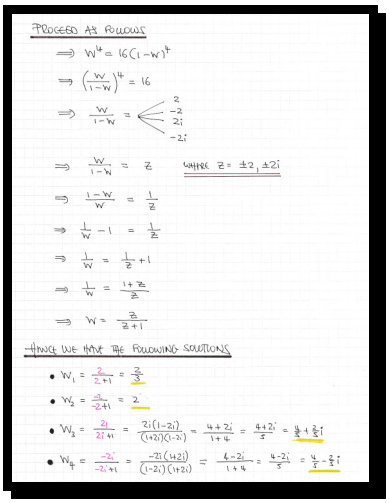
Question 69 **(*****)**

Find the solutions of the equation

$$w^4 = 16(1-w)^4,$$

giving the answers in the form $x+\mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$\square$, $\boxed{z_1 = 2, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{4}{5}+\mathrm{i}\frac{2}{5}, \quad z_4 = \frac{4}{5}-\mathrm{i}\frac{2}{5}}$

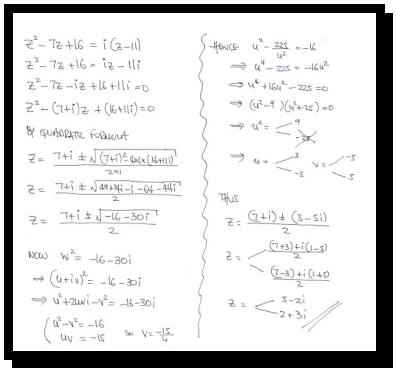


Question 70 **(*****)**

Solve the quadratic equation

$$z^2 - 7z + 16 = \mathrm{i}(z-11), \ z \in \mathbb{C}.$$

$\square$, $\boxed{z = 2+3\mathrm{i}, \ z = 5-2\mathrm{i}}$

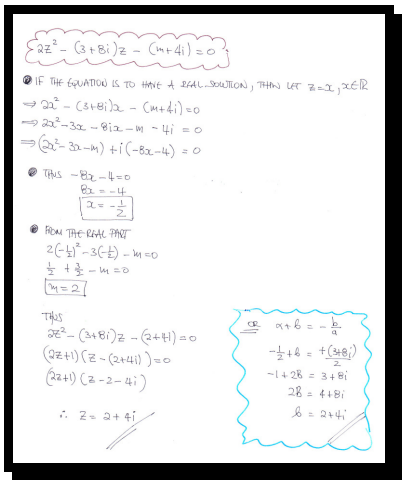


Question 71 (*****)

$$2z^2-(3+8\mathrm{i})z-(m+4\mathrm{i})=0,\ z\in\mathbb{C}.$$

Given that m is a real constant, find the two solutions of the above equation given further that one of these solutions is real.

$\boxed{}$, $\boxed{z=\frac{1}{2}}$, $\boxed{z=2+4\mathrm{i}}$

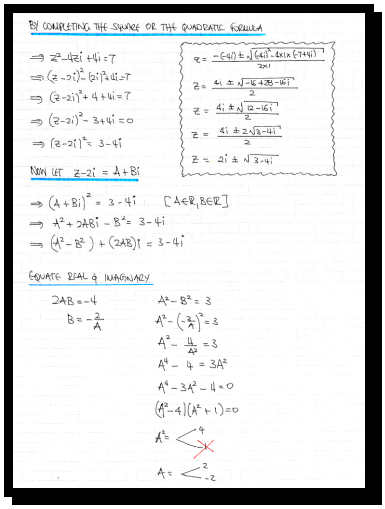


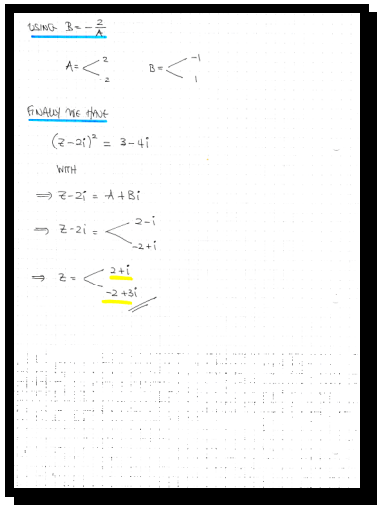
Question 72 (*****)

Solve the quadratic equation

$$z^2-4z\mathrm{i}+4\mathrm{i}=7\,,\ z\in\mathbb{C}\,.$$

$\square$, $\boxed{z=-2+3\mathrm{i},\ z=2+\mathrm{i}}$



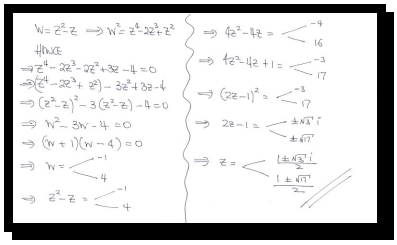


Question 73 (*****)

$$z^4-2z^3-2z^2+3z-4=0\,,\ z\in\mathbb{C}\,.$$

By using the substitution $w=z^2-z$, or otherwise, find in exact form the four solutions of the above equation.

$\square$, $\boxed{z=\dfrac{1\pm\sqrt{17}}{2},\ \dfrac{1\pm\mathrm{i}\sqrt{3}}{2}}$



Question 74 (*****)

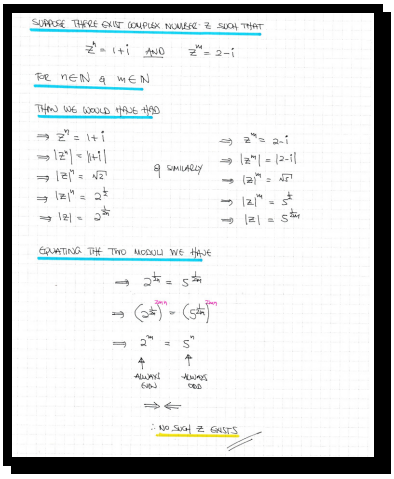
Show that if n and m are natural numbers, then the equations

$$z^n = 1+\mathrm{i}$$

$$z^m = 2-\mathrm{i},$$

have no common solution for $z \in \mathbb{C}$.

, proof

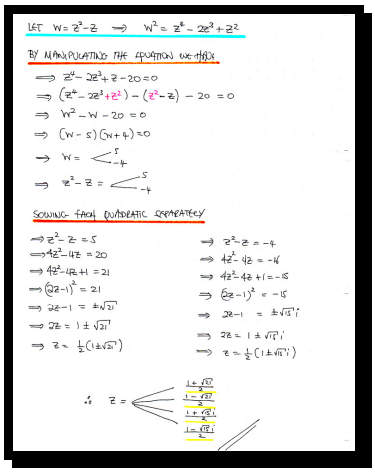


Question 75 (*****)

$$z^4-2z^3+z-20=0\,,\ z\in\mathbb{C}\,.$$

By using the substitution $w=z^2-z$, or otherwise, find in exact form the four solutions of the above equation.

$\square$, $\boxed{z=\dfrac{1\pm\sqrt{21}}{2},\ \dfrac{1\pm i\sqrt{15}}{2}}$



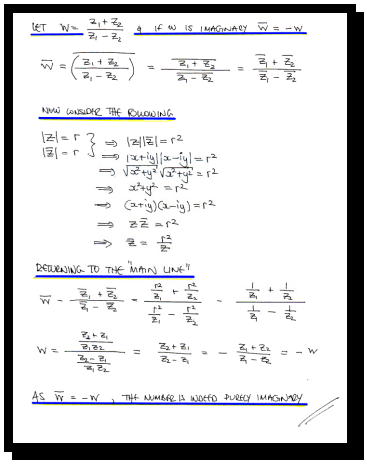
Question 76 (*****)

Two distinct complex numbers z_1 and z_2 are such so that $|z_1| = |z_2| = r \neq 0$.

Show clearly that $\dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

You may find the result $z\overline{z} = |z|^2 = r^2$ useful.

[ ], [proof]



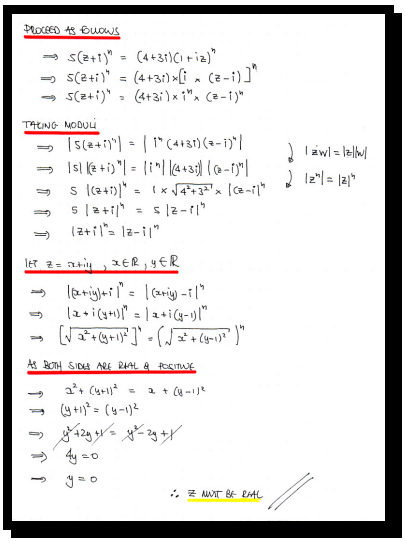
Question 77 (*****)

The complex number z satisfies the relationship

$$5(z+\mathrm{i})^n=(4+3\mathrm{i})(1+\mathrm{i}z)^n,\ n\in\mathbb{R}.$$

Show that z is a real number.

[ ], [proof]

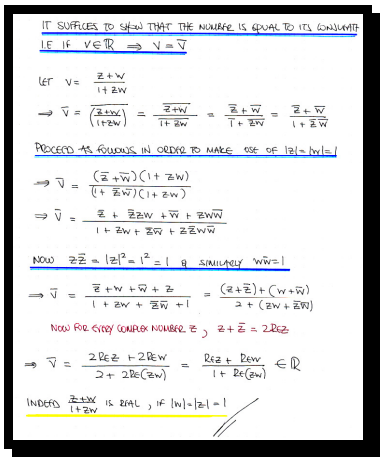


Question 78 (*****)

The complex numbers z and w are such so that $|z|=|w|=1$.

Show clearly that $\dfrac{z+w}{1+zw}$ is real.

[ ], [proof]

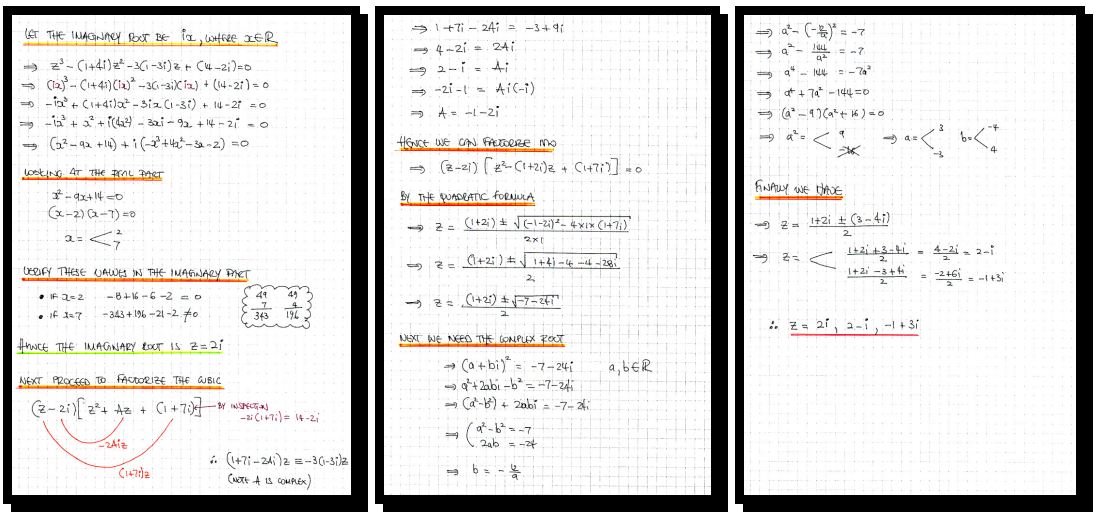


Question 79 **(*****)**

$$z^3-2(2-\mathrm{i})z^2+(8-3\mathrm{i})z-5+\mathrm{i}=0,\quad z\in\mathbb{C}.$$

Find the three solutions of the above equation given that one of these solutions is real.

$\boxed{}$, $\boxed{z=1}$, $\boxed{z=2-3\mathrm{i}}$, $\boxed{z=2+\mathrm{i}}$



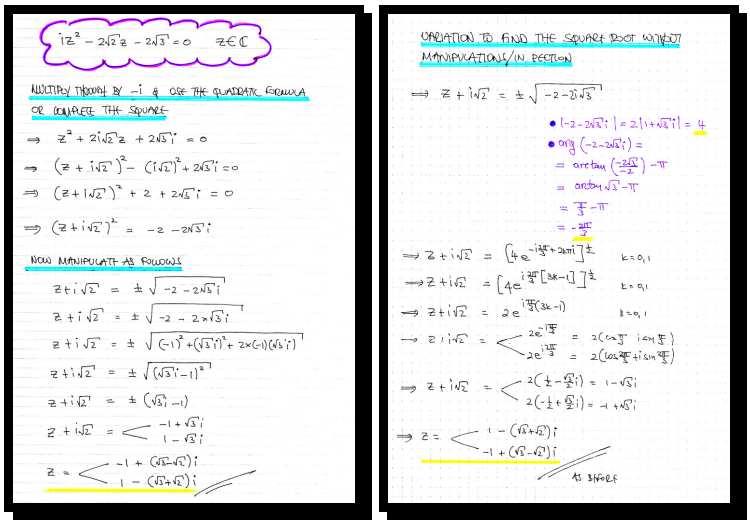
Question 80 (*****)

Solve the quadratic equation

$$\mathrm{i}z^2-2\sqrt{2}\,z-2\sqrt{3}=0\,,\ z\in\mathbb{C}\,.$$

Give the answers in the form $x+\mathrm{i}y$, where x and y are exact real constants.

$\square$, $\boxed{z=-1+\mathrm{i}\left(\sqrt{3}-\sqrt{2}\right),\quad z=1-\mathrm{i}\left(\sqrt{3}+\sqrt{2}\right)}$



Question 81 (*****)

The complex number z satisfies the equation

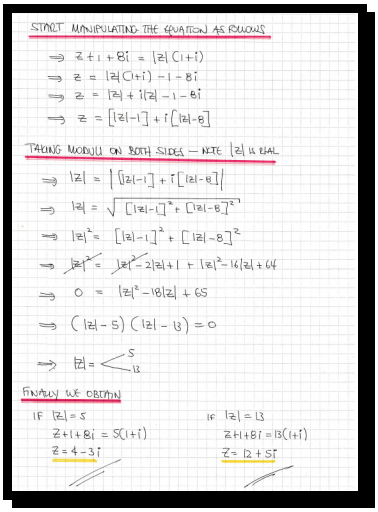
$$z+1+8\mathrm{i}=|z|(1+\mathrm{i}).$$

Show clearly that

$$|z|^2-18|z|+65=0,$$

and hence find the possible values of z.

$\boxed{}$, $\boxed{z=4-3\mathrm{i}}$, $\boxed{z=12+5\mathrm{i}}$

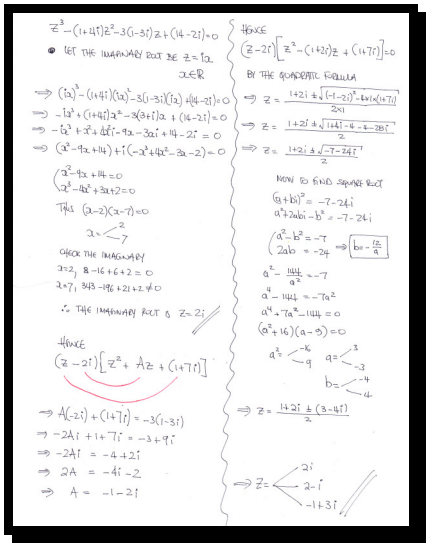


Question 82 (*****)

$$z^3-(2+4\mathrm{i})z^2-3(1-3\mathrm{i})z+14-2\mathrm{i}=0\,,\ z\in\mathbb{C}\,.$$

Find the three solutions of the above equation given that one of these solutions is purely imaginary.

$\boxed{}$, $\boxed{z=2\mathrm{i}}$, $\boxed{z=2-\mathrm{i}}$, $\boxed{z=-1+3\mathrm{i}}$



Question 83 (*****)

It is given that

$$z+w=|z|w,$$

where $z\in\mathbb{C}$, $w\in\mathbb{C}$, and $|w|>1$.

Determine an exact simplified expression for $|z|$, in terms of $|w|$.

$$\boxed{},\quad \boxed{|z|=\frac{\mp|w|}{1\mp|w|}=\frac{\pm|w|}{\pm|w|-1}}$$

