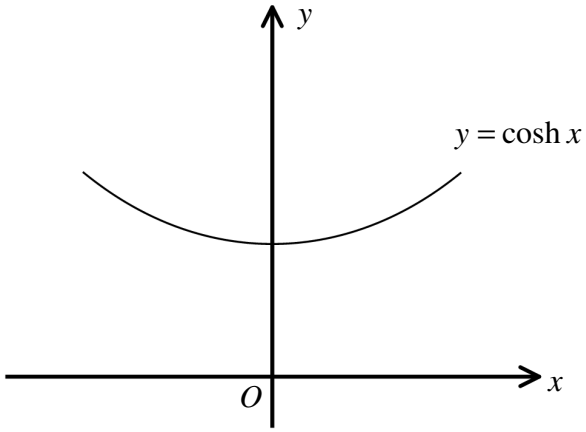
INTEGRATION
ARCLENGTHS
&
SURFACES

ARCLENGTH

Question 1 (**)

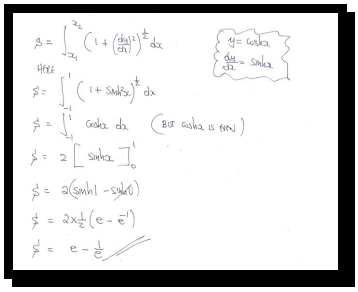


The figure above shows the graph of the curve with equation

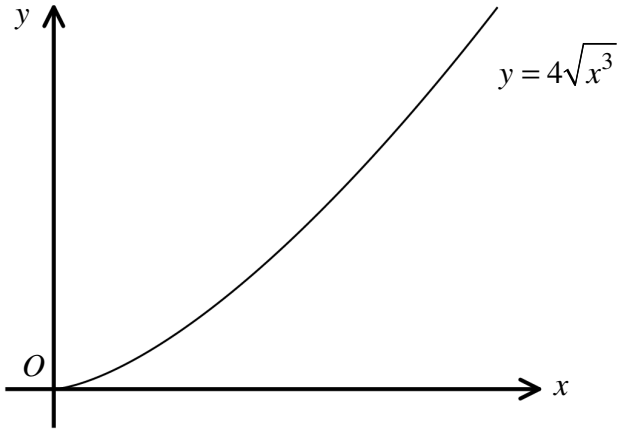
$$y=\cosh x\text{, for } -1\le x\le 1\text{.}$$

Find the length of the curve, in terms of e.

$$\mathrm{e}-\frac{1}{\mathrm{e}}$$



Question 2 (**)

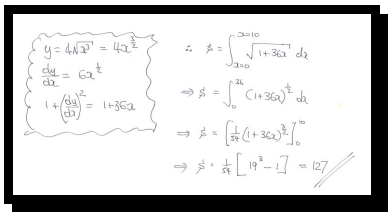


The figure above shows the graph of the curve with equation

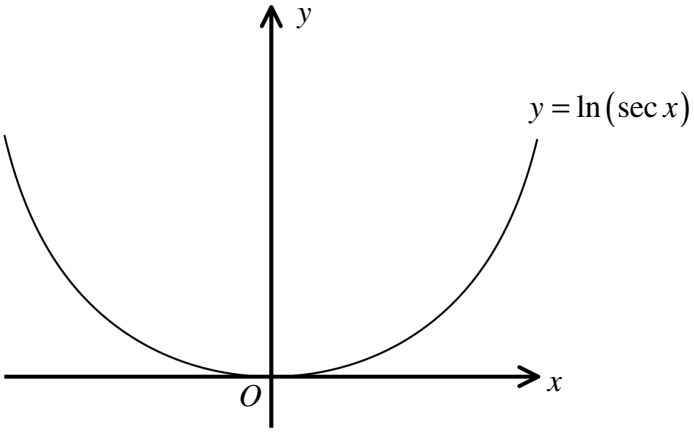
$$y = 4\sqrt{x^3}\ ,\ x \geq 0\ .$$

Find the length of the arc of the curve for $0 \leq x \leq 10$.

127



Question 3 (+)**



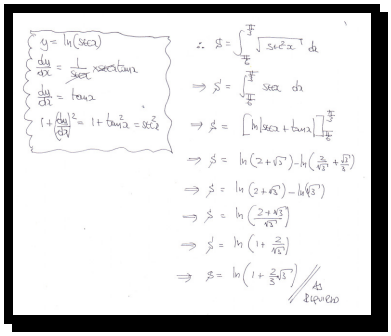
The figure above shows the graph of the curve with equation

$$y=\ln\left(\sec x\right),\ -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

Show that the length of the curve for $\frac{\pi}{6}\le x\le\frac{\pi}{3}$ is

$$\ln\left(1+\frac{2}{3}\sqrt{3}\right).$$

proof



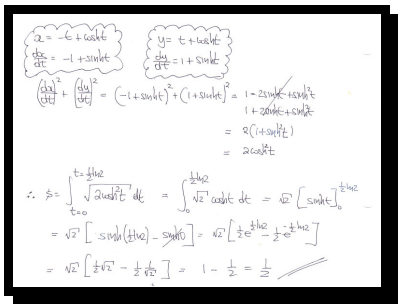
Question 4 (***)

A curve is given parametrically by

$$x=-t+\cosh t\ ,\ \ y=t+\cosh t\ ,\ \ 0\le t\le\frac{1}{2}\ln 2\ .$$

Show that the length the curve is $\frac{1}{2}$.

proof



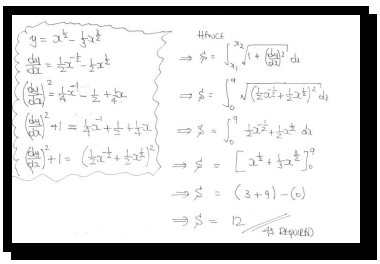
Question 5 (*+)**

A curve C has equation

$$y=x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{3}{2}}\ ,\ \ x\ge 0$$

Show that the length of the arc of C from $A(0,0)$ to $B(9,-6)$ is 12 units.

proof



Question 6 (*+)**

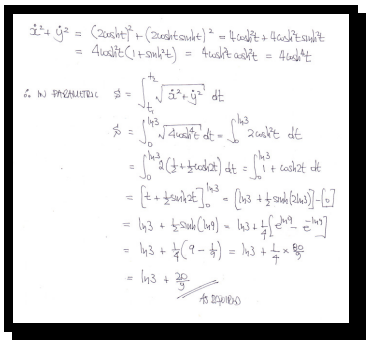
A curve is given parametrically by

$$x = 2\sinh t\,,\ \ y = \cosh^2 t\,,\ \ 0 \le t \le \ln 3\,.$$

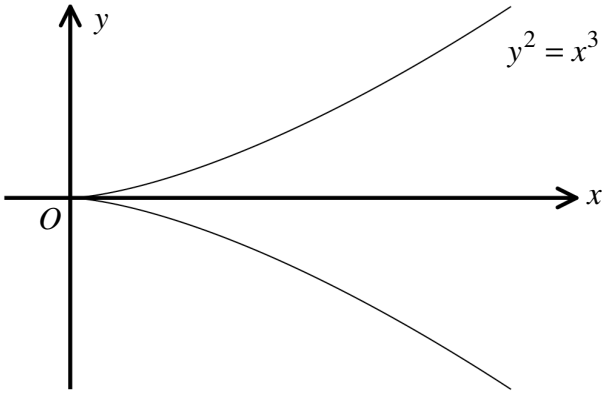
Show that the length the curve is exactly

$$\frac{20}{9} + \ln 3\,.$$

proof



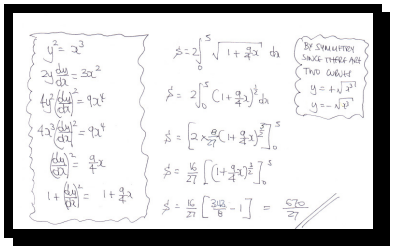
Question 7 (*+)**



A curve C has equation by $y^2 = x^3$ and its graph is shown in the figure above.

Show that the length of the arc of C from $A\left(5,-5\sqrt{5}\right)$ to $B\left(5,5\sqrt{5}\right)$ is exactly $\dfrac{670}{27}$.

proof



Question 8 (****)

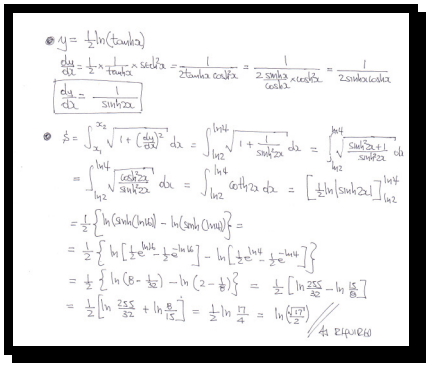
A curve C has equation

$$y=\frac{1}{2}\ln\left(\tanh x\right),\quad x\in\mathbb{R},\ x>0.$$

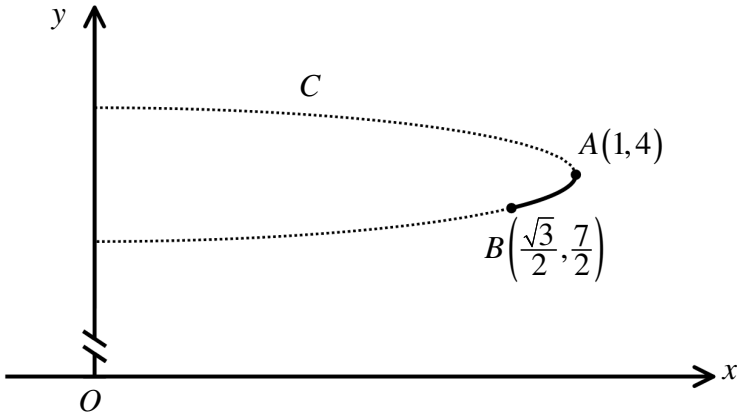
Show that the length of C from the point where $x=\ln 2$ to the point where $x=\ln 4$ is exactly

$$\ln\left(\frac{\sqrt{17}}{4}\right).$$

proof



Question 9 **(****)**



The curve C, shown above, is given parametrically by the equations

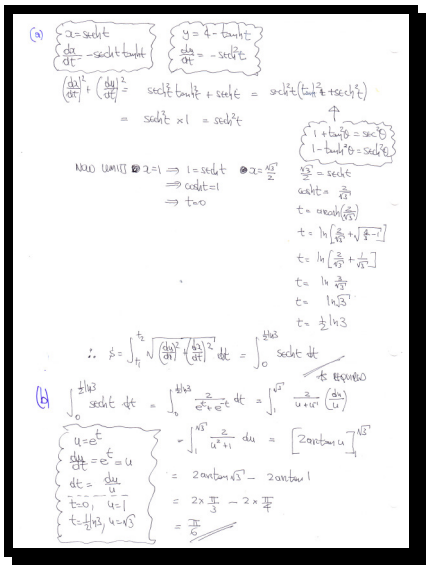
$$x = \operatorname{sech} t, \quad y = 4 - \tanh t, \quad t \in \mathbb{R}$$

a) Show that the length of the arc of C from $A(1,4)$ to $B\left(\frac{\sqrt{3}}{2}, \frac{7}{2}\right)$ is given by

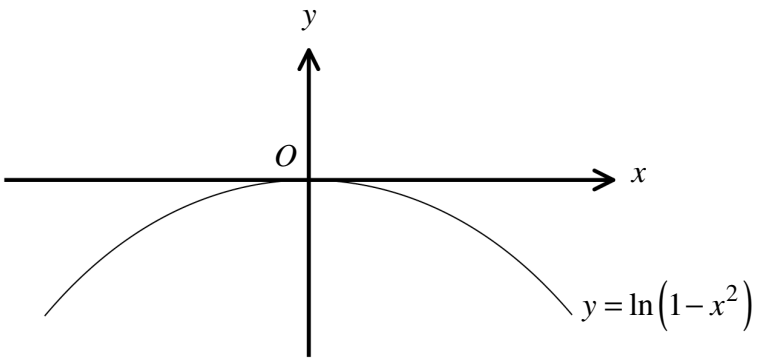
$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \ dx .$$

b) Use the substitution $u = \mathrm{e}^t$, to find the exact value of s.

$$\boxed{\frac{\pi}{6}}$$



Question 10 **(****)**



The figure above shows the graph of the curve with equation

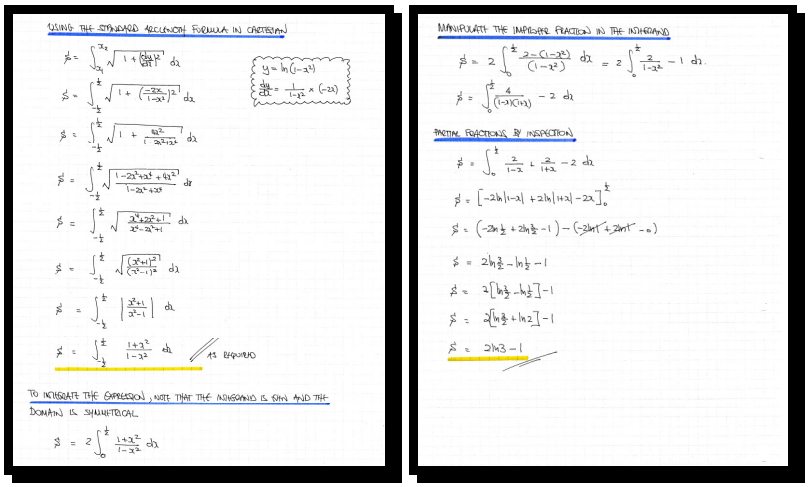
$$y=\ln\left(1-x^2\right),\quad \frac{1}{2}\le x\le\frac{1}{2}.$$

a) Show that the length s of the curve is given by

$$s=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1+x^2}{1-x^2}\,dx.$$

b) Hence find the exact length of the curve.

$\boxed{}$, $\boxed{-1+2\ln 3}$



Question 11 **(****)**

$$I(a,x)\equiv\int\sqrt{x^2+a^2}\,dx,\quad x\in\mathbb{R},\quad a\in\mathbb{R},\quad x>0.$$

a) Use a suitable hyperbolic substitution to show that

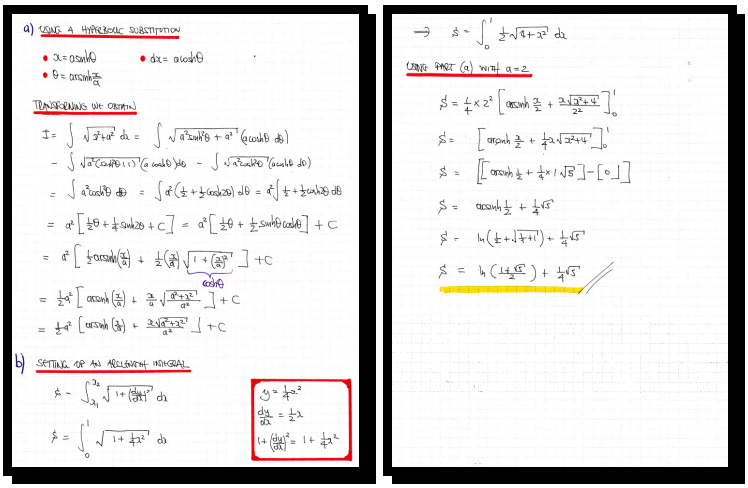
$$I(a,x)=\frac{1}{2}a^2\left[\operatorname{arsinh}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2+a^2}}{a^2}\right]+\text{constant}.$$

b) Hence find in exact form the length of the curve with equation

$$y=\frac{1}{4}x^2,$$

from the origin O to the point with coordinates $\left(1,\frac{1}{4}\right)$.

$\square$, $\boxed{s=\frac{1}{4}\sqrt{5}+\ln\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]}$



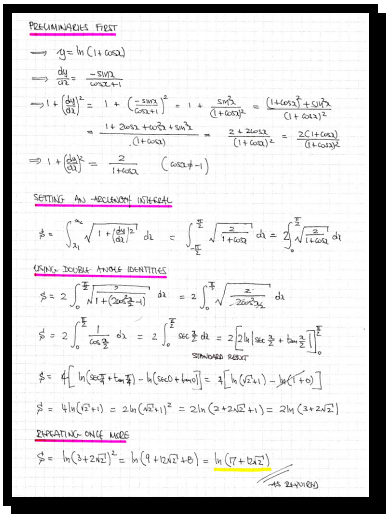
Question 12 (****)

A curve has equation

$$y=\ln\left(1+\cos x\right),\quad x\in\left[-\frac{1}{2}\pi,\frac{1}{2}\pi\right]$$

Show that the length this curve is $\ln\left(17+12\sqrt{2}\right)$ units.

[ ], [proof]



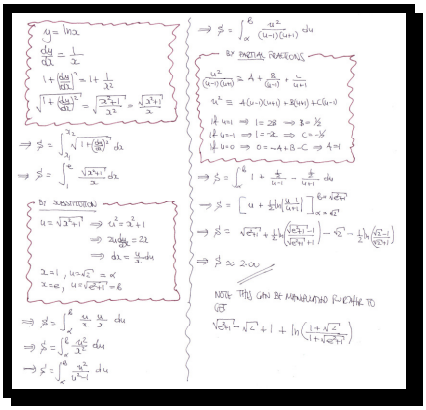
Question 13 (**+)**

Find an exact value for the length of the curve with equation

$$y=\ln x\,,\quad 1\le x\le \mathrm{e}\,.$$

$$\sqrt{\mathrm{e}^2+1}-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{\mathrm{e}^2+1}-1}{\sqrt{\mathrm{e}^2+1}+1}\right)-\frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\approx 2.00$$

$$\sqrt{\mathrm{e}^2+1}-\sqrt{2}+1+\ln\left(\frac{1+\sqrt{2}}{1+\sqrt{\mathrm{e}^2+1}}\right)\approx 2.00$$



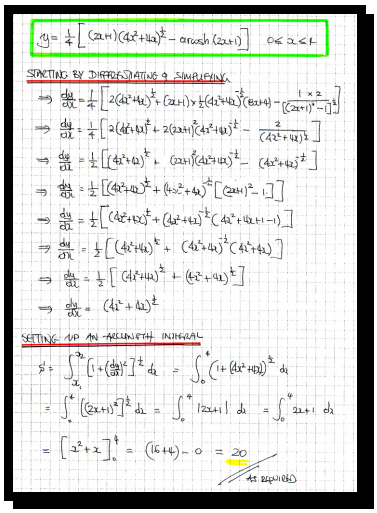
Question 14 (**+)**

A curve has equation

$$y=\frac{1}{4}\left[(2x+1)\sqrt{4x^2+4x}-\operatorname{arcosh}(2x+1)\right],\quad 0\le x\le 4.$$

Show that the length of the curve is 20 units.

$\boxed{}$, proof



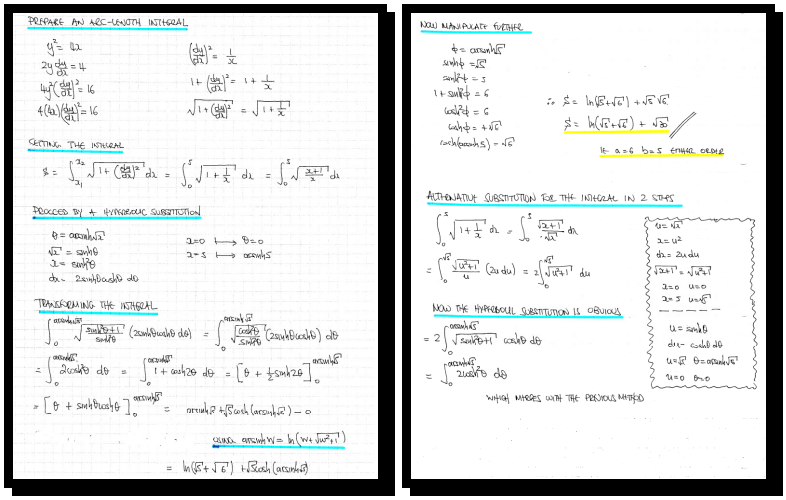
Question 15 (*****)

A parabola has equation

$$y^2 = 4x, \quad 0 \le x \le 5$$

Show that the length this parabola is exactly $\ln\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{ab}$ where a and b are positive integers.

$\boxed{}$, $\boxed{(a,b)=(6,5)=(5,6)}$



SURFACES OF REVOLUTION

Question 1 (***)

The part of the curve with equation

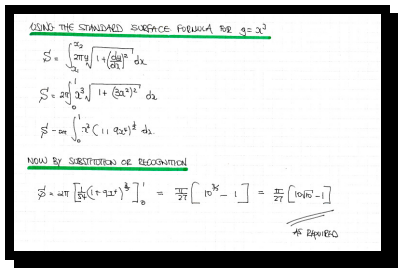
$$y = x^3, \quad 0 \le x \le 1$$

is rotated through 2π radians about the x axis.

Show that the area surface generated is

$$\frac{\pi}{27}\left[10\sqrt{10}-1\right].$$

[ ], proof



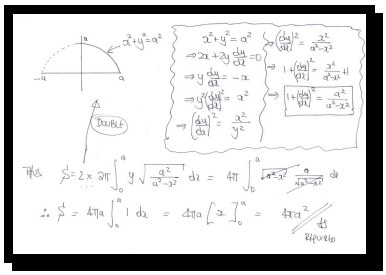
Question 2 (***)

By considering the top half of the circle with equation

$$x^2 + y^2 = a^2, \quad y \ge 0$$

show that the surface generated when the circle's top half is rotated through 2π radians about the x axis has an area of $4\pi a^2$ square units.

proof



Question 3 (***)

A parabola has equation

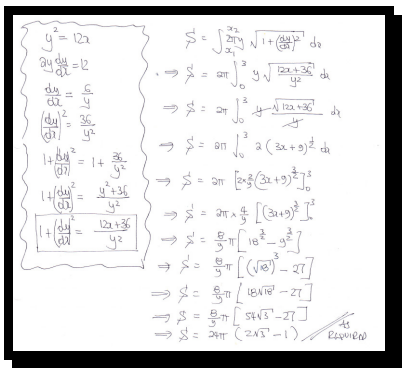
$$y^2 = 12x\,,\ x \geq 0\,.$$

The arc of the parabola from the point $A(0,0)$ to the point $B(3,6)$ is rotated through 2π radians about the x axis, to form a solid of revolution.

Show clearly that the area of the curved surface of the solid produced is exactly

$$24\pi\left(2\sqrt{2}-1\right).$$

proof



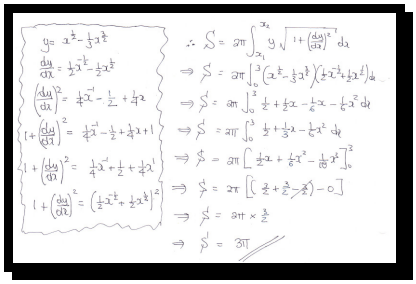
Question 4 (*+)**

A curve C has equation given by

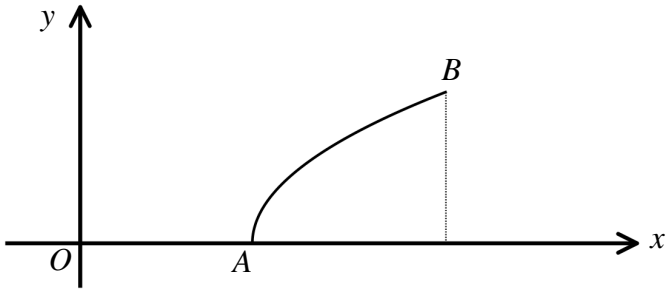
$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}, \quad x \geq 0 .$$

Show that the area of the surface generated when the arc of C for which $0 \leq x \leq 3$ is rotated through 2π radians about the x axis is 3π square units.

proof



Question 5 (****)



The figure above shows the curve C, given parametrically by the equations

$$x=\frac{1}{2}\cosh 2t\,,\quad y=2\sinh t\,,\quad t\in\mathbb{R}\,.$$

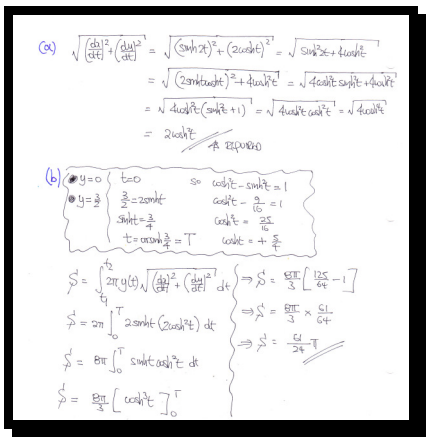
a) Show that

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=2\cosh^2 t$$

The arc of C from the point $A\left(\frac{1}{2},0\right)$ to the point $B\left(\frac{17}{16},\frac{3}{2}\right)$ is rotated through 2π radians about the x axis.

b) Show that the area of the surface generated is $\frac{61}{24}\pi$ square units.

proof



Question 6 (****)

The curve C has parametric equations

$$x=\cos\theta\,,\quad y=\ln\left(\sec\theta+\tan\theta\right)-\sin\theta\,,\ 0\le\theta\le\frac{\pi}{3}$$

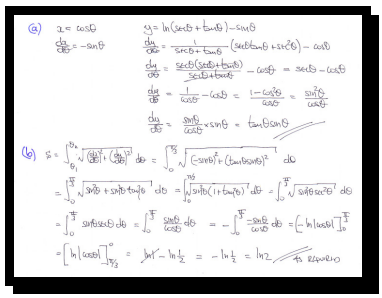
a) Show that

$$\frac{dy}{d\theta}=f\left(\theta\right)g\left(\theta\right),$$

where $f\left(\theta\right)$ and $g\left(\theta\right)$ are simple trigonometric functions.

b) Hence show that the length of C is $\ln 2$.

$$\boxed{\frac{dy}{d\theta}=\sin\theta\tan\theta}$$



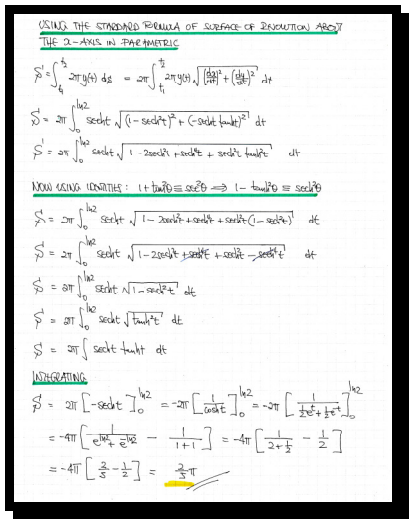
Question 7 **(****)**

A curve has parametric equations

$$x = t - \tanh t \,, \qquad y = \operatorname{sech} t \,, \qquad 0 \le t < \ln 2 \,.$$

Determine, in exact simplified form, the area of the surface of a complete revolution of the curve, about the x axis.

V, [ ], $\boxed{\frac{2}{5}\pi}$



Question 8 (**+)**

The curve C has equation given by

$$y^2 = x^2 + 32\,,\ x \in \mathbb{R}\,,\ 0 \le x \le 4$$

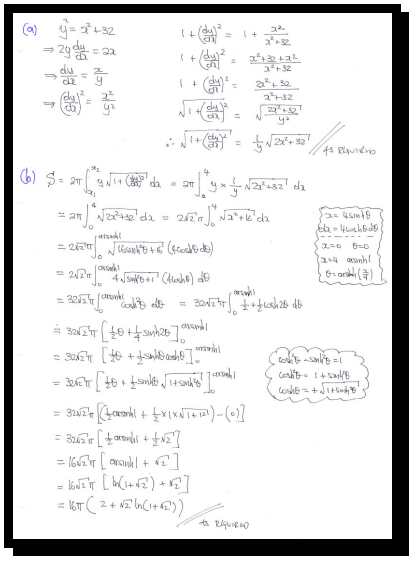
a) Show that

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{2x^2+32}}{y}\,.$$

b) Hence show further that the area of the surface generated when C is rotated by 2π radians in the x axis is given by

$$16\pi\left[2+\sqrt{2}\ln\left(1+\sqrt{2}\right)\right].$$

proof



Question 9 **(****+)**

A curve is defined parametrically by the following equations.

$$x=2\ln t\ , \quad y=t+\frac{1}{t}, \quad t\in\mathbb{R}, \quad 1\leq t\leq 4.$$

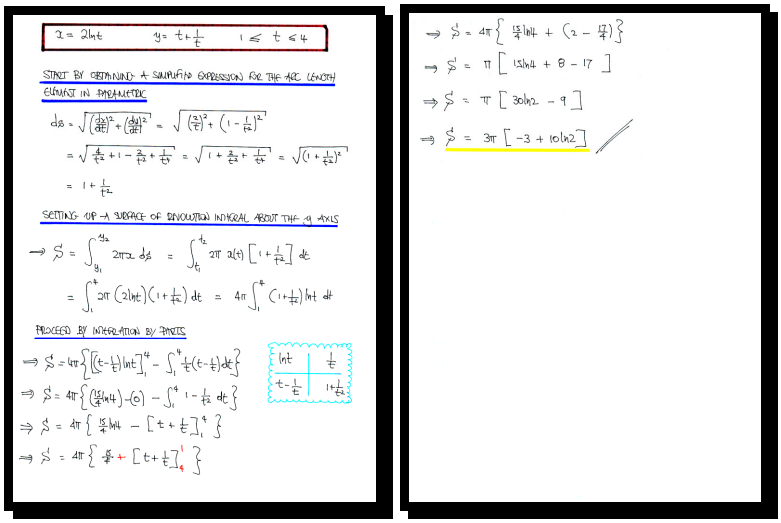
The curve is fully revolved about the y axis forming a surface of revolution.

Show that the area of this surface is

$$k\pi\left[-3+10\ln 2\right],$$

where k is a positive integer to be found.

$\boxed{}$, $\boxed{k=3}$



Question 10 (**+)**

A curve has parametric equations

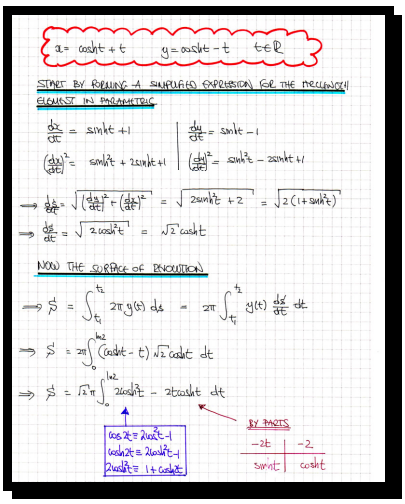
$$x=\cosh t+t\,,\quad y=\cosh t-t\,,\quad t\in\mathbb{R}\,.$$

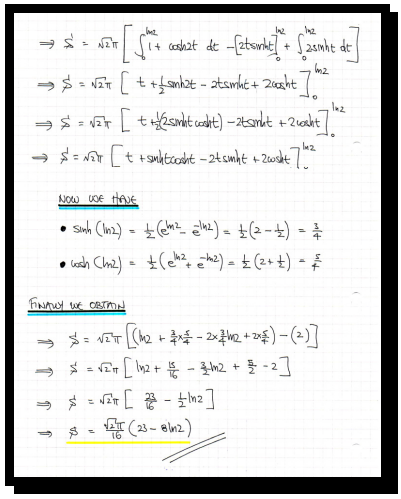
The part of the curve, for which $0\le t\le\ln 2$, is rotated through 2π radians about the x axis.

Show that the exact area of the surface generated is

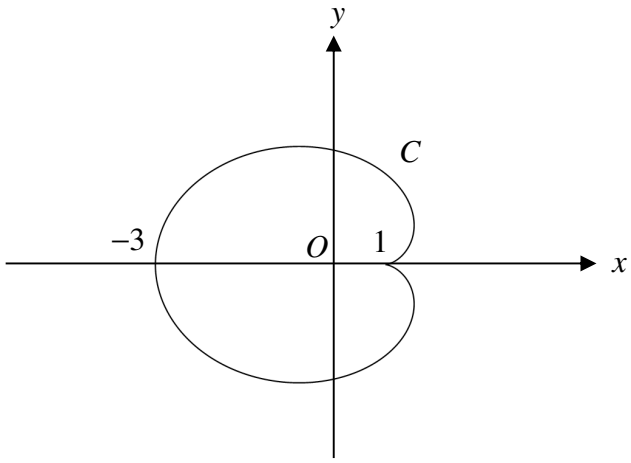
$$\frac{\pi\sqrt{2}}{16}(23-8\ln 2)\,.$$

[ ], proof





Question 11 **(****+)**



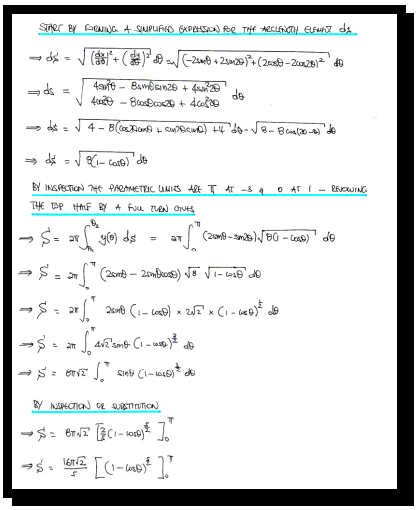
The figure above shows the cardioid C with parametric equations

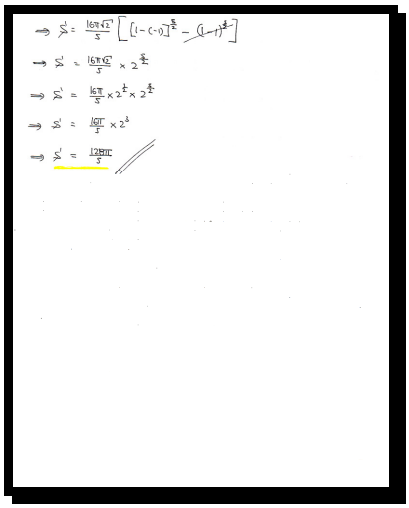
$$x = 2\cos\theta - \cos 2\theta, \quad y = 2\sin\theta - \sin 2\theta, \quad 0 \le \theta < 2\pi .$$

The curve is revolved by a full turn in the x axis, forming a surface of revolution.

Find in exact simplified form the area of this surface.

$\square$, $\boxed{\dfrac{128\pi}{5}}$





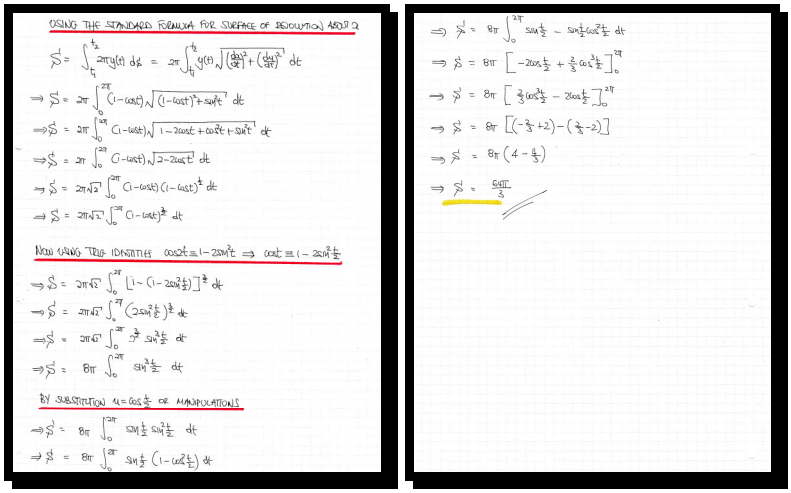
Question 12 **(*****)**

A curve has parametric equations

$$x = t - \sin t , \qquad y = 1 - \cos t , \qquad 0 \le t < 2\pi .$$

Determine, in exact simplified form, the area of the surface of a complete revolution of the curve, about the x axis.

V, , $\dfrac{64\pi}{3}$



Question 13 (*****)

Gabriel's horn is the geometric figure which is formed by revolving the graph of

$$y=\frac{1}{x},\quad x\in[1,\infty),$$

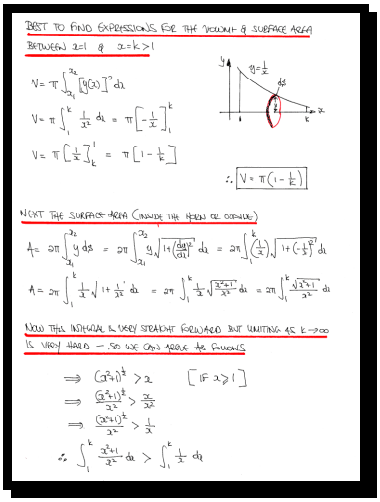
by 2π radians about the x axis.

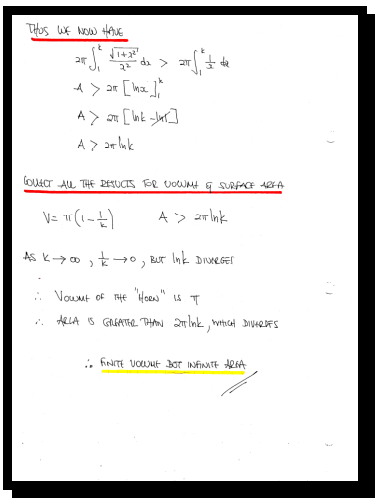
Gabriel's horn gives rise to the "Painter's Paradox", that the "horn" could be filled with a finite quantity of paint and yet that paint would not be sufficient to coat its inner or outer surface.

Use calculus to verify the validity of the apparent paradox, however you need **not** resolve the flaw in the paradox.

You must show any limiting processes and further advised NOT to find $\displaystyle\int\frac{\sqrt{1+x^2}}{x^2}\,dx$.

$\boxed{}$, proof





Question 14 (*****)

The part of the graph of the exponential curve

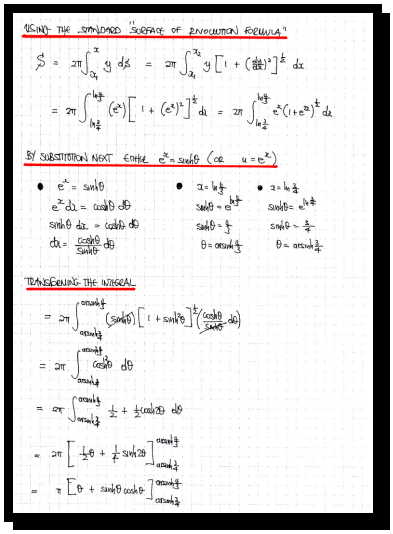
$$y = \mathrm{e}^x \ , \ \ln\left(\frac{3}{4}\right) \le x \le \ln\left(\frac{4}{3}\right),$$

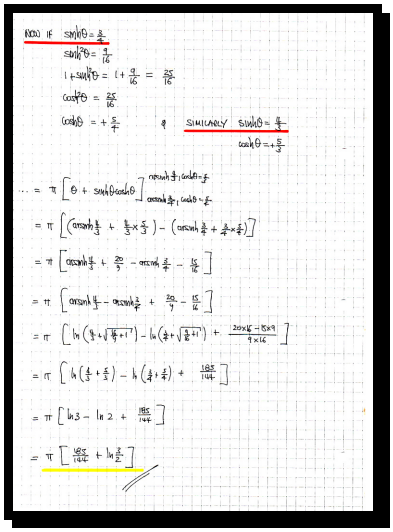
is rotated by 2π radians in the x axis, forming a surface of revolution S.

Show that area of S is

$$\pi\left[\frac{185}{144} + \ln\left(\frac{3}{2}\right)\right].$$

[ ], [proof]





Question 15 (*****)

The part of the curve with equation

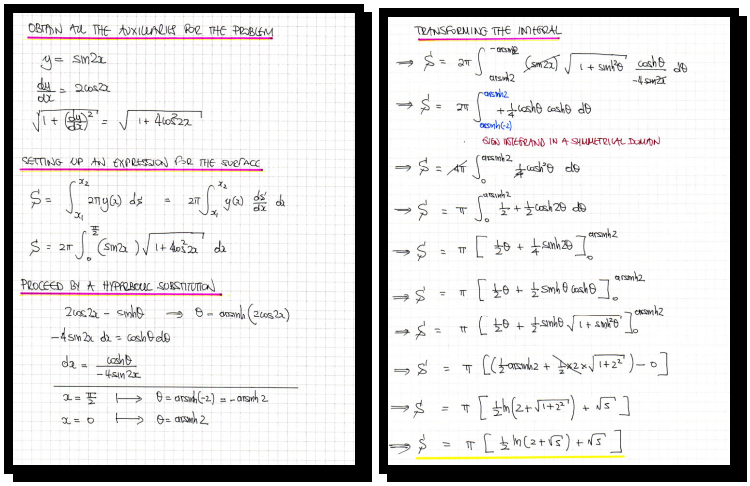
$$y = \sin 2x\,, \quad 0 \le x \le \frac{\pi}{2}$$

is rotated by $360°$ about the x axis.

Show that the area of the surface generated is

$$\pi\left[\frac{1}{2}\ln\left(2+\sqrt{5}\right)+\sqrt{5}\right].$$

, proof



Question 16 (*****)

A curve has parametric equations

$$x = 2 + \tanh t\,, \quad y = \operatorname{sech} t\,, \quad t \in \mathbb{R}$$

The part of the curve for which

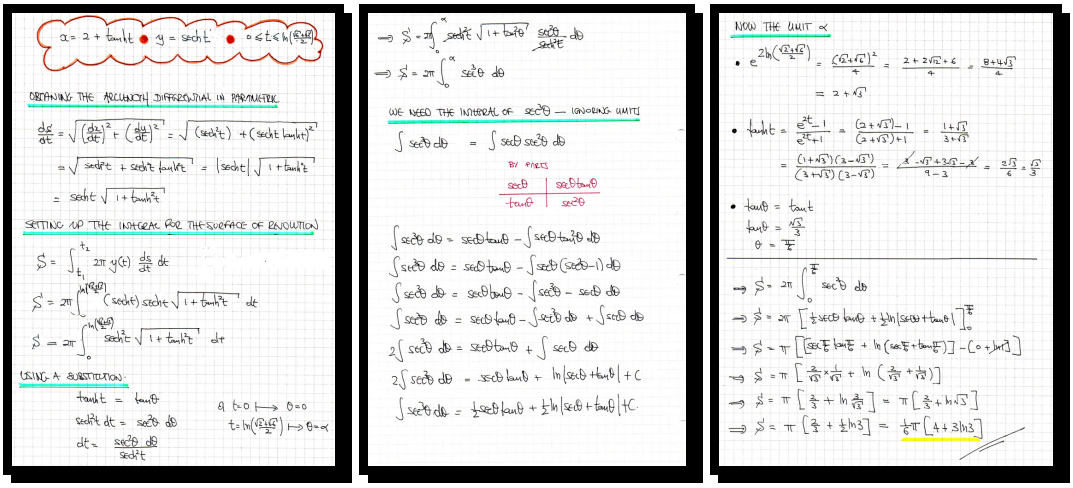
$$0 \le t \le \ln\left[\frac{\sqrt{2}+\sqrt{6}}{2}\right],$$

is rotated through 2π radians in the x axis.

Show that the exact area of the surface generated is

$$\frac{1}{6}\pi\left[4+3\sqrt{3}\right].$$

[ ], [proof]



Question 17 **(*****)**

A curve is defined parametrically by the following equations.

$$x = 2\ln t \ , \quad y = t + \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \geq 1.$$

The curve is fully revolved about the y axis forming a surface of revolution.

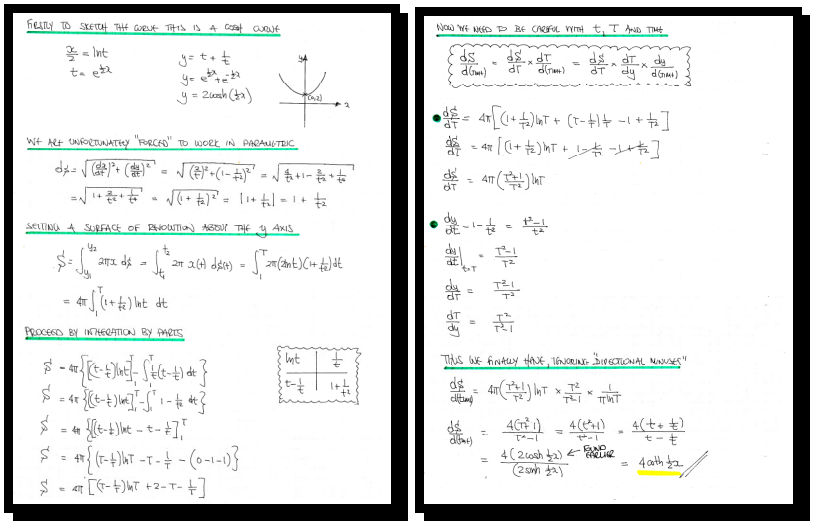
The surface is modelling the casing of a rocket

The vertex of the surface is held just above a container full of paint, with its line of symmetry vertical.

Its line of symmetry is vertically lowered into the paint, at a rate of $\dfrac{1}{\pi \ln t}$, $t > 1$.

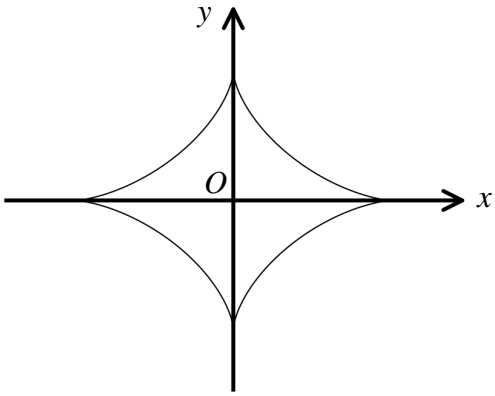
Show that the outer section of the surface is covered in paint at the rate $4\coth\left(\frac{1}{2}x\right)$.

$\boxed{}$, proof



MIXED QUESTIONS

Question 1 **(****)**



The parametric equations of an astroid are

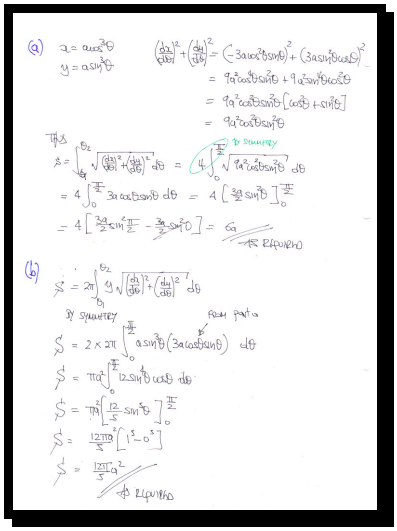
$$x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta < 2\pi$$

a) Show that the total length of the curve is $6a$ units.

The curve is rotated by $360°$ about the x axis forming a solid of revolution.

b) Show further that the surface area of the solid is $\frac{12}{5}\pi a^2$.

proof



Question 2 (****)

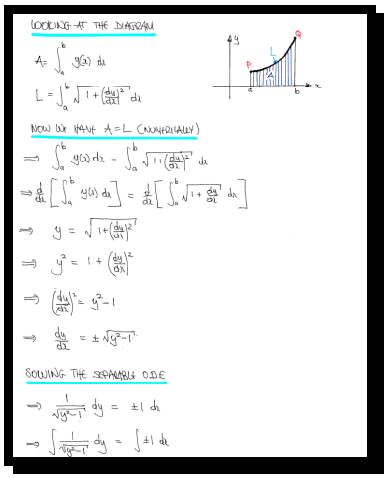
The curve with equation $y=f(x)$ satisfies $y>0$, for $x\in[a,b]$.

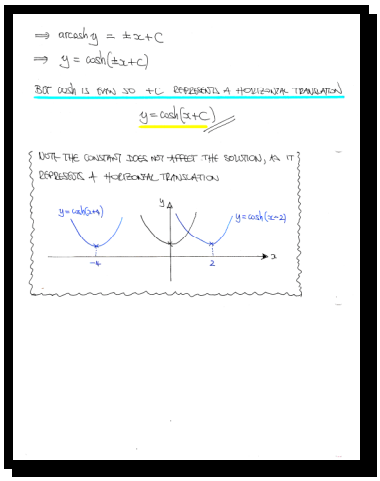
- The area of the region bounded by the curve with equation $y=f(x)$ and the x axis, for $a\le x\le b$, is denoted by A.

- The length along the curve from the point $P\left[a,f(a)\right]$ to the point $Q\left[b,f(b)\right]$, is denoted by L.

If A is **numerically equal** to L, determine the equation of the curve.

$\boxed{}$, $\boxed{y=\cosh(x+C)}$





Question 3 (**+)**

A cycloid has parametric equations

$$x=\theta+\sin\theta\,,\quad y=1+\cos\theta\,,\quad 0\le\theta\le\pi$$

a) Show that the total length of the curve is 4 units.

The cycloid is rotated by $360°$ about the x axis, forming a solid of revolution.

b) Show further that the **total** surface area of the solid is $\dfrac{44\pi}{3}$.

proof

