2^{nd} ORDER O.D.E.s
SUBSTITUTIONS

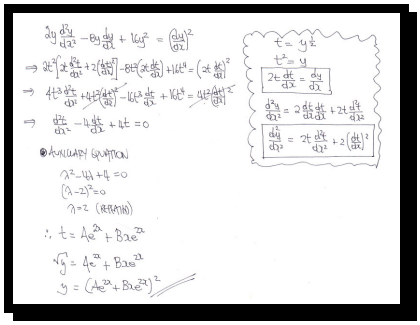
Question 1 **(***+)**

$$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2,\ y\neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t=\sqrt{y}$.

Give the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\left(A\mathrm{e}^{2x}+Bx\mathrm{e}^{2x}\right)^2}$$



Question 2 (*+)**

The differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x\,,\ x\neq 0\,,$$

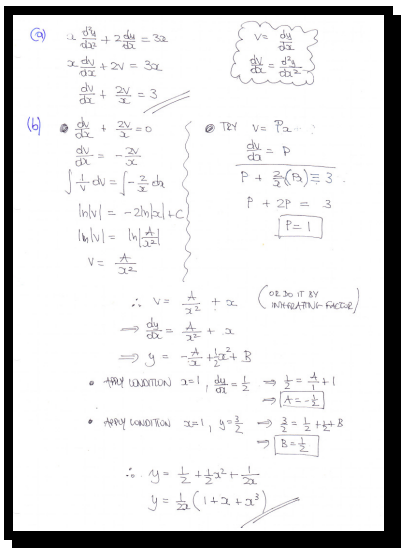
is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

a) Show that the substitution $v=\frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx}+\frac{2v}{x}=3\,.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left(x^2+\frac{1}{x}+1\right)}$$



Question 3 **(***+)**

The curve C has equation $y=f(x)$ and satisfies the differential equation

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-2y\left(2x^2-1\right)=3x^3\,\mathrm{e}^x,\ x\neq 0$$

is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

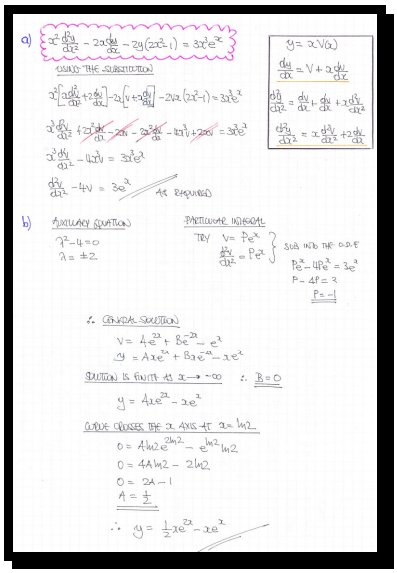
a) Show that the substitution $y=xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2v}{dx^2}-4v=3\mathrm{e}^x.$$

It is further given that C meets the x axis at $x=\ln 2$ and has a finite value for y as x gets infinitely negatively large.

b) Express the equation of C in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}x\mathrm{e}^{2x}-x\mathrm{e}^x}$$



Question 4 (*+)**

Given that if $x=\mathrm{e}^t$ and $y=f\left(x\right)$, show clearly that …

a) … $x\dfrac{dy}{dx}=\dfrac{dy}{dt}$.

b) … $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}$.

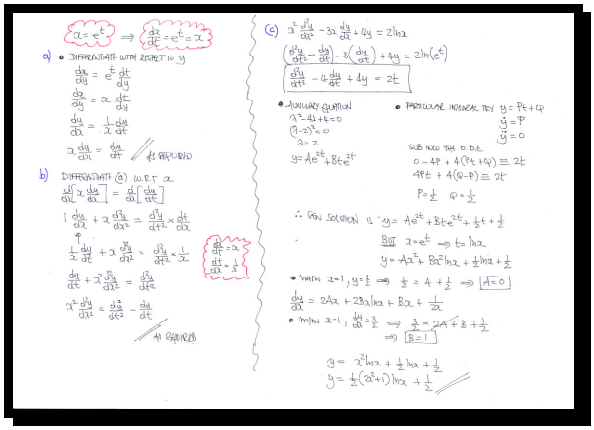
The following differential equation is to be solved

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=2\ln x$$

subject to the boundary conditions $y=\dfrac{1}{2}, \dfrac{dy}{dx}=\dfrac{3}{2}$ at $x=1$.

c) Use the substitution $x=\mathrm{e}^t$ to solve the above differential equation.

$$\boxed{y=\frac{1}{2}+\frac{1}{2}\left(2x^2+1\right)\ln x}$$



Question 5 (****)

The differential equation

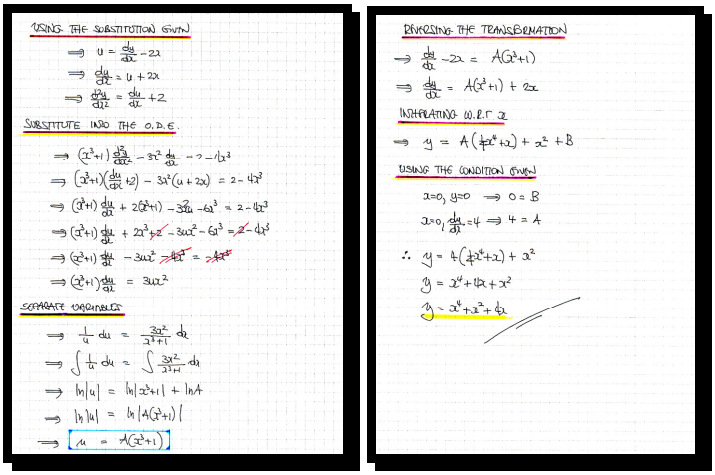
$$\left(x^3+1\right)\frac{d^2y}{dx^2}-3x^2\frac{dy}{dx}=2-4x^3,$$

is to be solved subject to the boundary conditions $y=0$, $\frac{dy}{dx}=4$ at $x=0$.

Use the substitution $u=\frac{dy}{dx}-2x$, where u is a function of x, to show that the solution of the above differential equation is

$$y=x^4+x^2+4x.$$

[ ], [proof]

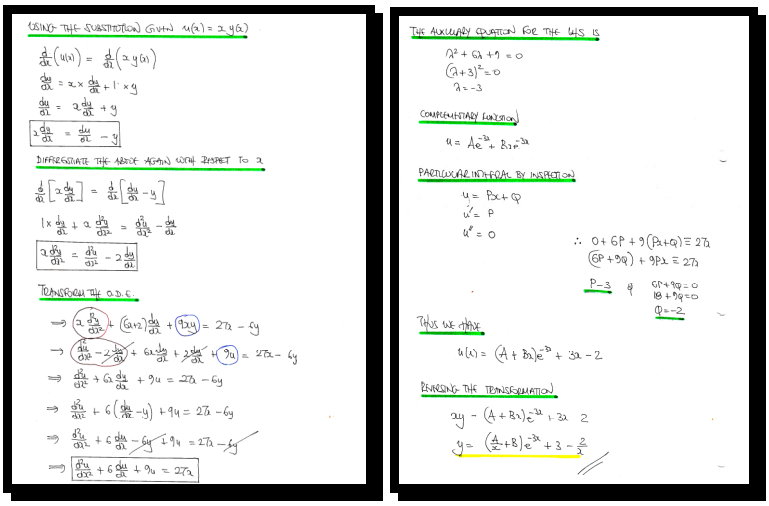


Question 6 (****)

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y\,.$$

Use the substitution $u=xy$, where u is a function of x, to find a general solution of the above differential equation.

$\boxed{}$, $\boxed{y=\frac{A}{x}\mathrm{e}^{-3x}+B\mathrm{e}^{-3x}+3-\frac{2}{x}}$



USING THE SUBSTITUTION GIVEN $u(x)=x\,y(x)$

$\frac{d}{dx}\left(u(x)\right)=\frac{d}{dx}\left(x\,y(x)\right)$

$\frac{du}{dx}=x\times\frac{dy}{dx}+1\times y$

$\frac{du}{dx}=x\frac{dy}{dx}+y$

$\boxed{x\frac{dy}{dx}=\frac{du}{dx}-y}$

DIFFERENTIATE THE ABOVE AGAIN WITH RESPECT TO x

$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=\frac{d}{dx}\left[\frac{du}{dx}-y\right]$

$1\times\frac{dy}{dx}+x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-\frac{dy}{dx}$

$\boxed{x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-2\frac{dy}{dx}}$

TRANSFORM THE O.D.E.

$\Rightarrow x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y$

$\rightarrow \left(\frac{d^2u}{dx^2}-2\frac{dy}{dx}\right)+6x\frac{dy}{dx}+2\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6x\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\left(\frac{du}{dx}-y\right)+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\frac{du}{dx}-6y+9u=27x-6y$

$\Rightarrow \boxed{\frac{d^2u}{dx^2}+6\frac{du}{dx}+9u=27x}$

THE AUXILIARY EQUATION FOR THE LHS IS

$\lambda^2+6\lambda+9=0$

$(\lambda+3)^2=0$

$\lambda=-3$

COMPLEMENTARY FUNCTION

$u=Ae^{-3x}+Bxe^{-3x}$

PARTICULAR INTEGRAL BY INSPECTION

$u=Px+Q$

$u'=P$

$u''=0$

$\therefore\ 0+6P+9(Px+Q)\equiv 27x$

$(6P+9Q)+9Px\equiv 27x$

$P=3$ & $6P+9Q=0$, $18+9Q=0$, $Q=-2$

THUS WE HAVE

$u(x)=(A+Bx)e^{-3x}+3x-2$

REVERSING THE TRANSFORMATION

$xy=(A+Bx)e^{-3x}+3x-2$

$y=\left(\frac{A}{x}+B\right)e^{-3x}+3-\frac{2}{x}$

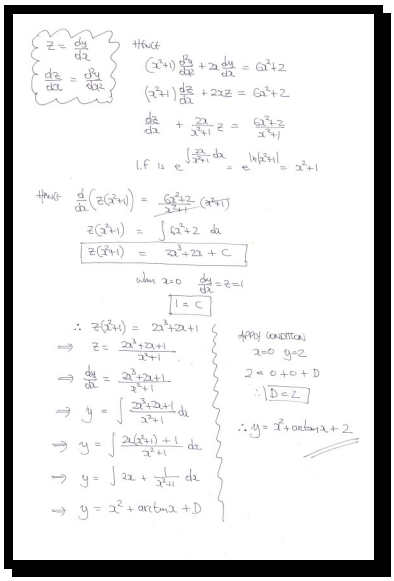
Question 7 (****)

By using the substitution $z=\dfrac{dy}{dx}$, or otherwise, solve the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=6x^2+2\,,$$

subject to the conditions $x=0$, $y=2$, $\dfrac{dy}{dx}=1$

$$\boxed{y=x^2+2+\arctan x}$$



Question 8 **(****)**

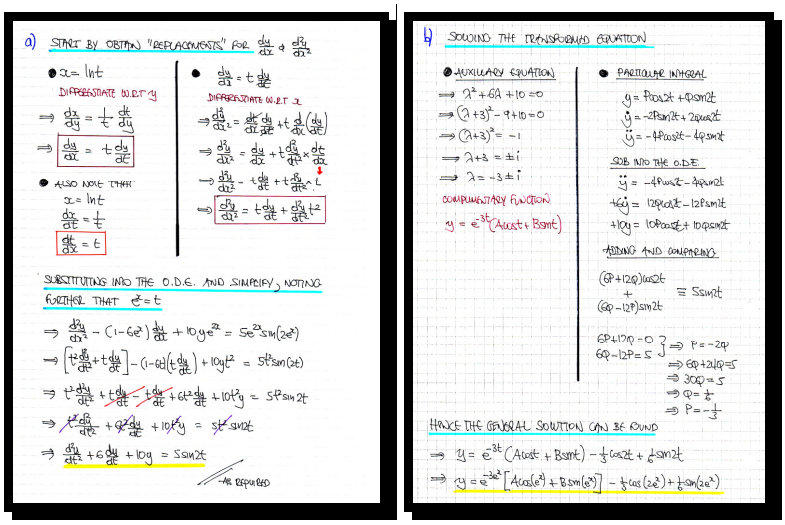
$$\frac{d^2y}{dx^2}-\left(1-6\mathrm{e}^{x}\right)\frac{dy}{dx}+10y\,\mathrm{e}^{2x}=5\mathrm{e}^{2x}\sin\left(2\mathrm{e}^{x}\right).$$

a) By using the substitution $x=\ln t$ or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t\,.$$

b) Hence find a general solution for the original differential equation.

$$\boxed{},\quad \boxed{y=\mathrm{e}^{-3\mathrm{e}^{x}}\left[A\cos\left(\mathrm{e}^{x}\right)+B\sin\left(\mathrm{e}^{x}\right)\right]+\frac{1}{6}\sin\left(2\mathrm{e}^{x}\right)-\frac{1}{3}\cos\left(2\mathrm{e}^{x}\right)}$$



Question 9 (****)

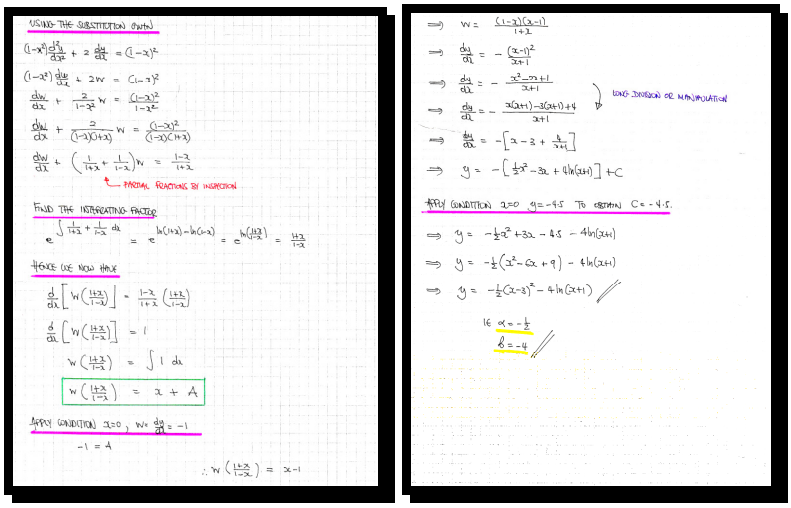
Use the substitution $w=\dfrac{dy}{dx}$ to solve the following differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=\left(1-x\right)^2,\ |x|<1$$

subject to the boundary conditions $y=-4.5$ and $\dfrac{dy}{dx}=-1$ at $x=0$.

Give the answer in the form $y=\alpha\left(x-3\right)^2+\beta\ln\left(x+1\right)$, where α and β are constants to be found.

$\boxed{}$, $\boxed{y=-\frac{1}{4}\left(x-3\right)^2-4\ln\left(x+1\right)}$



Question 10 (****)

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}\tan x-y\sec^4 x=0.$$

The above differential equation is to be solved by a substitution.

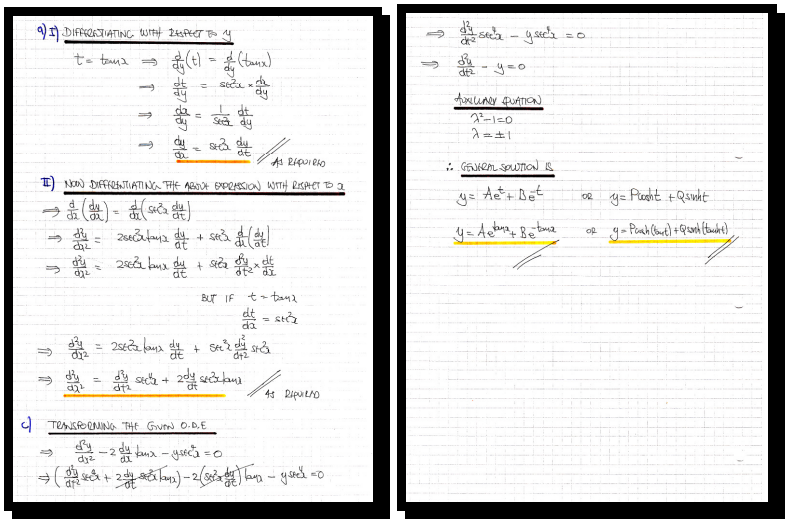
a) If $t=\tan x$ show that …

i. … $\dfrac{dy}{dx}=\dfrac{dy}{dt}\sec^2 x$

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dt^2}\sec^4 x+2\dfrac{dy}{dt}\sec^2 x\tan x$

b) Use the results obtained in part **(a)** to find a general solution of the differential equation in the form $y=f\left(x\right)$.

$\boxed{}$, $\boxed{y=A\mathrm{e}^{\tan x}+B\mathrm{e}^{-\tan x}}$



Question 11 (****)

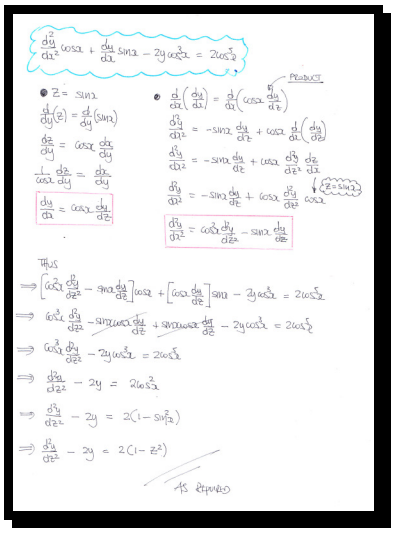
Show clearly that the substitution $z = \sin x$, transforms the differential equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2\left(1 - z^2\right)$$

proof



Question 12 (**+)**

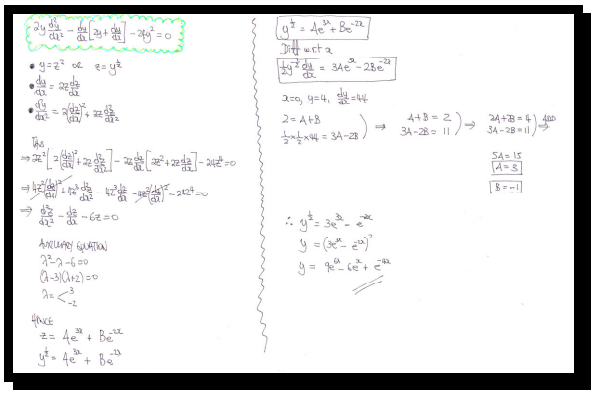
Use the substitution $z=\sqrt{y}$, where $y=f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+2y=0,$$

subject to the boundary conditions $y=4$, $\frac{dy}{dx}=44$ at $x=0$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=9\mathrm{e}^{6x}-6\mathrm{e}^{x}+\mathrm{e}^{-4x}}$$



Question 13 **(****+)**

$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0\,.$$

The above differential equation is to be solved by a substitution.

a) Given that $y=f\left(x\right)$ and $t=x^{\frac{1}{2}}$, show clearly that …

i. … $\dfrac{dy}{dx}=\dfrac{1}{2t}\dfrac{dy}{dt}$.

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{1}{4t^2}\dfrac{d^2y}{dt^2}-\dfrac{1}{4t^3}\dfrac{dy}{dt}$.

b) Hence show further that the differential equation

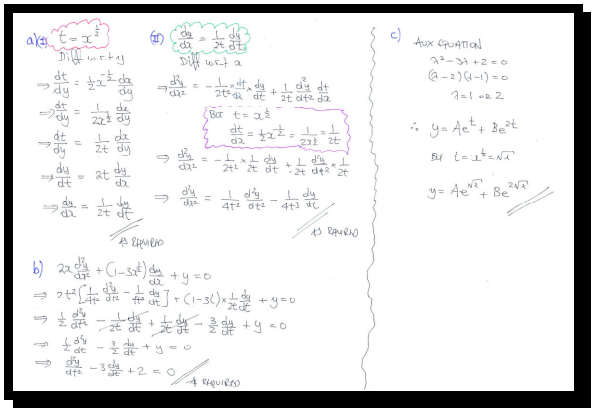
$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0\,,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0\,.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=A\mathrm{e}^{\sqrt{x}}+B\mathrm{e}^{2\sqrt{x}}}$$



Question 14 (**+)**

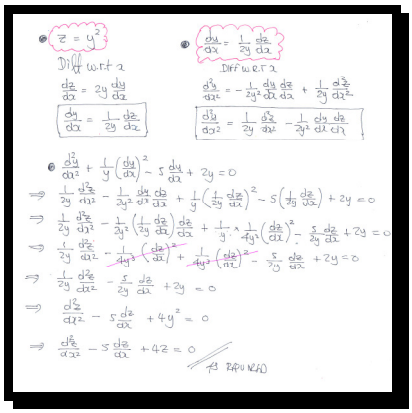
Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

proof



Question 15 **(****+)**

Given that if $x=t^{\frac{1}{2}}$, where $y=f\left(x\right)$, show clearly that

a) $\dfrac{dy}{dx}=2t^{\frac{1}{2}}\dfrac{dy}{dt}$.

b) $\dfrac{d^2y}{dx^2}=4t\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}$.

The following differential equation is to be solved

$$x\frac{d^2y}{dx^2}-\left(8x^2+1\right)\frac{dy}{dx}+12x^3y=12x^5,$$

subject to the boundary conditions $y=\dfrac{10}{3}$, $\dfrac{d^2y}{dx^2}=10$ at $x=0$.

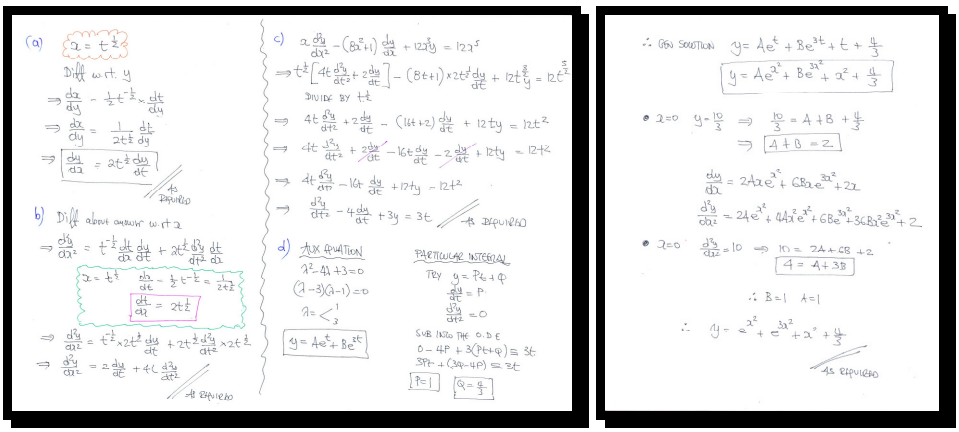
c) Show further that the substitution $x=t^{\frac{1}{2}}$, where $y=f\left(x\right)$, transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=3t.$$

d) Show that a solution of the **original** differential equation is

$$y=\mathrm{e}^{3x^2}+\mathrm{e}^{x^2}+x^2+\frac{4}{3}.$$

proof



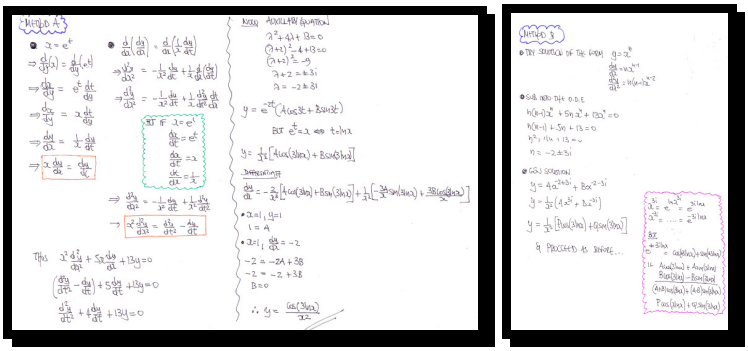
Question 16 (**+)**

The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \ x > 0.$$

By using the substitution $x = \mathrm{e}^t$, or otherwise, determine an equation for $y = f(x)$, given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$\boxed{y = \frac{\cos(3\ln x)}{x^2}}$$



Question 17 **(****+)**

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}\cot x+2y\operatorname{cosec}^2 x=2\cos x-2\cos^3 x\,.$$

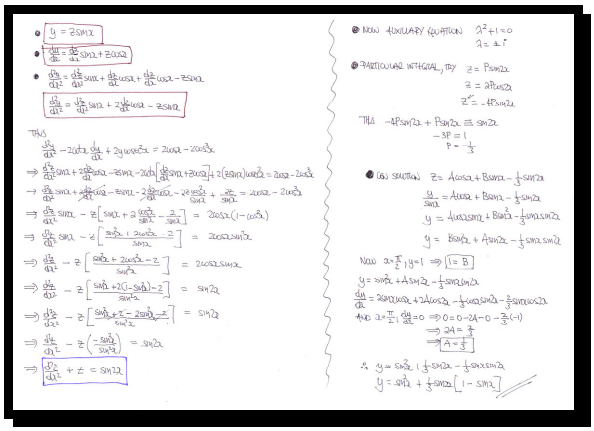
Use the substitution $y=z\sin x$, where z is a function of x, to solve the above differential equation subject to the boundary conditions $y=1$, $\dfrac{dy}{dx}=0$ at $x=\dfrac{\pi}{2}$.

Give the answer in the form

$$y=a\sin^2 x+b\left(1-\sin x\right)\sin 2x\,,$$

where a and b are constants to be found.

$$\boxed{a=1}\,,\quad \boxed{b=\frac{1}{3}}$$



Question 18 (**+)**

The function $y = f(x)$ satisfies the following relationship.

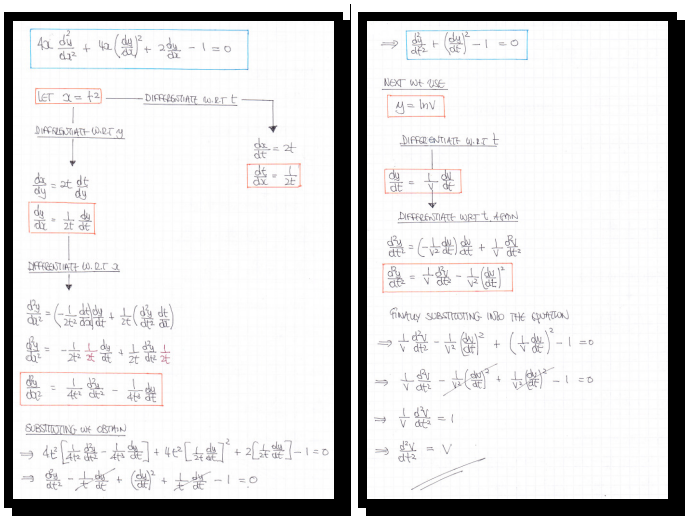
$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0 .$$

It is further given that $x = t^2$ and $y = \ln v$.

Show that

$$\frac{d^2v}{dt^2} = v .$$

proof

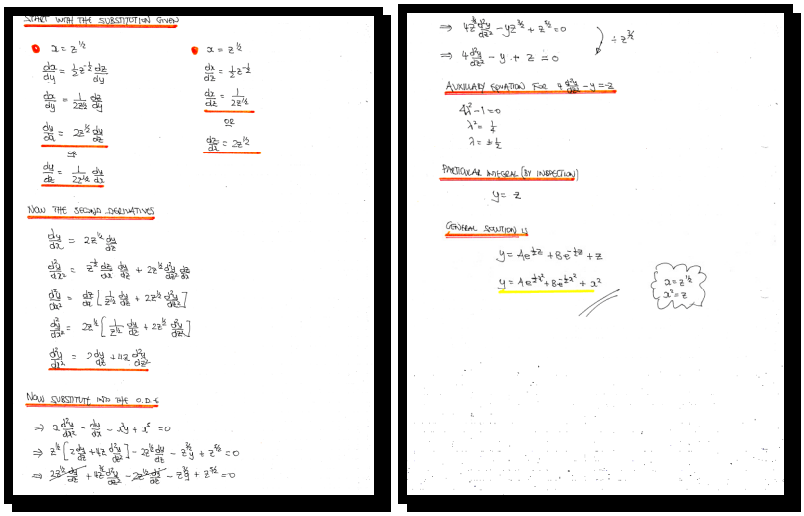


Question 19 **(****+)**

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-x^3y+x^5=0\,.$$

Use the substitution $x=z^{\frac{1}{2}}$, where $y=f\left(x\right)$, to find a general solution of the above differential equation.

V, [ ], $\boxed{y=A\mathrm{e}^{\frac{1}{2}x^2}+B\mathrm{e}^{-\frac{1}{2}x^2}+x^2}$



Question 20 (*****)

Find the solution of following differential equation

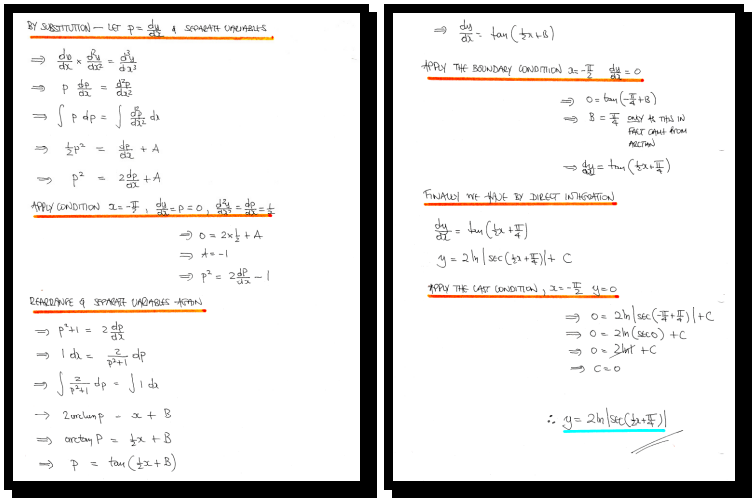
$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\tfrac{1}{2}\pi\right)=y'\left(-\tfrac{1}{2}\pi\right)=0, \qquad y''\left(-\tfrac{1}{2}\pi\right)=\tfrac{1}{2}.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=2\ln\left|\sec\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right|}$



Question 21 (*****)

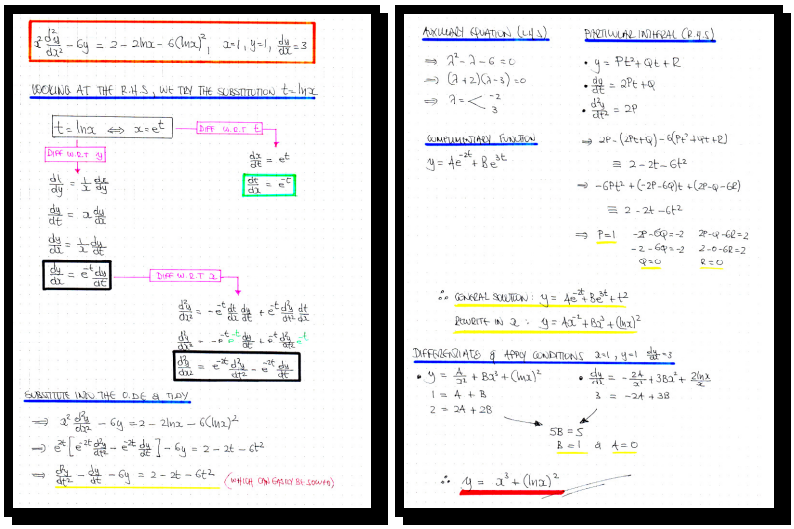
Use a suitable substitution to solve the differential equation

$$x^2\frac{d^2y}{dx^2}-6y=2-2\ln x-6(\ln x)^2,$$

subject to the boundary conditions $y(1)=1$, $\dfrac{dy}{dx}(1)=3$

Give a simplified answer in the form $y=f(x)$.

$\square$, $\boxed{y=x^3+(\ln x)^2}$



Question 22 (*****)

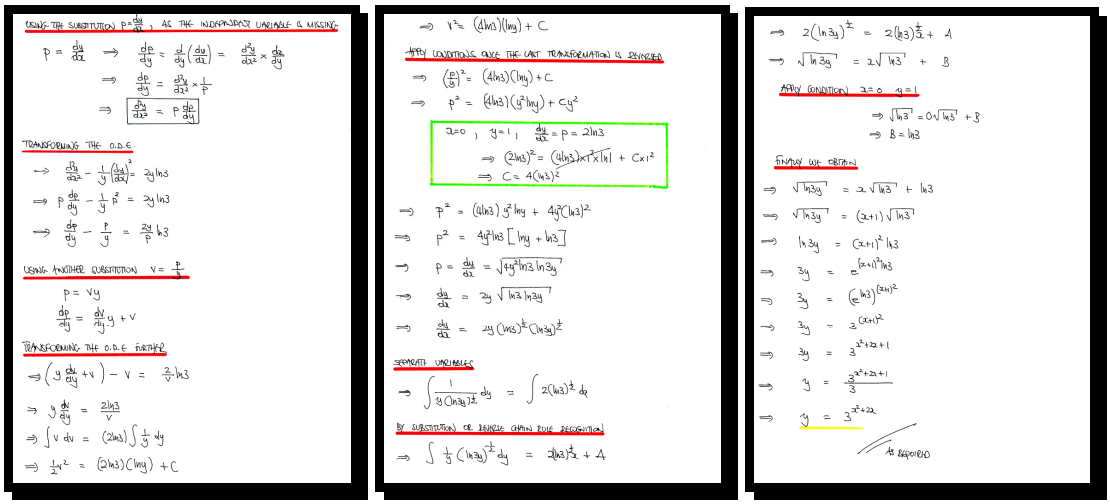
The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation by using the substitution $p = \dfrac{dy}{dx}$, to show that

$$y = 3^{x^2+2x}.$$

[ ], [proof]



Question 23 **(*****)**

$$4x\frac{d^2y}{dx^2}+4x\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}=1.$$

By using the substitution $t=\sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y=A-\sqrt{x}+\ln\left[1+B\mathrm{e}^{2\sqrt{x}}\right],$$

where A and B are arbitrary constants.

, proof

