1^{st} ORDER O.D.E.

SOLUTIONS BY SUBSTITUTIONS

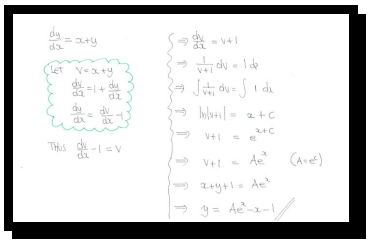
Question 1 **(**+)**

By using the substitution $u = x + y$ find a general solution of the differential equation

$$\frac{dy}{dx} = x + y,$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A\mathrm{e}^{x} - x - 1}$$



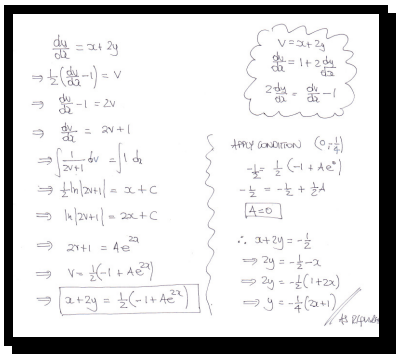
Question 2 **(***)**

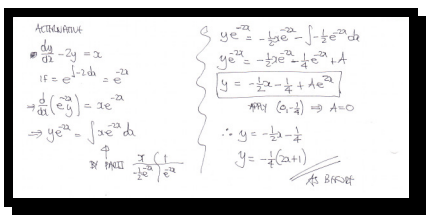
$$\frac{dy}{dx} = x + 2y, \text{ with } y = -\frac{1}{4} \text{ at } x = 0.$$

By using the substitution $v = x + 2y$, show that the solution of the differential equation is given by

$$y = -\frac{1}{4}(2x + 1).$$

$$\boxed{\text{proof}}$$





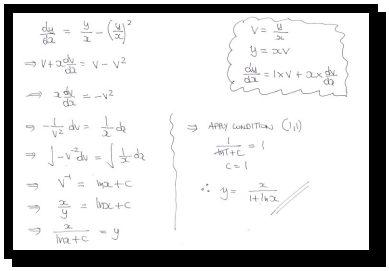
Question 3 (***)

By using the substitution $v=\dfrac{y}{x}$, where $v=f\left(x\right)$, solve the differential equation

$$\frac{dy}{dx}=\frac{y}{x}-\left(\frac{y}{x}\right)^2,\ x>0$$

subject to the condition $y=1$ at $x=1$.

$$\boxed{y=\frac{x}{1+\ln x}}$$



Question 4 (***)

$$\frac{dy}{dx}=\frac{(4x+y)(x+y)}{x^2},\ x>0.$$

a) Use the substitution $y=xv$, where $v=f(x)$, to show that the above differential equation can be transformed to

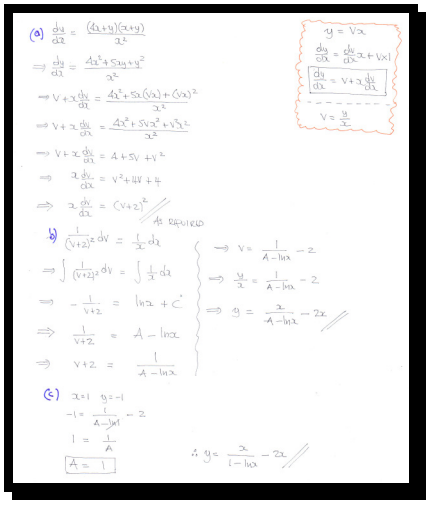
$$x\frac{dv}{dx}=(v+2)^2.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y=f(x)$.

c) Use the boundary condition $y=-1$ at $x=1$, to show that a specific solution of the original differential equation is

$$y=\frac{x}{1-\ln x}-2x.$$

$$\boxed{y=\frac{x}{A-\ln x}-2x}$$



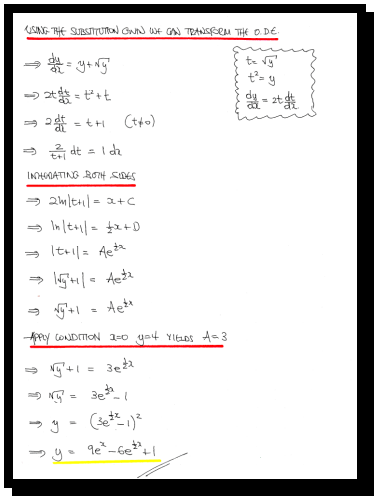
Question 5 (***)

Use the substitution $t=\sqrt{y}$ to solve the following differential equation.

$$\frac{dy}{dx}=y+\sqrt{y}\,,\quad y>0,\quad y(0)=4\,.$$

Given the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=9\mathrm{e}^{x}-6\mathrm{e}^{\frac{1}{2}x}+1}$



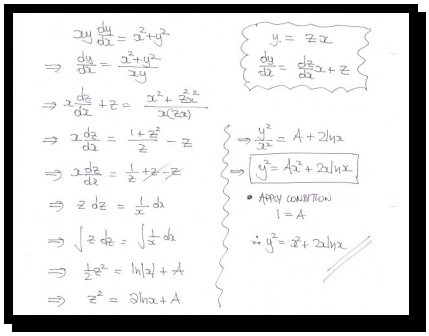
Question 6 (*+)**

By using the substitution $y = xz$, where $z = f(x)$, solve the differential equation

$$xy\frac{dy}{dx} = x^2 + y^2, \ x > 0$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{y = x^2(1 + 2\ln x)}$$



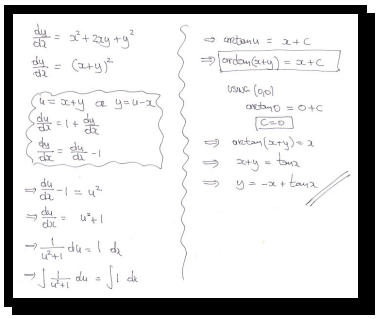
Question 7 (*+)**

By using the substitution $u = x + y$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = x^2 + 2xy + y^2,$$

subject to the condition $y(0) = 0$.

$$\boxed{y = -x + \tan x}$$



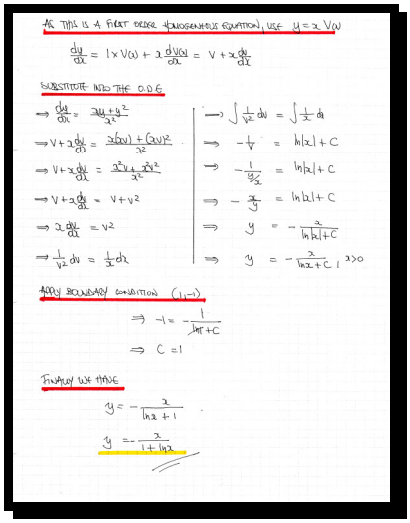
Question 8 **(***+)**

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}, \quad x > 0$$

subject to the condition $y = -1$ at $x = 1$.

$\square$, $\boxed{y = -\dfrac{x}{1 + \ln x}}$

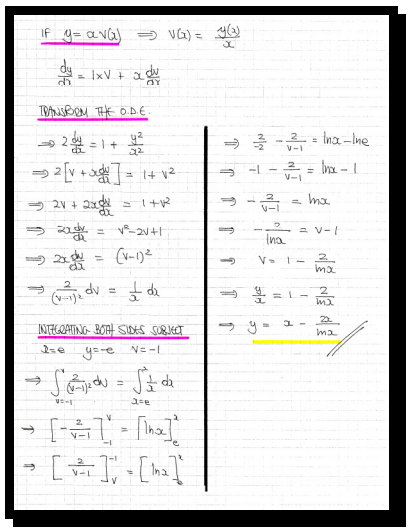


Question 9 (*+)**

Use the substitution $y=xv$, where $v=v(x)$, to solve the following differential equation

$$2\frac{dy}{dx}=1+\frac{y^2}{x^2}, \quad y(\mathrm{e})=-\mathrm{e}.$$

$\square$, $\boxed{y=x-\dfrac{2x}{\ln x}}$



Question 10 **(***+)**

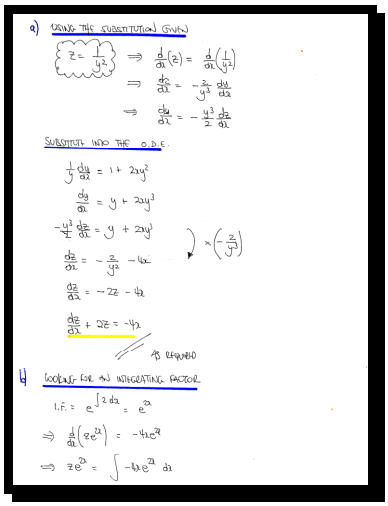
$$\frac{1}{y}\frac{dy}{dx}=1+2xy^2,\ \ y>0.$$

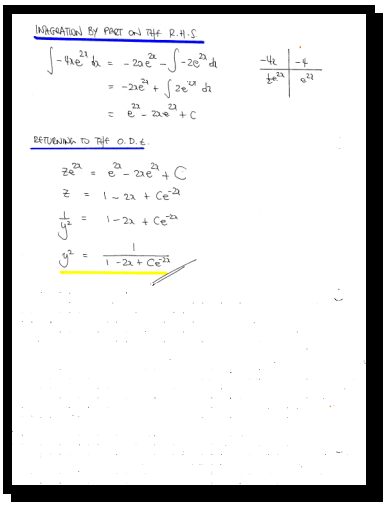
a) Show that the substitution $z=\dfrac{1}{y^2}$ transforms the above differential equation into the new differential equation

$$\frac{dz}{dx}+2z=-4x.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y^2=f(x)$.

$\square$, $\boxed{y^2=\dfrac{1}{Ae^{-2x}-2x+1}}$





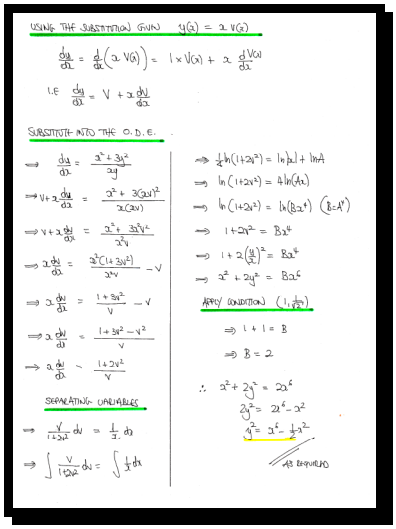
Question 11 (*+)**

$$\frac{dy}{dx}=\frac{x^2+3y^2}{xy},\ x>0,\ y>0.$$

Use the substitution $y=xv$, where $v=f\left(x\right)$, and the boundary condition $y=\frac{1}{\sqrt{2}}$ at $x=1$, to show that

$$y^2=x^6-\frac{1}{2}x^2.$$

$\boxed{}$, $\boxed{\text{proof}}$



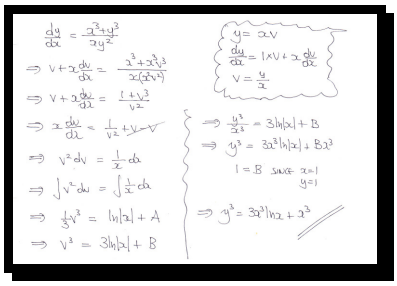
Question 12 (****)

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y^3 = x^3(3\ln x + 1)}$$



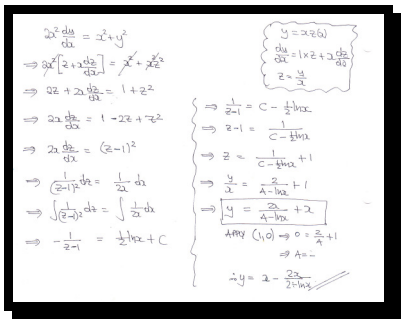
Question 13 (****)

By using the substitution $y = xz$, where $z = f(x)$, solve the differential equation

$$2x^2 \frac{dy}{dx} = x^2 + y^2, \ x > 0$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{y = x - \frac{2x}{2 + \ln x}}$$



Question 14 (****)

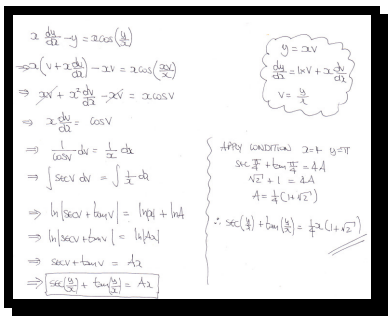
By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right), \ x \neq 0$$

subject to the condition $y = \pi$ at $x = 4$.

The final answer may not involve natural logarithms.

$$\boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x\left(1+\sqrt{2}\right)}$$



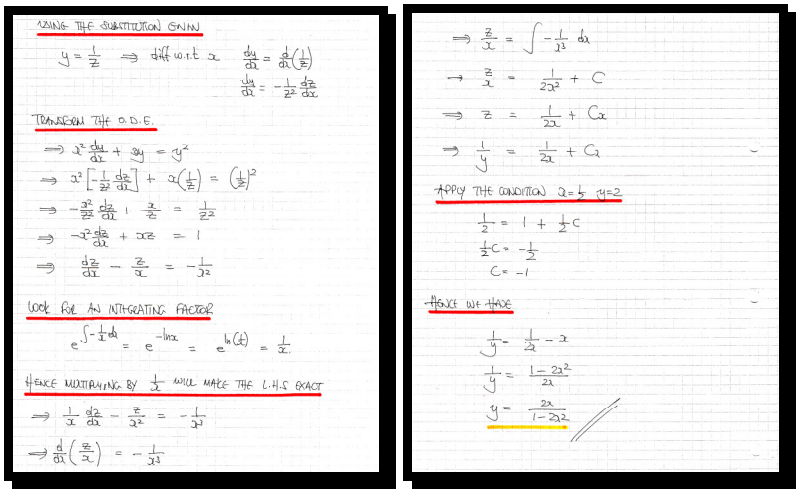
Question 15 (****)

By using the substitution $y=\dfrac{1}{z}$, or otherwise, solve the differential equation

$$x^2\frac{dy}{dx}+xy=y^2$$

subject to the condition $y=2$ at $x=\dfrac{1}{2}$.

$\square$, $\boxed{y=\dfrac{2x}{1-2x^2}}$



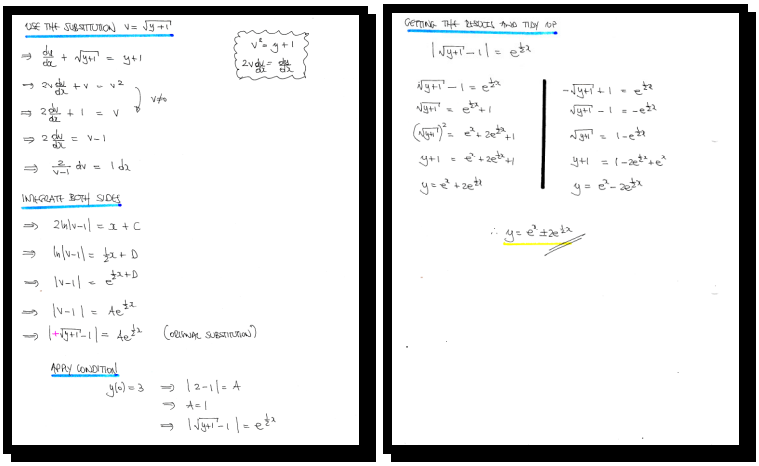
Question 16 (****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+\sqrt{y+1}=y+1, \quad y>-1, \quad y(0)=3.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=e^{x}\pm 2e^{\frac{1}{2}x}}$



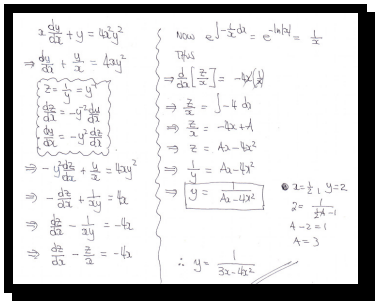
Question 17 (****)

By using the substitution $z=\dfrac{1}{y}$, or otherwise, solve the differential equation

$$x\frac{dy}{dx}+y=4x^2y^2$$

subject to the condition $y=2$ at $x=\dfrac{1}{2}$.

$$\boxed{y=\frac{1}{3x-4x^2}}$$



Question 18 (****)

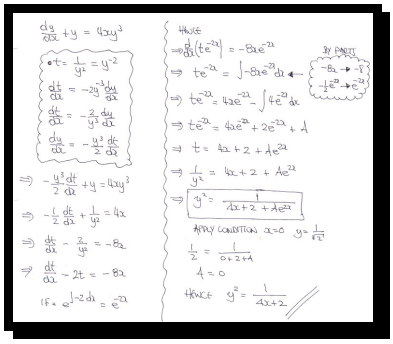
By using the substitution $t=\dfrac{1}{y^2}$, or otherwise, solve the differential equation

$$\frac{dy}{dx}+y=4xy^3$$

subject to the boundary condition $y=\dfrac{1}{\sqrt{2}}$ at $x=0$.

Give the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=\frac{1}{4x+2}}$$



Question 19 (****)

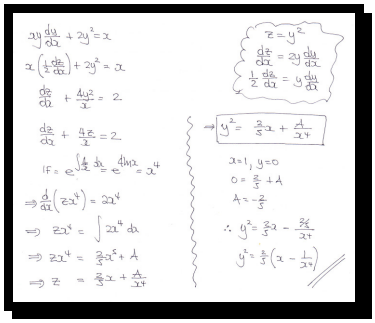
By using the substitution $z = y^2$ or otherwise, solve the differential equation

$$xy\frac{dy}{dx} + 2y^2 = x$$

subject to the boundary condition $y = 0$ at $x = 1$.

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)}$$



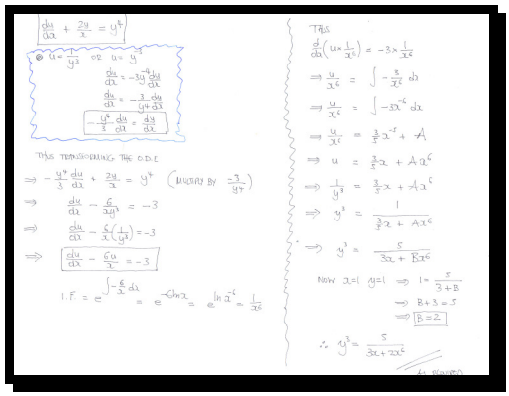
Question 20 (****)

$$\frac{dy}{dx}+\frac{2y}{x}=y^4,\ x>0,\ y>0.$$

Use the substitution $u=y^{-3}$ and the boundary condition $y=1$ at $x=1$, to show that

$$y^3=\frac{5}{3x+2x^6}.$$

proof



Question 21 (****)

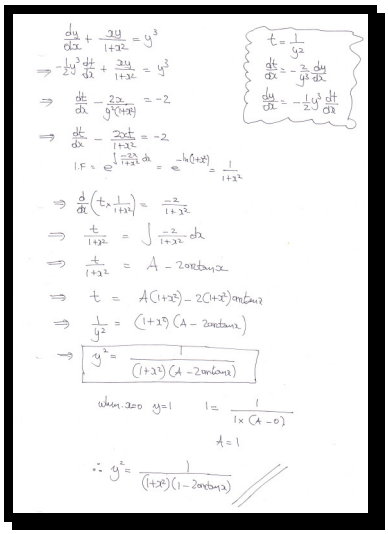
By using the substitution $t=\dfrac{1}{y^2}$ or otherwise, solve the differential equation

$$\frac{dy}{dx}+\frac{xy}{1+x^2}=y^3$$

subject to the boundary condition $y=1$ at $x=0$.

Give the answer in the form $y^2=f\left(x\right)$.

$$\boxed{y^2=\frac{1}{\left(1+x^2\right)\left(1-2\arctan x\right)}}$$



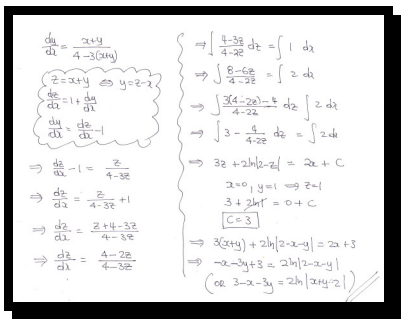
Question 22 (****)

By using the substitution $z = x + y$ solve the differential equation

$$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)},$$

subject to the boundary condition $y = 1$ at $x = 0$.

$$\boxed{2\ln|x+y-2| = 3-x-3y}$$



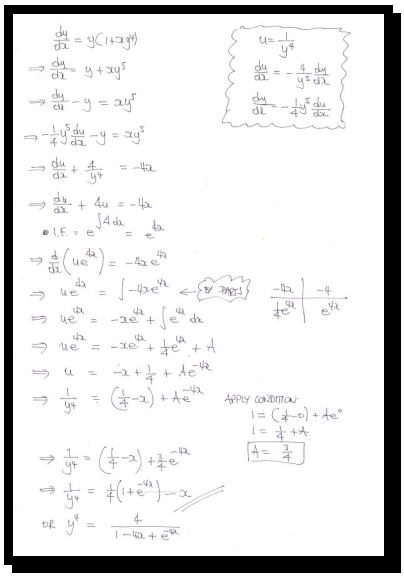
Question 23 (****)

By using the substitution $u=\dfrac{1}{y^4}$, or otherwise, solve the differential equation

$$\frac{dy}{dx}=y\left(1+xy^4\right)$$

subject to the condition $y=1$ at $x=0$.

$$\boxed{\frac{1}{y^4}=\frac{1}{4}\left(1+3\mathrm{e}^{-4x}\right)-x}$$



Question 24 (****)

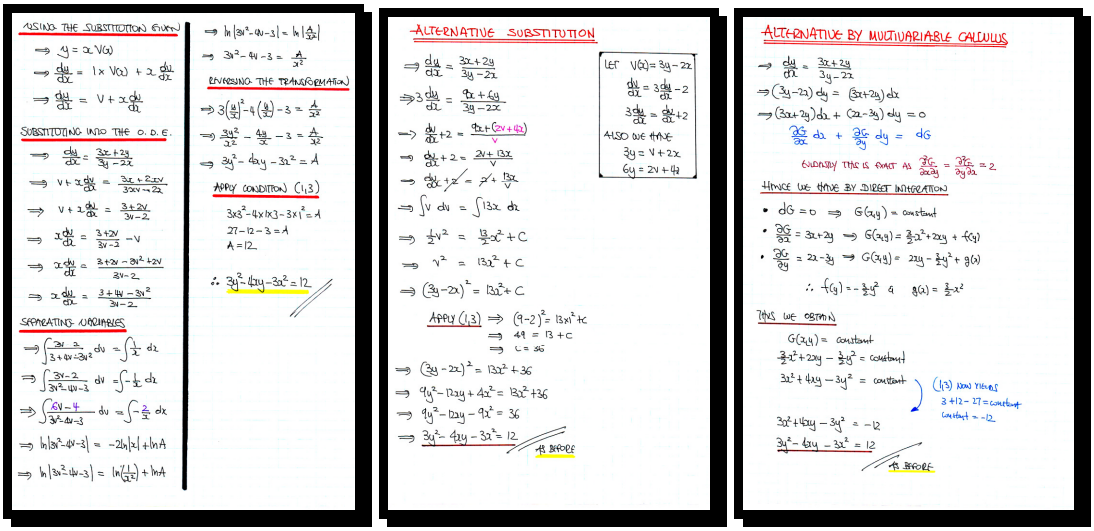
By using the substitution $y = xv$, where $v = f(x)$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = \frac{3x + 2y}{3y - 2x},$$

subject to the condition $y = 3$ at $x = 1$.

Give the final answer in the form $F(x, y) = 12$

$\square$, $\boxed{3y^2 - 4xy - 3x^2 = 12}$



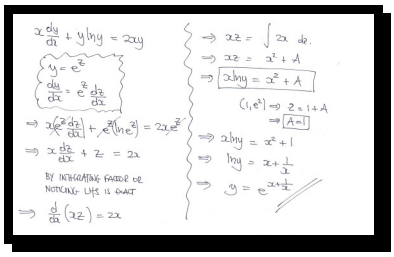
Question 25 (****)

By using the substitution $y = \mathrm{e}^z$, or otherwise, solve the differential equation

$$x\frac{dy}{dx} + y\ln y = 2xy,$$

subject to the condition $y = \mathrm{e}^2$ at $x = 1$.

$$\boxed{y = \mathrm{e}^{x+\frac{1}{x}}}$$



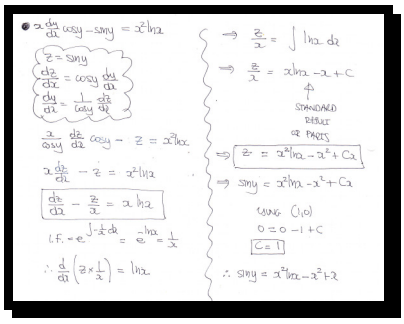
Question 26 (****)

By using the substitution $z = \sin y$, or otherwise, solve the differential equation

$$x\frac{dy}{dx}\cos y - \sin y = x^2 \ln x,$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{\sin y = x^2 \ln x - x^2 + x}$$



Question 27 (****)

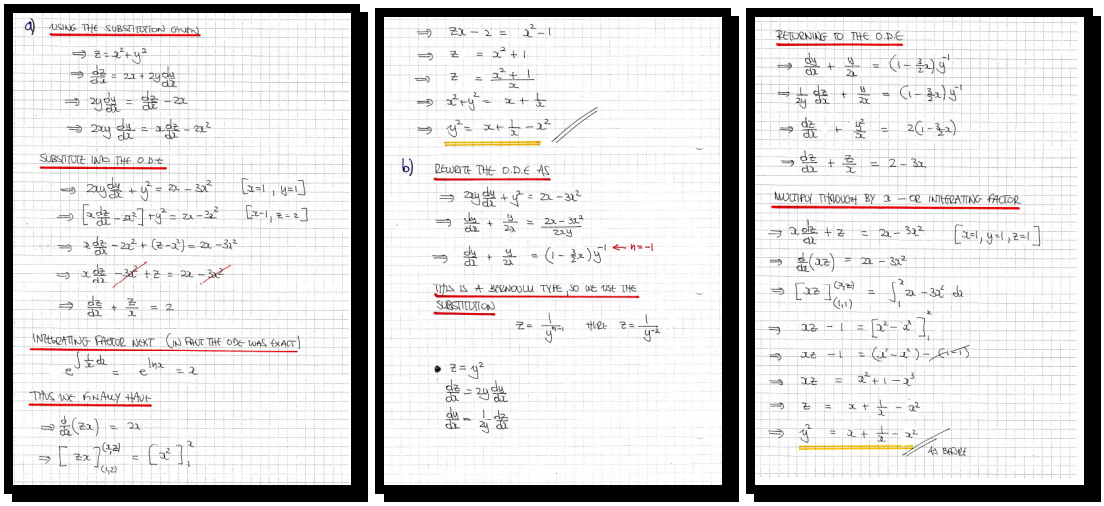
a) By using the substitution $z = x^2 + y^2$, solve the following differential equation

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

subject to the condition $y = 1$ at $x = 1$.

b) Verify the answer to part **(a)** by using the substitution $z = y^2$ to solve the same differential equation and subject to the same condition.

$$\boxed{}, \quad \boxed{y^2 = x - x^2 + \frac{1}{x}}$$



Question 28 (**+)**

A curve C passes through the point $(1,1)$ and satisfies the differential equation

$$\frac{dy}{dx}-\frac{y}{x}=\frac{x^3}{4y^3},\ x>0,\ y>0,$$

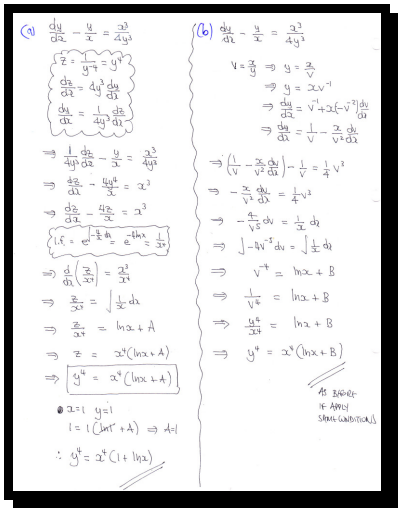
subject to the condition $y=1$ at $x=1$.

a) Find an equation of C by using the substitution $z=y^4$.

b) Find an equation of C by using the substitution $v=\frac{x}{y}$.

Give the answer in the form $y^4=f(x)$.

$$\boxed{y^4=x^4(1+\ln x)}$$



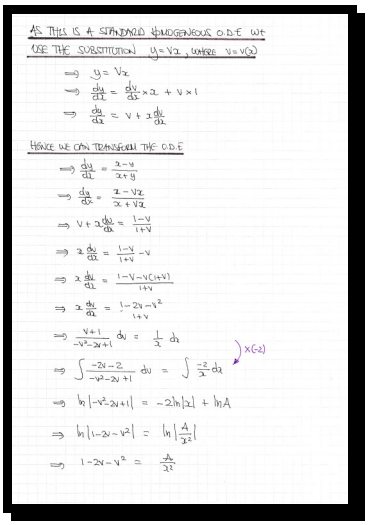
Question 29 (**+)**

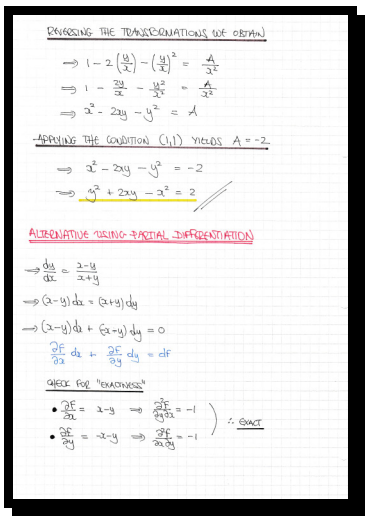
By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

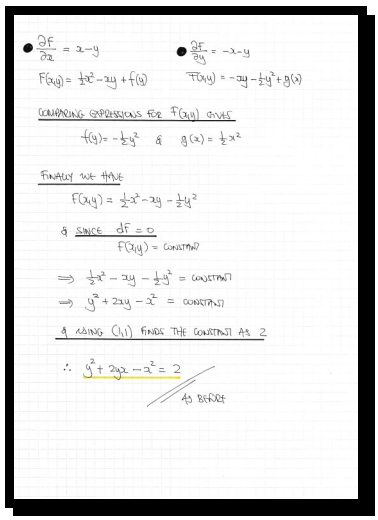
$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

subject to the condition $y = 1$ at $x = 1$.

$\boxed{}$, $\boxed{y^2 + 2xy - x^2 = 2}$







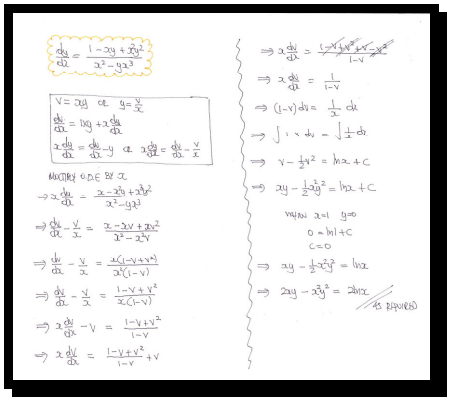
Question 30 (**+)**

By using the substitution $v = xy$, where $y = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{1 - xy + x^2 y^2}{x^2 - yx^3}, \ x > 0,$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{2xy - x^2 y^2 = 2\ln x}$$

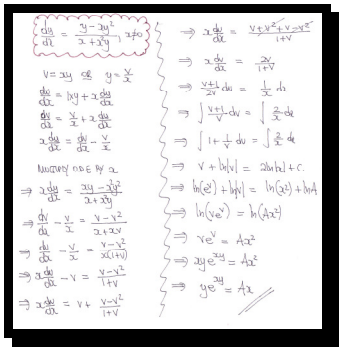


Question 31 (**+)**

Use the substitution $v = xy$, where $y = f(x)$, to find a general solution for the differential equation

$$\frac{dy}{dx} = \frac{y - xy^2}{x + yx^2}, \ x \neq 0.$$

$$\boxed{y\mathrm{e}^{xy} = Cx}$$



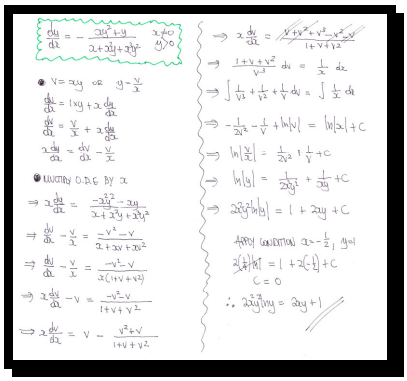
Question 32 (**+)**

By using the substitution $v = xy$, where $y = f(x)$, solve the differential equation

$$\frac{dy}{dx} = -\frac{xy^2 + y}{x + yx^2 + x^3 y^2}, \ x \neq 0, \ y > 0,$$

subject to the condition $y = 1$ at $x = \frac{1}{2}$.

$$\boxed{2x^2 y^2 \ln y = 2xy + 1}$$



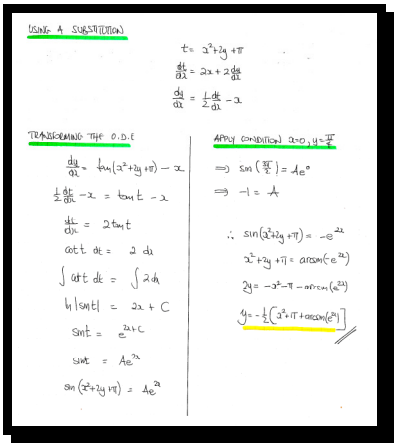
Question 33 (**+)**

$$\frac{dy}{dx}=\tan\left(x^2+2y+\pi\right)\ -\ x,\quad y(0)=\frac{1}{4}\pi\ .$$

Solve the above differential equation to show that

$$y=-\frac{1}{2}\left[x^2+\pi+\arcsin\left(\mathrm{e}^{2x}\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



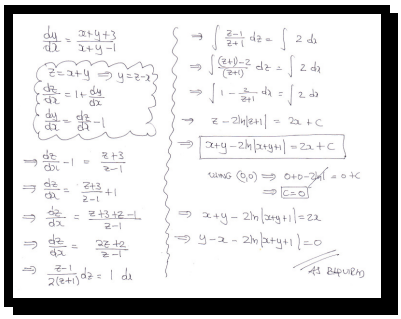
Question 34 (**+)**

$$\frac{dy}{dx}=\frac{x+y+3}{x+y-1},\ y(0)=0.$$

Use the substitution $z=x+y$, to show that the solution of the above differential equation is

$$y-x-2\ln(x+y+1)=0.$$

proof



Question 35 (**+)**

$$\frac{dy}{dx}=\frac{3x-y+1}{x+y+1},\ y(1)=2.$$

a) Show that the transformation equations

$$x=X-\frac{1}{2}$$
$$y=Y-\frac{1}{2}$$

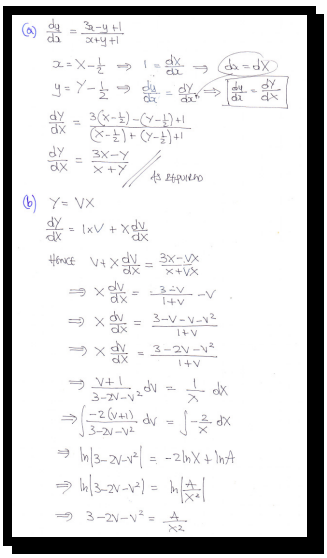
transform the above differential equation to

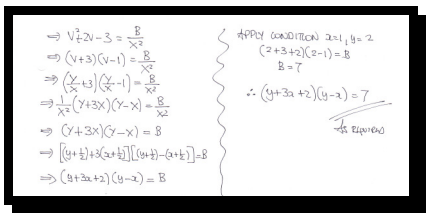
$$\frac{dY}{dX}=\frac{3X-Y}{X+Y}.$$

b) Use the substitution $Y=XV$, where $V=f(x)$, to show that the solution of the original differential equation is

$$(y-x)(y+3x+2)=7.$$

proof





Question 36 (**+)**

$$\frac{dy}{dx}=\frac{2x+5y+3}{4x+y-3},\ y(1)=1.$$

a) Show that the transformation equations

$$x=X+1$$
$$y=Y-1$$

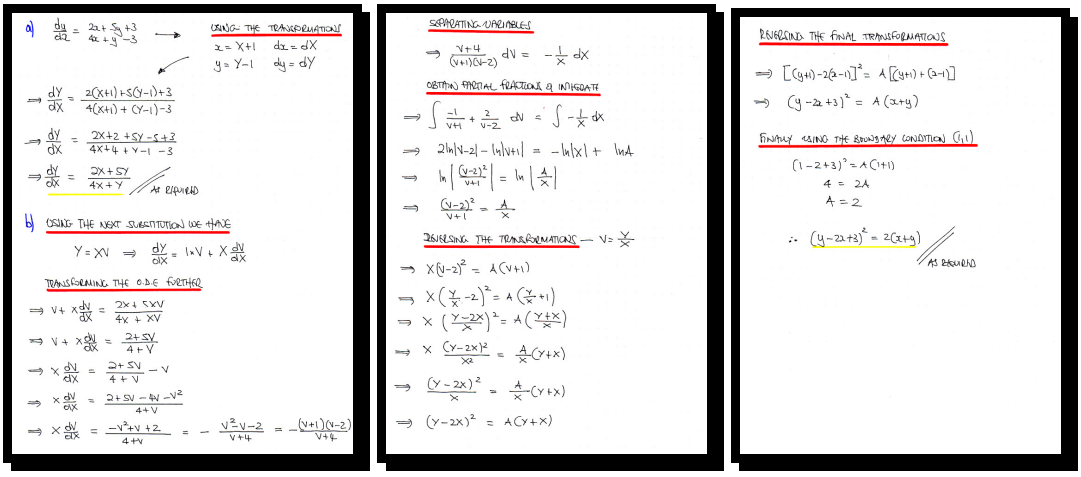
transform the above differential equation to

$$\frac{dY}{dX}=\frac{2X+5Y}{4X+Y}.$$

b) Use the substitution $Y=XV$, where $V=f(x)$, to show that the solution of the original differential equation is

$$(y-2x+3)^2=2(x+y).$$

[ ], [proof]



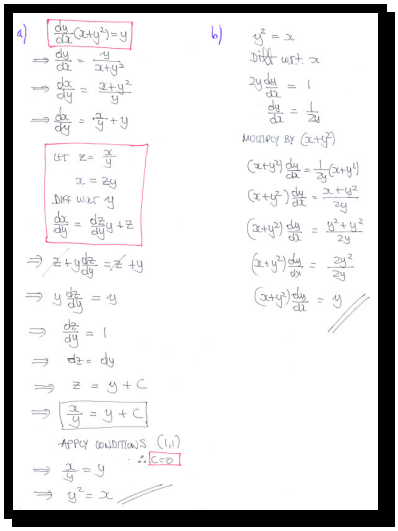
Question 37 **(****+)**

$$\frac{dy}{dx}\left(x+y^2\right)=y\,.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$, by considering $\dfrac{dx}{dy}$, followed by a suitable substitution.

b) Verify the validity of the answer obtained in part **(a)**.

$$\boxed{y^2=x}$$



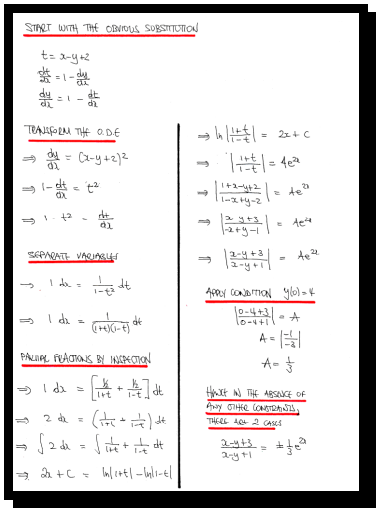
Question 38 **(*****)**

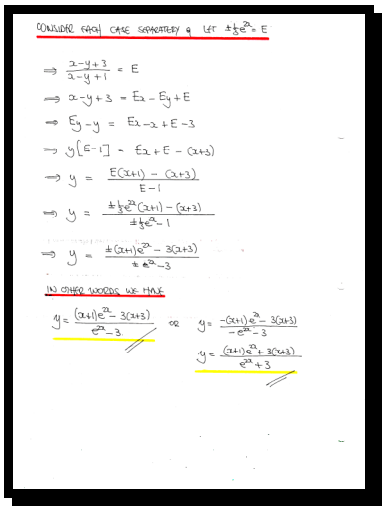
Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}=(x-y+2)^2, \quad y(0)=4.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=\dfrac{(x+1)\mathrm{e}^{2x}\pm 3(x+3)}{\mathrm{e}^{2x}\pm 3}=\dfrac{\pm(x+1)\mathrm{e}^{2x}-3(x+3)}{\pm\mathrm{e}^{2x}-3}}$





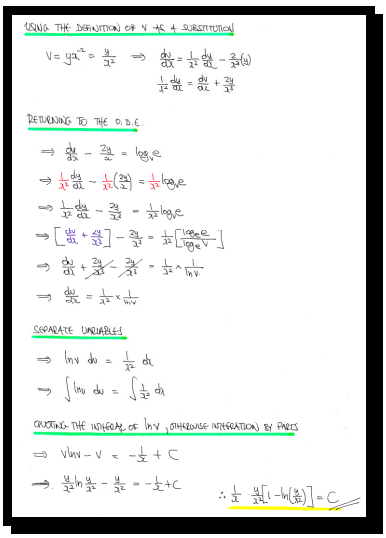
Question 39 **(****+)**

Given that $v = yx^{-2}$ find a general solution for the following differential equation.

$$\frac{dy}{dx} - \frac{2y}{x} = \log_v \mathrm{e}, \quad u > 0, \quad u \neq 1.$$

Given the answer in the form $f(x, y) = \text{constant}$.

$\square$, $\boxed{\dfrac{1}{x} - \dfrac{y}{x^2}\left[1 - \ln\left(yx^{-2}\right)\right] = \text{constant}}$



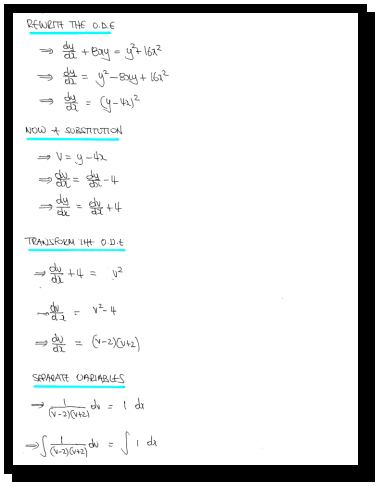
Question 40 **(*****)**

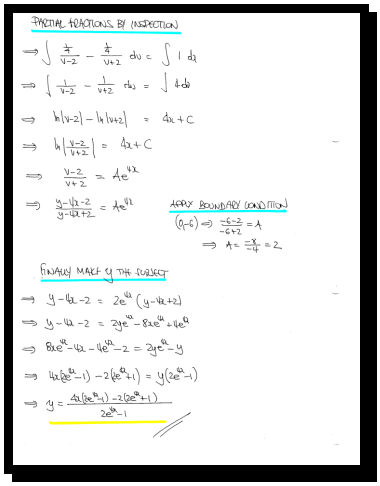
Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+8xy=y^2+16x^2\,,\quad y(0)=-6\,.$$

Given the answer in the form $y=f(x)$.

$$\boxed{}\,,\quad \boxed{y=\frac{4x\left(2\mathrm{e}^{4x}-1\right)-2\left(2\mathrm{e}^{4x}+1\right)}{2\mathrm{e}^{4x}-1}}$$



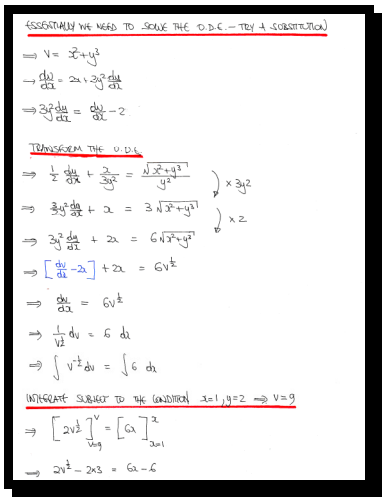


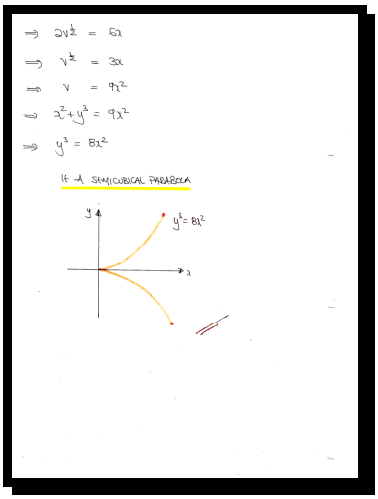
Question 41 **(*****)**

Sketch the curve which passes through the point with coordinates $(1,2)$ and satisfies

$$\frac{1}{2}\frac{dy}{dx}+\frac{x}{3y^2}=\frac{\sqrt{x^2+y^3}}{y^2}.$$

[ ], [graph]





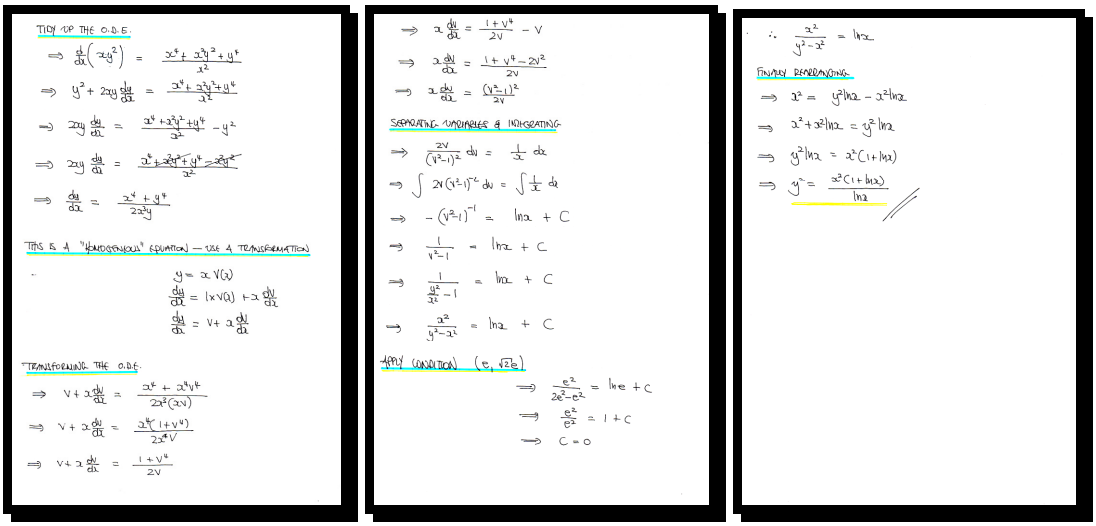
Question 42 (*****)

Solve the differential equation

$$\frac{d}{dx}\left(xy^2\right)=\frac{x^4+x^2y^2+y^4}{x^2},\ y(\mathrm{e})=\sqrt{2}\,\mathrm{e}\,.$$

Give the answer in the form $y^2=f(x)$.

$\square$, $\boxed{y^2=\dfrac{x^2(1+\ln x)}{\ln x}}$

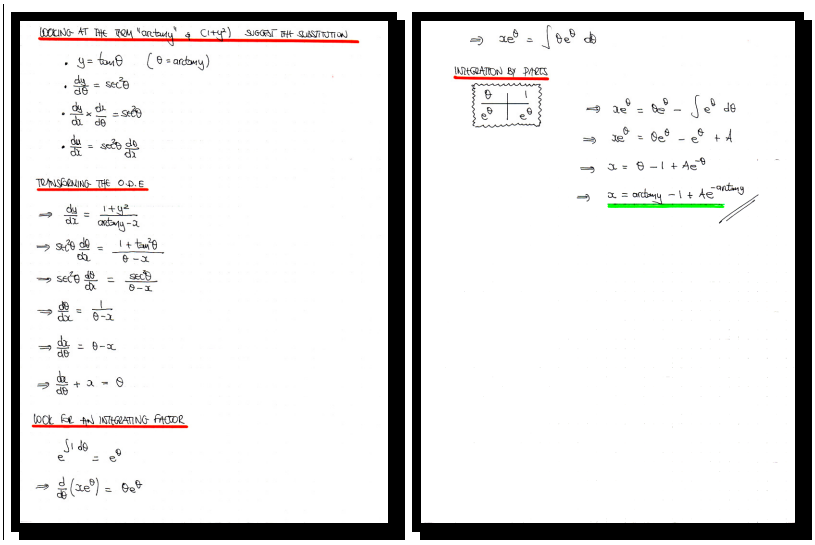


Question 43 (*****)

By using a suitable transformation, or otherwise, find a general solution for the following differential equation

$$\frac{dy}{dx}=\frac{1+y^2}{-x+\arctan y}.$$

$\boxed{}$, $\boxed{x=-1+\arctan y+A\mathrm{e}^{-\arctan y}}$



Question 44 (*****)

The curve with equation $y=f\left(x\right)$ has as asymptote the line $y=1$ and satisfies the differential equation

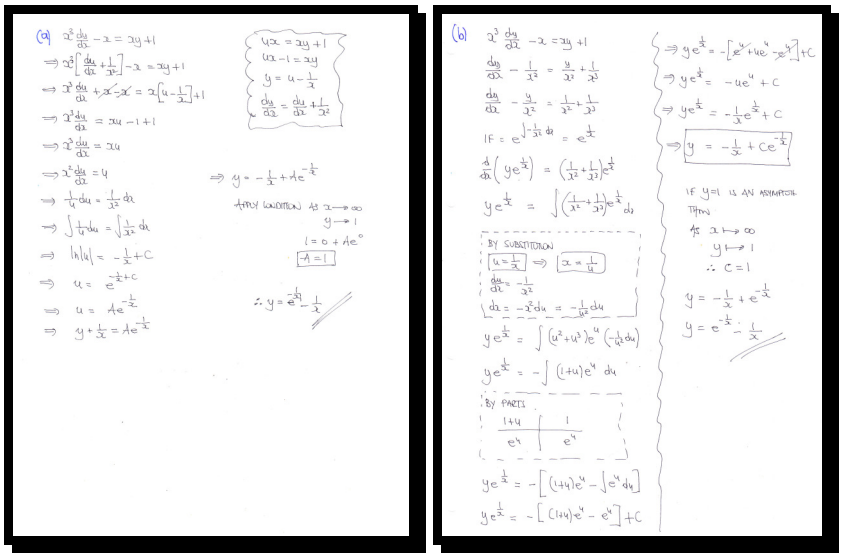
$$x^3\frac{dy}{dx}-x=xy+1,\ x\neq 0.$$

Solve the above differential equation by using …

a) … the substitution $ux=xy+1$, where $u=g\left(x\right)$.

b) … an integrating factor.

$$\boxed{y=\mathrm{e}^{-\frac{1}{x}}-\frac{1}{x}}$$



Question 45 (*****)

Use the substitution $v=\dfrac{y-x}{y+x}$, $y+x\neq 0$, to solve the following differential equation

$$x\frac{dy}{dx}-y=\frac{(1-x)\left(x^2-y^2\right)}{x^3+x^2+x+1},\quad y(0)=1.$$

Give the answer in the form $y=f(x)$.

$\square$, $\boxed{y=x^2+x+1}$

