MISCELLANEOUS SEQUENCES & SERIES QUESTIONS

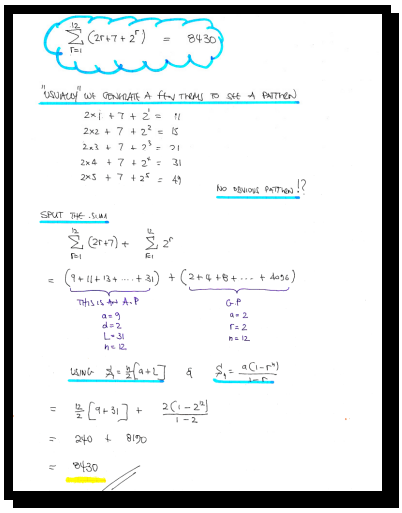
Question 1 **(***+)**

Show that

$$\sum_{r=1}^{12}\left[2r+7+2^r\right] = 8430.$$

Detailed workings must be shown in this question.

, proof



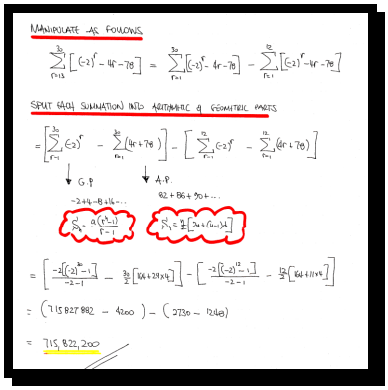
Question 2 (*+)**

Evaluate the following sum

$$\sum_{r=13}^{30}\left[(-2)^r-4r-78\right].$$

Detailed workings must support the answer.

$\square$, $\boxed{715,822,200}$

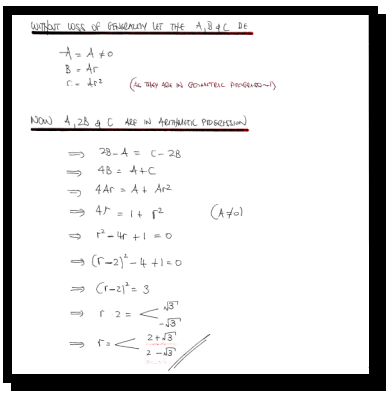


Question 3 (*+)**

Three numbers, A, B, C in that order, are in geometric progression with common ratio r.

Given further that A, $2B$, C in that order are in arithmetic progression, determine the possible values of r.

$\square$, $\boxed{r=2\pm\sqrt{3}}$



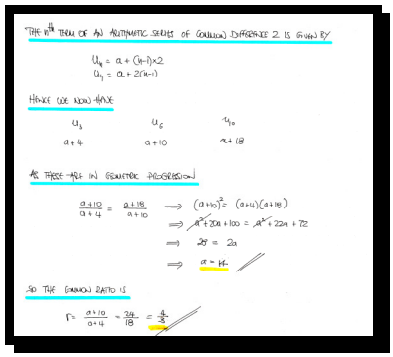
Question 4 (*+)**

An arithmetic series has common difference 2.

The 3^{rd}, 6^{th} and 10^{th} terms of the arithmetic series are the respective first three terms of a geometric series.

Determine in any order the first term of the arithmetic series and the common ratio of the geometric series.

$\square$, $\boxed{a=14}$, $\boxed{r=\frac{4}{3}}$



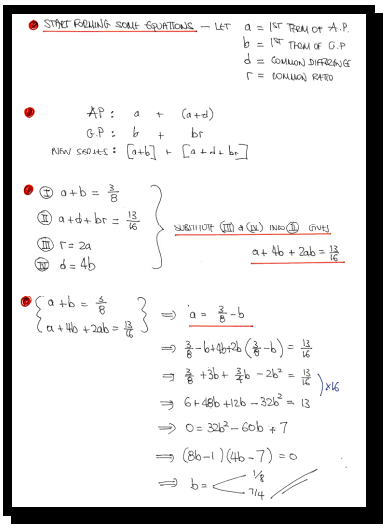
Question 5 (*+)**

Each of the terms of an arithmetic series is added to the corresponding terms of a geometric series, forming a new series with first term $\frac{3}{8}$ and second term $\frac{13}{16}$.

The common difference of the arithmetic series is four times as large as the first term of the geometric series. The common ratio of the geometric series is twice as large as the first term of the arithmetic series.

Determine the possible values of the first term of the geometric series.

$\square$, $\boxed{\frac{1}{8} \cup \frac{7}{4}}$



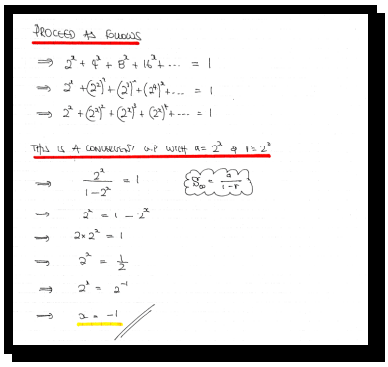
Question 6 (****)

Solve the following equation

$$2^x + 4^x + 8^x + 16^x + 32^x + \ldots = 1.$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x=-1}$



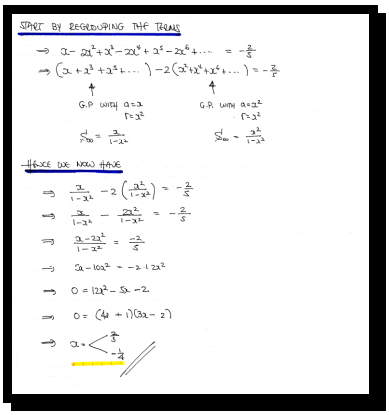
Question 7 (****)

Solve the following equation

$$x - 2x^2 + x^3 - 2x^4 + x^5 - 2x^6 + \ldots = -\frac{2}{5}.$$

You may assume that the left hand side of the equation converges.

$\square$, $\boxed{x=\frac{2}{3} \cup x=-\frac{1}{4}}$

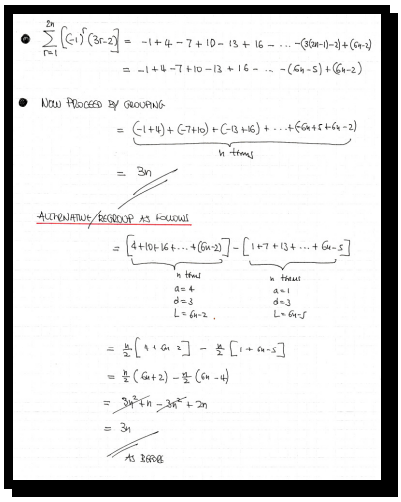


Question 8 (****)

Find in simplified form, in terms of n, the value of

$$\sum_{r=1}^{2n}\left[(3r-2)(-1)^r\right].$$

$\square$, $\boxed{3n}$



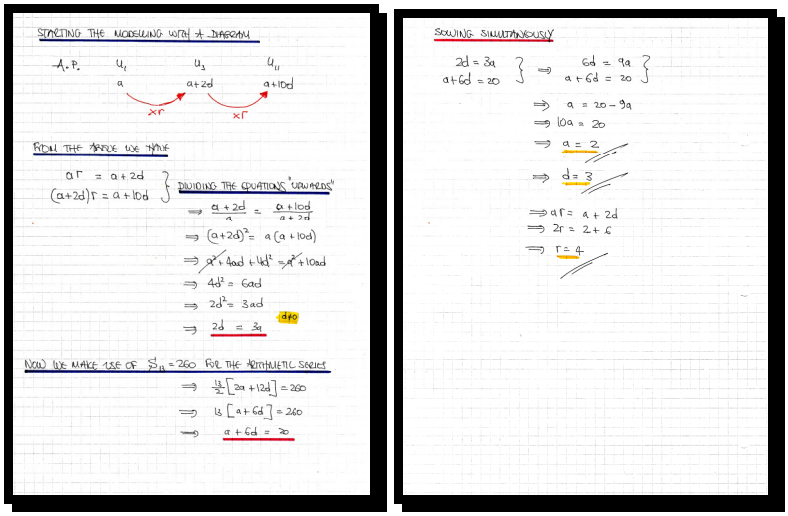
Question 9 (****)

The 1^{st} , 3^{rd} and 11^{th} term of an arithmetic progression are the first three terms of a geometric progression.

It is further given that the sum of the first 13 terms of the arithmetic progression is 260 .

Find, in any order, the common ratio of the geometric progression and the first term and common difference of the arithmetic progression.

$\boxed{}$, $\boxed{r=4}$

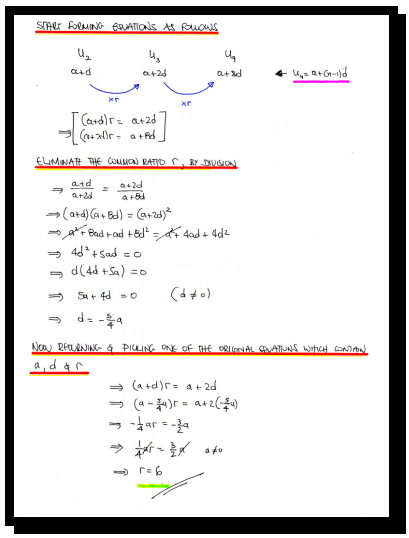


Question 10 (****)

The 2^{nd}, 3^{rd} and 9^{th} term of an arithmetic progression are three consecutive terms of a geometric progression.

Find the common ratio of the geometric progression.

$\boxed{}$, $\boxed{r=6}$



Question 11 (**+)**

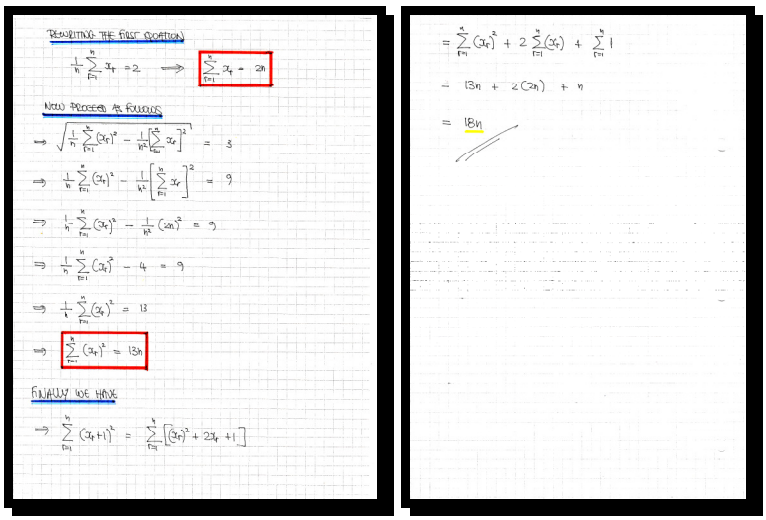
It is given that

$$\frac{1}{n}\sum_{r=1}^{n} x_r = 2 \qquad \text{and} \qquad \sqrt{\frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} x_r\right)^2} = 3 .$$

Determine, in terms of n, the value of

$$\sum_{r=1}^{n}(x_r+1)^2 .$$

$\square$, $\boxed{\displaystyle\sum_{r=1}^{n}(x_r+1)^2 = 18n}$



Question 12 (**+)**

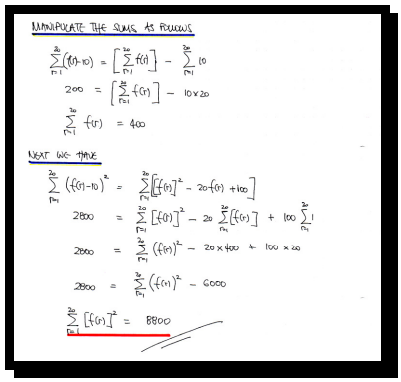
It is given that

$$\sum_{r=1}^{20}\left[f(r)-10\right]=200 \quad \text{and} \quad \sum_{r=1}^{20}\left[f(r)-10\right]^2=2800\,.$$

Find the value of

$$\sum_{r=1}^{20}\left[f(r)\right]^2\,.$$

$\square$, $\boxed{\displaystyle\sum_{r=1}^{20}\left[f(r)\right]^2=8800}$



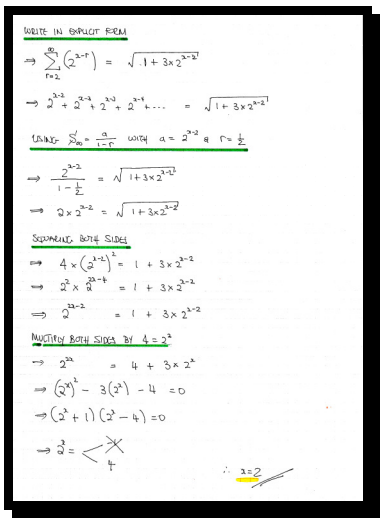
Question 13 (**+)**

Solve the following equation

$$\sum_{r=2}^{\infty}\left(2^{x-r}\right)=\sqrt{1+3\times 2^{x-2}}\ .$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x=2}$



Question 14 (**+)**

The sum of the first 2 terms of an arithmetic progression is 40.

The sum of the first 4 terms of the same arithmetic progression is 130.

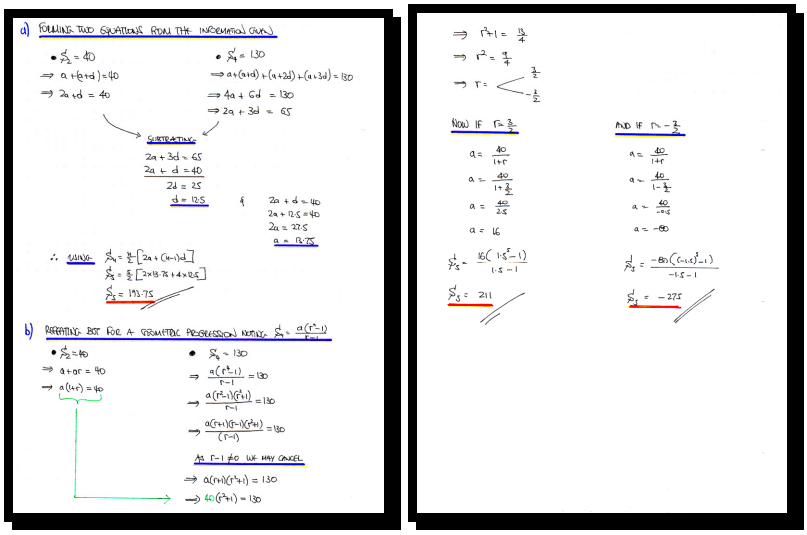
a) Determine the sum of the first 5 terms of the arithmetic progression.

The sum of the first 2 terms of a geometric progression is 40.

The sum of the first 4 terms of the same geometric progression is 130.

b) Find the two possible values of the sum of the first 5 terms of the geometric progression.

$\boxed{}$, $\boxed{S_5=\frac{775}{4}=193.75}$, $\boxed{S_5=211 \quad \text{or} \quad S_5=-275}$



Question 15 (**+)**

Consider the following 2 sequences.

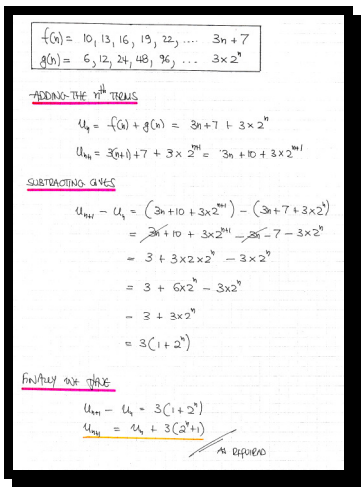
$$10, 13, 16, 19, 22, \ldots \qquad \text{and} \qquad 6, 12, 24, 48, 96, \ldots$$

The sum of the n^{th} term of the first sequence and the n^{th} term of the second sequence is denoted by U_n.

Show algebraically that

$$U_{n+1} = U_n + 3\left(1+2^n\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



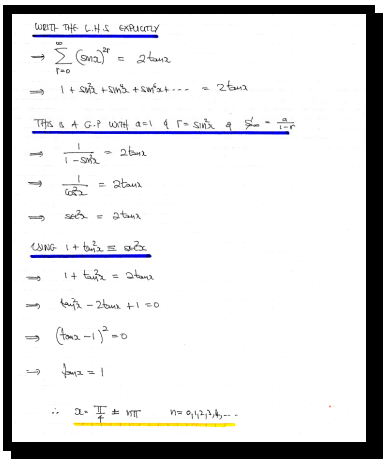
Question 16 (**+)**

Solve the following equation

$$\sum_{r=0}^{\infty}(\sin x)^{2r} = 2\tan x .$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x=\frac{1}{4}\pi\pm n\pi,\quad n=0,1,2,3,...}$



Question 17 (*****)

The product operator $\prod$, is defined as

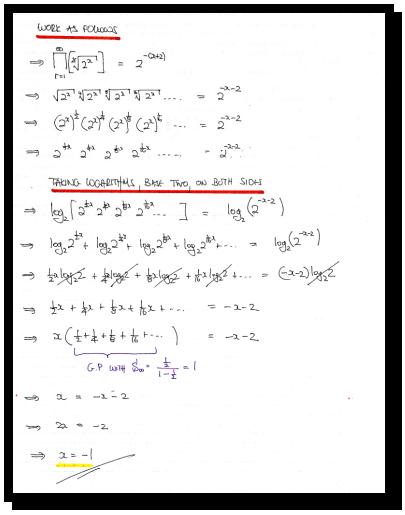
$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k \, .$$

Solve the equation

$$\prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)} .$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x=-1}$



Question 18 (*****)

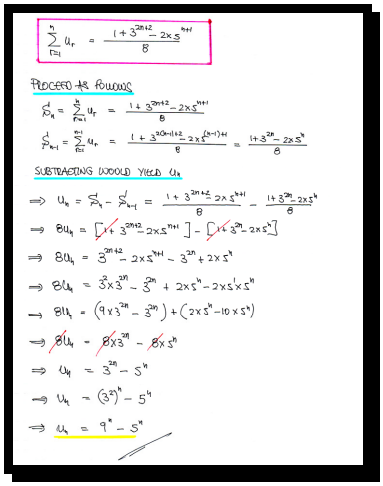
It is given that

$$\sum_{r=1}^{n} u_r = \frac{1+3^{2n+2}-2\times 5^{n+1}}{8},$$

where u_n is the n^{th} term of a sequence.

Find a simplified expression for u_n.

$\square$, $\boxed{u_n = 9^n - 5^n}$



Question 19 (*****)

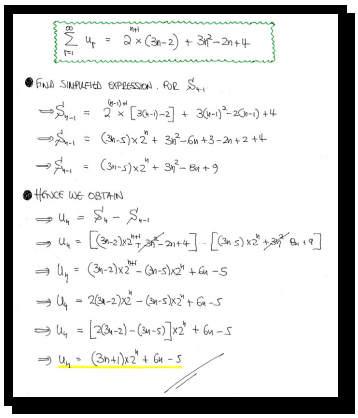
It is given that

$$\sum_{r=1}^{n} u_r = 3n^2 - 2n + 4 + (3n-2)\times 2^{n+1},$$

where u_n is the n^{th} term of a sequence.

Find a simplified expression for u_n.

$\square$, $\boxed{u_n = (3n+1)\times 2^n + 6n - 5}$



Question 20 (*****)

It is given that

$$\sum_{r=1}^{n} u_r = 6^{n+1} - 10\times 2^n + 4,$$

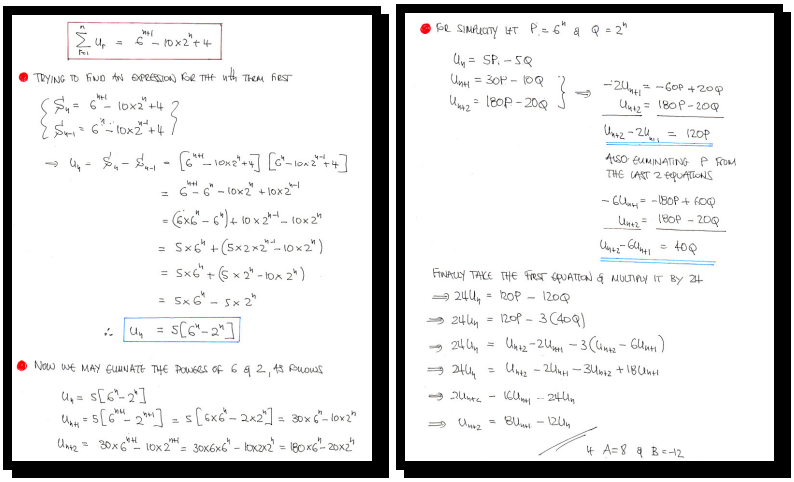
where u_n is the n^{th} term of a sequence.

Show clearly that

$$u_{n+2} = Au_{n+1} + Bu_n,$$

where A and B are integers to be found.

$\boxed{}$, $\boxed{u_{n+2} = 8u_{n+1} - 12u_n}$



$\sum_{r=1}^{n} u_r = 6^{n+1} - 10\times 2^n + 4$

TRYING TO FIND AN EXPRESSION FOR THE n^{th} TERM FIRST

$\left\{ \begin{array}{l} S_n = 6^{n+1} - 10\times 2^n + 4 \\ S_{n-1} = 6^{n} - 10\times 2^{n-1} + 4 \end{array} \right\}$

$\Rightarrow u_n = S_n - S_{n-1} = \left[6^{n+1} - 10\times 2^n + 4\right] - \left[6^n - 10\times 2^{n-1} + 4\right]$

$= 6^{n+1} - 6^n - 10\times 2^n + 10\times 2^{n-1}$

$= \left(6\times 6^n - 6^n\right) + 10\times 2^{n-1} - 10\times 2^n$

$= 5\times 6^n + \left(5\times 2\times 2^{n-1} - 10\times 2^n\right)$

$= 5\times 6^n + \left(5\times 2^n - 10\times 2^n\right)$

$= 5\times 6^n - 5\times 2^n$

$\therefore \boxed{u_n = 5\left[6^n - 2^n\right]}$

NOW WE MAY ELIMINATE THE POWERS OF 6 & 2, AS FOLLOWS

$u_n = 5\left[6^n - 2^n\right]$

$u_{n+1} = 5\left[6^{n+1} - 2^{n+1}\right] = 5\left[6\times 6^n - 2\times 2^n\right] = 30\times 6^n - 10\times 2^n$

$u_{n+2} = 30\times 6^{n+1} - 10\times 2^{n+1} = 30\times 6\times 6^n - 10\times 2\times 2^n = 180\times 6^n - 20\times 2^n$

FOR SIMPLICITY LET $P = 6^n$ & $Q = 2^n$

$u_n = 5P - 5Q$

$u_{n+1} = 30P - 10Q$

$u_{n+2} = 180P - 20Q$

$\Rightarrow \left\{ \begin{array}{l} -2u_{n+1} = -60P + 20Q \\ u_{n+2} = 180P - 20Q \end{array} \right.$

$u_{n+2} - 2u_{n+1} = 120P$

ALSO ELIMINATING P FROM THE LAST 2 EQUATIONS

$-6u_{n+1} = -180P + 60Q$

$u_{n+2} = 180P - 20Q$

$u_{n+2} - 6u_{n+1} = 40Q$

FINALLY TAKE THE FIRST EQUATION & MULTIPLY IT BY 24

$\Rightarrow 24u_n = 120P - 120Q$

$\Rightarrow 24u_n = 120P - 3(40Q)$

$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3\left(u_{n+2} - 6u_{n+1}\right)$

$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3u_{n+2} + 18u_{n+1}$

$\Rightarrow 2u_{n+2} - 16u_{n+1} - 24u_n$

$\Rightarrow u_{n+2} = 8u_{n+1} - 12u_n$

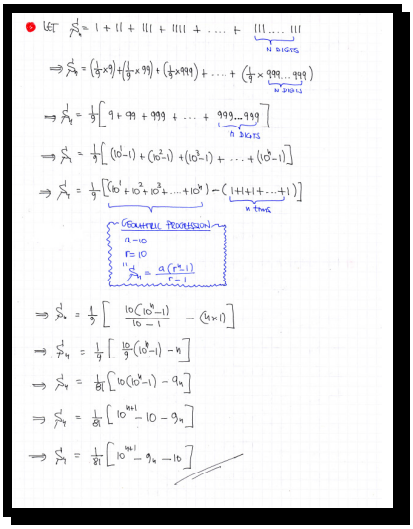
$\therefore A = 8$ & $B = -12$

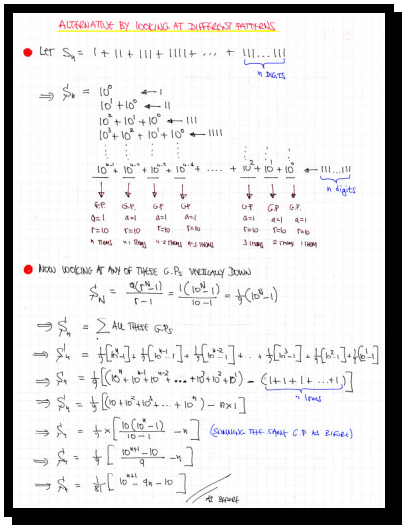
Question 21 (*****)

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + \ldots$$

$$\boxed{}, \quad \boxed{S_n = \frac{1}{81}\left[10^{n+1} - 10 - 9n\right]}$$



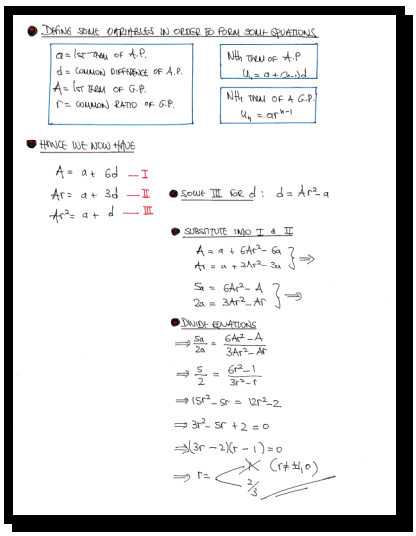


Question 22 (*****)

The first three terms of a geometric progression are the respective 7^{th} term, 4^{th} term, and 2^{nd} term of an arithmetic progression.

Determine the common ratio of the geometric progression.

$\square$, $\boxed{r=\frac{2}{3}}$



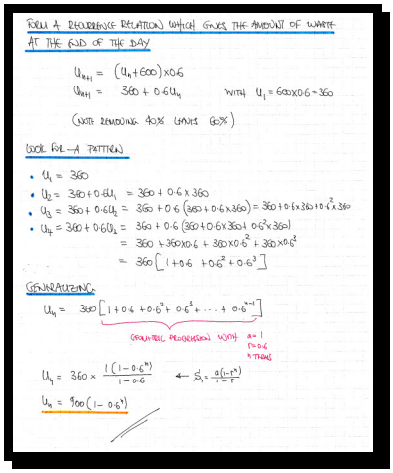
Question 23 (*****)

A factory gets permission to dispose, at the start of every day, 600 kg of waste into a stream of water.

The running stream removes 40% of the any waste present, by the end of the day.

Determine a simplified expression for the amount of waste present in the stream at the end of the n^{th} day.

$\square$, $\boxed{u_n = 900\left[1-\left(\frac{3}{5}\right)^n\right]}$



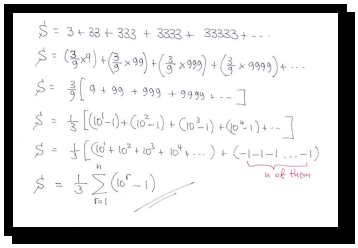
Question 24 (*****)

$$3 + 33 + 333 + 3333 + 33333 + ...$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$S_n = \sum_{r=1}^{n} \left[\frac{1}{3}\left(10^r - 1\right) \right]$$



Question 25 (*****)

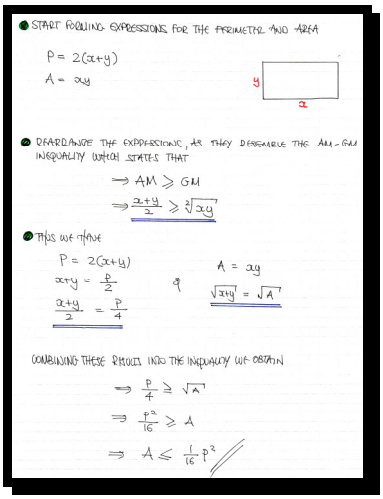
A rectangle has perimeter P and area A.

Show that

$$A \leq f(P),$$

where $f(P)$ is a simplified expression to be found.

$\square$, $\boxed{A \leq \frac{1}{16}P^2}$



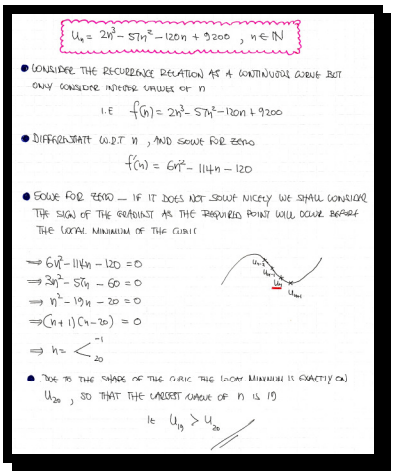
Question 26 (*****)

A sequence of positive integers is generated by the formula

$$u_n = 2n^3 - 57n^2 - 120n + 9200, \quad n \in \mathbb{N}.$$

Determine the largest value of n, such that $u_n > u_{n+1}$.

$\boxed{}$, $\boxed{n=19}$



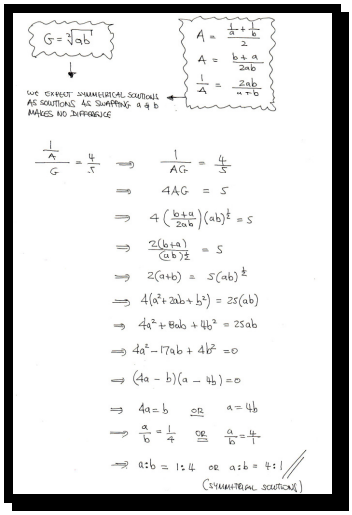
Question 27 (*****)

The geometric mean of two positive numbers a and b is denoted by G.

The arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$ is denoted by A.

Given further that the ratio $\frac{1}{A}:G=4:5$, determine the ratio between a and b.

$\boxed{}$, $\boxed{4:1}$



Question 28 (*****)

A function is defined as

$$[x] \equiv \{\text{the greatest integer less or equal to } x\}.$$

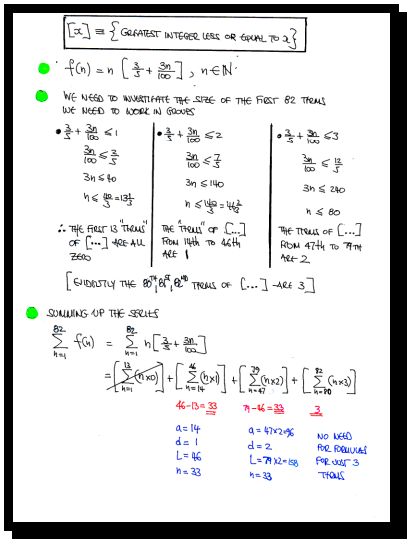
The function f is defined as

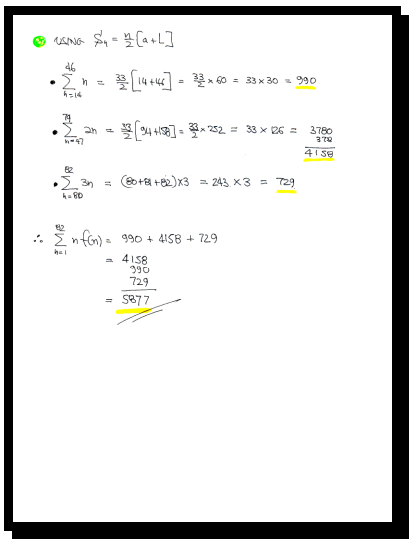
$$f(n) = n\left[\frac{3}{5} + \frac{3n}{100}\right], \quad n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

$\square$, $\boxed{5877}$





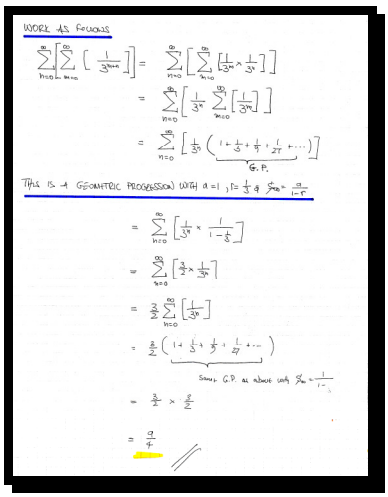
Question 29 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right].$$

Detailed workings must be shown.

$\square$, $\boxed{\frac{9}{4}}$



Question 30 (*****)

The sum to infinity S of the convergent geometric series is given by

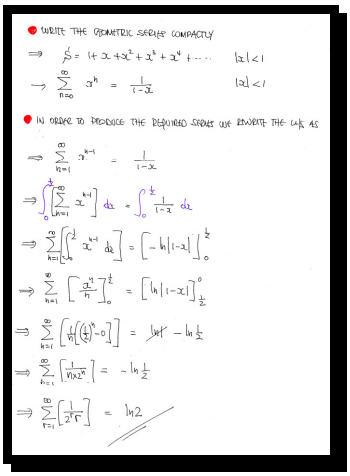
$$S=1+x+x^2+x^3+x^4+\ldots, \quad |x|<1,$$

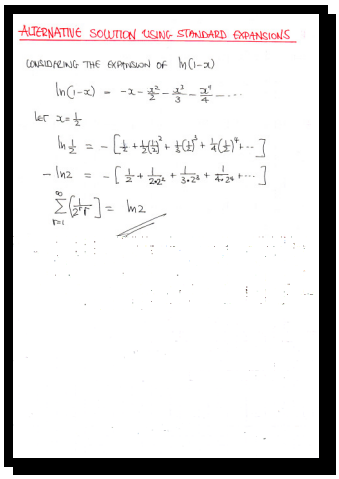
By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty}\left[\frac{1}{r\times 2^r}\right].$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\ln 2}$



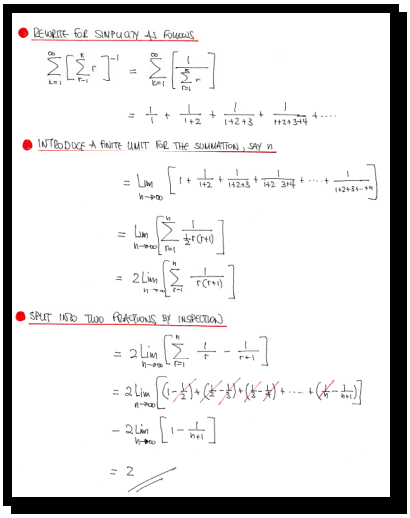


Question 31 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}.$$

[ ], [2]



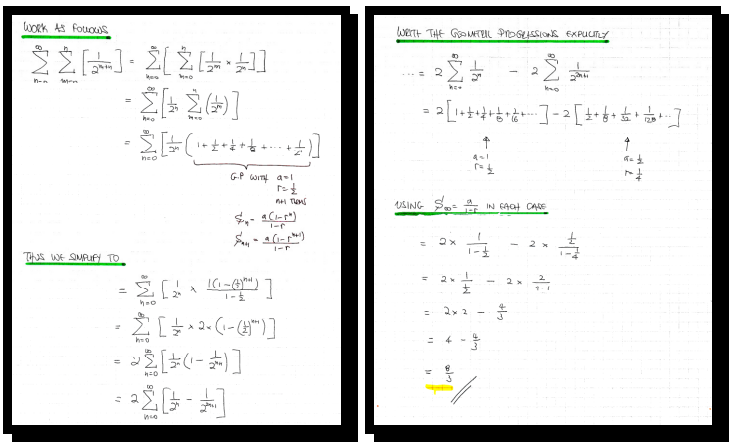
Question 32 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{8}{3}}$



Question 33 (*****)

It is given that L is the sum to infinity of the following convergent series

$$\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right].$$

Use this fact to find, in terms of L, the sum to infinity of this convergent series

$$\sum_{r=1}^{\infty}\left[\frac{r^3}{r!}\right].$$

$\boxed{}$, $\boxed{5L}$

