VECTOR OPERATORS

GRADIENT

$$\text{grad}\,\varphi \equiv \nabla \varphi$$

Question 1

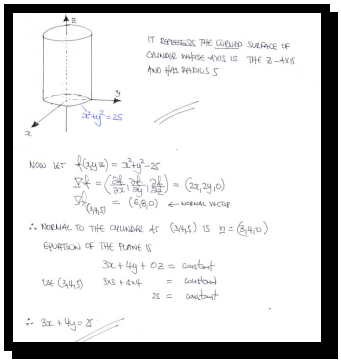
A surface S is given by the Cartesian equation

$$x^2 + y^2 = 25 .$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3,4,5)$.

$$\boxed{3x + 4y = 25}$$



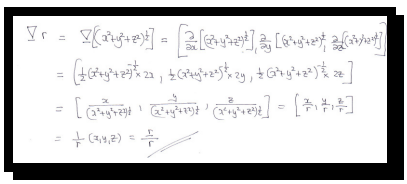
Question 2

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r = \frac{\mathbf{r}}{r} .$$

proof



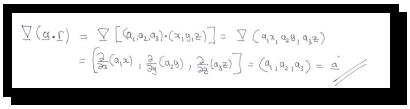
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector find

$$\nabla(\mathbf{a}\cdot\mathbf{r}).$$

$$\boxed{\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}}$$



Question 4

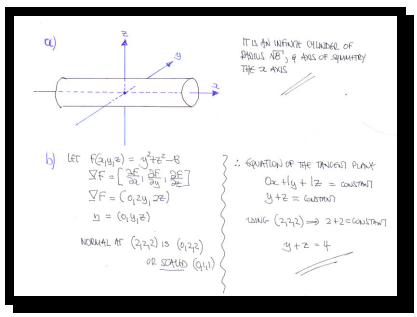
A surface S is defined by the Cartesian equation

$$y^2+z^2=8.$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2,2,2)$.

$$\boxed{y+z=4}$$



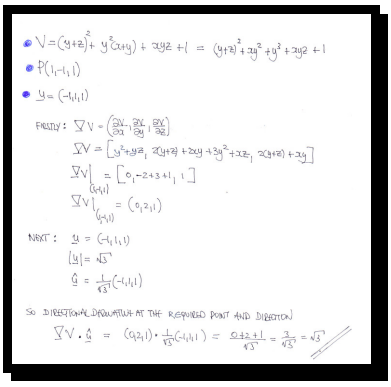
Question 5

The scalar function V is defined as

$$V(x,y,z)=(y+z)^2+y^2(x+y)+xyz+1.$$

Determine the value of the directional derivative of V at the point $P(1,-1,1)$, in the direction $-\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$$\boxed{\sqrt{3}}$$



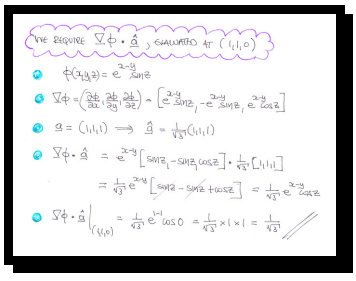
Question 6

The scalar function φ is defined as

$$\varphi(x,y,z)=\mathrm{e}^{x-y}\sin z.$$

Determine the value of the directional derivative of φ at the point $P(1,1,0)$, in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$$\boxed{\frac{1}{\sqrt{3}}}$$



Question 7

The point $P(1,2,3)$ lies on the surface with Cartesian equation

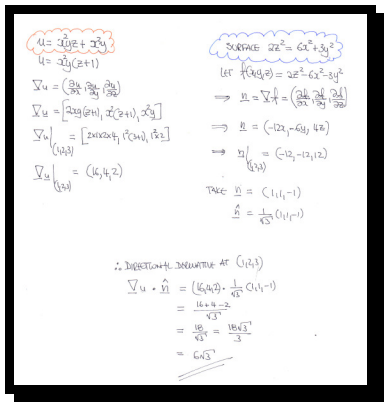
$$2z^2 = 6x^2 + 3y^2 .$$

The scalar function u is defined as

$$u(x, y, z) = x^2 yz + x^2 y .$$

Determine the value of the directional derivative of u at the point P in the direction to the normal at P.

$$\boxed{6\sqrt{3}}$$



Question 8

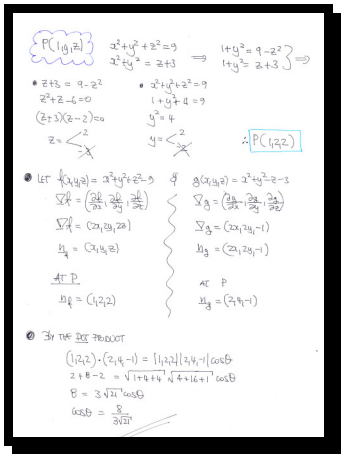
The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3 .$$

The two surfaces intersect each other at an angle θ, at the point P.

Given further that P lies in the first octant, determine the exact value of $\cos\theta$.

$$\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$



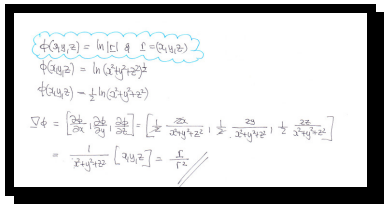
Question 9

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\varphi(r)=\ln r$, show that

$$\nabla\varphi(r)=\frac{\mathbf{r}}{r^2}.$$

proof



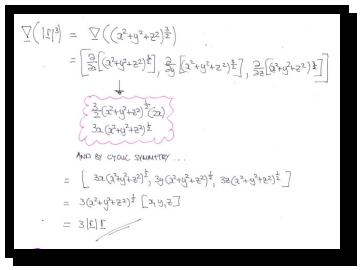
Question 10

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Show by detailed workings that

$$\nabla r^3\equiv 3r\mathbf{r}.$$

proof



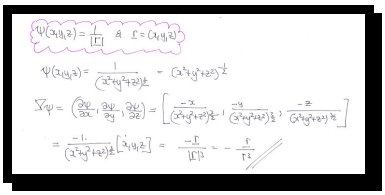
Question 11

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\psi(r)=\dfrac{1}{r}$, show that

$$\nabla\psi(r)=-\frac{\mathbf{r}}{r^3}.$$

proof



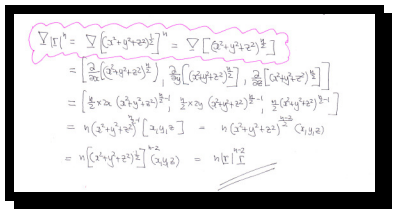
Question 12

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Show clearly that

$$\nabla\left(r^n\right)=nr^{n-2}\mathbf{r}.$$

proof



Question 13

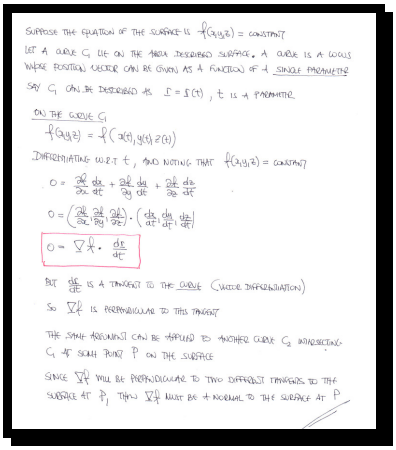
The surface S has Cartesian equation

$$f(x, y, z) = \text{constant},$$

where f is a smooth function.

Given that $\nabla f \neq \mathbf{0}$, show that ∇f is a normal to S.

proof



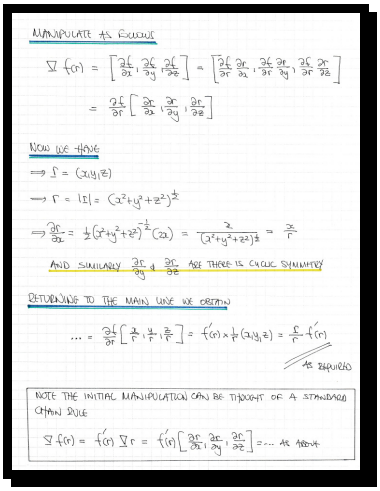
Question 14

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla f(r)=\frac{\mathbf{r}}{r}f'(r).$$

, proof



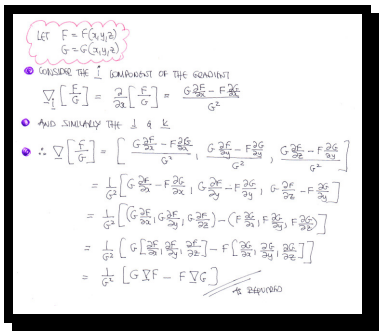
Question 15

The smooth functions $F(x,y,z)$ and $G(x,y,z)$ are given.

Show that

$$\nabla\left[\frac{F}{G}\right]=\frac{G(\nabla F)-F(\nabla G)}{G^2}.$$

proof



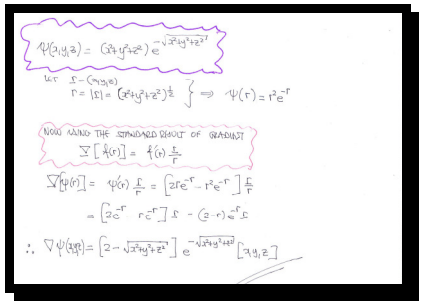
Question 16

$$\Psi(x,y,z)=\left(x^2+y^2+z^2\right)\mathrm{e}^{-\sqrt{x^2+y^2+z^2}}$$

Show that

$$\nabla\left[\Psi(x,y,z)\right]=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\left(2-\sqrt{x^2+y^2+z^2}\right)\mathrm{e}^{-\sqrt{x^2+y^2+z^2}}.$$

proof



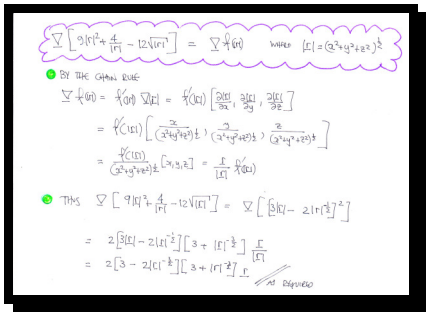
Question 17

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Show that

$$\nabla\left(9r^2+\frac{4}{r}-12\sqrt{r}\right)=2\left(3-2r^{-\frac{3}{2}}\right)\left(3+r^{-\frac{3}{2}}\right)\mathbf{r}.$$

proof



Question 18

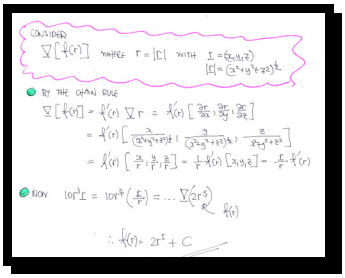
A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

The smooth function $f(r)$ satisfies

$$\nabla\left[f(r)\right]=10r^3\mathbf{r}.$$

Determine a simplified expression for $f(r)$.

$$f(r)=2r^5+C$$



DIVERGENCE

$$\text{div}\,\mathbf{F} \equiv \nabla \cdot \mathbf{F}$$

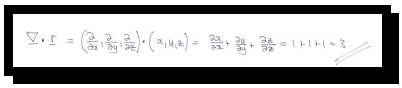
Question 1

A Cartesian position vector is denoted by $\mathbf{r}$.

Determine the value of

$$\nabla \cdot \mathbf{r}.$$

$$\boxed{\nabla \cdot \mathbf{r} = 3}$$



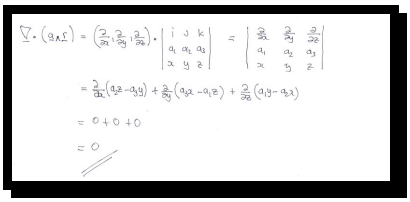
Question 2

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, find

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}).$$

$$\boxed{\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = 0}$$



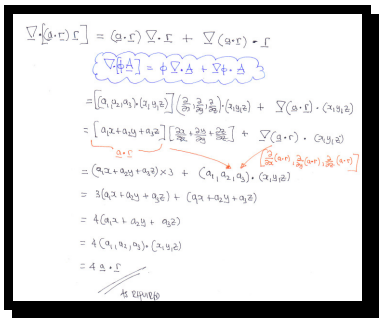
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla\cdot\left[(\mathbf{a}\cdot\mathbf{r})\mathbf{r}\right]=4\mathbf{a}\cdot\mathbf{r}\,.$$

proof



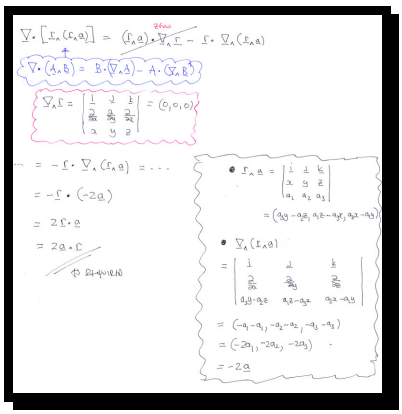
Question 4

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla\cdot\left[\mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{a})\right]=2\mathbf{a}\cdot\mathbf{r}\,.$$

proof



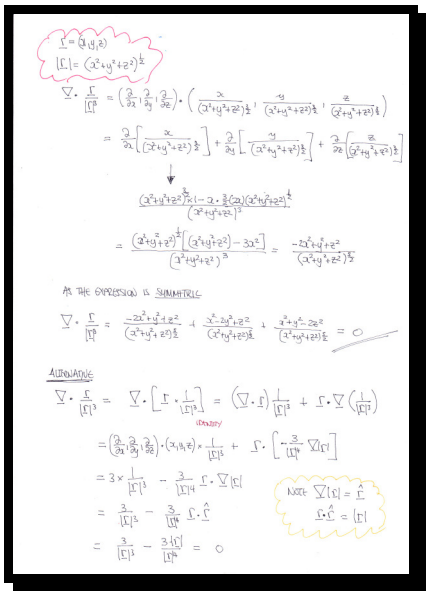
Question 5

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show clearly that

$$\nabla\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right)=0\,.$$

proof



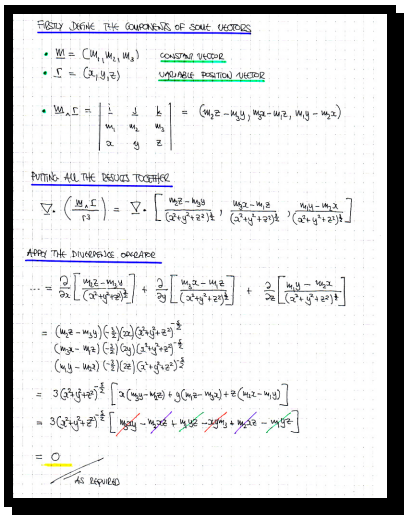
Question 6

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{m}$ is a constant vector, show that

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{|\mathbf{r}|^3} \right) = 0 .$$

, proof



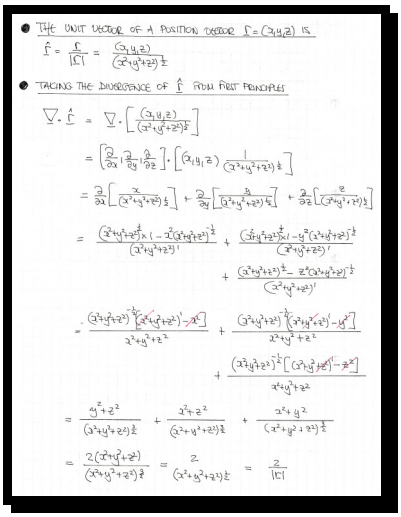
Question 7

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

Show clearly that

$$\nabla\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{2}{|\mathbf{r}|}\,.$$

proof



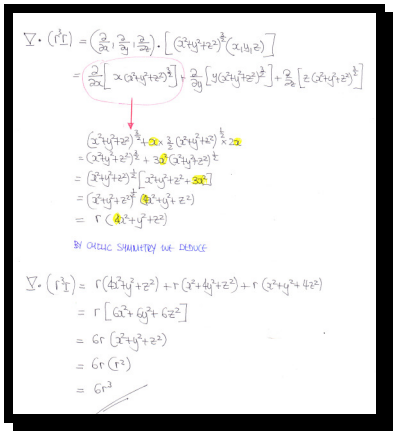
Question 8

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show, with a detailed method, that

$$\nabla\cdot\left(\mathbf{r}|\mathbf{r}|^3\right)=6|\mathbf{r}|^3\,.$$

proof



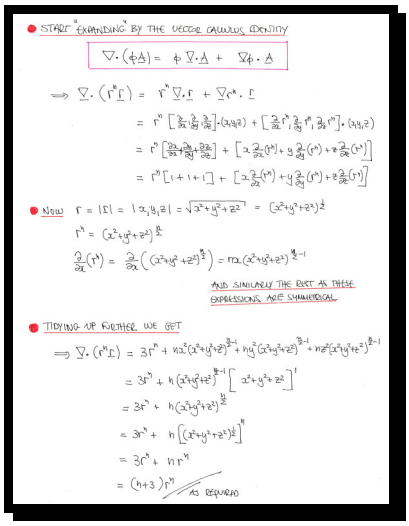
Question 9

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Show that

$$\nabla\cdot\left(r^n\mathbf{r}\right)=(n+3)r^n.$$

[ ], proof



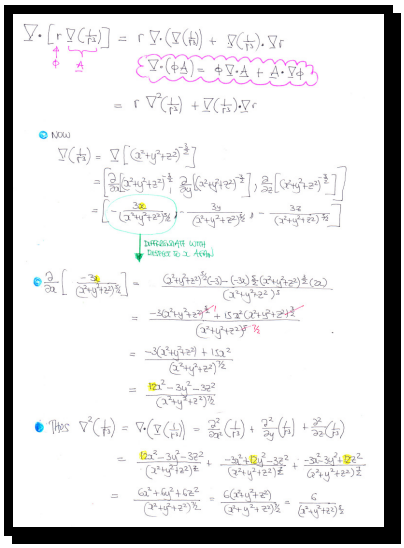
Question 10

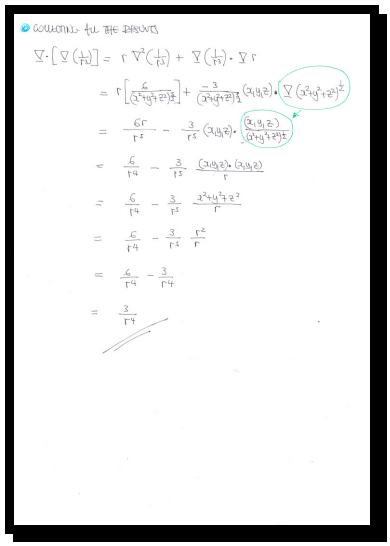
$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show, with a detailed method, that

$$\nabla\cdot\left[|\mathbf{r}|\nabla\left(\frac{1}{|\mathbf{r}|^3}\right)\right]=\frac{3}{|\mathbf{r}|^4}\,.$$

proof





CURL

$$\text{curl}\,\mathbf{F} \equiv \nabla \wedge \mathbf{F}$$

Question 1

A Cartesian vector is denoted by

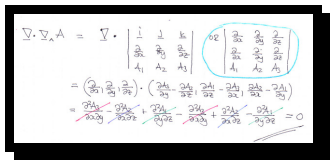
$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} ,$$

where $A_i = f\left(x, y, z\right)$, $i = 1, 2, 3$

Given that A_i are differentiable functions, show that

$$\nabla \cdot \nabla \wedge \mathbf{A} = 0 .$$

proof



Question 2

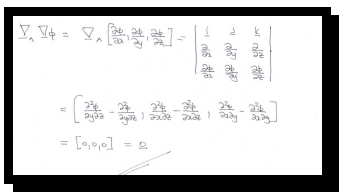
A function φ is denoted by

$$\varphi = \varphi\left(x, y, z\right) .$$

Given that φ is differentiable show that

$$\nabla \wedge \nabla \varphi = \mathbf{0} .$$

proof



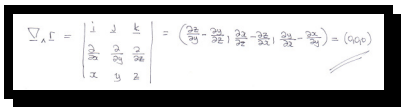
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Determine the value of

$$\nabla\wedge\mathbf{r}\,.$$

$$\boxed{\nabla\wedge\mathbf{r}=\mathbf{0}}$$



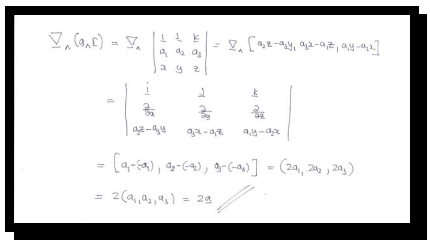
Question 4

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, find

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{r})\,.$$

$$\boxed{\nabla\wedge(\mathbf{a}\wedge\mathbf{r})=2\mathbf{a}}$$



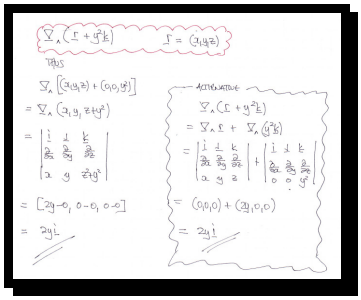
Question 5

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Determine

$$\nabla\wedge\left(\mathbf{r}+y^2\mathbf{k}\right).$$

$$\boxed{\nabla\wedge\left(\mathbf{r}+y^2\mathbf{k}\right)=2y\mathbf{i}}$$



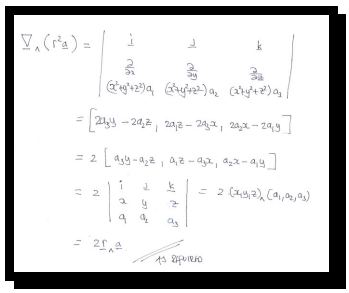
Question 6

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla\wedge\left(r^2\mathbf{a}\right)\equiv 2\mathbf{r}\wedge\mathbf{a}\,.$$

proof



Question 7

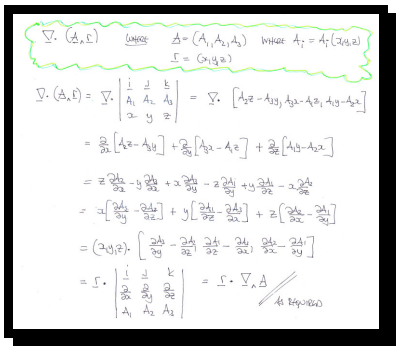
A vector function $\mathbf{A}$ is defined as

$$\mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}.$$

Given that the standard Cartesian position vector is denoted by $\mathbf{r}$, show that

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{r}) \equiv \mathbf{r} \cdot \nabla \wedge \mathbf{A}.$$

proof

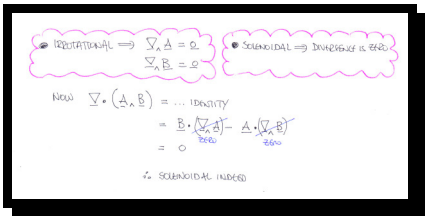


Question 8

The smooth vector functions, $\mathbf{A}$ and $\mathbf{B}$, are both irrotational.

Show that $\mathbf{A} \wedge \mathbf{B}$ is solenoidal.

proof



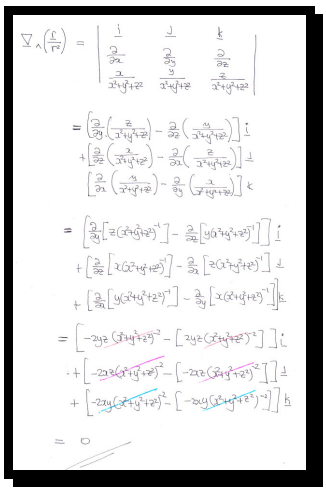
Question 9

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Find the value of

$$\nabla\wedge\left(\frac{\mathbf{r}}{|\mathbf{r}|^2}\right).$$

$$\boxed{\mathbf{0}}$$

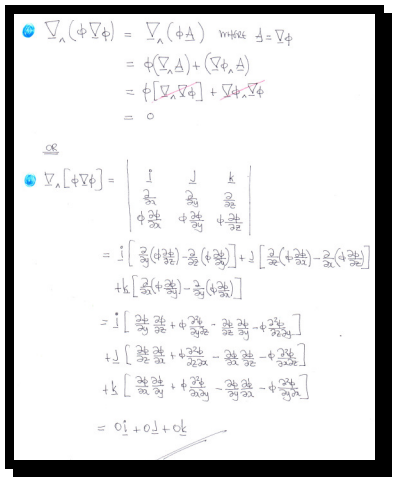


Question 10

If $\varphi = \varphi(x, y, z)$ is a smooth function, prove that.

$$\nabla \wedge (\varphi \nabla \varphi) = \mathbf{0}.$$

proof



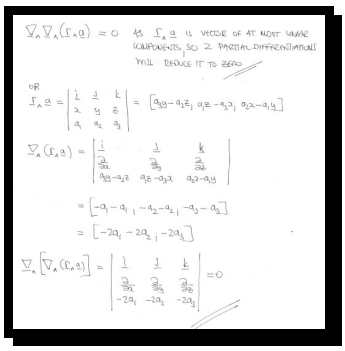
Question 11

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Given that $\mathbf{a}$ is a constant vector, find the value of

$$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}).$$

0



Question 12

The irrotational vector field $\mathbf{F}$ is given by

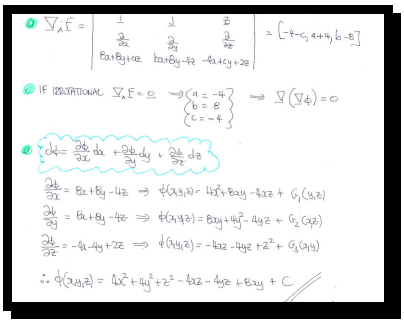
$$\mathbf{F}=(8x+8y+az)\mathbf{i}+(bx+8y-4z)\mathbf{j}+(-4x+cy+2z)\mathbf{k}\,,$$

where a, b and c are scalar constants.

Determine a smooth scalar function $\varphi(x,y,z)$ such that

$$\nabla\varphi=\mathbf{F}\,.$$

$$\boxed{\varphi(x,y,z)=(2x+2y-x)^2+\text{constant}=4x^2+4y^2+z^2-4xz-4yz+8xy+\text{constant}}$$



Question 13

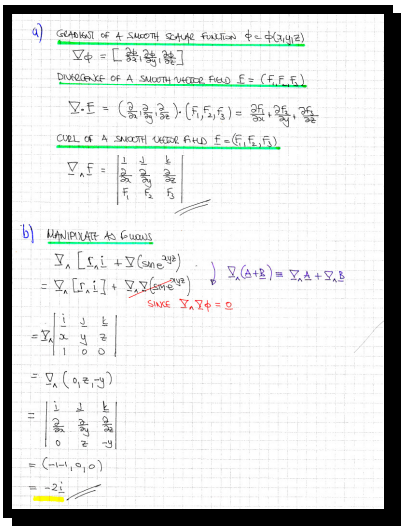
a) Define the vector calculus operators grad, div and curl.

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

b) Determine the vector

$$\nabla\wedge\left[\mathbf{r}\wedge\mathbf{i}+\nabla\left(\sin\mathrm{e}^{xyz}\right)\right].$$

$\square$, $\boxed{\nabla\wedge\left[\mathbf{r}\wedge\mathbf{i}+\nabla\left(\sin\mathrm{e}^{xyz}\right)\right]=-2\mathbf{i}}$



Question 14

The smooth functions f and $\mathbf{A}$ are defined as

$$f = f(x,y,z) \quad \text{and} \quad \mathbf{A} = A_1(x,y,z)\mathbf{i} + A_2(x,y,z)\mathbf{j} + A_3(x,y,z)\mathbf{k}\,.$$

a) Define the vector calculus operators grad, div and curl with reference to f and $\mathbf{A}$.

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

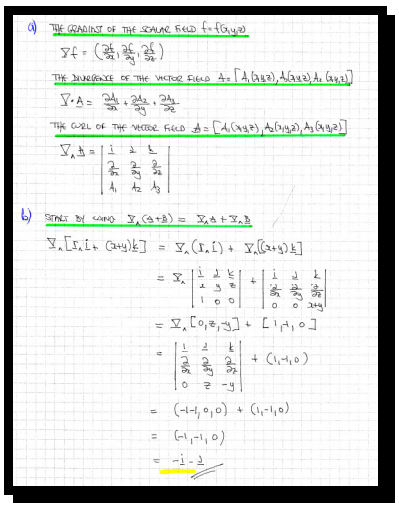
b) Determine the vector

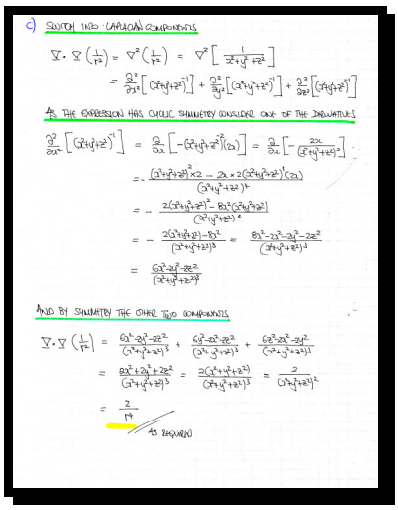
$$\nabla \wedge \left[\mathbf{r} \wedge \mathbf{i} + (x+y)\mathbf{k} \right].$$

c) Show that

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \frac{2}{r^4}\,.$$

$\boxed{}$, $\boxed{-\mathbf{i}-\mathbf{j}}$





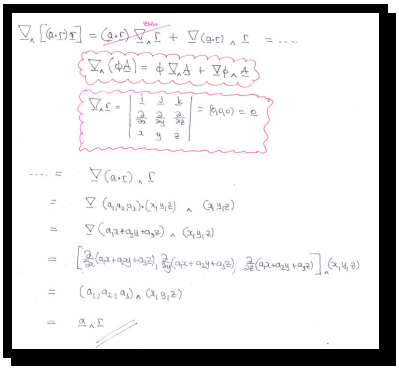
Question 15

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla\wedge\left[(\mathbf{a}\cdot\mathbf{r})\mathbf{r}\right]=\mathbf{a}\wedge\mathbf{r}\,.$$

proof



Question 16

The smooth functions $\mathbf{A}=\mathbf{A}(x,y,z)$, $\mathbf{B}=\mathbf{B}(x,y,z)$ and $\varphi=\varphi(x,y,z)$ are defined as

$$\mathbf{A}=yz\,\mathbf{i}-x^2y\,\mathbf{j}+xz^2\,\mathbf{k}\,,\qquad \mathbf{B}=x^2\,\mathbf{i}+yz\,\mathbf{j}-xy\,\mathbf{k}\qquad\text{and}\qquad \varphi=x\,y\,z\,.$$

Find, in simplified form, expressions for

a) $(\mathbf{A}\cdot\nabla)\varphi$.

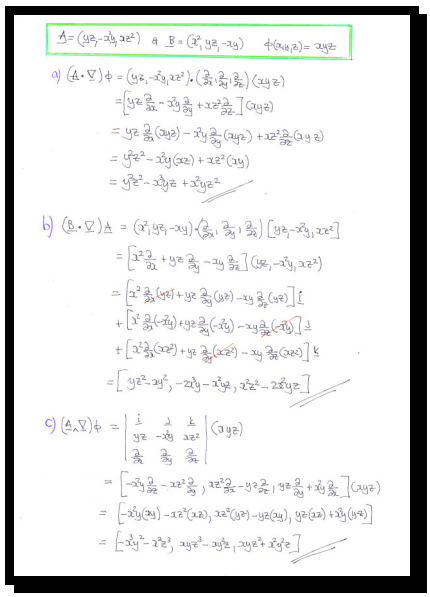
b) $(\mathbf{B}\cdot\nabla)\mathbf{A}$.

c) $(\mathbf{A}\wedge\nabla)\varphi$

$$\boxed{(\mathbf{A}\cdot\nabla)\varphi=y^2z^2-x^3yz+x^2yz^2}\,,$$

$$\boxed{(\mathbf{B}\cdot\nabla)\mathbf{A}=\left(yz^2-xy^2\right)\mathbf{i}+\left(-2x^3y-x^2yz\right)\mathbf{j}+\left(x^2z^2-2x^2yz\right)\mathbf{k}}\,,$$

$$\boxed{(\mathbf{A}\wedge\nabla)\varphi=\left(-x^3y^2-x^2z^3\right)\mathbf{i}+\left(xyz^3-xy^2z\right)\mathbf{j}+\left(xyz^2+x^2y^2z\right)\mathbf{k}}$$



Question 17

a) Define the vector calculus operators grad, div and curl.

b) Given that $\varphi(x,y,z)$ and $\psi(x,y,z)$ are smooth functions, show that

$$\nabla\wedge[\varphi\nabla\psi]\equiv\nabla\varphi\wedge\nabla\psi\,.$$

c) Evaluate

$$\nabla\wedge\left[\nabla\wedge\left[\nabla\wedge\left[\nabla\wedge\left((x+y+z)^3\mathbf{i}+\left(4x^3-yz\right)\mathbf{j}+(xyz)\mathbf{k}\right)\right]\right]\right].$$

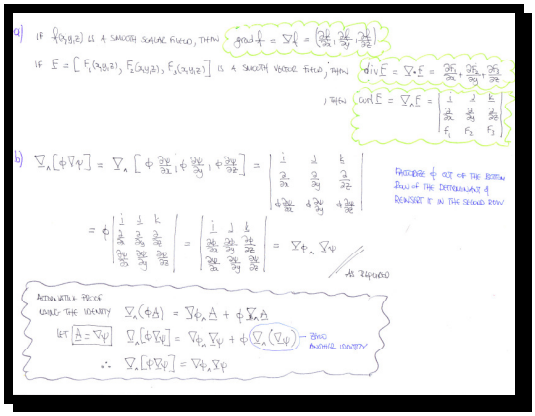
d) Use the vector function

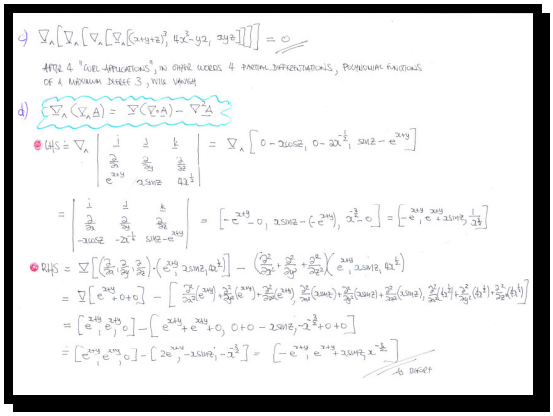
$$\mathbf{A}(x,y,z)=\left(\mathrm{e}^{x+y}\right)\mathbf{i}+(x\sin z)\,\mathbf{j}+\left(4\sqrt{x}\right)\mathbf{k}$$

to verify the validity of the identity

$$\nabla\wedge(\nabla\wedge\mathbf{A})\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}\,.$$

$$\boxed{\nabla\wedge\left[\nabla\wedge\left[\nabla\wedge\left[\nabla\wedge\left((x+y+z)^3\mathbf{i}+\left(4x^3-yz\right)\mathbf{j}+(xyz)\mathbf{k}\right)\right]\right]\right]=\mathbf{0}}$$





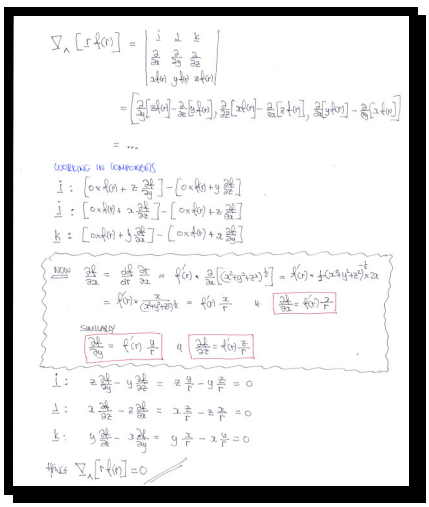
Question 18

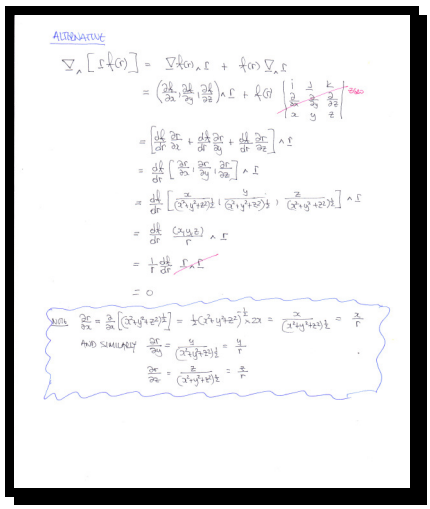
A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla\wedge\left[\mathbf{r}\,f(r)\right]=\mathbf{0}.$$

proof



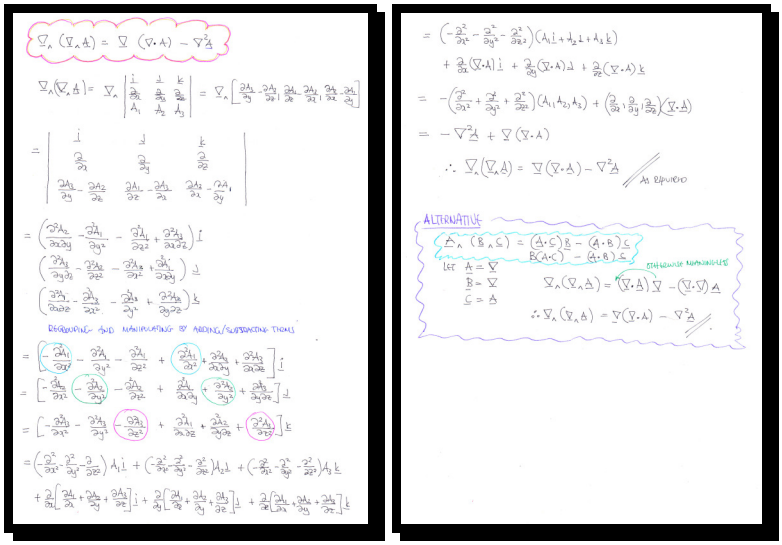


Question 19

Given that $\mathbf{A}=\mathbf{A}(x,y,z)$ is a twice differentiable vector function, show that

$$\nabla\wedge(\nabla\wedge\mathbf{A})\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}\,.$$

proof

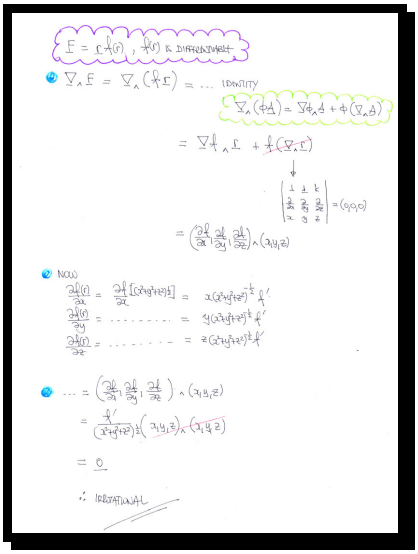


Question 20

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show that $\mathbf{r}\,f\left(|\mathbf{r}|\right)$ is irrotational, where $f\left(|\mathbf{r}|\right)$ is differentiable function of $|\mathbf{r}|$.

proof



Question 21

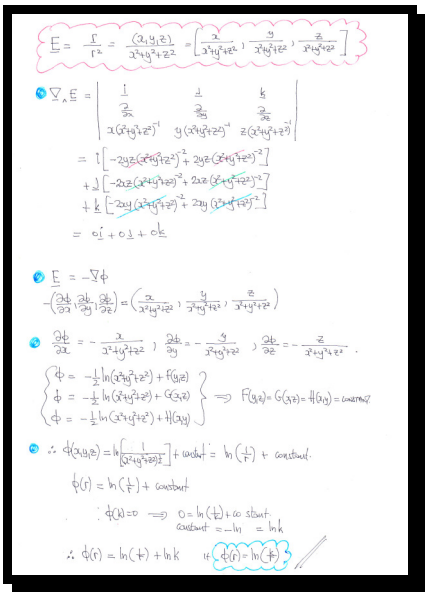
The vector function $\mathbf{E}$ satisfies

$$\mathbf{E}=\frac{\mathbf{r}}{|\mathbf{r}|^2},\ |\mathbf{r}|\neq 0,$$

where $\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Show that $\mathbf{E}$ is irrotational, and find a smooth scalar function $\varphi(|\mathbf{r}|)$, with $\varphi(k)=0$, so that $\mathbf{E}=-\nabla\varphi(|\mathbf{r}|)$.

$$\boxed{\varphi(|\mathbf{r}|)=\ln\left(\frac{k}{|\mathbf{r}|}\right)}$$



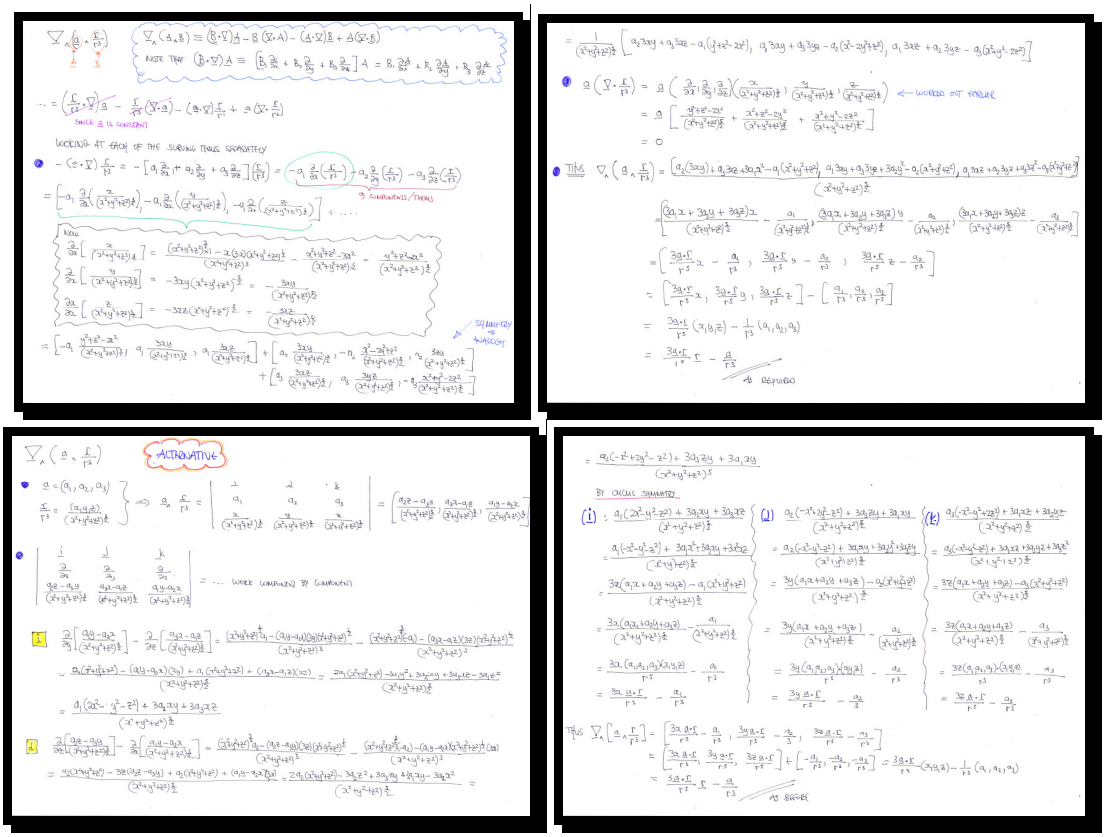
Question 22

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla\wedge\left(\mathbf{a}\wedge\frac{\mathbf{r}}{r^3}\right)=\frac{3\mathbf{a}\cdot\mathbf{r}}{r^5}\mathbf{r}-\frac{\mathbf{a}}{r^3}\,.$$

proof



Question 23

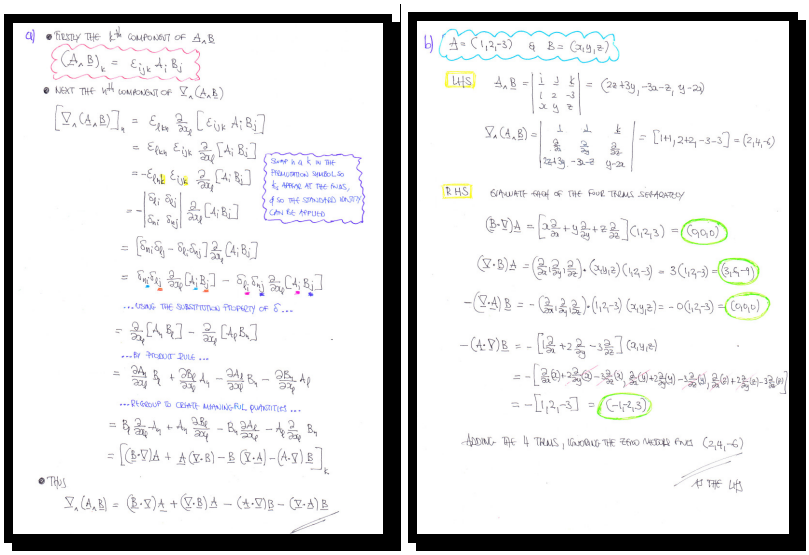
$$\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla)\mathbf{A} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\nabla \cdot \mathbf{A})\mathbf{B} \, .$$

a) Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions, use index summation notation to prove the validity of the above vector identity.

b) Verify the validity of the vector identity if

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{B} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

both sides yield $2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$



LAPLACIAN

$$\nabla \cdot \nabla \varphi \equiv \nabla^2 \varphi$$

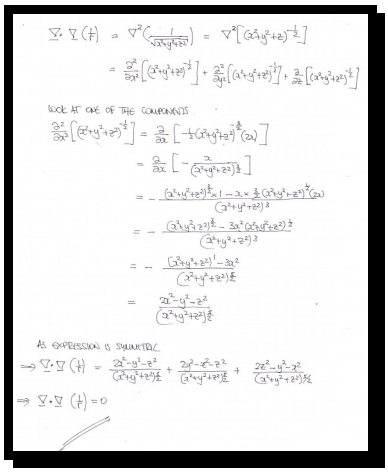
Question 1

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Determine the value of

$$\text{div}\left[\text{grad}\left(\frac{1}{r}\right)\right],\ r \neq 0.$$

$$\boxed{\text{div}\left[\text{grad}\left(\frac{1}{r}\right)\right] = 0}$$



Question 2

A smooth scalar field is denoted by $\varphi = \varphi(x, y, z)$ and a smooth vector field is denoted by $\mathbf{A} = \mathbf{A}(x, y, z)$.

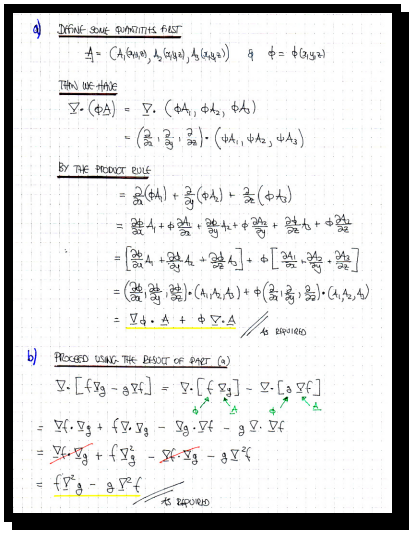
a) Use the standard definitions of vector operators to show that

$$\nabla \cdot (\varphi \mathbf{A}) = \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A} .$$

b) Given further that f and g are functions of x, y and z, whose second partial derivatives exist, deduce that

$$\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f .$$

, proof



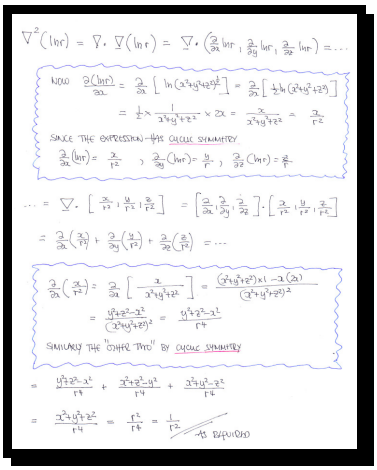
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$, $r\neq 0$.

Show that

$$\nabla^2\left(\ln r\right)=\frac{1}{r^2}.$$

proof



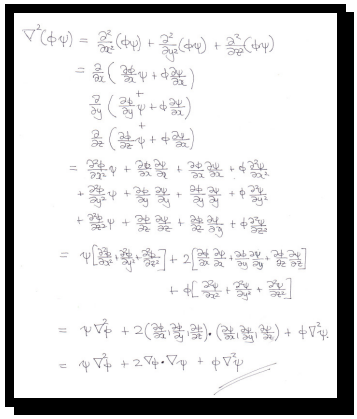
Question 4

It is given that $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ are twice differentiable functions.

Show, with a detailed method, that

$$\nabla^2(\varphi\psi) = \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi + \varphi\nabla^2\psi .$$

proof



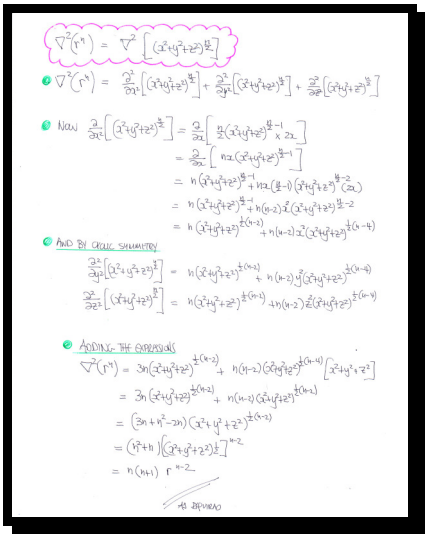
Question 5

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show clearly that

$$\nabla^2\left(|\mathbf{r}|^n\right)=n(n+1)|\mathbf{r}|^{n-2}\,.$$

proof



Question 6

It is given that

$$\varphi(x, y, z) = z + \sinh x \sin y .$$

a) Verify that φ is a solution of Laplace's equation

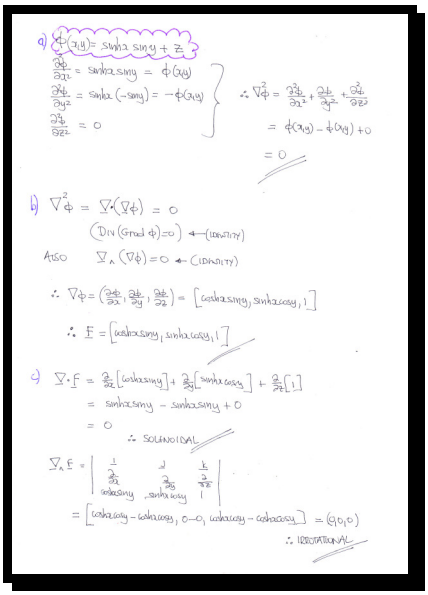
$$\nabla^2 \varphi = 0 .$$

b) Hence find a vector field $\mathbf{F}$, so that

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{F} = \mathbf{0} .$$

c) Verify that $\mathbf{F}$ found in part **(b)** is solenoidal and irrotational.

$$\boxed{\mathbf{F} = (\cosh x \sin y)\mathbf{i} + (\sinh x \cos y)\mathbf{j} + \mathbf{k}}$$



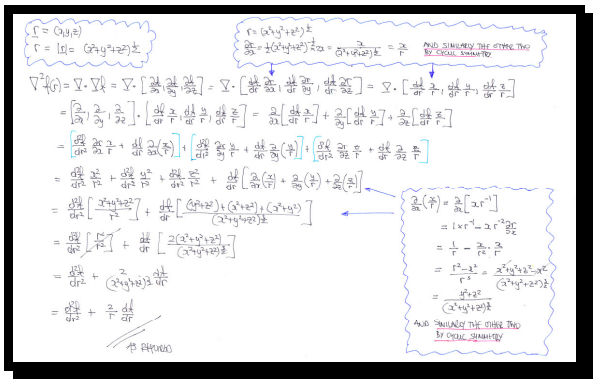
Question 7

A Cartesian position vector is denoted by $\mathbf{r}$ and $r=|\mathbf{r}|$.

Given that $f(r)$ is a twice differentiable function, show that

$$\nabla^2 f=\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr},\ r\neq 0.$$

proof



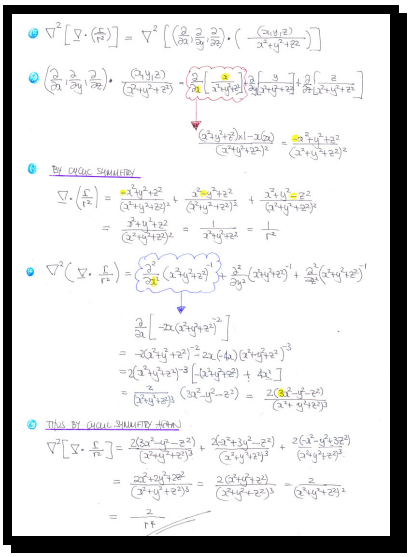
Question 8

$$\mathbf{r}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Show, with a detailed method, that

$$\nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|^2}\right)\right]=\frac{2}{|\mathbf{r}|^4}\,.$$

proof



Question 9

The scalar function φ is given below in terms of the non zero constants λ and μ.

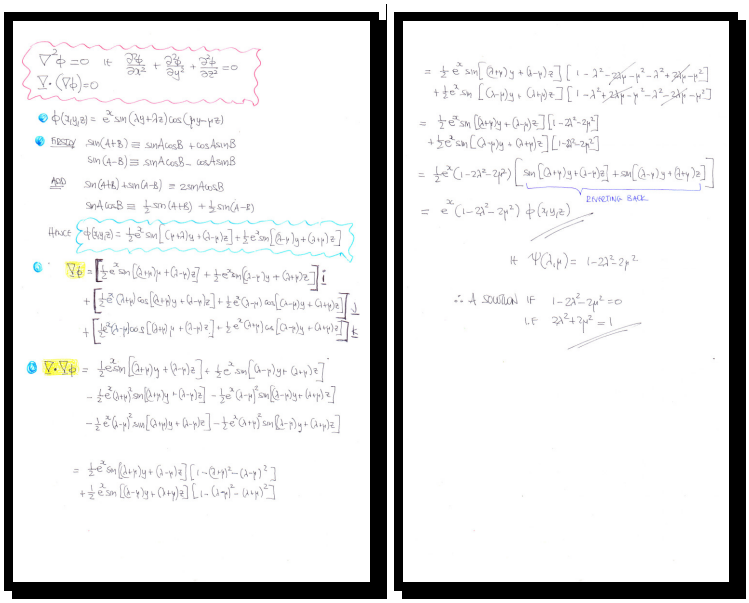
$$\varphi(x,y,z)=\mathrm{e}^{x}\sin(\lambda y+\lambda z)\cos(\mu y-\mu z).$$

Given that

$$\nabla\cdot(\nabla\varphi)=0,$$

show, with a detailed method, that $\lambda^2+\mu^2=\frac{1}{2}$.

proof



Question 10

It is given that

$$\mathbf{A}\wedge\left(\mathbf{B}\wedge\mathbf{C}\right)\equiv\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\left(\mathbf{A}\cdot\mathbf{B}\right)\mathbf{C}\,.$$

a) Use the above identity to find a simplified expression for $\nabla\wedge\left(\nabla\wedge\mathbf{F}\right)$, where $\mathbf{F}$ is a smooth vector field.

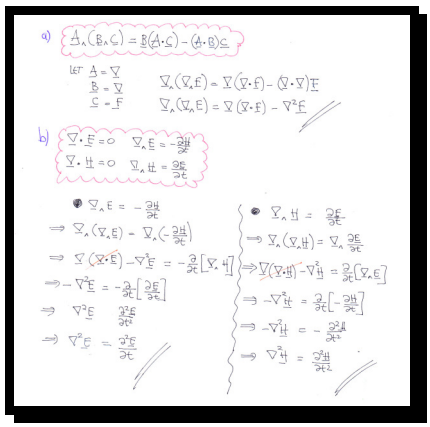
The smooth vector fields $\mathbf{E}$ and $\mathbf{H}$, satisfy the following relationships.

- $\nabla\cdot\mathbf{E}=0$
- $\nabla\wedge\mathbf{E}=-\dfrac{\partial\mathbf{H}}{\partial t}$
- $\nabla\cdot\mathbf{H}=0$
- $\nabla\wedge\mathbf{H}=\dfrac{\partial\mathbf{E}}{\partial t}$

b) Show that $\mathbf{E}$ and $\mathbf{H}$, satisfy the wave equation .

$$\nabla^2\mathbf{U}=\frac{\partial^2\mathbf{U}}{\partial t^2}\,.$$

$$\boxed{\nabla\wedge\left(\nabla\wedge\mathbf{F}\right)\equiv\nabla\left(\nabla\cdot\mathbf{F}\right)-\nabla^2\mathbf{F}}$$



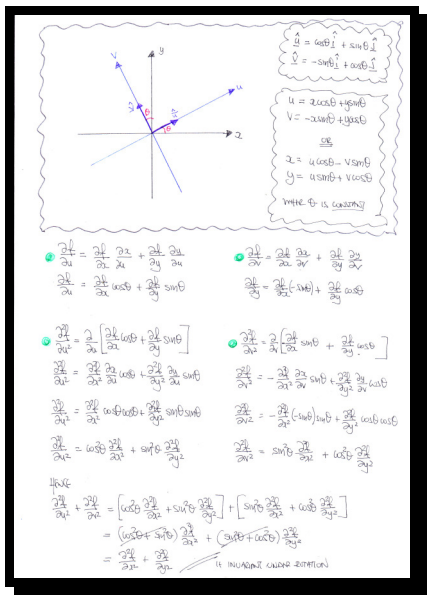
Question 11

The Laplacian operator ∇^2 in the standard two dimensional Cartesian system of coordinates is defined as

$$\nabla^2(\ ) \equiv \frac{\partial^2}{\partial x^2}(\ ) + \frac{\partial^2}{\partial y^2}(\ ).$$

Show that the two dimensional Laplacian operator is invariant under rotation.

proof



Question 12

It is given that if $\mathbf{F}$ is a smooth vector field, then

$$\nabla\wedge\left(\nabla\wedge\mathbf{F}\right)=\nabla\left(\nabla\cdot\mathbf{F}\right)-\nabla^2\mathbf{F}\,.$$

a) By using index summation notation, or otherwise, prove the validity of the above vector identity.

Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, satisfy

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0},\qquad \nabla\wedge\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \nabla\cdot\mathbf{B}=0\quad\text{and}\quad \nabla\wedge\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t},$$

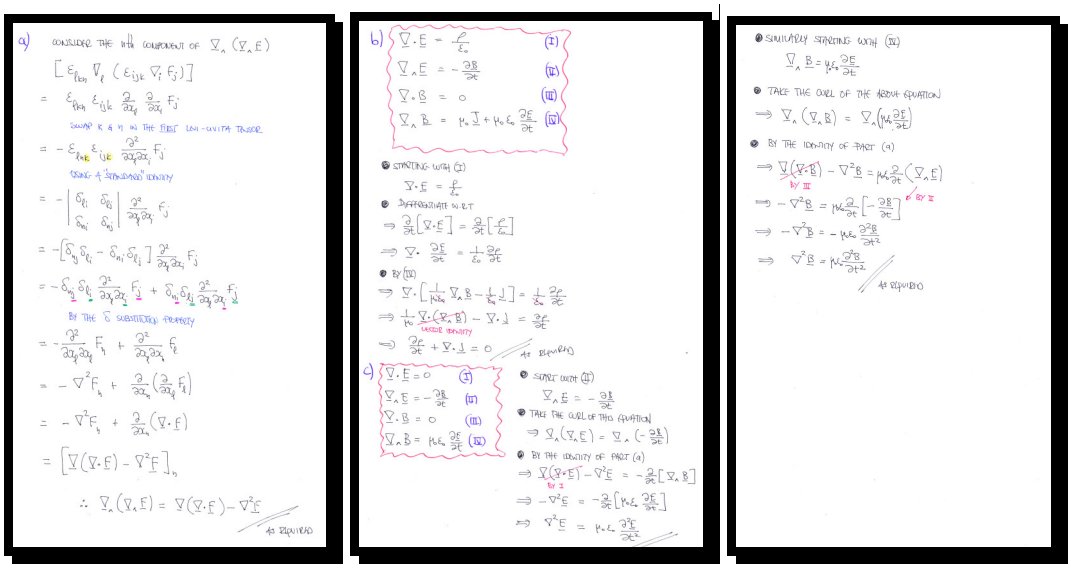
where ρ is the electric charge density, $\mathbf{J}$ is the current density, and μ_0 and ε_0 are positive constants.

b) Show that $\dfrac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0$.

c) Given further that $\rho=0$ and $\mathbf{J}=\mathbf{0}$, show also that

$$\nabla^2\mathbf{E}=\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\quad\text{and}\quad \nabla^2\mathbf{B}=\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}\,.$$

proof



Question 13

Show that the Laplacian operator in the standard two dimensional Polar system of coordinates is given by

$$\nabla^2(\ ) \equiv \frac{\partial^2}{\partial r^2}(\ ) + \frac{1}{r}\frac{\partial}{\partial r}(\ ) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\ ).$$

proof

