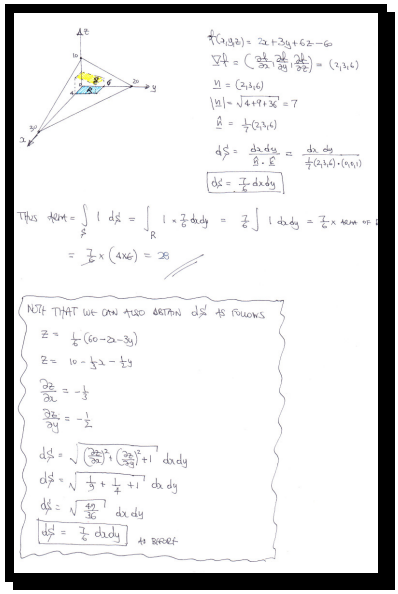
SURFACE INTEGRALS

Question 1

Find the area of the plane with equation

$$2x+3y+6z=60,\ 0\le x\le 4,\ 0\le y\le 6.$$

28



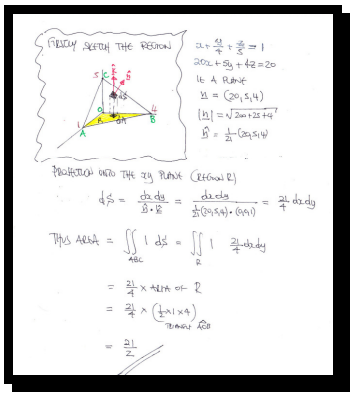
Question 2

A surface has Cartesian equation

$$x+\frac{y}{4}+\frac{z}{5}=1\,.$$

Determine the area of the surface which lies in the first octant.

$$\frac{21}{2}$$



Question 3

The plane with equation

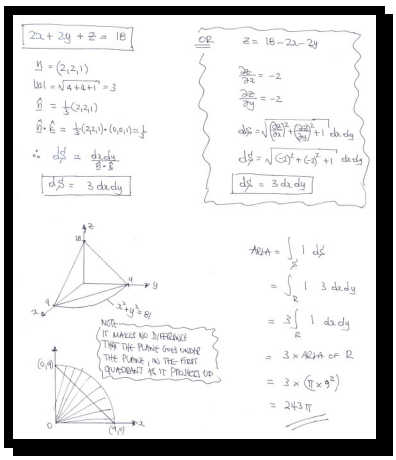
$$2x+2y+z=18,$$

intersects the cylinder with equation

$$x^2+y^2=81.$$

Determine the area of the cross-sectional cut.

$\boxed{243\pi}$



Question 4

A tube in the shape of a right circular cylinder of radius 4 m and height 0.5 m, emits heat from its curved surface only.

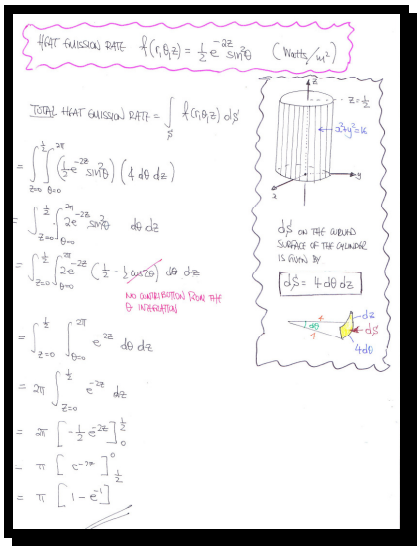
The heat emission rate, in Wm^{-2}, is given by

$$\frac{1}{2}\text{e}^{-2z}\sin^2\theta,$$

where θ and z are standard cylindrical polar coordinates, whose origin is at the centre of one of the flat faces of the cylinder.

Given that the cylinder is contained in the part of space for which $z \geq 0$, determine the total heat emission rate from the tube.

$$\boxed{\pi\left(1-\text{e}^{-1}\right)}$$



Question 5

A surface has Cartesian equation

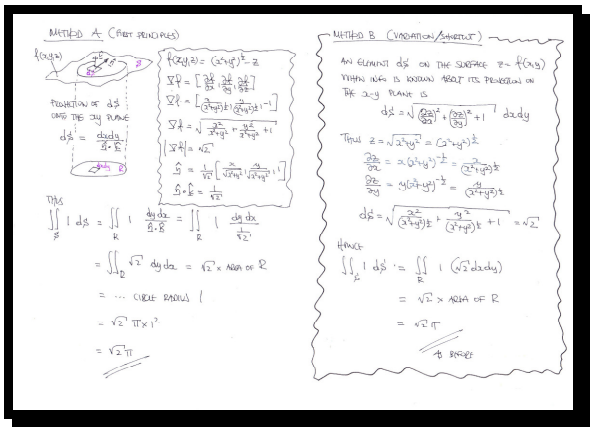
$$z = \sqrt{x^2 + y^2} \, .$$

The projection in the x-y plane of the region S on this surface, is the region R with Cartesian equation

$$x^2 + y^2 = 1 \, .$$

Find the area of S.

$\boxed{\pi\sqrt{2}}$



Question 6

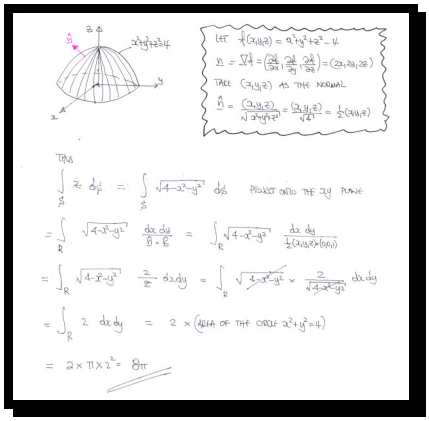
$$I = \int_S z \, dS \, .$$

Find the exact value of I, if S is the surface of the hemisphere with equation

$$x^2 + y^2 + z^2 = 4 \, , \; z \geq 0 \, .$$

You may only use Cartesian coordinates in this question.

8π



Question 7

A hemispherical surface, of radius a m, is electrically charged.

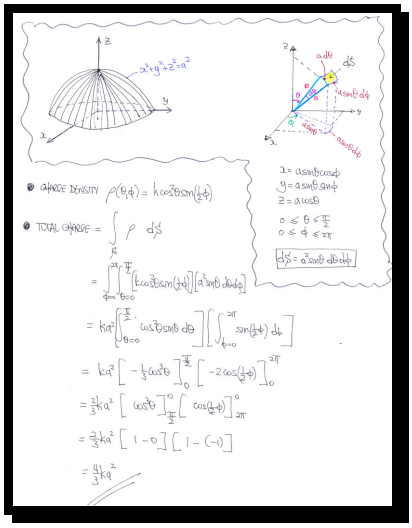
The electric charge density $\rho(\theta,\varphi)$, in C m^{-2}, is given by

$$\rho(\theta,\varphi)=k\cos^2(\theta)\sin\left(\frac{1}{2}\varphi\right),$$

where k is a positive constant, and θ and φ are standard spherical polar coordinates, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z\geq 0$, determine the total charge on its surface.

$$\boxed{\frac{4}{3}ka^2}$$



Question 8

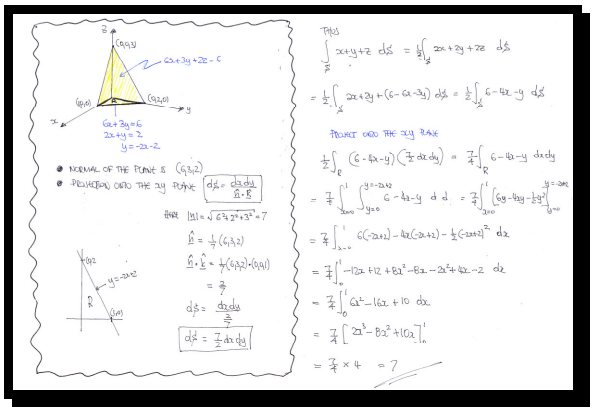
Evaluate the integral

$$\int_S (x+y+z)\, dS\,,$$

where S is the plane with Cartesian equation

$$6x+3y+2z=6, \quad x\geq 0, \quad y\geq 0, \quad z\geq 0.$$

7



Question 9

A hemispherical surface, of radius a m, has small indentations due to particle bombardment.

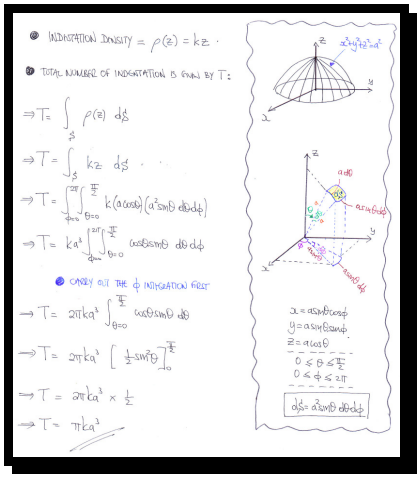
The indentation density $\rho(z)$, in m^{-2}, is given by

$$\rho(z) = k\,z\,,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z \geq 0$, determine the total number of indentations on its surface.

$$\boxed{\pi k a^3}$$



Question 10

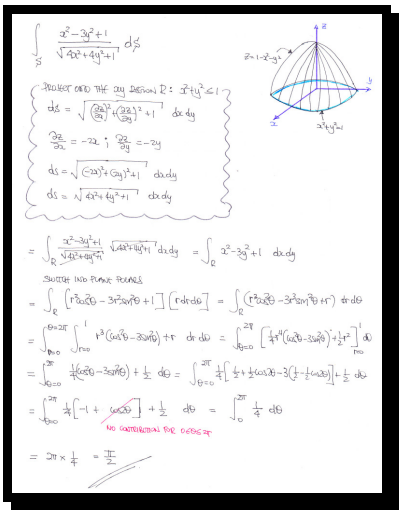
Evaluate the integral

$$\int_S \frac{x^2-3y^2+1}{\sqrt{4x^2+4y^2+1}}\,dS\,,$$

where S is the surface with Cartesian equation

$$z=1-x^2-y^2,\quad z\geq 0\,.$$

$$\frac{\pi}{2}$$



Question 11

Evaluate the integral

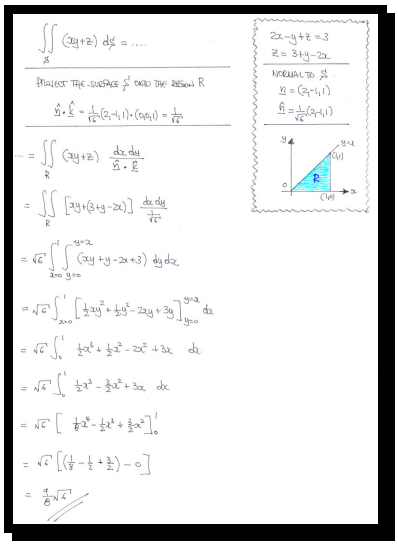
$$\int_S (xy+z)\,dS\,,$$

where S is the plane with Cartesian equation

$$2x-y+z=3\,,$$

whose projection onto the plane with equation $z=0$ is the rectilinear triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$.

$$\frac{9\sqrt{6}}{8}$$



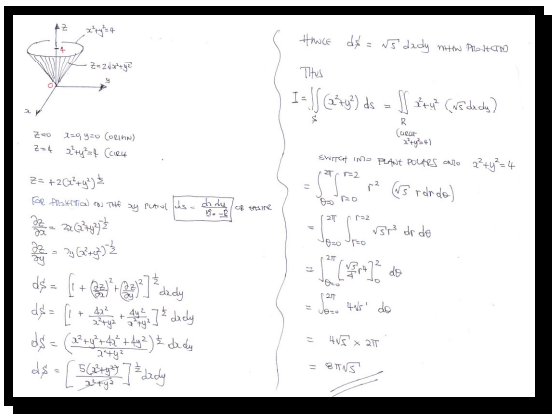
Question 12

$$I = \int_S x^2 + y^2 \, dS \, .$$

Find the exact value of I, if S is the surface of the cone with equation

$$z^2 = 4\left(x^2 + y^2\right), \; 0 \le z \le 4 \, .$$

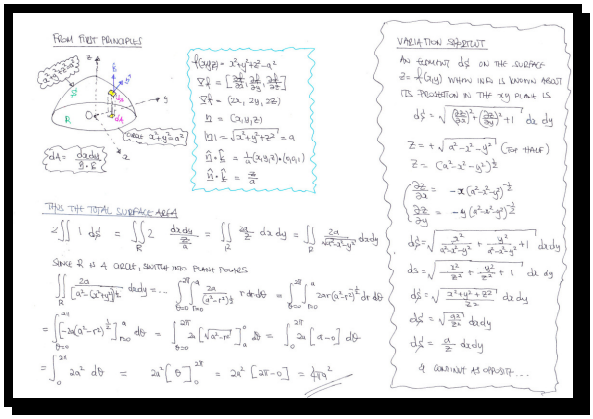
$\boxed{8\pi\sqrt{5}}$



Question 13

Show clearly, by a **Cartesian projection** onto the x-y plane, that the surface area of a sphere of radius a, is $4\pi a^2$.

proof



Question 14

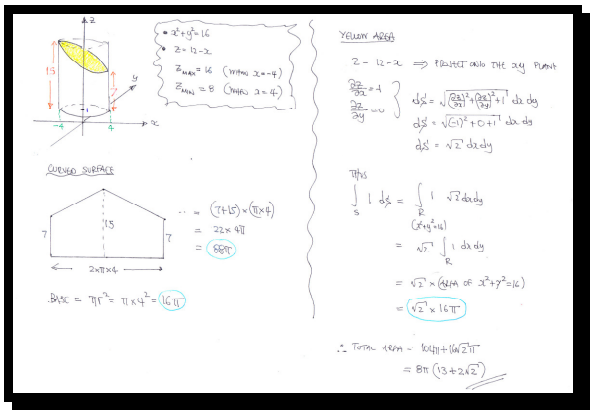
Find in exact form the **total** surface area of the cylinder with equation

$$x^2 + y^2 = 16\,,\ z \geq 1\,,$$

cut of by the plane the plane with equation

$$z = 12 - x\,.$$

$$\boxed{8\pi\left(13 + 2\sqrt{2}\right)}$$



Question 15

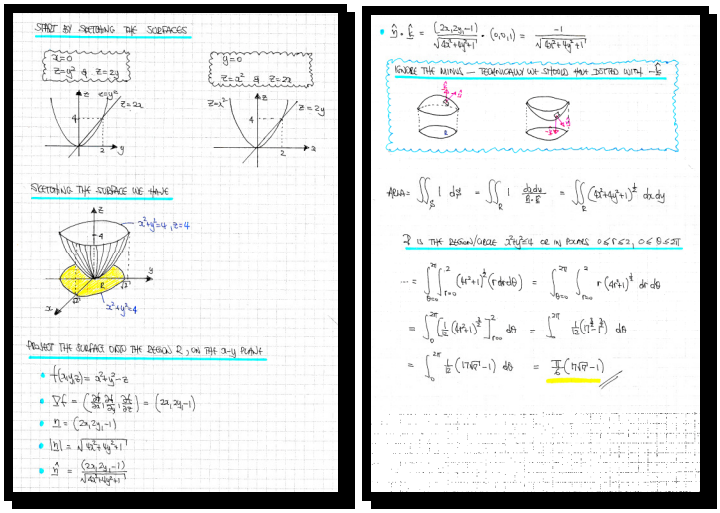
Find the area of finite region on the paraboloid with equation

$$z = x^2 + y^2,$$

cut off by the cone with equation

$$\frac{1}{2}z = \sqrt{x^2 + y^2}\,.$$

$\square$, $\boxed{\frac{1}{6}\pi\left[17\sqrt{17}-1\right]}$



Question 16

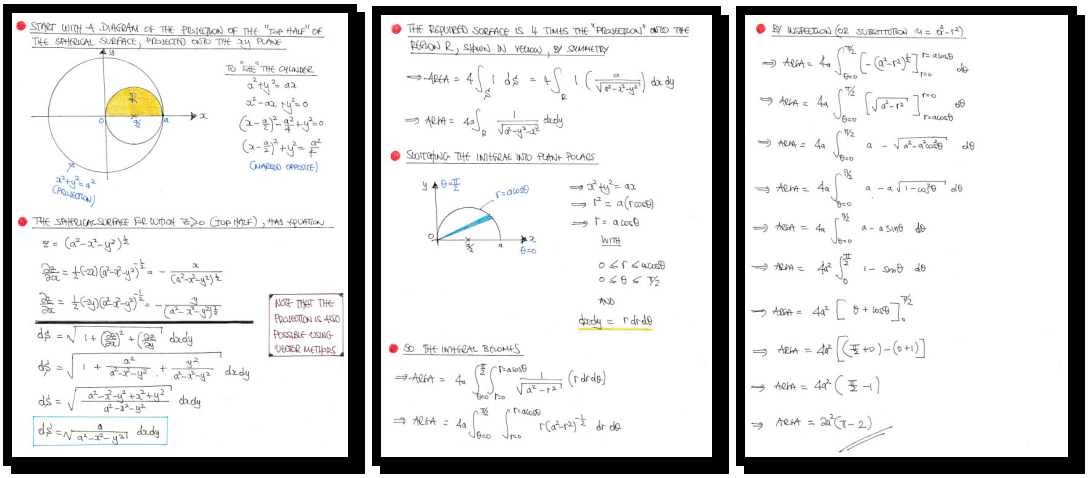
Find a simplified expression for the surface area cut out of the sphere with equation

$$x^2 + y^2 + z^2 = a^2 \, , \; a > 0 \, ,$$

when it is intersected by the cylinder with equation

$$x^2 + y^2 = ax \, , \; a > 0 \, .$$

$\square$, $\boxed{A = 2a^2 [\pi - 2]}$



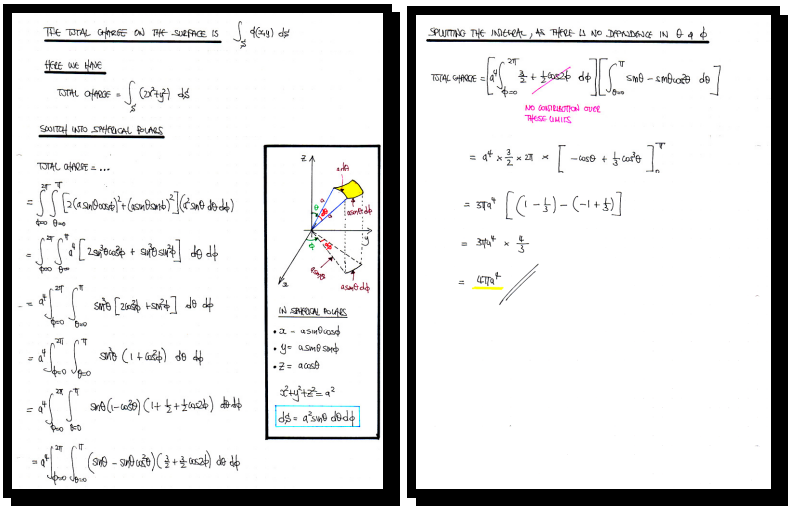
Question 17

Electric charge q is thinly distributed on the surface of a spherical shell with equation

$$x^2 + y^2 + z^2 = a^2 \, , \; a > 0 \, .$$

Given that $q(x, y) = 2x^2 + y^2$, determine the total charge on the shell.

$\boxed{}$, $\boxed{q = 4\pi a^4}$



Question 18

A inverted right circular cone, whose vertex is at the origin of a Cartesian axes, lies in the region for which $z \geq 0$. The z axis is the axis of symmetry of the cone. Both the radius and the height of the cone is 6 units.

Electric charge Q is thinly distributed on the **curved** surface of the cone.

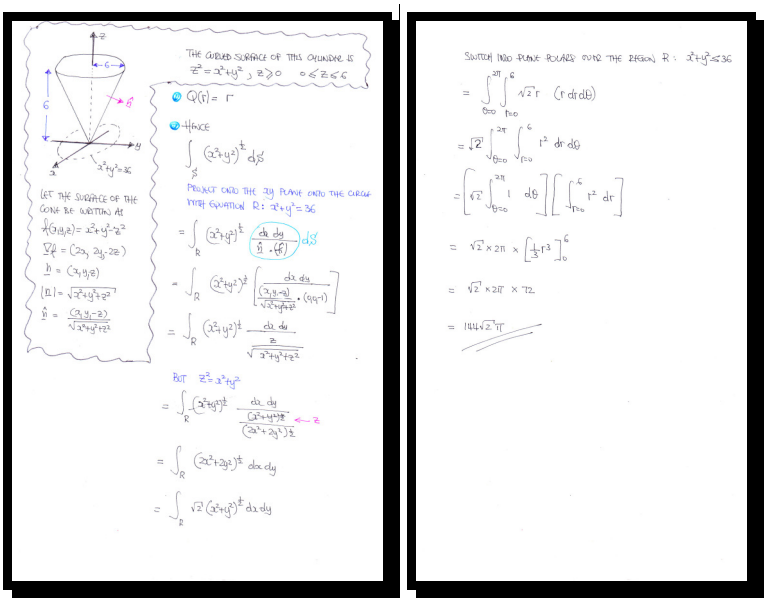
The charge at a given point on the curved surface of the cone satisfies

$$Q(r) = r,$$

where r is the shortest of the point from the z axis.

Determine the total charge on the cone.

$$\boxed{Q = 144\pi\sqrt{2}}$$



Question 19

A surface S has Cartesian equation

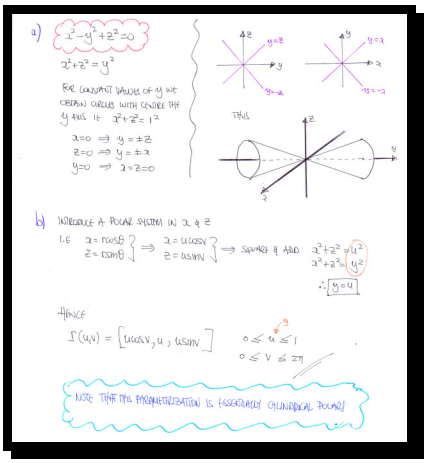
$$x^2 - y^2 + z^2 = 0 .$$

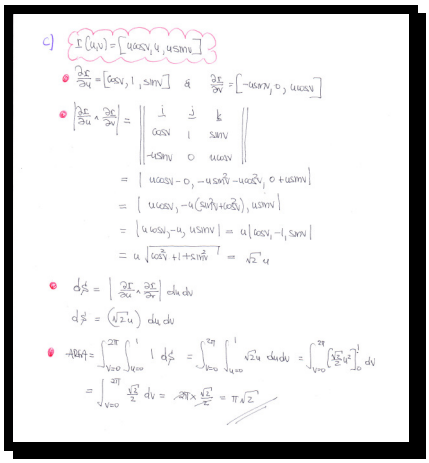
a) Sketch the graph of S.

b) Find a parameterization for the equation of S, in terms of the parameters u and v.

c) Use the parameterization of part **(b)** to find the area of S, for $0 \le y \le 1$.

$\boxed{\mathbf{r}(u,v) = \langle u\cos v, u, u\sin v \rangle}$, $\boxed{\text{area} = \pi\sqrt{2}}$





Question 20

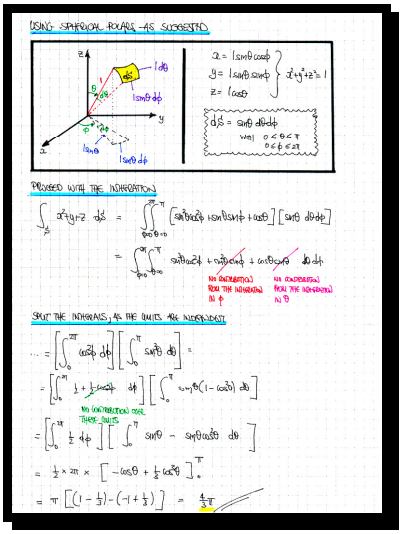
The surface S is the sphere with Cartesian equation

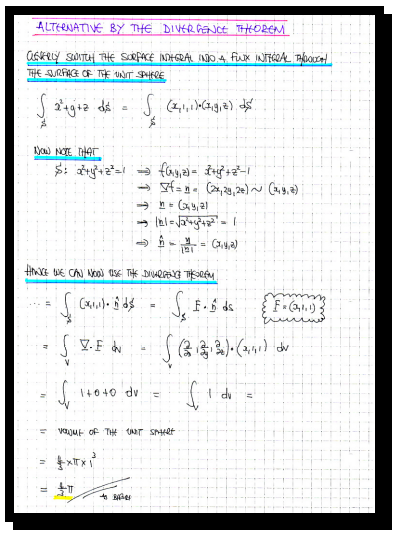
$$x^2+y^2+z^2=1.$$

By using Spherical Polar coordinates (r,θ,φ), or otherwise, evaluate

$$\iint_S \left(x^2+y+z\right) dS .$$

$\square$, $\frac{4}{3}\pi$





Question 21

A bead is modelled as a sphere with a cylinder, whose axis is a diameter of the sphere, removed from the sphere.

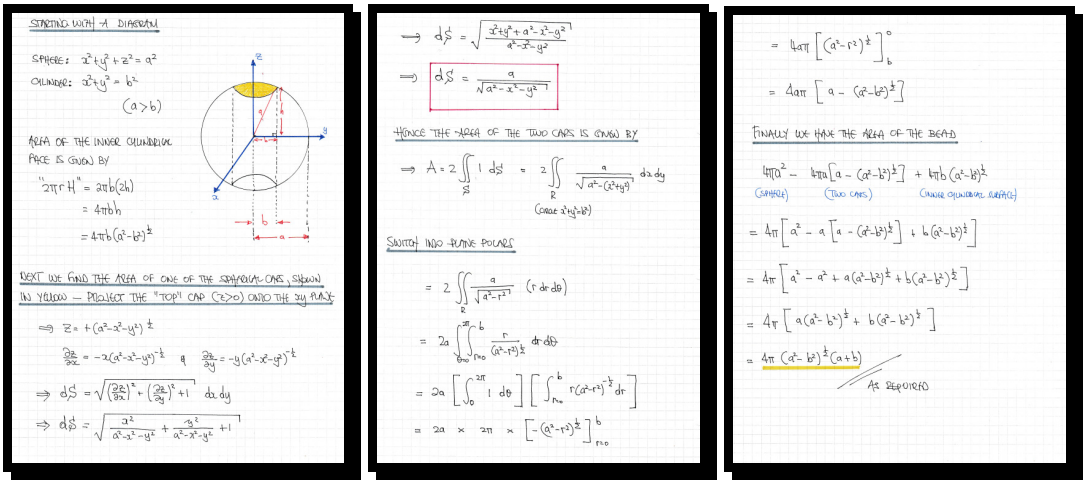
If the respective equations of the sphere and the cylinder are

$$x^2+y^2+z^2=a^2 \quad \text{and} \quad x^2+y^2=b^2, \quad 0<b<a.$$

Show that the total surface area of the bead is

$$4\pi(a+b)\sqrt{a^2-b^2}\,.$$

[ ], [proof]



Question 22

A surface S has Cartesian equation

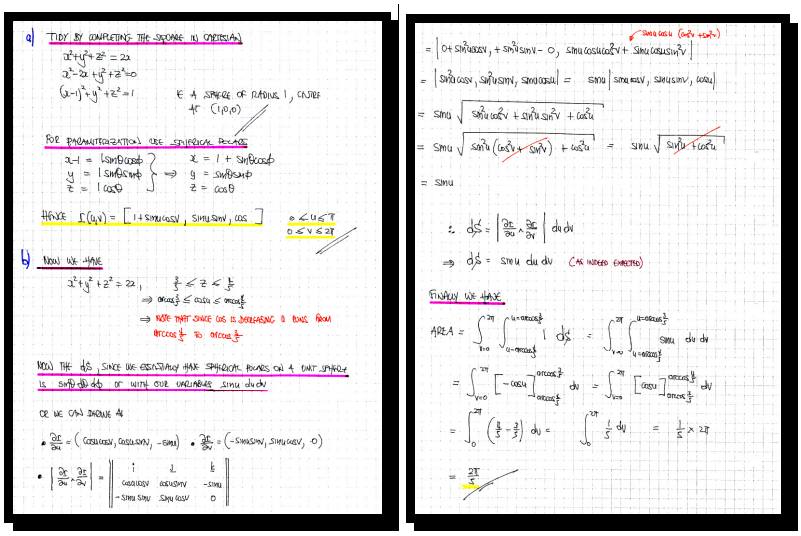
$$x^2+y^2+z^2=2x\,.$$

a) Describe fully the graph of S, and hence find a parameterization for its equation in terms of the parameters u and v.

b) Use the parameterization of part **(a)** to find the area for the part of S, for which $\frac{3}{5}\le z\le\frac{4}{5}$.

$\boxed{}$, $\boxed{\mathbf{r}(u,v)=\langle 1+\sin u\cos v,\sin u\sin v,\cos u\rangle,\quad 0\le u\le\pi,\ 0\le v\le 2\pi}$,

$\boxed{\text{area}=\frac{2}{5}\pi}$



Question 23

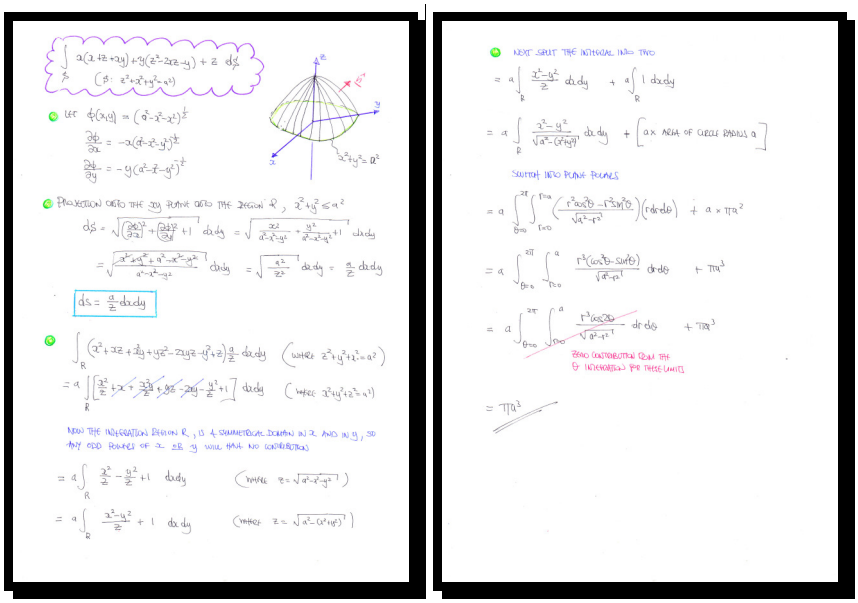
Evaluate the integral

$$\int_S x\left(x+z+xy\right)+y\left(z^2-2xz-y\right)+z\ dS\,,$$

where S is the surface with Cartesian equation

$$x^2+y^2+z^2=a^2\,,\quad a>0\,,\quad z\geq 0\,.$$

$$\boxed{\pi a^3}$$



Question 24

A surface S has Cartesian equation

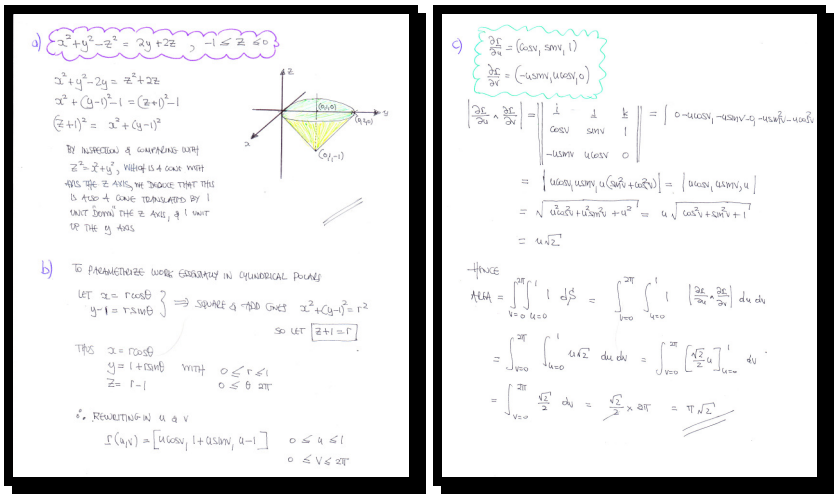
$$x^2+y^2-z^2=2y+2z\,,\quad -1\le z\le 0\,.$$

a) Sketch the graph of S.

b) Find a parameterization for the equation of S, in terms of the parameters u and v.

c) Use the parameterization of part **(b)** to find the area of S.

$$\boxed{\mathbf{r}(u,v)=\langle u\cos v,1+u\sin v,u-1\rangle,\ \ 0\le u\le 1,\ 0\le v\le 2\pi}\,,\quad \boxed{\text{area}=\pi\sqrt{2}}$$

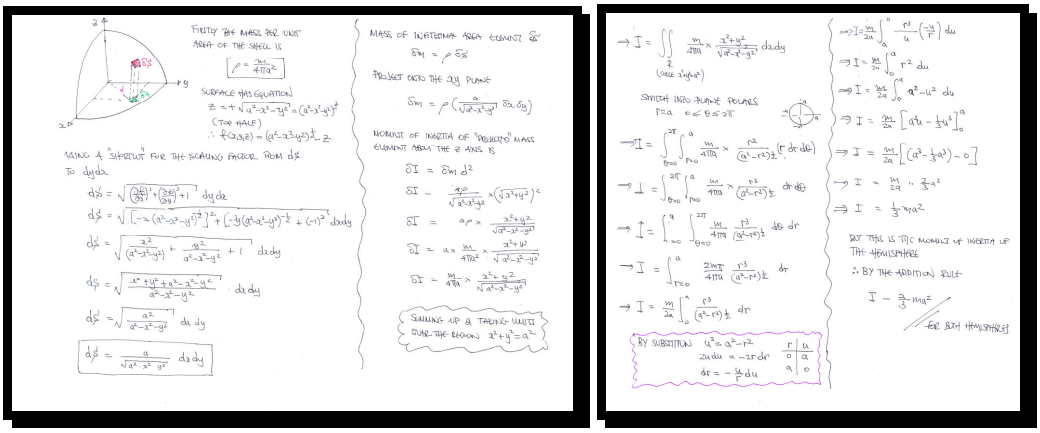


Question 25

A thin uniform spherical shell has mass m and radius a.

Use surface integral projection techniques in x-y plane, to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



Question 26

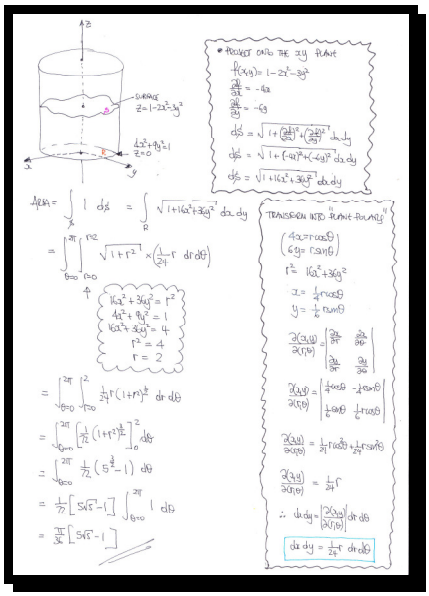
Find the area of the surface S which consists of the part of the surface with Cartesian equation

$$z = 1 - 2x^2 - 3y^2 ,$$

contained within the elliptic cylinder with Cartesian equation

$$4x^2 + 9y^2 = 1 .$$

$$\boxed{\frac{\pi}{36}\left[5\sqrt{5}-1\right]}$$



Question 27

The surface S is the hemisphere with Cartesian equation

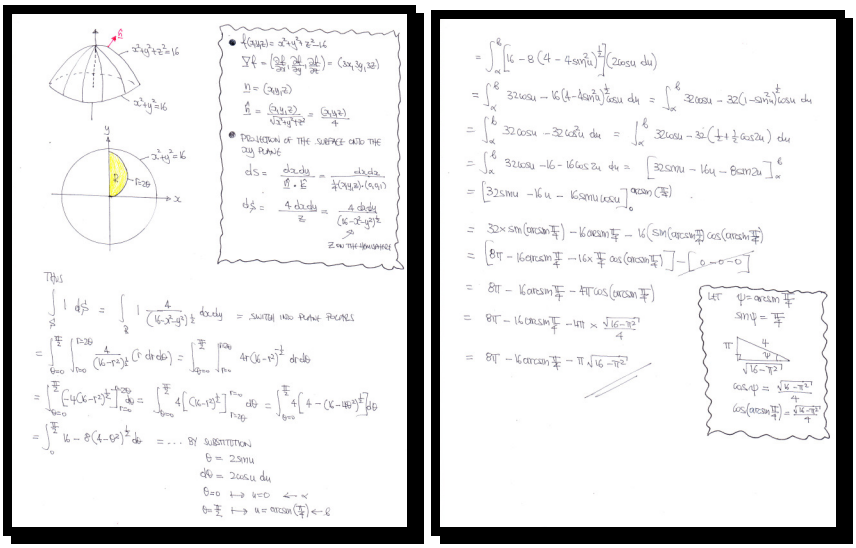
$$x^2 + y^2 + z^2 = 16,\ z \geq 0$$

The projection of S onto the x-y plane is the area within the curve with polar equation

$$r = 2\theta,\ 0 \leq \theta \leq \frac{\pi}{2}.$$

Find, in exact form, the area of S.

$$\boxed{8\pi - \pi\sqrt{16-\pi^2} - 16\arcsin\frac{\pi}{4}}$$



Question 28

The surface S is the sphere with Cartesian equation

$$x^2+y^2+z^2=4$$

a) By using Spherical Polar coordinates, (r,θ,φ), evaluate by direct integration the following surface integral

$$I=\iint_S \left(x^4+xy^2+z\right)dS\,.$$

b) Verify the answer of part **(a)** by using the Divergence Theorem.

$$\frac{256\pi}{5}$$



Question 29

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta,\varphi)=(R+r\cos\theta)\cos\varphi,\qquad y(\theta,\varphi)=(R+r\cos\theta)\sin\varphi,\qquad z(\theta,\varphi)=r\sin\theta,$$

where $0\le\theta\le 2\pi$ and $0\le\varphi\le 2\pi$.

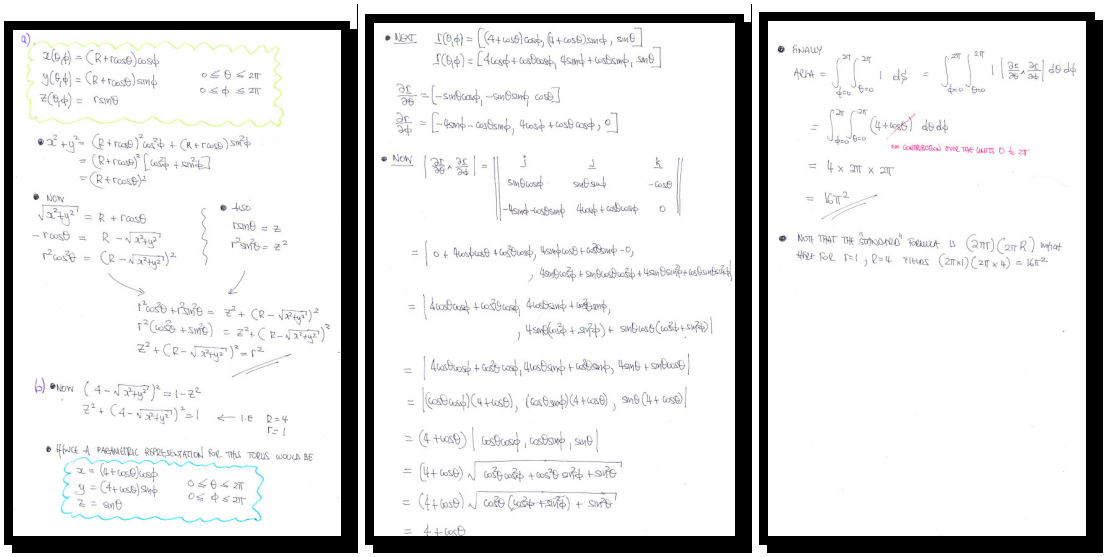
a) Find a general Cartesian equation for the surface of a torus.

A torus T has Cartesian equation

$$\left(4-\sqrt{x^2+y^2}\right)^2=1-z^2.$$

b) Use a suitable parameterization of T to find its surface area.

$$\boxed{z^2+\left(R-\sqrt{x^2+y^2}\right)^2=r^2},\qquad \boxed{\text{area}=(2\pi r)(2\pi R)=16\pi^2}$$



Question 30

A spiral ramp is modelled by the surface S defined by the vector function

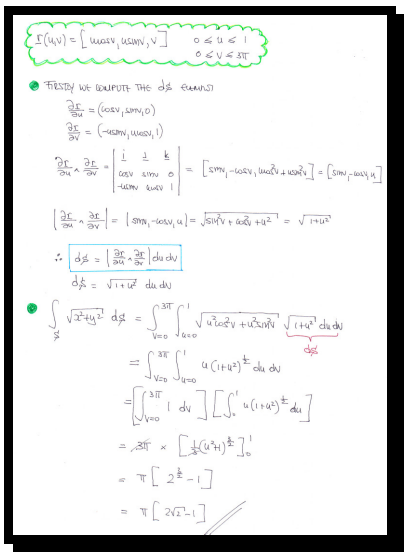
$$\mathbf{r}(u,v)=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(u\cos v)\mathbf{i}+(u\sin v)\mathbf{j}+v\mathbf{k}\,,$$

where $0\le u\le 1$, $0\le v\le 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2+y^2}\;dS$$

$$\boxed{\pi\left[\sqrt{8}-1\right]}$$



Question 31

The surface S is defined by the vector equation

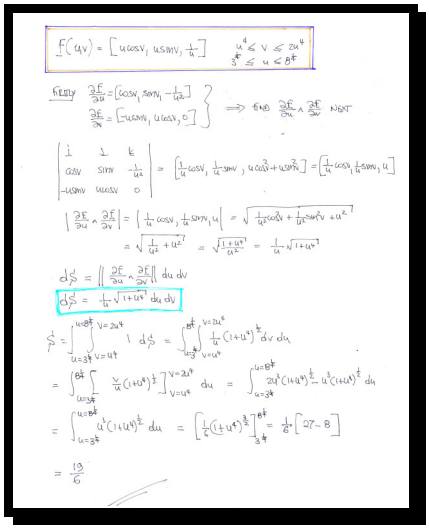
$$\mathbf{F}(u,v)=\left[u\cos v, u\sin v, \frac{1}{u}\right]^T,\ u\neq 0.$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v=u^4,\ v=2u^4,$$

and the straight lines with equations $u=3^{\frac{1}{4}}$ and $u=8^{\frac{1}{4}}$.

$$\boxed{\frac{19}{6}}$$



Question 32

The surface S is defined by the parametric equations

$$x = t\cosh\theta,\ y = t\sinh\theta,\ z = \frac{1}{2}\left(1-t^2\right),$$

where t and θ are real parameters such that $0 \le t \le 1$ and $0 \le \theta \le 1$.

Find, in exact form, the value of

$$\int_S xy\ dS .$$

$$\boxed{\frac{1}{30}\left[\frac{(\cosh 2+1)^{\frac{5}{2}}-1}{\cosh 2}+1-4\sqrt{2}\right] \approx 0.274397...}$$

