SPECIAL FUNCTIONS

CHEBYSHEV POLYNOMIALS

Question 1

Chebyshev's differential equation is given below.

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\qquad n=0,\,1,\,2,\,3,\,4,\,...$$

a) Use the substitution $x=\cos t$, to show that a general solution for Chebyshev's differential equation is

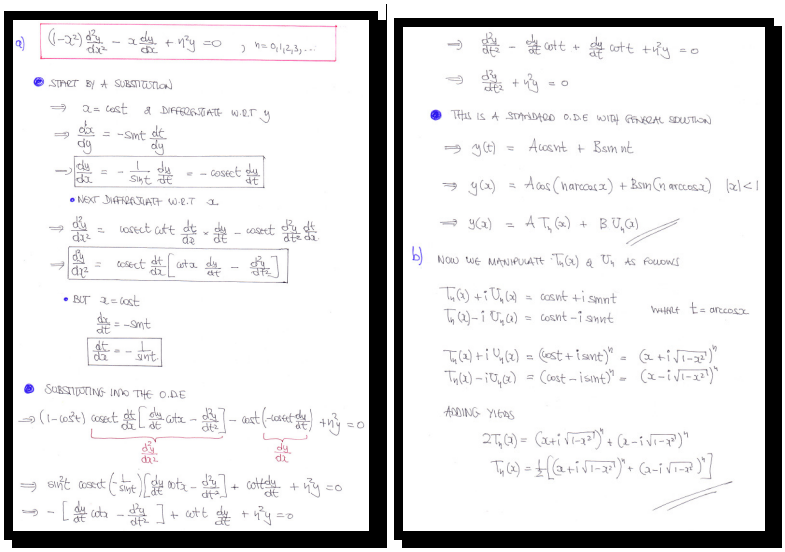
$$y=A\,\mathrm{T}_n\left(x\right)+B\,\mathrm{U}_n\left(x\right),\ \left|x\right|<1,$$

with $\mathrm{T}_n\left(x\right)=\cos\left[n\arccos\left(x\right)\right]$, $\mathrm{U}_n\left(x\right)=\sin\left[n\arccos\left(x\right)\right]$, and A and B are arbitrary constants.

b) Show further that $\mathrm{T}_n\left(x\right)$ can be written as

$$\mathrm{T}_n\left(x\right)=\frac{1}{2}\left[\left(x+\mathrm{i}\sqrt{1-x^2}\right)+\left(x-\mathrm{i}\sqrt{1-x^2}\right)\right].$$

proof



Question 2

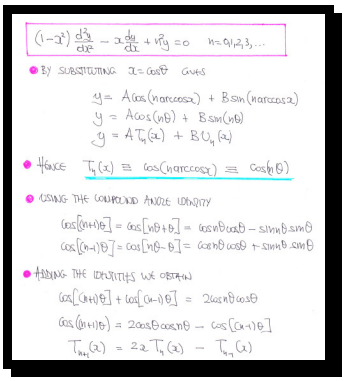
Chebyshev's polynomials of the first kind $\mathrm{T}_n(x)$ are defined as

$$\mathrm{T}_n(x)=\cos\left[n\arccos(x)\right],\ n=0,\,1,\,2,\,3,\,4,\,...$$

By writing $\theta=\arccos(x)$ and considering the compound angle trigonometric identities for $\cos\left[(n\pm1)\theta\right]$, obtain the recurrence relation

$$\mathrm{T}_{n+1}(x)=2x\mathrm{T}_n(x)-\mathrm{T}_{n-1}(x),\ |x|<1.$$

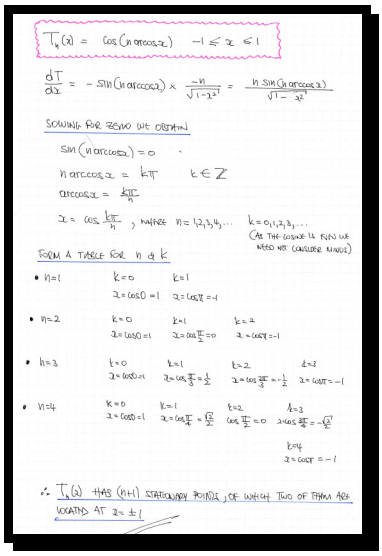
proof



Question 3

Investigate the stationary points of the Chebyshev polynomials of the first kind $T_n(x)$.

$$\boxed{n+1 \text{ stationary points, two them at } x=\pm 1}$$



Question 4

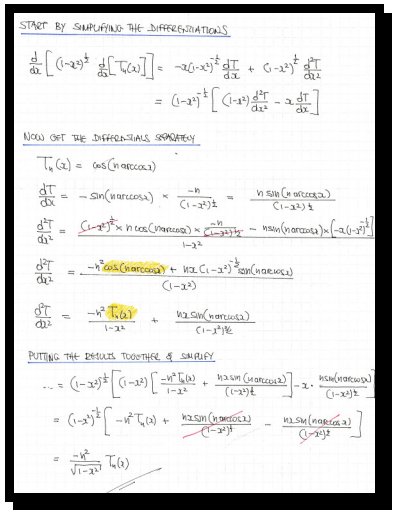
The Chebyshev polynomials of the first kind $T_n(x)$ are defined as

$$T_n(x)=\cos\left(n\arccos x\right),\ -1\le x\le 1,\ n\in\mathbb{N}.$$

Show that

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}\left[T_n(x)\right]\right]\ =\ \frac{-n^2\,T_n(x)}{\sqrt{1-x^2}}.$$

proof



Question 5

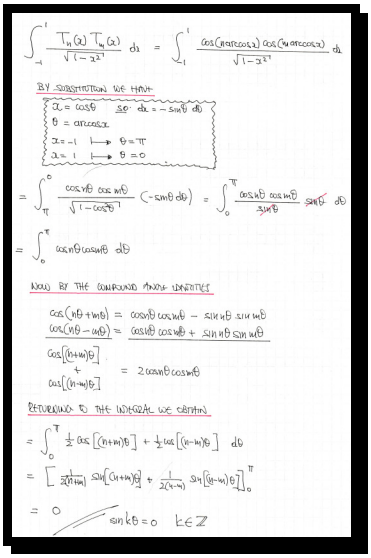
The Chebyshev polynomials of the first kind $T_n(x)$ are defined as

$$T_n(x)=\cos(n\arccos x)\,,\ -1\le x\le 1\,,\ n\in\mathbb{N}\,.$$

Show that

$$\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx\ =\ 0\,.$$

proof



Question 6

Chebyshev's differential equation is given below.

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\qquad n=0,\,1,\,2,\,3,\,4,\,\ldots$$

a) By using the substitution $x=\cosh t$, show that a general solution for Chebyshev's differential equation is

$$y=A\,\mathrm{T}_n\left(x\right)+B\,\mathrm{U}_n\left(x\right),\ \left|x\right|>1\,,$$

with $\mathrm{T}_n\left(x\right)=\cosh\left[n\operatorname{arcosh}\left(x\right)\right]$, $\mathrm{U}_n\left(x\right)=\sinh\left[n\operatorname{arcosh}\left(x\right)\right]$, and A and B are arbitrary constants.

By using the substitution $x=\cos t$, Chebyshev's differential equation has general solution

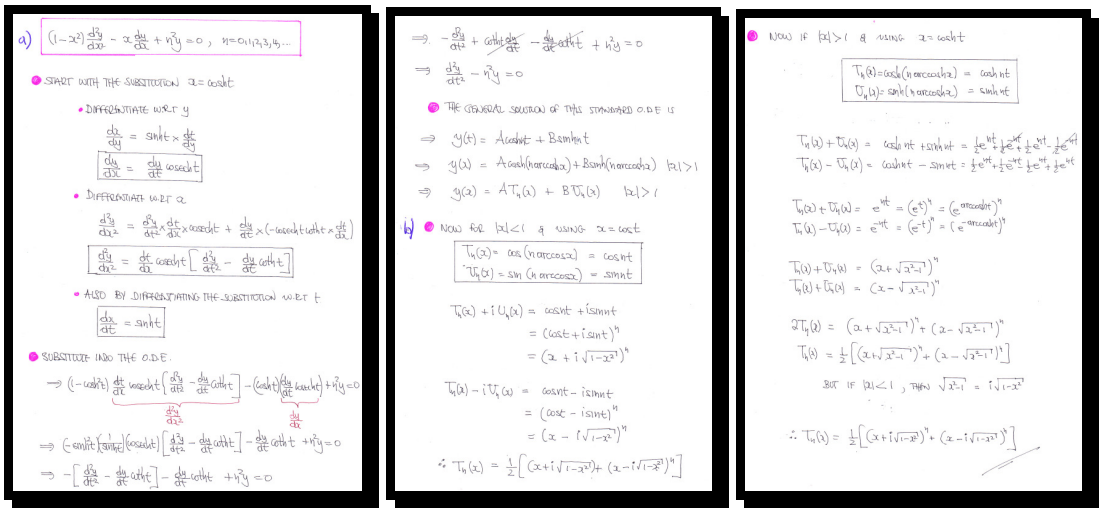
$$y=A\,\mathrm{T}_n\left(x\right)+B\,\mathrm{U}_n\left(x\right),\ \left|x\right|<1,$$

with $\mathrm{T}_n\left(x\right)=\cos\left[n\arccos\left(x\right)\right]$, $\mathrm{U}_n\left(x\right)=\sin\left[n\arccos\left(x\right)\right]$, and as above A and B are arbitrary constants.

b) Show that $\mathrm{T}_n\left(x\right)$, for both the general solutions obtained via the substitutions $x=\cos t$ and $x=\cosh t$, can be written as

$$\mathrm{T}_n\left(x\right)=\frac{1}{2}\left[\left(x+\mathrm{i}\sqrt{1-x^2}\right)+\left(x-\mathrm{i}\sqrt{1-x^2}\right)\right].$$

proof



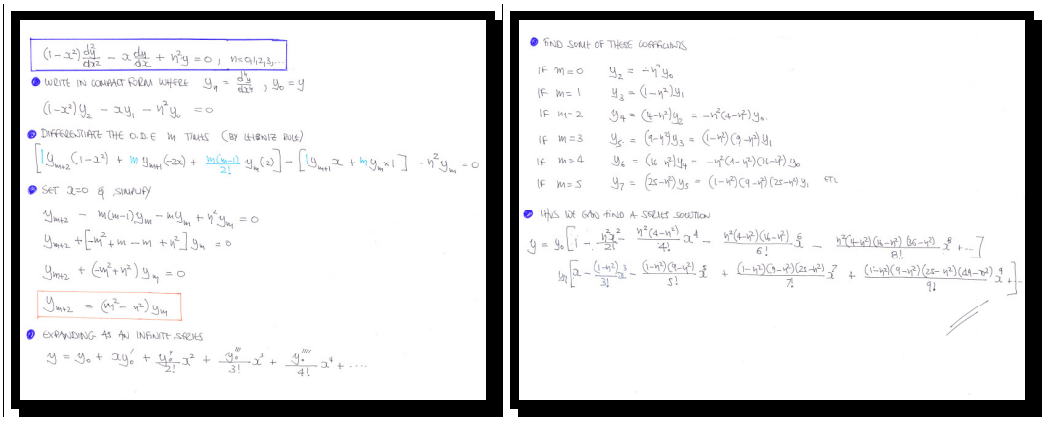
Question 7

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ n=0,\,1,\,2,\,3,\,...$$

Find a series solution for Chebyshev's equation, by using the Leibniz method

$$y=$$
$$A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-...\right]$$
$$+$$
$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-...\right]$$



Question 8

Chebyshev's equation is shown below

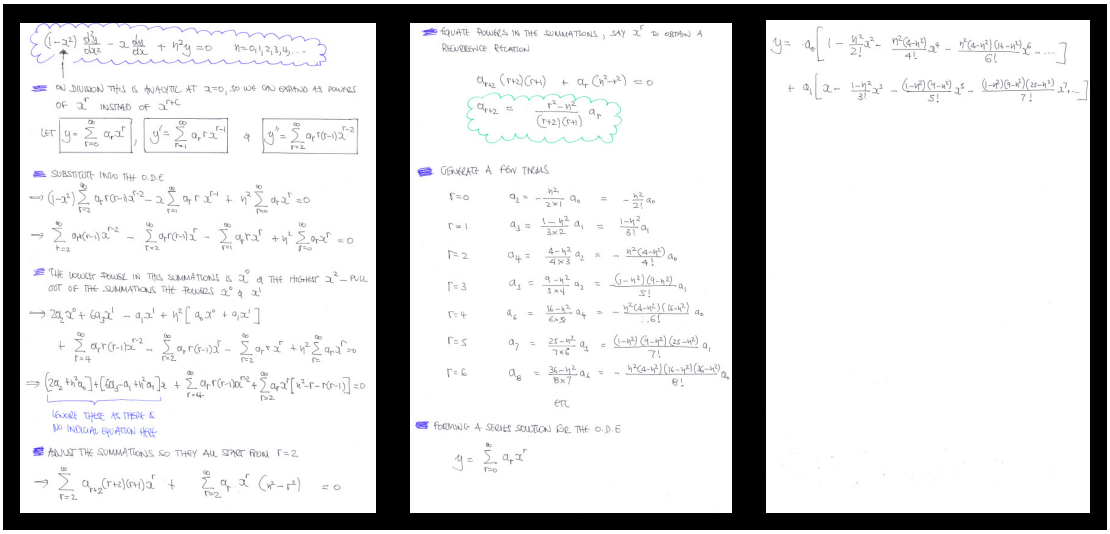
$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ n=0,\,1,\,2,\,3,\,...$$

Find a series solution for Chebyshev's equation, by using the Frobenius method.

$$y=A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-...\right]$$

$$+$$

$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-...\right]$$



LAMBERT FUNCTIONS

Question 1

The Lambert W function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(x\mathrm{e}^{x}\right)\equiv x\,,$$

and hence if $x\mathrm{e}^{x}=y$ then $x=W\left(y\right)$.

For example

$$-x\mathrm{e}^{-x}=2\ \ \Rightarrow\ \ -x=W\left(2\right),\ \ \left(x+\pi\right)\mathrm{e}^{x+\pi}=\frac{1}{2}\ \ \Rightarrow\ \ x+\pi=W\left(\frac{1}{2}\right)\text{ and so on.}$$

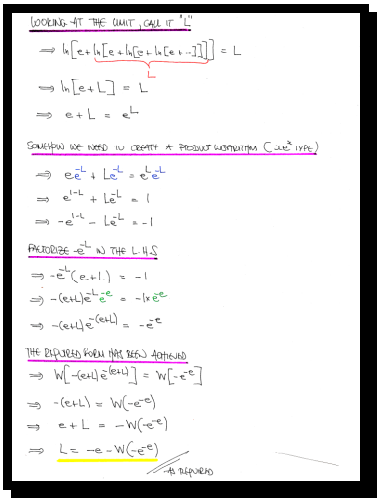
Use this result to show that the limit of

$$\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+...\right)\right)\right)\right)$$

is given by

$$-\mathrm{e}-W\left[-\mathrm{e}^{-\mathrm{e}}\right].$$

[ ], [proof]



Question 2

The Lambert W function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(x\mathrm{e}^{x}\right)\equiv x\,,\quad x\in\left[-\mathrm{e}^{-1},\infty\right)$$

and hence if $x\mathrm{e}^{x}=y$ then $x=W\left(y\right)$.

For example

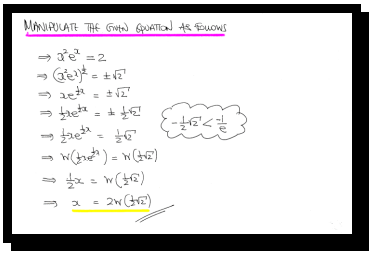
$$-x\mathrm{e}^{-x}=2\quad\Rightarrow\quad -x=W\left(2\right),\quad \left(x+\pi\right)\mathrm{e}^{x+\pi}=\frac{1}{2}\quad\Rightarrow\quad x+\pi=W\left(\frac{1}{2}\right)\text{ and so on.}$$

Use this result to find the exact solution of

$$x^{2}\mathrm{e}^{x}=2\,.$$

Give the answer in the form $x=\lambda W\left(\mu\right)$, where λ and μ are constants.

$\square$, $\boxed{x=2W\left(\frac{1}{2}\sqrt{2}\right)}$



Question 3

The Lambert W function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(x\mathrm{e}^{x}\right)\equiv x\,,\quad x\in\left[-\mathrm{e}^{-1},\infty\right)$$

and hence if $x\mathrm{e}^{x}=y$ then $x=W\left(y\right)$.

For example

$$-x\mathrm{e}^{-x}=2\ \Rightarrow\ -x=W\left(2\right),\quad \left(x+\pi\right)\mathrm{e}^{x+\pi}=\frac{1}{2}\ \Rightarrow\ x+\pi=W\left(\frac{1}{2}\right)\text{ and so on.}$$

Use this result to find the exact solution of

$$x+\mathrm{e}^{x}=2\,.$$

Give the answer in the form $x=k-W\left(\mathrm{e}^{k}\right)$, where k is a constant.

$\boxed{}$, $\boxed{x=2-W\left(\mathrm{e}^{2}\right)}$

