SERIES

EXAM QUESTIONS

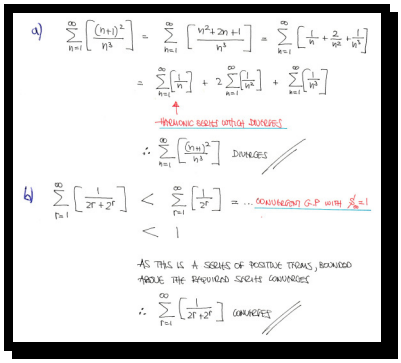
Question 1 (**)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3}$.

b) $\displaystyle\sum_{r=1}^{\infty}\frac{1}{2r+2^r}$.

[ ], [divergent], [convergent]



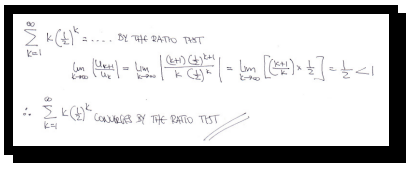
Question 2 (**)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k.$$

Show a full method, justifying every step in the workings.

[convergent]



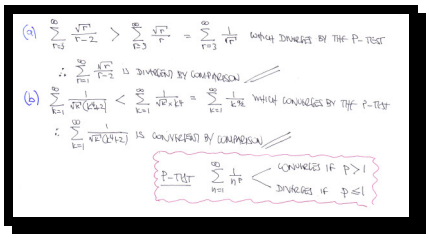
Question 3 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{r=3}^{\infty}\frac{\sqrt{r}}{r-2}$.

b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\left(k^4+2\right)\sqrt{k}}$.

divergent, convergent



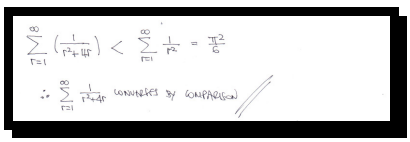
Question 4 (**)

Determine whether the following series converges or diverges.

$$\sum_{r=1}^{\infty}\frac{1}{r^2+4r}.$$

Show a full method, justifying every step in the workings.

convergent



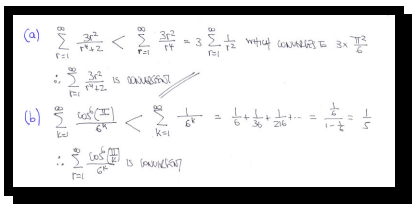
Question 5 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{r=1}^{\infty}\frac{3r^2}{r^4+2}.$$

b) $$\sum_{k=1}^{\infty}\left[\frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k}\right].$$

convergent , convergent



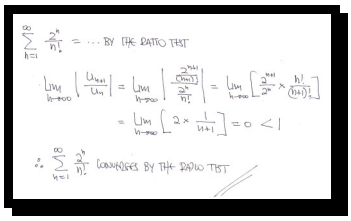
Question 6 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{2^n}{n!}.$$

Show a full method, justifying every step in the workings.

convergent



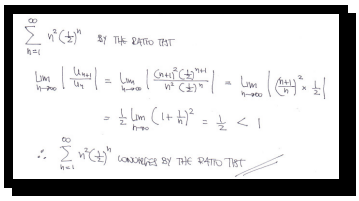
Question 7 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n .$$

Show a full method, justifying every step in the workings,

convergent



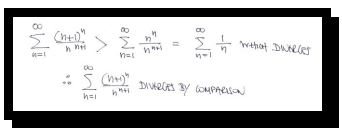
Question 8 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} .$$

Show a full method, justifying every step in the workings.

divergent



Question 9 (**)

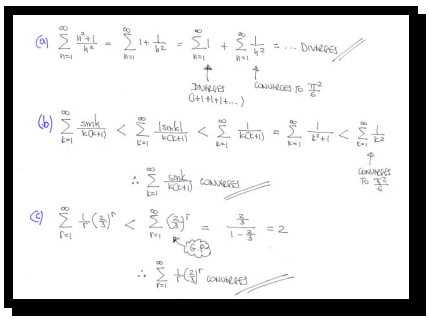
By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{n^2}$.

b) $\displaystyle\sum_{k=1}^{\infty}\frac{\sin k}{k(k+1)}$.

c) $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{2}{3}\right)^r$.

$\boxed{\text{divergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{convergent}}$



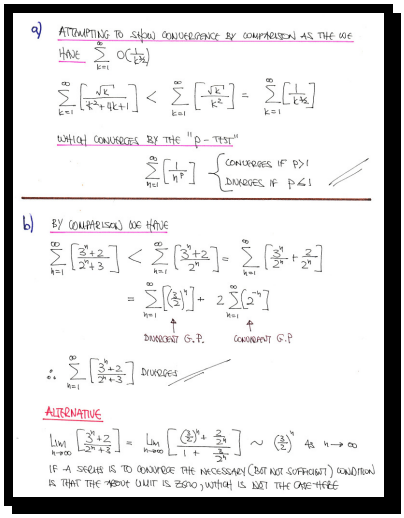
Question 10 (**)

Investigate the convergence or divergence of each of the following series justifying every step in the workings.

a) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2+4k+1}\right]$.

b) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right]$.

[ ], [convergent], [divergent]

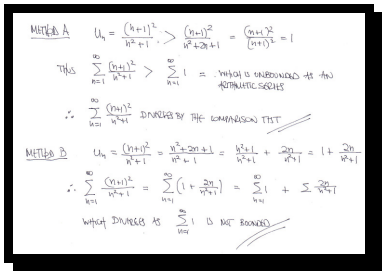


Question 11 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^2+1}.$$

divergent

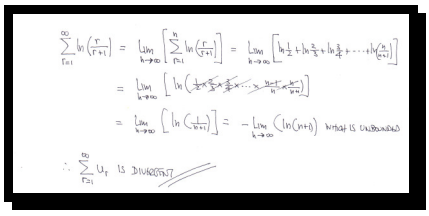


Question 12 (+)**

$$u_r=\ln\left(\frac{r}{r+1}\right),\ r\in\mathbb{N}.$$

Show clearly that $\sum_{r=1}^{\infty}u_r$ is divergent.

proof

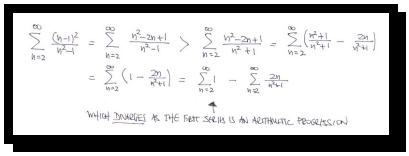


Question 13 (+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=2}^{\infty}\frac{(n-1)^2}{n^2-1}.$$

divergent



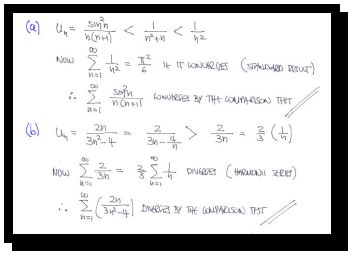
Question 14 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{n=1}^{\infty}\left[\frac{\sin^2 n}{n(n+1)}\right].$$

b) $$\sum_{n=1}^{\infty}\left[\frac{2n}{3n^2-4}\right].$$

convergent , divergent



Question 15 (***)

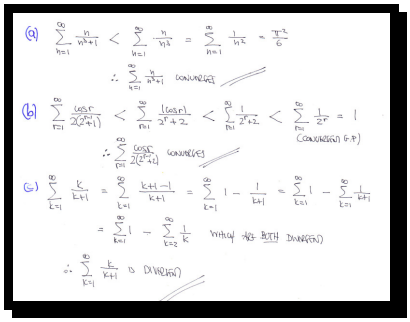
By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$.

b) $\displaystyle\sum_{r=1}^{\infty}\frac{\cos r}{2\left(2^{r-1}+1\right)}$.

c) $\displaystyle\sum_{k=1}^{\infty}\frac{k}{k+1}$.

$\boxed{\text{convergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{divergent}}$



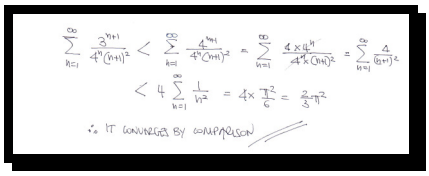
Question 16 (***)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n\left(n+1\right)^2}.$$

Show a full method, justifying every step in the workings.

convergent



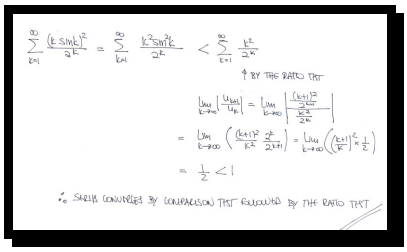
Question 17 (***)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty}\frac{\left(k\sin k\right)^2}{2^k}.$$

Show a full method, justifying every step in the workings,

convergent

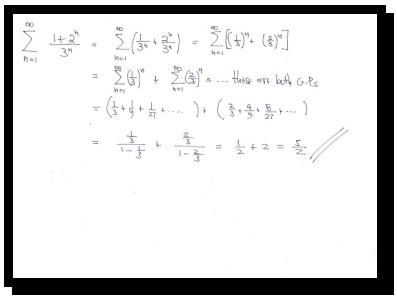


Question 18 (***)

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}.$$

$\boxed{\frac{5}{2}}$



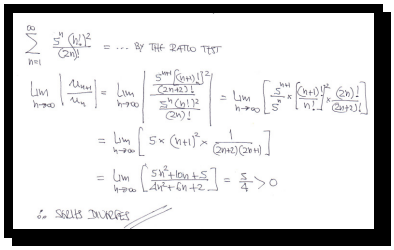
Question 19 (***)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{5^n(n!)^2}{(2n)!}.$$

Show a full method, justifying every step in the workings.

$\boxed{\text{divergent}}$



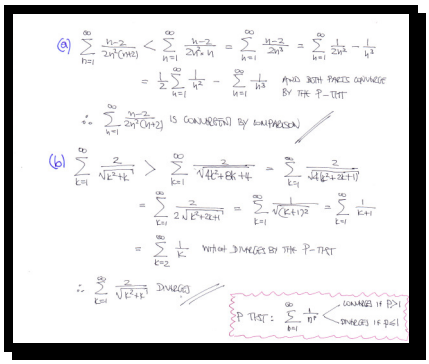
Question 20 **(***)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n-2}{2n^2(n+2)}$.

b) $\displaystyle\sum_{k=1}^{\infty}\frac{2}{\sqrt{k^2+k}}$.

convergent , divergent



Question 21 (***)

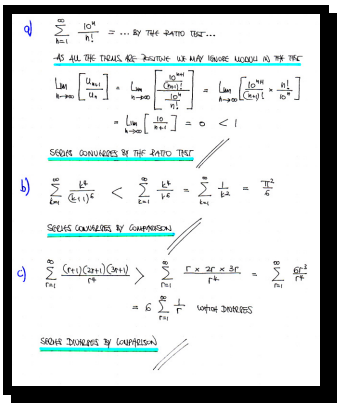
By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{n!}$.

b) $\displaystyle\sum_{k=1}^{\infty}\frac{k^4}{(k+1)^6}$.

c) $\displaystyle\sum_{r=1}^{\infty}\frac{(r+1)(2r+1)(3r+1)}{r^4}$.

$\boxed{}$, $\boxed{\text{convergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{divergent}}$



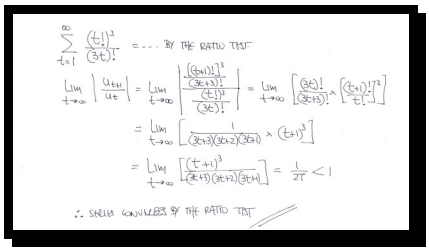
Question 22 (***)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty}\frac{(t!)^3}{(3t)!}.$$

Show a full method, justifying every step in the workings.

convergent



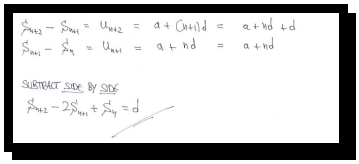
Question 23 (***)

The sum of the first n terms of an arithmetic series with first term a and common difference d, is denoted by S_n.

Simplify fully

$$S_n-2S_{n+1}+S_{n+2}.$$

$$\boxed{S_n-2S_{n+1}+S_{n+2}=d}$$



Question 24 (***)

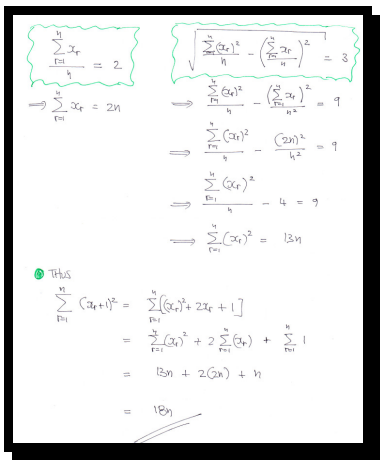
It is given that

$$\frac{1}{n}\sum_{r=1}^{n} x_r = 2 \qquad \text{and} \qquad \sqrt{\frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} x_r\right)^2} = 3 .$$

Determine, in terms of n, the value of

$$\sum_{r=1}^{n}(x_r+1)^2 .$$

$\square$, $\boxed{\displaystyle\sum_{r=1}^{n}(x_r+1)^2 = 18n}$



Question 25 (***)

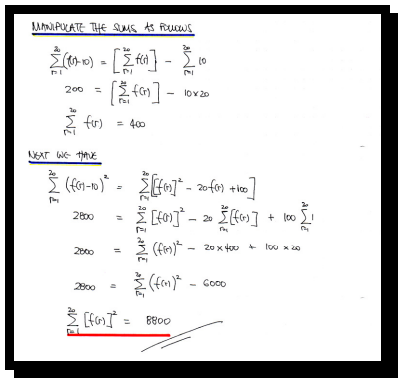
It is given that

$$\sum_{r=1}^{20}\left[f(r)-10\right]=200 \quad \text{and} \quad \sum_{r=1}^{20}\left[f(r)-10\right]^2=2800\,.$$

Find the value of

$$\sum_{r=1}^{20}\left[f(r)\right]^2\,.$$

$\square$, $\boxed{\sum_{r=1}^{20}\left[f(r)\right]^2=8800}$



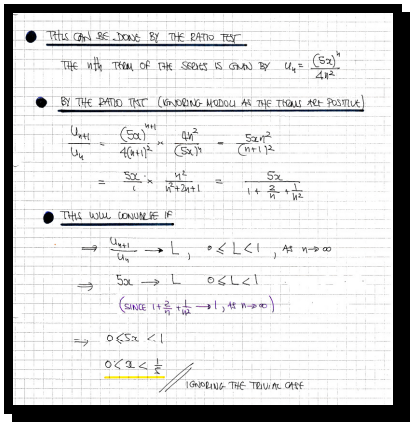
Question 26 (***)

It is given that the following series converges.

$$\sum_{n=1}^{\infty}\frac{(5x)^n}{4n^2},\ x\in\mathbb{R},\ x>0.$$

Determine the range of possible values of x.

$\boxed{\phantom{0<x<\frac{1}{5}}}$, $\boxed{0<x<\frac{1}{5}}$

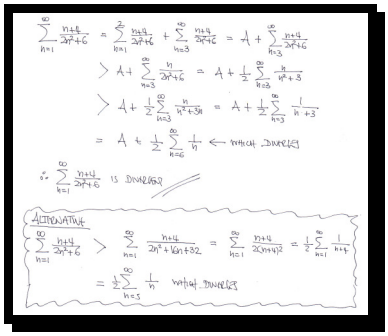


Question 27 (*+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty}\frac{n+4}{2n^2+6}.$$

divergent



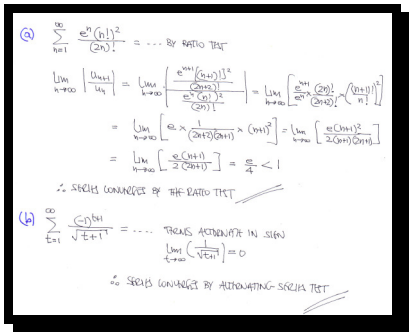
Question 28 (*+)**

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{\mathrm{e}^n\left(n!\right)^2}{(2n)!}$.

b) $\displaystyle\sum_{t=1}^{\infty}\frac{(-1)^{t+1}}{\sqrt{t+1}}$.

convergent , convergent

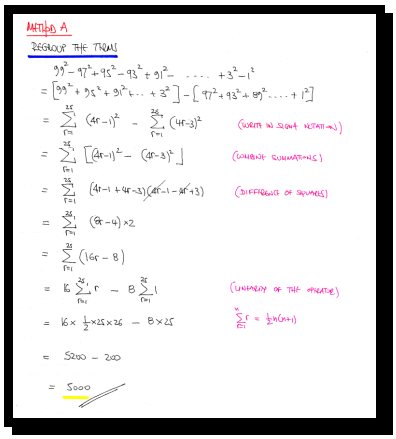


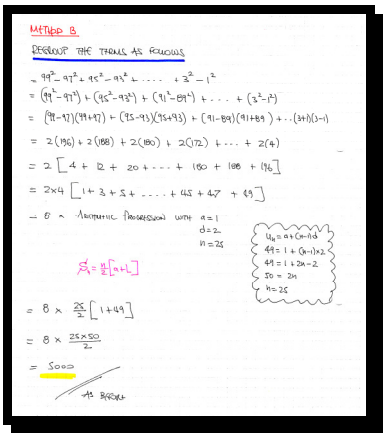
Question 29 (*+)**

By using an algebraic method, find the value of

$$99^2-97^2+95^2-93^2+...+3^2-1^2$$

$\square$, $\boxed{5000}$



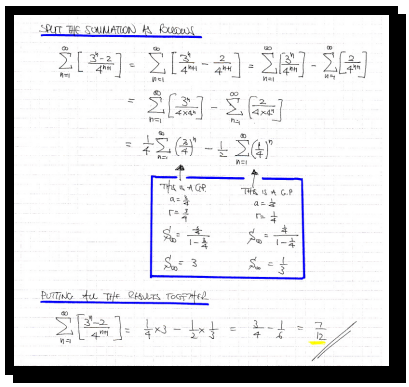


Question 30 (*+)**

Evaluate, showing clearly your method

$$\sum_{n=1}^{\infty}\frac{3^n-2}{4^{n+1}}.$$

$\boxed{}$, $\boxed{\frac{7}{12}}$

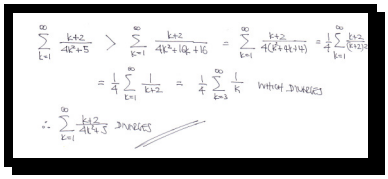


Question 31 (*+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{k=1}^{\infty}\frac{k+2}{4k^2+5}.$$

$\boxed{\text{divergent}}$

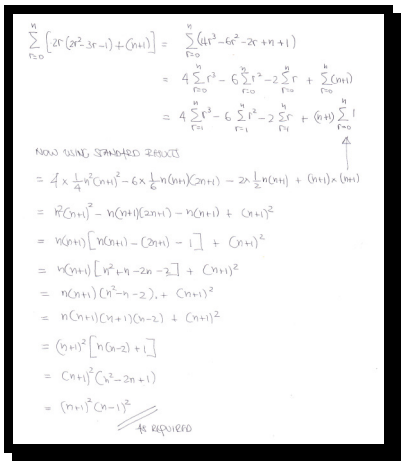


Question 32 (*+)**

Show clearly that

$$\sum_{r=1}^{n}\left[2r\left(2r^2-3r-1\right)+n+1\right]=\left(n+1\right)^2\left(n-1\right)^2.$$

proof



Question 33 (*+)**

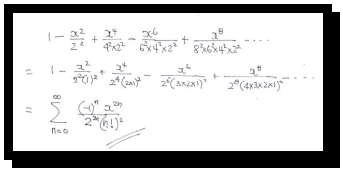
Consider the infinite series

$$1-\frac{x^2}{2^2}+\frac{x^4}{4^2\times 2^2}-\frac{x^6}{6^2\times 4^2\times 2^2}+\frac{x^8}{8^2\times 6^2\times 4^2\times 2^2}-\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}\left(n!\right)^2}x^{2n}$$

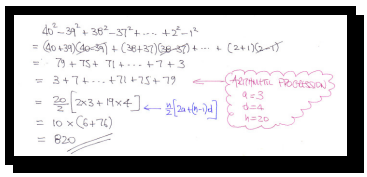


Question 34 (*+)**

Show clearly by an algebraic method that

$$40^2-39^2+38^2-37^2+\ldots+2^2-1^2=820\,.$$

proof

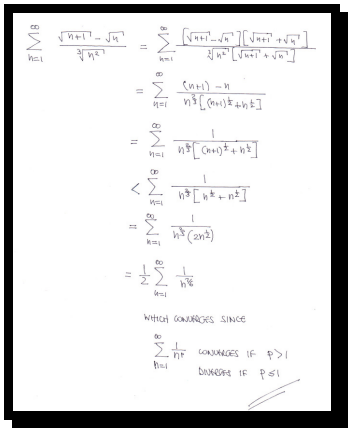


Question 35 (****)

By justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n^2}}.$$

convergent



Question 36 (****)

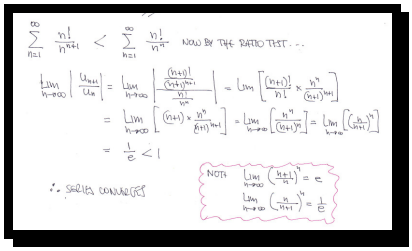
Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n!}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

You may assume without proof the value of $\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n\right]$.

convergent



Question 37 (****)

The sum of the first n terms of a series with general term u_n is given by the expression

$$S_n = n^2(n+1)(n+2).$$

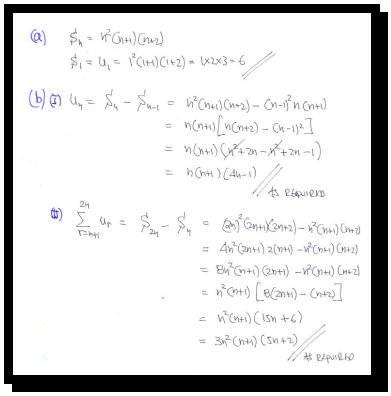
a) Find the first term of the series.

b) Show clearly that …

i. … $u_n = n(n+1)(4n-1)$

ii. … $\displaystyle\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$.

$\boxed{u_1 = 6}$



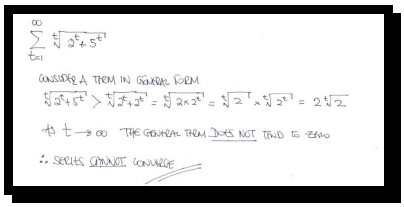
Question 38 (****)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty}\sqrt[t]{2^t+5^t}\,.$$

Show a full method, justifying every step in the workings.

$\boxed{\text{divergent}}$



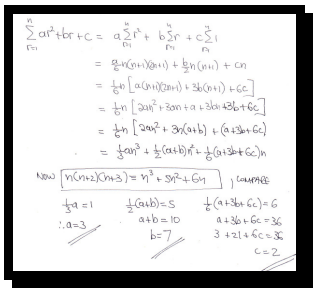
Question 39 (****)

$$\sum_{r=1}^{n}\left(ar^2+br+c\right)\equiv n^3+5n^2+6n\,,$$

where a, b and c are integer constants.

Determine the value of a, b and c.

$\boxed{a=3}$, $\boxed{b=7}$, $\boxed{c=2}$



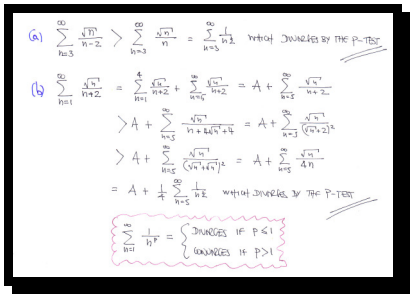
Question 40 (****)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\displaystyle\sum_{n=3}^{\infty}\left[\frac{\sqrt{n}}{n-2}\right]$.

b) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{\sqrt{n}}{n+2}\right]$.

$\boxed{\text{divergent}}$, $\boxed{\text{divergent}}$

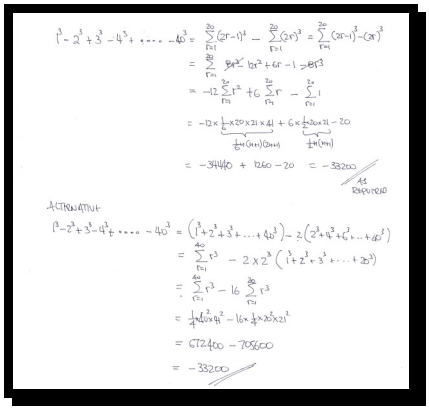


Question 41 (****)

Show clearly that

$$1^3-2^3+3^3-4^3+\ldots-40^3=-33200\,.$$

proof

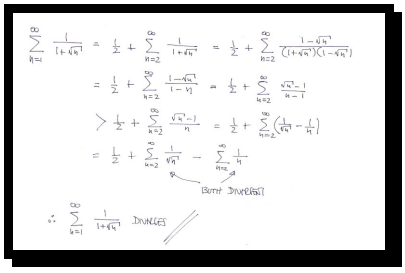


Question 42 (****)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}\,.$$

divergent



Question 43 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

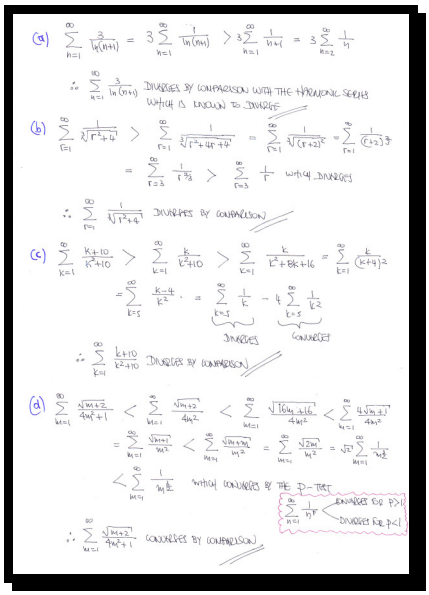
a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}$.

b) $\displaystyle\sum_{r=1}^{\infty}\frac{1}{\sqrt[3]{r^2+4}}$.

c) $\displaystyle\sum_{k=1}^{\infty}\frac{k+10}{k^2+10}$.

d) $\displaystyle\sum_{m=1}^{\infty}\frac{\sqrt{m+2}}{4m^2+1}$

$\boxed{\text{divergent}}$, $\boxed{\text{divergent}}$, $\boxed{\text{divergent}}$, $\boxed{\text{convergent}}$



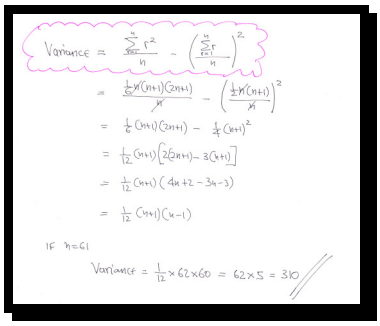
Question 44 **(****)**

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n)=\frac{1}{n}\sum_{r=1}^{n}r^2-\left[\frac{1}{n}\sum_{r=1}^{n}r\right]^2.$$

Determine a simplified expression $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n)=\frac{1}{12}\left(n^2-1\right)},\ \boxed{\text{Var}(61)=310}$$



Question 45 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

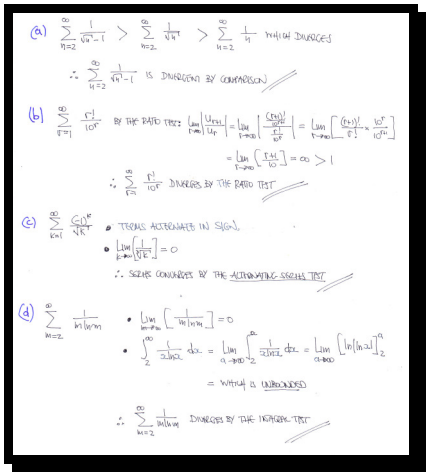
a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}-1}$.

b) $\displaystyle\sum_{r=1}^{\infty}\frac{r!}{10^r}$.

c) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$.

d) $\displaystyle\sum_{m=2}^{\infty}\frac{1}{m\ln m}$

$\boxed{\text{divergent}}$, $\boxed{\text{divergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{divergent}}$



Question 46 (****)

Consider the infinite series

$$1-\frac{x^2}{2}+\frac{x^4}{4\times 2}-\frac{x^6}{6\times 4\times 2}+\frac{x^8}{8\times 6\times 4\times 2}-\ldots$$

a) Write the above series in Sigma notation, in its simplest form.

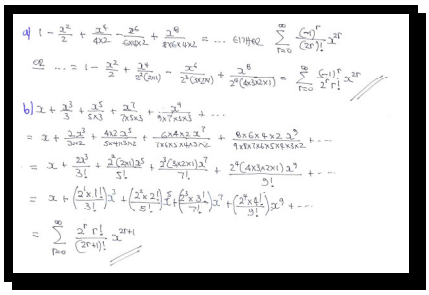
Next consider another infinite series

$$x+\frac{x^3}{3}+\frac{x^5}{5\times 3}+\frac{x^6}{7\times 5\times 3}+\frac{x^9}{9\times 7\times 5\times 3}+\ldots$$

b) Also, write this series in Sigma notation, in its simplest form.

[*You are not required to investigate the convergence or the sum of these series.*]

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{2^r r!}x^{2r}\,,\qquad \sum_{r=0}^{\infty}\frac{2^r r!}{(2r+1)!}x^{2r}$$



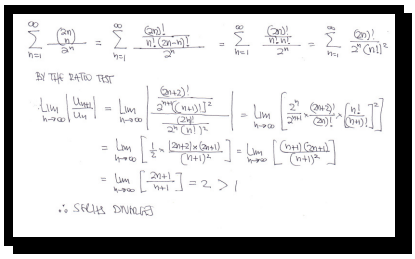
Question 47 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{2^n}$$

Show a full method, justifying every step in the workings.

$$\boxed{\text{divergent}}$$

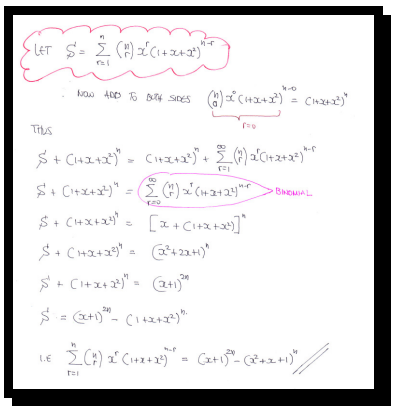


Question 48 (****)

$$\sum_{r=1}^{n}\left[\binom{n}{r}x^r\left(1+x+x^2\right)^{n-r}\right].$$

Simplify fully the above sum, into a summation free expression

$$\boxed{(x+1)^{2n}-\left(x^2+x-1\right)^n}$$



Question 49 **(****+)**

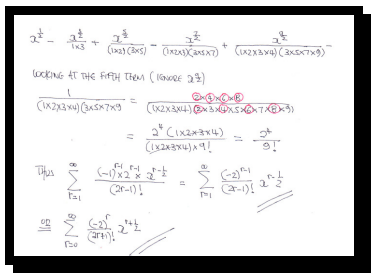
Consider the infinite series

$$x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{1\times 3}+\frac{x^{\frac{5}{2}}}{(1\times 2)(3\times 5)}-\frac{x^{\frac{7}{2}}}{(1\times 2\times 3)(3\times 5\times 7)}+\frac{x^{\frac{9}{2}}}{(1\times 2\times 3\times 4)(3\times 5\times 7\times 9)}-\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{r=0}^{\infty}\left[\frac{(-2)^r}{(2r+1)!}x^{\frac{1}{2}+r}\right]$$



Question 50 **(****+)**

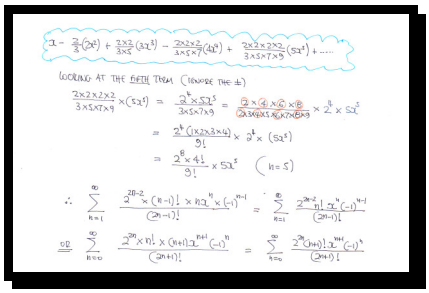
Consider the infinite series

$$x-\frac{2}{3}\left(2x^2\right)+\frac{2\times 2}{3\times 5}\left(3x^3\right)-\frac{2\times 2\times 2}{3\times 5\times 7}\left(4x^4\right)+\frac{2\times 2\times 2\times 2}{3\times 5\times 7\times 9}\left(5x^5\right)-\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=0}^{\infty}\left[\frac{2^{2n}\times(-1)^n\times(n+1)!\times x^{n+1}}{(2n+1)!}\right]$$

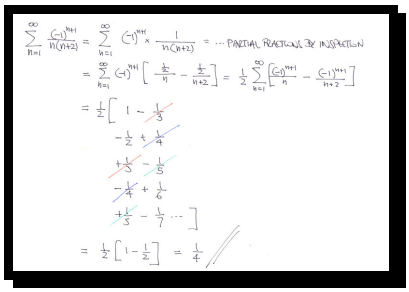


Question 51 **(****+)**

Use a suitable method to sum the following series.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+2)}.$$

$$\boxed{\frac{1}{4}}$$



Question 52 **(****+)**

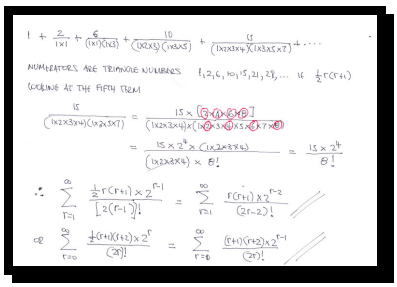
Consider the infinite series

$$1+\frac{2}{1\times 1}+\frac{6}{(1\times 2)(1\times 3)}+\frac{10}{(1\times 2\times 3)(1\times 3\times 5)}+\frac{15}{(1\times 2\times 3\times 4)(1\times 3\times 5\times 7)}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{r=0}^{\infty}\frac{(r+1)(r+2)2^{r-1}}{(2r)!}}$$

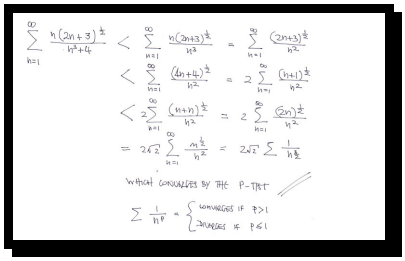


Question 53 (**+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty}\frac{n\left(2n+3\right)^{\frac{1}{2}}}{n^3+4}.$$

convergent



Question 54 (**+)**

A sequence is generated by the function

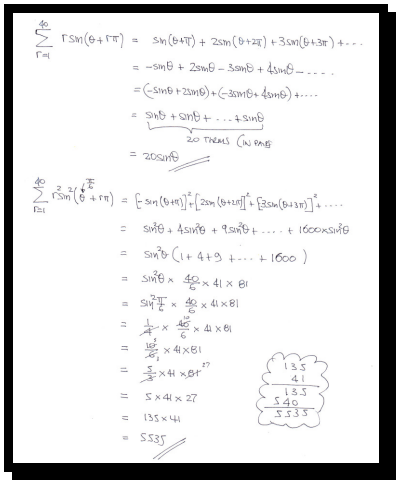
$$u_r(\theta) \equiv r\sin(\theta + r\pi),\ r \in \mathbb{N}.$$

Find an expression or the value, whichever is appropriate, for each of the series

a) $\displaystyle\sum_{r=1}^{40} u_r(\theta)$.

b) $\displaystyle\sum_{r=1}^{40}\left[u_r\left(\frac{\pi}{6}\right)\right]^2$.

$\boxed{20\sin\theta}$, $\boxed{5535}$



Question 55 **(****+)**

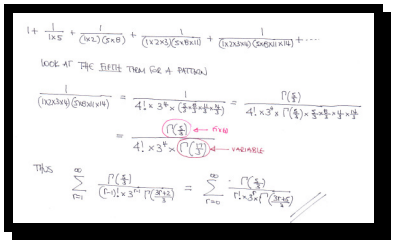
Consider the infinite series

$$1+\frac{1}{1\times 5}+\frac{1}{(1\times 2)(5\times 8)}+\frac{1}{(1\times 2\times 3)(5\times 8\times 11)}+\frac{1}{(1\times 2\times 3\times 4)(5\times 8\times 11\times 14)}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{r=1}^{\infty}\frac{\Gamma\left(\frac{5}{3}\right)}{3^{r-1}\times(r-1)!\times\Gamma\left(\frac{3r+2}{3}\right)}=\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{5}{3}\right)}{3^{r}\times r!\times\Gamma\left(\frac{3r+5}{3}\right)}$$



Question 56 (**+)**

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

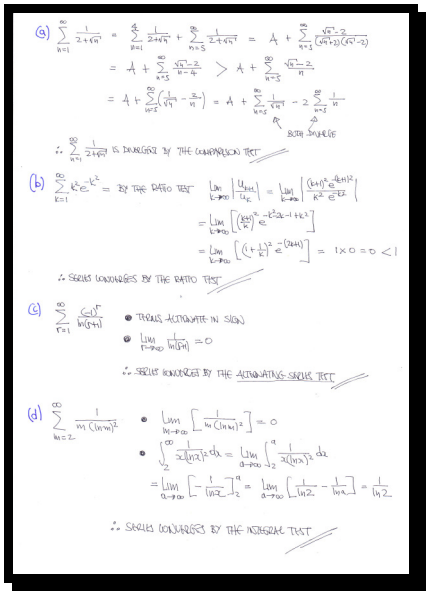
a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$.

b) $\displaystyle\sum_{k=1}^{\infty}k^2\,\mathrm{e}^{-k^2}$.

c) $\displaystyle\sum_{r=1}^{\infty}\frac{(-1)^r}{\ln(r+1)}$.

d) $\displaystyle\sum_{m=2}^{\infty}\frac{1}{m(\ln m)^2}$

$\boxed{\text{divergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{convergent}}$, $\boxed{\text{convergent}}$



Question 57 (**+)**

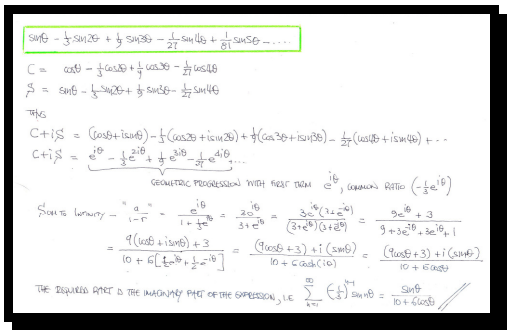
The following convergent series S is given below

$$S=\sin\theta-\frac{1}{3}\sin 2\theta+\frac{1}{9}\sin 3\theta-\frac{1}{27}\sin 4\theta\ ...$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S=\frac{\sin\theta}{10+6\cos\theta}.$$

proof



Question 58 (**+)**

A sequence of positive integers is generated by

$$u_n = 3^n - 1\,,\ n = 1,2,3,4,...$$

a) Write down the first seven terms of this sequence.

b) Verify that

$$u_{n+1} = 3u_n + 2\,.$$

c) Show clearly that …

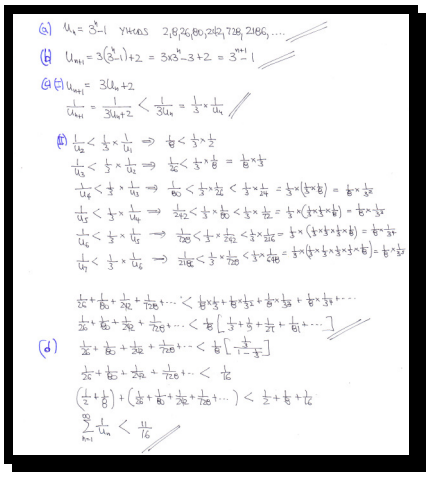
i. … $\dfrac{1}{u_{n+1}} < \dfrac{1}{3} \times \dfrac{1}{u_n}$.

ii. … $\dfrac{1}{26} + \dfrac{1}{80} + \dfrac{1}{242} + \dfrac{1}{728} + \dfrac{1}{2186} + ... < \dfrac{1}{8}\left[\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + ...\right]$

d) Deduce that

$$\sum_{n=1}^{\infty} u_n < \frac{11}{16}\,.$$

2, 8, 26, 80, 242, 728, 2186,...



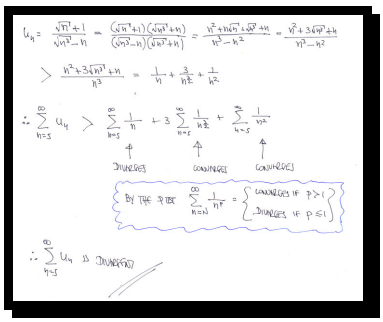
Question 59 (**+)**

$$u_n = \frac{\sqrt{n}+1}{\sqrt{n^3}-n},\ n \in \mathbb{N},\ n \geq 5.$$

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of

$$\sum_{n=5}^{\infty} u_n .$$

convergent

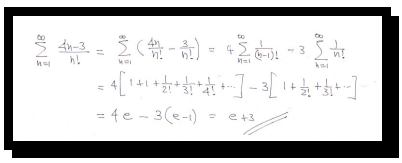


Question 60 (**+)**

Show clearly that

$$\sum_{n=1}^{\infty} \frac{4n-3}{n!} = \mathrm{e}+3 .$$

proof



Question 61 (**+)**

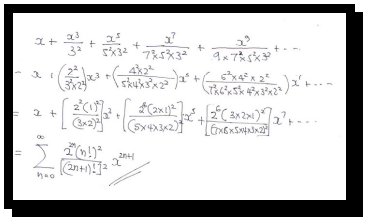
Consider the infinite series

$$x+\frac{x^3}{3^2}+\frac{x^5}{5^2\times 3^2}+\frac{x^7}{7^2\times 5^2\times 3^2}+\frac{x^9}{9^2\times 7^2\times 5^2\times 3^2}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=0}^{\infty}\left[\frac{2^{2n}(n!)^2}{\left[(2n+1)!\right]^2}x^{2n+1}\right]$$

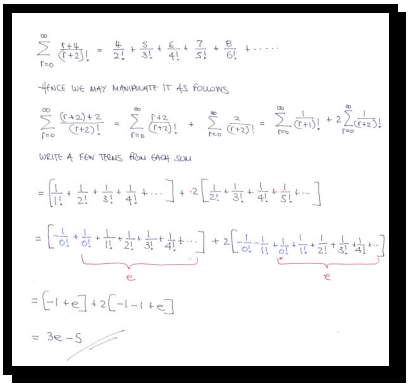


Question 62 (**+)**

Show clearly that

$$\sum_{r=0}^{\infty}\frac{r+4}{(r+2)!}=3\mathrm{e}-5.$$

proof



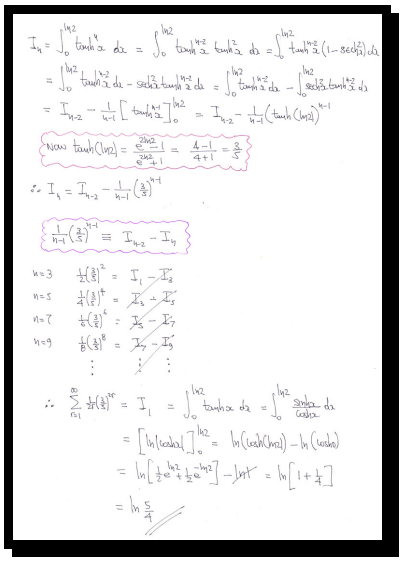
Question 63 (**+)**

$$I_n = \int_0^{\ln 2} \tanh^n x \ dx, \ n \in \mathbb{N}.$$

By considering a reduction formula for I_n, or otherwise, show clearly that

$$\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right).$$

proof



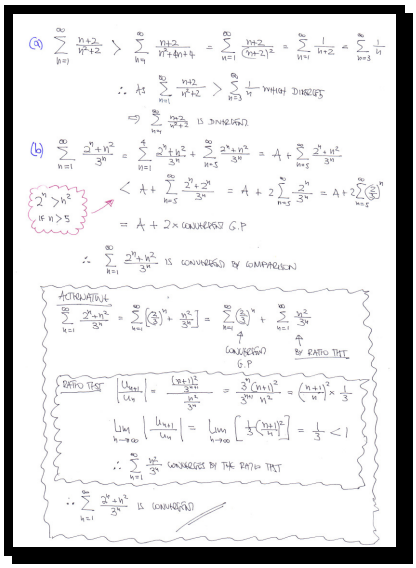
Question 64 **(****+)**

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $$\sum_{n=1}^{\infty}\left[\frac{n+2}{n^2+2}\right].$$

b) $$\sum_{n=1}^{\infty}\left[\frac{2^n+n^2}{3^n}\right].$$

$\boxed{\text{divergent}}$, $\boxed{\text{convergent}}$



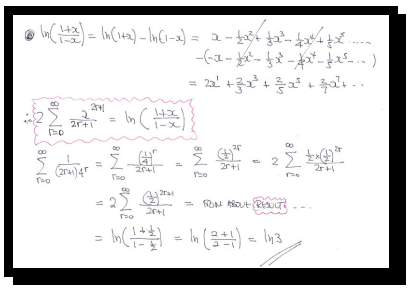
Question 65 (**+)**

By considering the Mclaurin expansion of $\ln\left(\frac{1+x}{1-x}\right)$ find the value of

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)4^r},$$

giving the final answer as the natural logarithm of an integer.

$\ln 3$



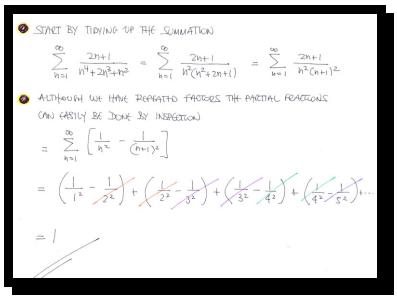
Question 66 (**+)**

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty}\frac{2n+1}{n^4+2n^3+n^2}.$$

You may assume the series converges.

$\square$, 1



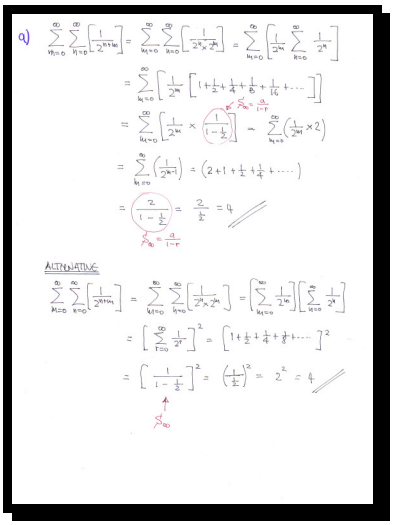
Question 67 **(****+)**

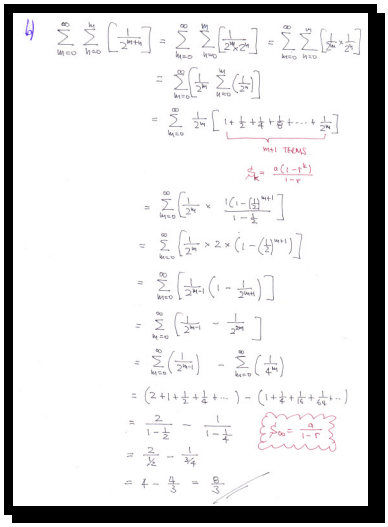
Sum each of the following double series.

a) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{2^{m+n}}\right]$.

b) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left[\frac{1}{2^{m+n}}\right]$.

$\boxed{}$, $\boxed{4}$, $\boxed{\frac{8}{3}}$





Question 68 **(****+)**

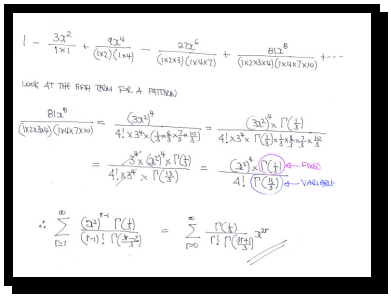
Consider the infinite series

$$1-\frac{3x^2}{1\times 1}+\frac{9x^4}{(1\times 2)(1\times 4)}-\frac{27x^6}{(1\times 2\times 3)(1\times 4\times 7)}-\frac{81x^8}{(1\times 2\times 3\times 4)(1\times 4\times 7\times 10)}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\Gamma\left(\frac{1}{3}\right)}{(r-1)!\times\Gamma\left(\frac{3r-2}{3}\right)}\left(x^2\right)^{r-1}=\sum_{r=0}^{\infty}\frac{(-1)^{r}\Gamma\left(\frac{1}{3}\right)}{r!\times\Gamma\left(\frac{3r+1}{3}\right)}x^{2r}$$

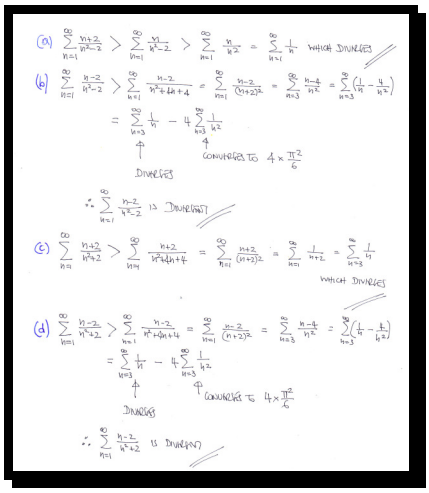


Question 69 (**+)**

$$\sum_{n=1}^{\infty}\left[\frac{n\pm 2}{n^2\pm 2}\right].$$

Use a comparison test to show that all four series described by the above expression are divergent.

proof

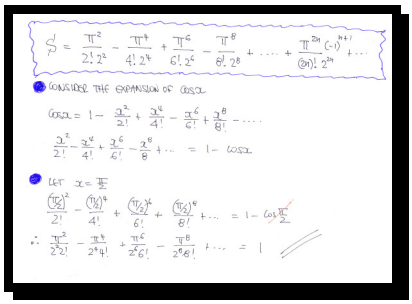


Question 70 (**+)**

By showing a detailed method, sum the following series.

$$\frac{\pi^2}{2^2\,2!}-\frac{\pi^4}{2^4\,4!}+\frac{\pi^6}{2^6\,6!}-\frac{\pi^8}{2^8\,8!}+\ldots+\frac{(-1)^{n+1}\,\pi^{2n}}{2^{2n}\,(2n)!}+\ldots$$

$\boxed{1}$

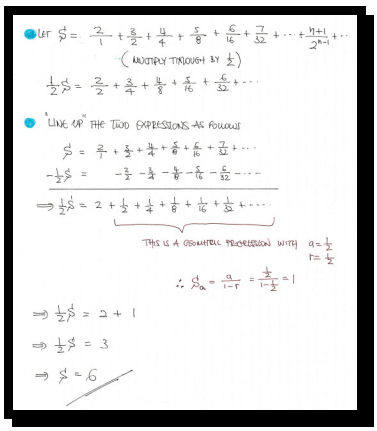


Question 71 (**+)**

By showing a detailed method, sum the following series.

$$\frac{2}{1}+\frac{3}{2}+\frac{4}{4}+\frac{5}{8}+\frac{6}{16}+\frac{7}{32}\ldots$$

$\boxed{}$, $\boxed{6}$



Question 72 (**+)**

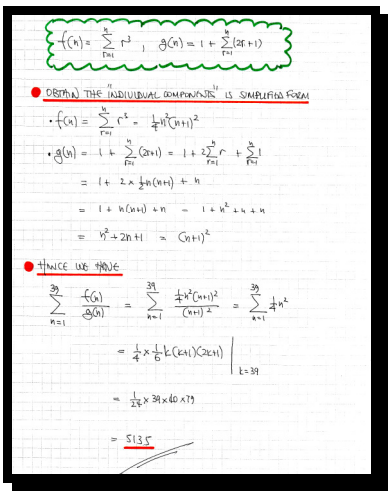
The positive integer functions f and g are defined as

$$f(n)=\sum_{r=1}^{n} r^3 \qquad \text{and} \qquad g(n)=1+\sum_{r=1}^{n}(2r+1).$$

Evaluate

$$\sum_{n=1}^{39}\left[\frac{f(n)}{g(n)}\right].$$

$\boxed{}$, $\boxed{5135}$



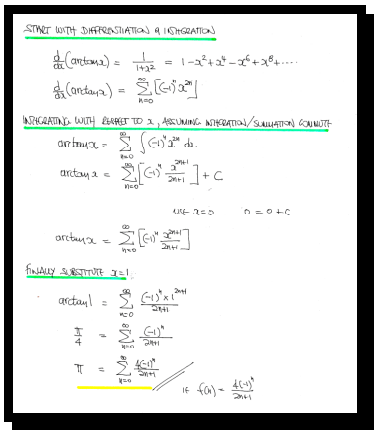
Question 73 **(****+)**

Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f.

$\square$, $\boxed{\pi = \sum_{n=0}^{\infty} \frac{4(-1)n}{2n+1}}$



Question 74 **(*****)**

It is given that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=k^n,$$

where n and k are positive integer constants.

a) By considering the binomial expansion of $(1+x)^n$, determine the value of k.

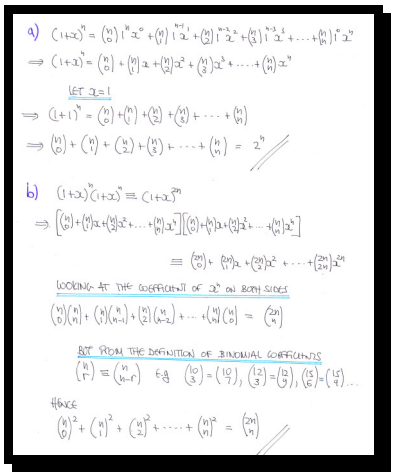
b) By considering the coefficient of x^n in

$$(1+x)^n(1+x)^n\equiv(1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

$\boxed{}$, $\boxed{k=2}$



Question 75 (*****)

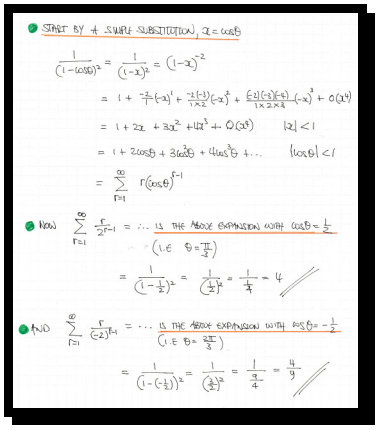
By considering the binomial expansion of

$$\frac{1}{\left(1-\cos\theta\right)^2},$$

sum each of the following series.

- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]$.
- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]$.

$\square$, $\boxed{\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]=4}$, $\boxed{\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]=\frac{4}{9}}$



Question 76 (*****)

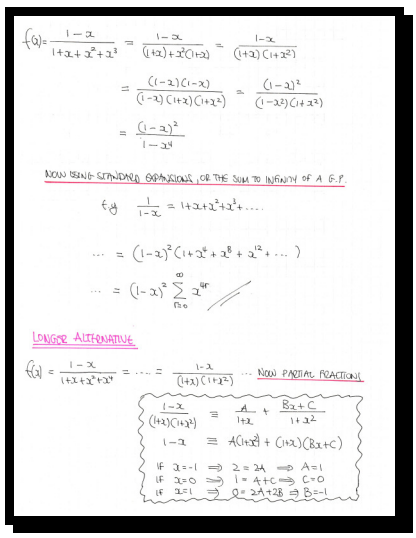
$$f(x)\equiv\frac{1-x}{1+x+x^2+x^3},\ -1<x<1.$$

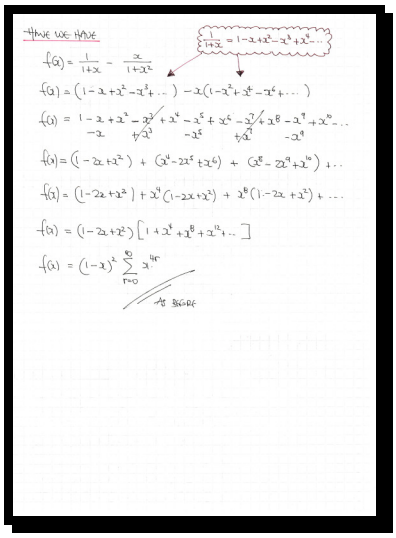
Show that $f(x)$ can be written in the form

$$f(x)=g(x)\sum_{r=0}^{\infty}\left(x^{4r}\right),$$

where $g(x)$ is a simplified function to be found.

$\square$, $\boxed{g(x)=(1-x)^2}$





Question 77 (*****)

The sum to infinity S of the convergent geometric series is given by

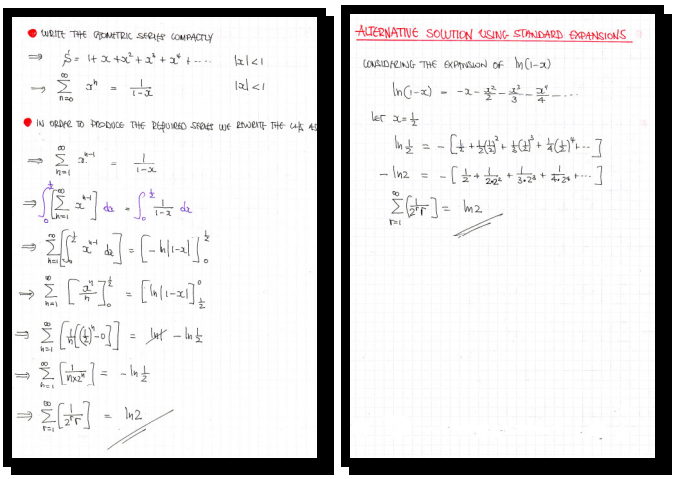
$$S=1+x+x^2+x^3+x^4+\ldots, \quad |x|<1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty}\left[\frac{1}{r\times 2^r}\right].$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\ln 2}$

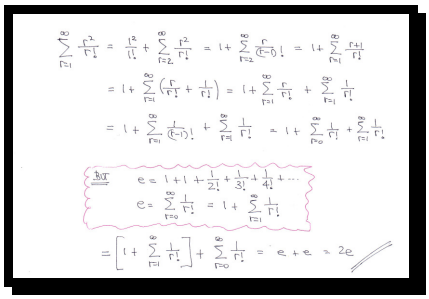


Question 78 (*****)

Show clearly that

$$\sum_{r=1}^{\infty}\frac{r^2}{r!}=2\mathrm{e}\ .$$

proof



Question 79 (*****)

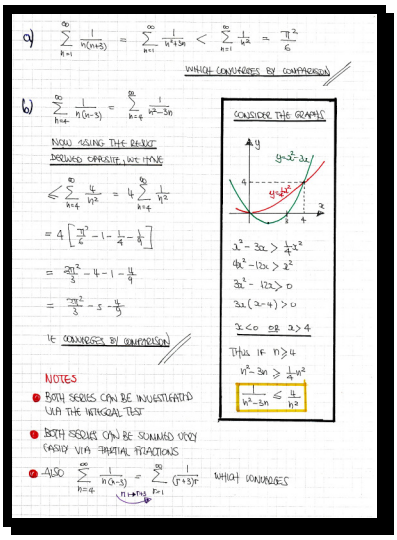
Investigate the convergence or divergence of each of the following two series using standard tests and justifying every step in the workings.

a) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n(n+3)}\right]$.

b) $\displaystyle\sum_{n=4}^{\infty}\left[\frac{1}{n(n-3)}\right]$.

You may not conclude simply by summing each the series.

[ ], [convergent], [convergent]



Question 80 **(*****)**

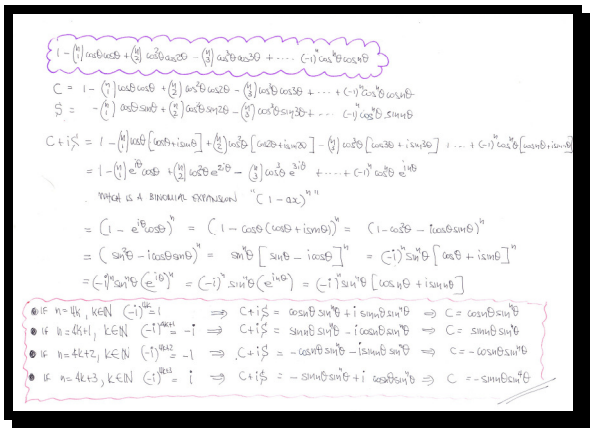
The finite sum C is given below.

$$C=\sum_{r=0}^{n}\left[\binom{n}{r}(-1)^n\cos^n\theta\cos n\theta\right].$$

Given that $n\in\mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact fully simplified form.

$\boxed{}$, $\boxed{n=4k,\ k\in\mathbb{N}:C=\cos n\theta\sin^n\theta}$, $\boxed{n=4k+1,\ k\in\mathbb{N}:C=\sin n\theta\sin^n\theta}$,
$\boxed{n=4k+2,\ k\in\mathbb{N}:C=-\cos n\theta\sin^n\theta}$, $\boxed{n=4k+3,\ k\in\mathbb{N}:C=-\sin n\theta\sin^n\theta}$



Question 81 (*****)

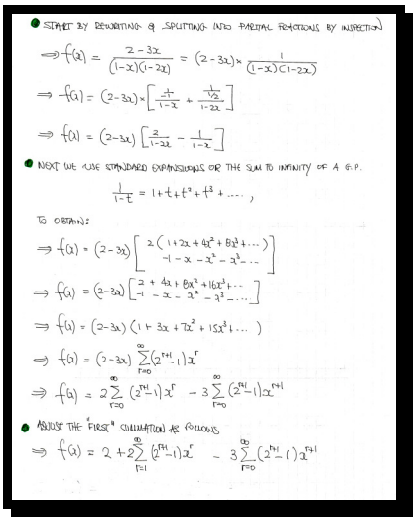
$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, \ -\frac{1}{2} < x < \frac{1}{2}.$$

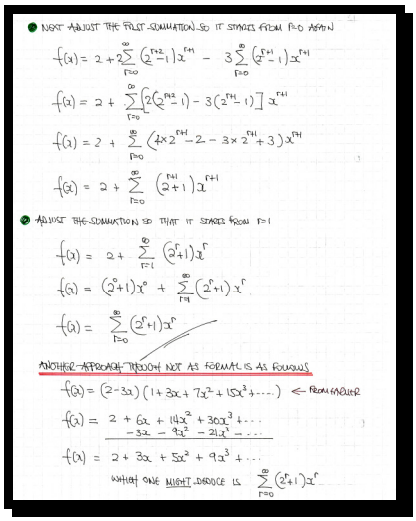
Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty} \left[x^r \, g(r) \right],$$

where $g(r)$ is a simplified function to be found.

$\square$, $\boxed{g(r) = 2^r + 1}$





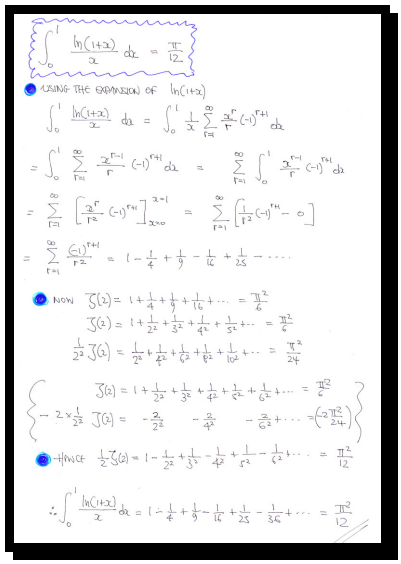
Question 82 **(*****)**

Show, by considering standard series, that

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$$

You may assume without proof that $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{1}{6}\pi^2$

[ ], [proof]



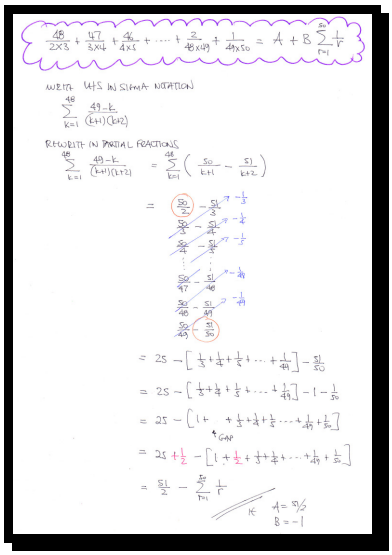
Question 83 **(*****)**

Show, by a detailed method, that

$$\frac{48}{2\times 3}+\frac{47}{3\times 4}+\frac{46}{4\times 5}\ldots+\frac{2}{48\times 49}+\frac{1}{49\times 50}=A+B\sum_{r=1}^{50}\frac{1}{r},$$

where A and B are constants to be found.

$\boxed{\phantom{A=\frac{51}{2},\quad B=-1}}$, $\boxed{A=\frac{51}{2},\quad B=-1}$

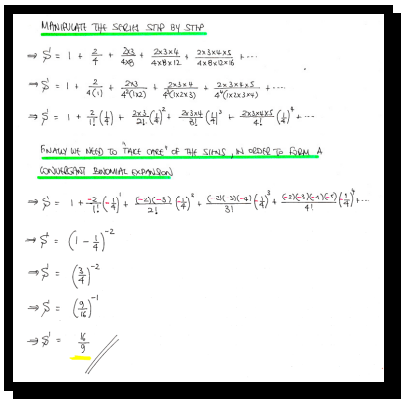


Question 84 **(*****)**

$$S=1+\frac{2}{4}+\frac{2\cdot 3}{4\cdot 8}+\frac{2\cdot 3\cdot 4}{4\cdot 8\cdot 12}+\frac{2\cdot 3\cdot 4\cdot 5}{4\cdot 8\cdot 12\cdot 16}+...$$

By considering a suitable binomial series, or other wise, find the sum to infinity of S.

$\square$, $\boxed{S_{\infty}=\frac{16}{9}}$



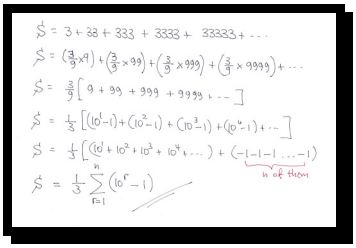
Question 85 **(*****)**

$$3 + 33 + 333 + 3333 + 33333 + ...$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$\boxed{S_n=\sum_{r=1}^{n}\left[\frac{1}{3}\left(10^r-1\right)\right]}$$



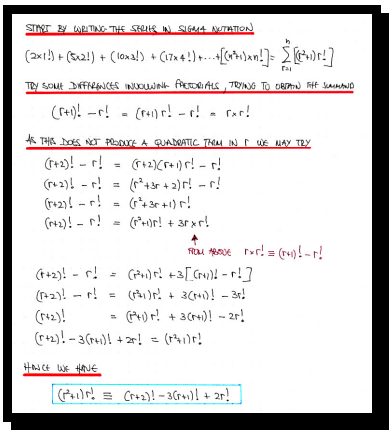
Question 86 **(*****)**

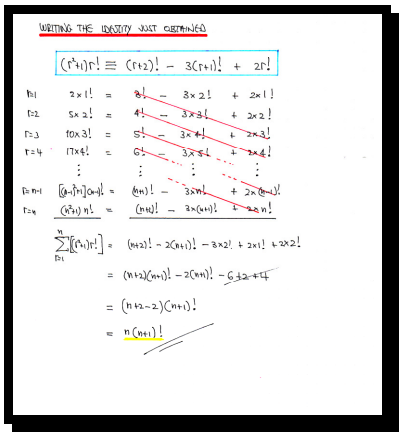
$$S_n = (2\times 1!)+(5\times 2!)+(10\times 3!)+(17\times 4!)+\ldots+\left(n^2+1\right)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

[ ], proof





Question 87 **(*****)**

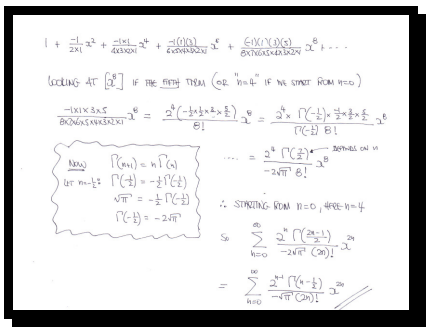
Consider the infinite series

$$1+\frac{-1}{2\times 1}x^2+\frac{-1\times 1}{4\times 3\times 2\times 1}x^4+\frac{-1\times 1\times 3}{6\times 5\times 4\times 3\times 2\times 1}x^6+\frac{-1\times 1\times 3\times 5}{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}x^8+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=0}^{\infty}\left[\frac{2^{n-1}\Gamma\left(n-\frac{1}{2}\right)}{-\sqrt{\pi}\times(2n)!}x^{2n}\right]$$

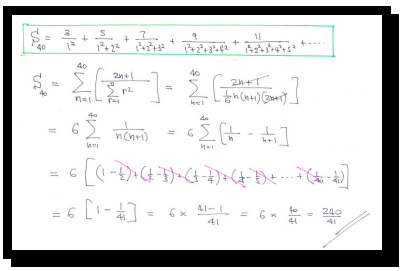


Question 88 **(*****)**

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\frac{11}{1^2+2^2+3^2+4^2+5^2}+\ldots,$$

Show, by a detailed method, that the sum of the first 40 terms of the series shown above is $\frac{240}{41}$.

[ ], proof

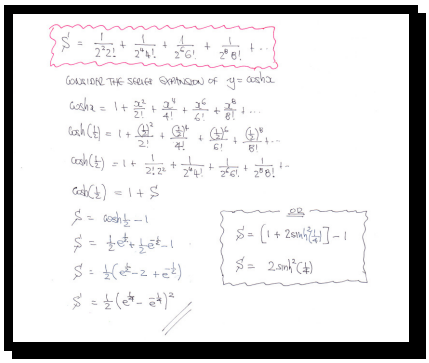


Question 89 (*****)

By showing a detailed method, sum the following series.

$$\frac{1}{2^2\,2!}+\frac{1}{2^4\,4!}+\frac{1}{2^6\,6!}+\frac{1}{2^8\,8!}+\ldots+\frac{1}{2^{2r}\,(2r)!}+\ldots$$

$$\boxed{\frac{1}{2}\left[e^{\frac{1}{4}}-e^{-\frac{1}{4}}\right]^2=2\sinh^2\left(\frac{1}{4}\right)}$$



Question 90 (*****)

A function is defined as

$$[x] \equiv \{\text{the greatest integer less or equal to } x\}.$$

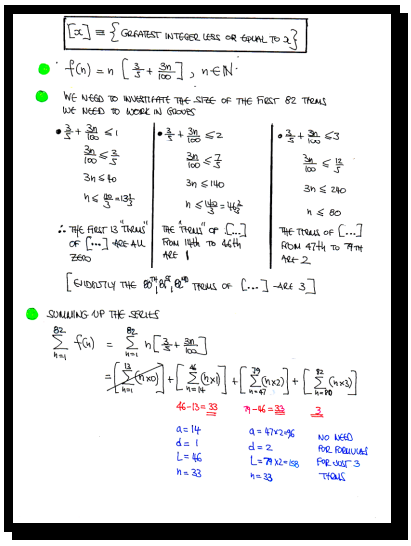
The function f is defined as

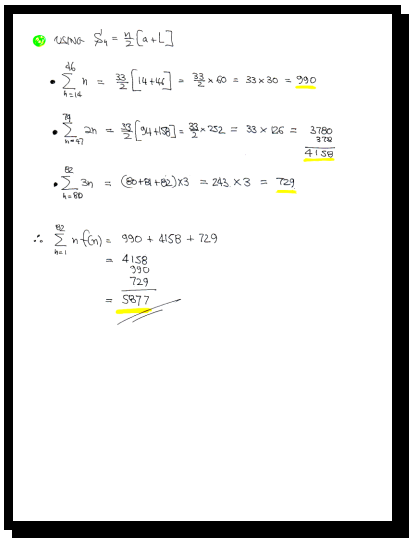
$$f(n) = n\left[\frac{3}{5}+\frac{3n}{100}\right],\ n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

$\square$, $\boxed{5877}$



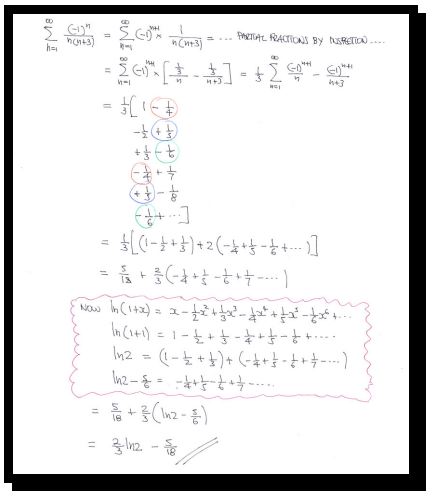


Question 91 (*****)

Use partial fractions and a suitable Mclaurin expansion to sum the following series.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+3)}.$$

$$\boxed{\frac{2}{3}\ln 2-\frac{5}{18}}$$



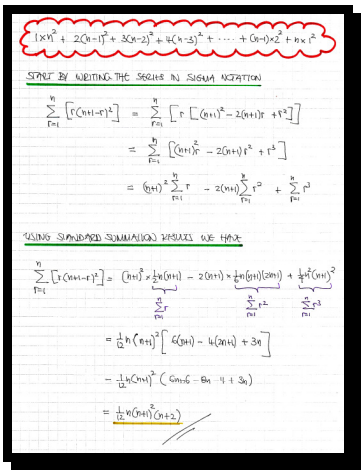
Question 92 (*****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1\times n^2+2(n-1)^2+3(n-2)^2+4(n-3)^2+\ldots+(n-1)\times 2^2+n\times 1^2 .$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

$\square$, $\boxed{f(n)=\frac{1}{12}n(n+2)(n+1)^2}$

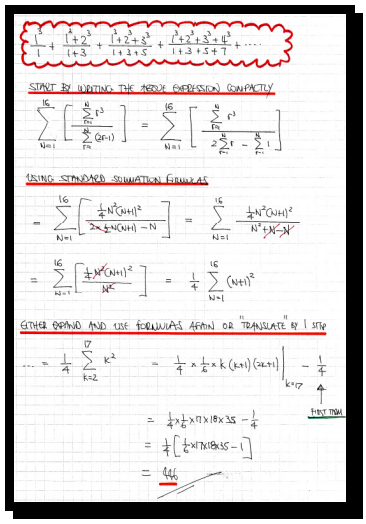


Question 93 (*****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\frac{1^3+2^3+3^3+4^3}{1+3+5+7}+...$$

$\square$, $\boxed{446}$



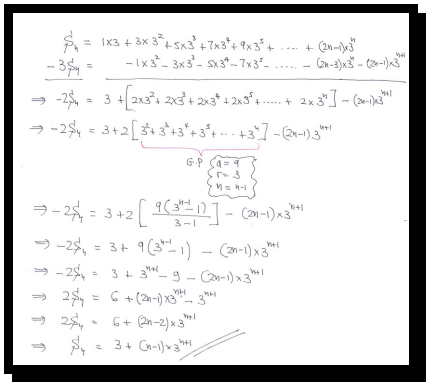
Question 94 **(*****)**

$$S_n = 1\times 3 + 3\times 3^2 + 5\times 3^3 + 7\times 3^4 + ... + (2n-1)\times 3^n$$

Find a simplified expression for S_n, giving the answer in the form $A + f(n)\times 3^{n+1}$, where A is an integer and $f(n)$ a linear function of n.

[*The standard techniques used for the summation of a geometric series are useful in this question*]

$$\boxed{S_n = 3 + (n-1)\times 3^{n+1}}$$

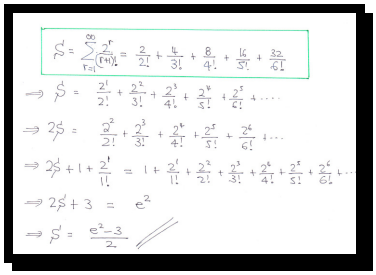


Question 95 **(*****)**

By showing a detailed method, sum the following series.

$$\sum_{r=1}^{\infty}\left[\frac{2^r}{(r+1)!}\right].$$

$$\boxed{\frac{1}{2}\left(e^2-3\right)}$$



Question 96 (*****)

The r^{th} term of a progression is given by

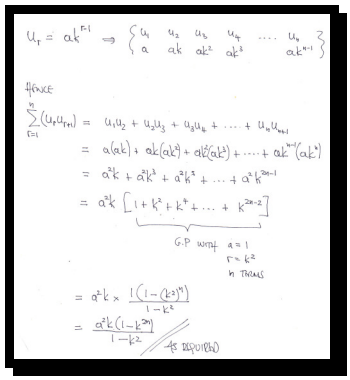
$$u_r = ak^{r-1},$$

where a and k are constants with $k \neq \pm 1$.

Show clearly that

$$\sum_{r=1}^{n}\left(u_r \times u_{r+1}\right) = \frac{a^2k\left(1-k^{2n}\right)}{1-k^2}.$$

proof



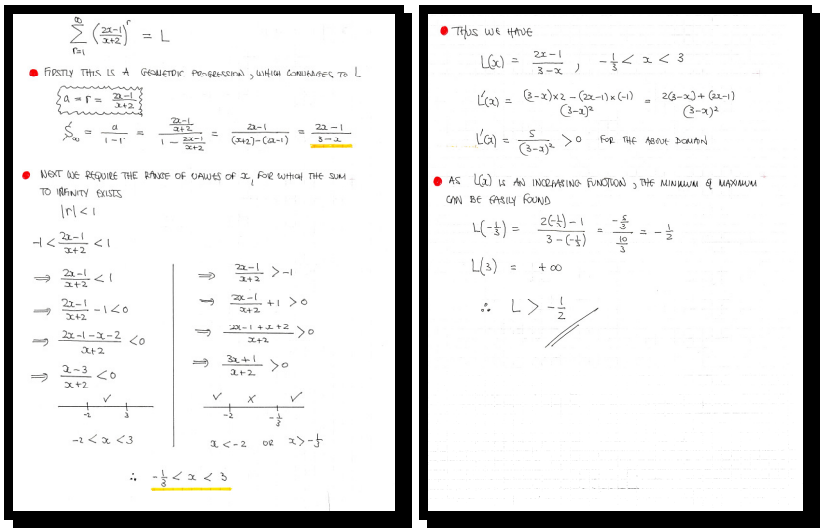
Question 97 (*****)

It is given that the following series converges to a limit L.

$$\sum_{r=1}^{\infty}\left[\frac{2x-1}{x+2}\right]^r .$$

Determine with full justification the range of possible values of L.

$\square$, $\boxed{L>-\frac{1}{2}}$



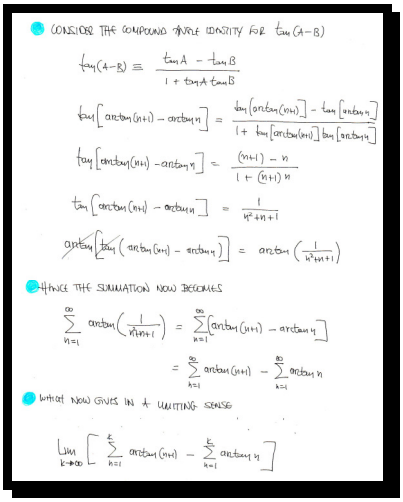
Question 98 **(*****)**

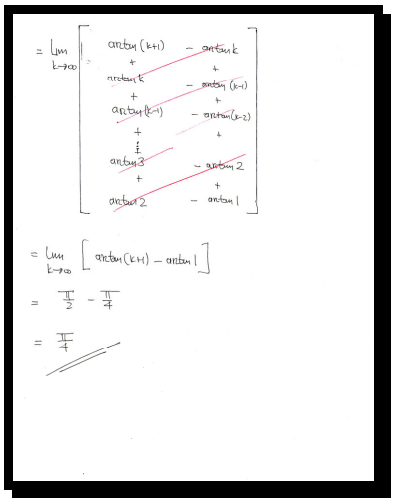
By considering the trigonometric identity for $\tan(A-B)$, with $A=\arctan(n+1)$ and $B=\arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\square$, $\boxed{\frac{\pi}{4}}$





Question 99 **(*****)**

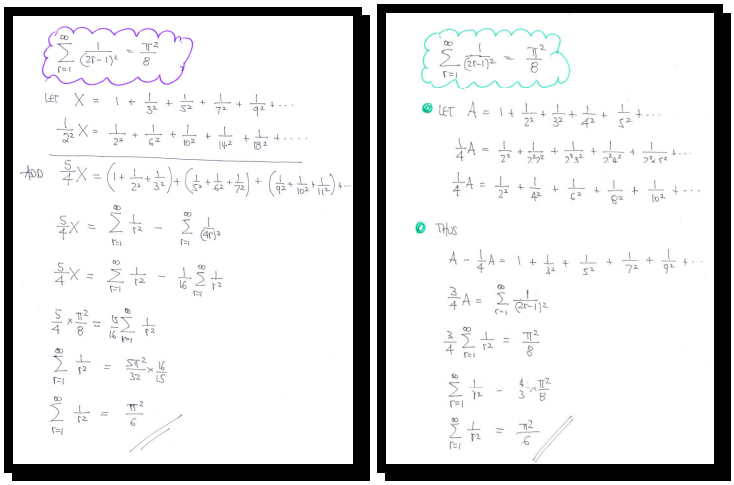
It is given that

$$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=\frac{\pi^2}{8}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty}\frac{1}{r^2}.$$

$$\boxed{\frac{\pi^2}{6}}$$



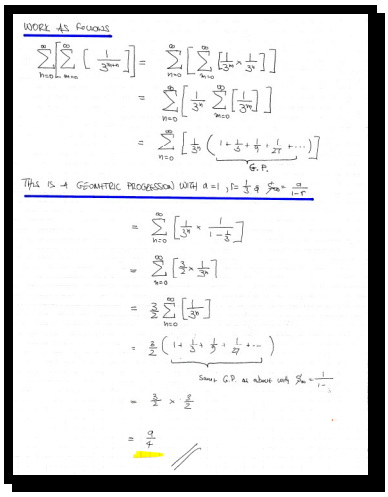
Question 100 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{9}{4}}$

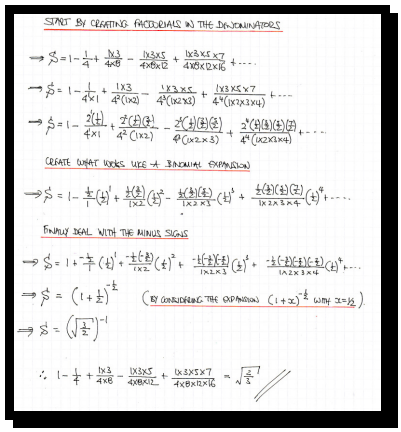


Question 101 (*****)

$$S=1-\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}-\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16}-...$$

Find the sum to infinity of S, by considering the binomial series expansion of $(1+x)^n$ for suitable values of x and n.

$\square$, $\boxed{S_{\infty}=\sqrt{\frac{2}{3}}}$

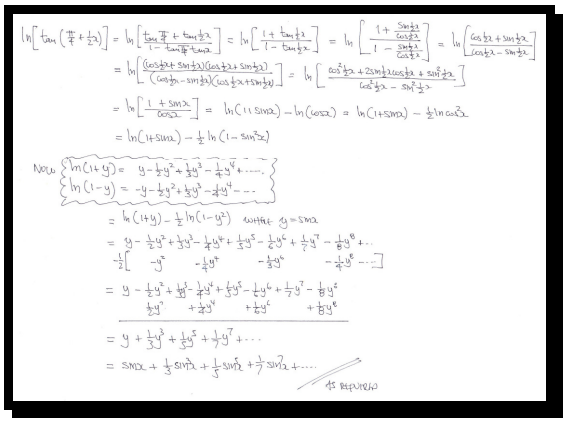


Question 102 (*****)

Show clearly that

$$\ln\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\sum_{n=1}^{\infty}\frac{\sin^{2n-1}x}{2n-1}.$$

proof



Question 103 **(*****)**

The function f is defined as

$$f(n)=\frac{\mathrm{e}^{-\lambda}\lambda^{n}}{n!},$$

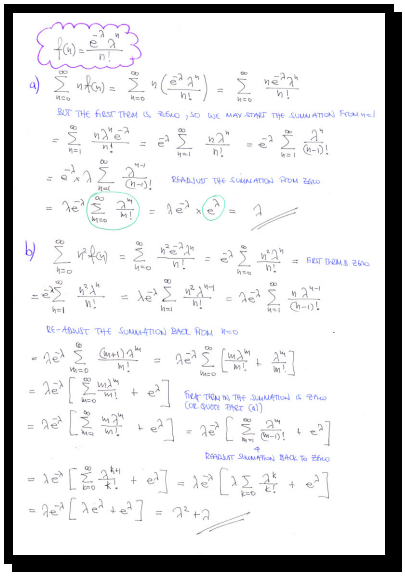
where $n=0,1,2,3,4,...$ and λ is a positive constant.

By showing a detailed method, prove that …

a) … $\displaystyle\sum_{n=0}^{\infty}\left[n f(n)\right]=\lambda$.

b) … $\displaystyle\sum_{n=0}^{\infty}\left[n^{2} f(n)\right]=\lambda^{2}+\lambda$.

proof

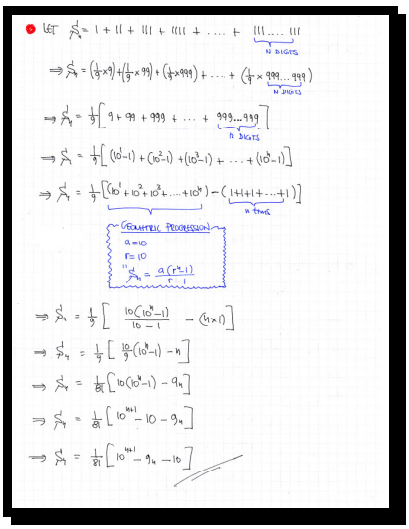


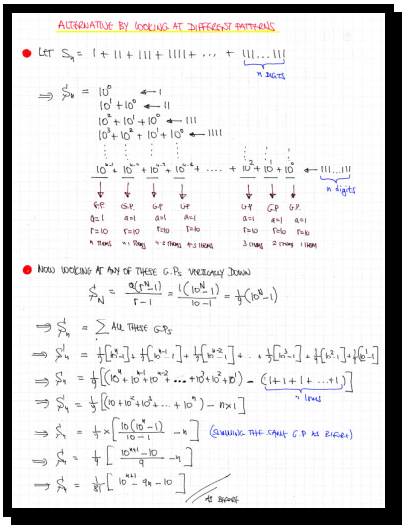
Question 104 (*****)

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + ...$$

$\square$, $\boxed{S_n = \frac{1}{81}\left[10^{n+1} - 10 - 9n\right]}$





Question 105 (*****)

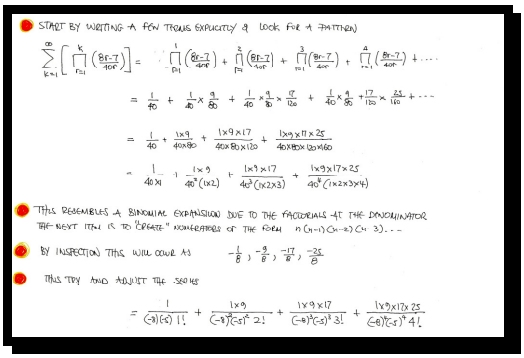
The product operator $\prod$, is defined as

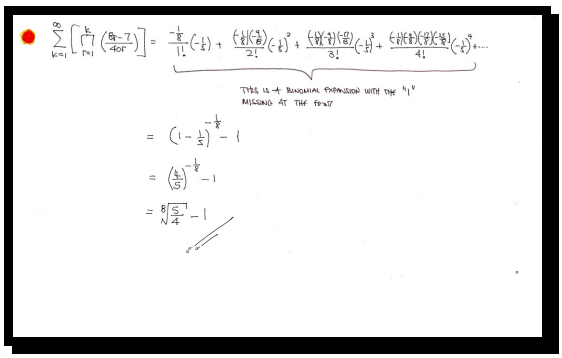
$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k \, .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

$\square$, $\sqrt[8]{\frac{5}{4}}-1$





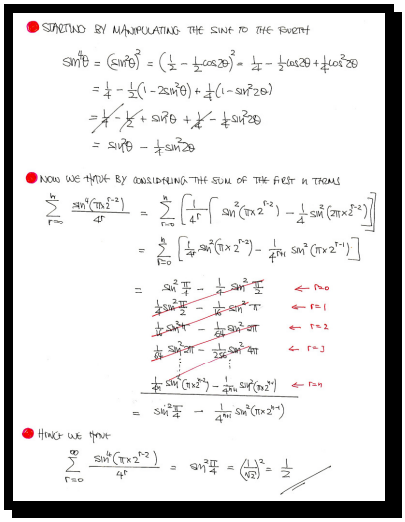
Question 106 (*****)

Find the value of

$$\sum_{r=0}^{\infty}\left[\frac{\sin^4\left(\pi\times 2^{r-2}\right)}{4^r}\right].$$

Hint: Express $\sin^4\theta$ in terms of $\sin^2\theta$ and $\sin^2 2\theta$ only.

$\square$, $\boxed{\frac{1}{2}}$



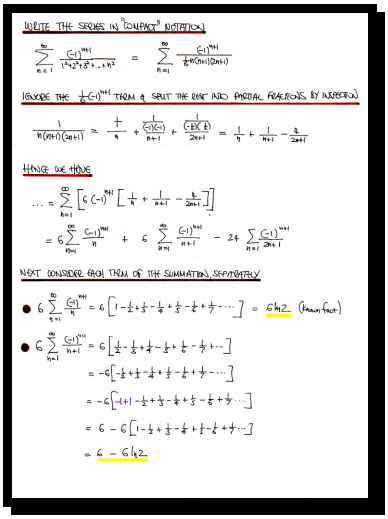
Question 107 (*****)

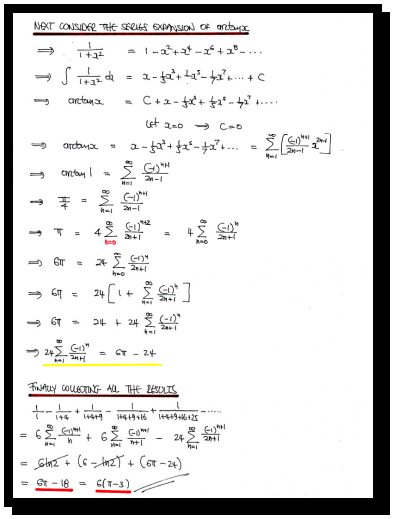
Find the sum to infinity of the following series.

$$\frac{1}{1}-\frac{1}{1+4}+\frac{1}{1+4+9}-\frac{1}{1+4+9+16}+\frac{1}{1+4+9+16+25}+...$$

You may find the series expansion of $\arctan x$ useful in this question.

$\square$, $\boxed{6(\pi-3)}$





Question 108 **(*****)**

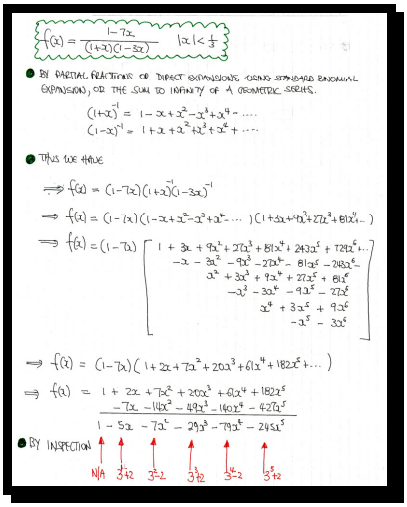
$$f(x)\equiv\frac{1-7x}{(1+x)(1-3x)},\ -\frac{1}{3}<x<\frac{1}{3}.$$

Show that $f(x)$ can be written in the form

$$f(x)=1-\sum_{r=1}^{\infty}\left[x^r\,g(r)\right],$$

where $g(r)$ is a simplified function to be found.

$\square$, $\boxed{g(r)=3^r+2\times(-1)^{r+1}}$



Question 109 **(*****)**

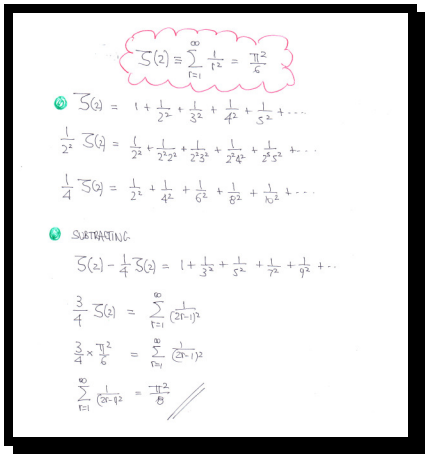
It is given that

$$\zeta(2)=\sum_{r=1}^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}.$$

$$\boxed{\frac{\pi^2}{8}}$$

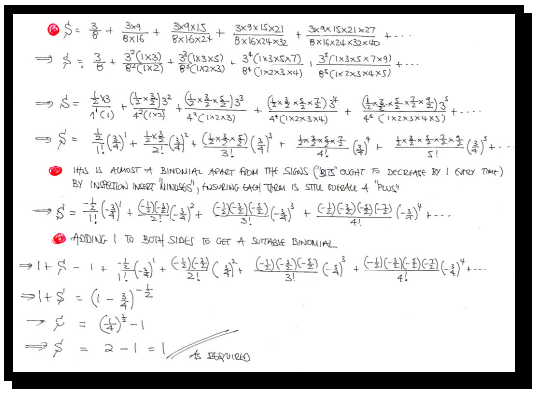


Question 110 **(*****)**

$$S=\frac{3}{8}+\frac{3\times 9}{8\times 16}+\frac{3\times 9\times 15}{8\times 16\times 24}+\frac{3\times 9\times 15\times 21}{8\times 16\times 24\times 32}+\frac{3\times 9\times 15\times 21\times 27}{8\times 16\times 24\times 32\times 40}\ldots$$

By considering a suitable binomial expansion, show that $S=1$.

$\boxed{}$, $\boxed{\text{proof}}$

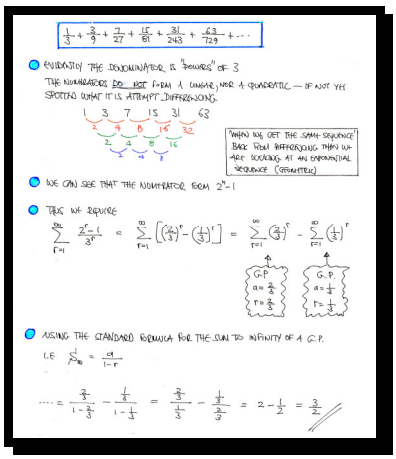


Question 111 **(*****)**

Sum the following series of infinite terms.

$$\frac{1}{3}+\frac{3}{9}+\frac{7}{27}+\frac{15}{81}+\frac{31}{243}+\frac{63}{729}+\ldots$$

$\boxed{}$, $\boxed{\frac{3}{2}}$

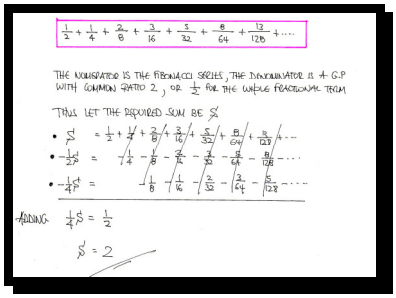


Question 112 (*****)

Sum the following series of infinite terms.

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+...$$

$\square$, $\boxed{2}$



Question 113 **(*****)**

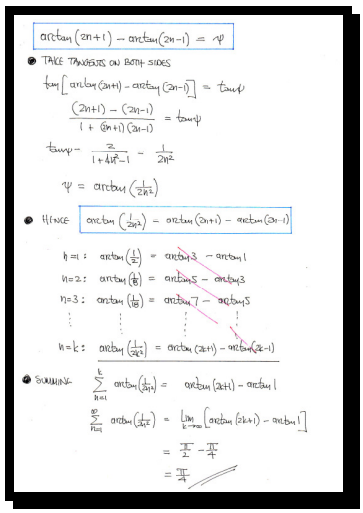
By considering the simplification of

$$\arctan\left(2n+1\right)-\arctan\left(2n-1\right),$$

determine the exact value of

$$\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{2n^2}\right)\right].$$

$\square$, $\boxed{\dfrac{\pi}{4}}$

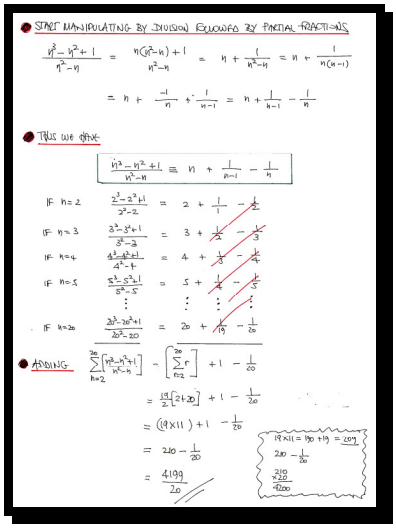


Question 114 (*****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20}\left[\frac{n^3-n^2+1}{n^2-n}\right].$$

$\square$, $\boxed{\frac{4199}{20}}$



Question 115 **(*****)**

$$\sum_{r=1}^{\infty}\left[\frac{1}{r^2}\right]=L\,.$$

It is given that the above infinite series converges to a limit L.

Find, in terms of L where appropriate, the limit of each of the following infinite series.

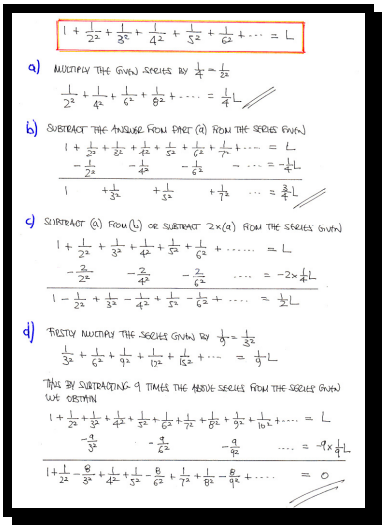
a) $\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}+\dfrac{1}{8^2}+\dfrac{1}{10^2}+\ldots$

b) $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\dfrac{1}{9^2}+\ldots$

c) $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dfrac{1}{5^2}-\dfrac{1}{6^2}+\ldots$

d) $\dfrac{1}{1^2}+\dfrac{1}{2^2}-\dfrac{8}{3^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}-\dfrac{8}{6^2}+\dfrac{1}{7^2}+\dfrac{1}{8^2}-\dfrac{8}{9^2}+\ldots$

$\boxed{}$, $\boxed{\frac{1}{4}L}$, $\boxed{\frac{3}{4}L}$, $\boxed{\frac{1}{2}L}$, $\boxed{0}$



Question 116 (*****)

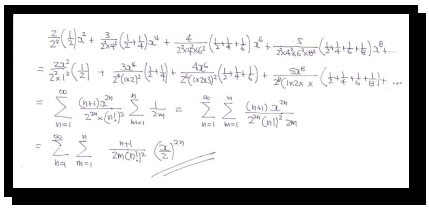
Consider the infinite series

$$\frac{2}{2^2}\left(\frac{1}{2}\right)x^2+\frac{3}{2^2\times4^2}\left(\frac{1}{2}+\frac{1}{4}\right)x^4+\frac{4}{2^2\times4^2\times6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)x^6+\frac{5}{2^2\times4^2\times6^2\times8^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)x^8+...$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=1}^{\infty}\left[\frac{n+1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\sum_{m=1}^{n}\frac{1}{2m}\right] \quad \text{or} \quad \sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{n+1}{2m(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]$$

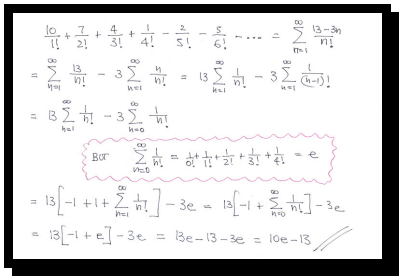


Question 117 (*****)

Determine the sum to infinity of the following series

$$\frac{10}{1!}+\frac{7}{2!}+\frac{4}{3!}+\frac{1}{4!}-\frac{2}{5!}-\frac{5}{6!}+...$$

$$10e-13$$



Question 118 (*****)

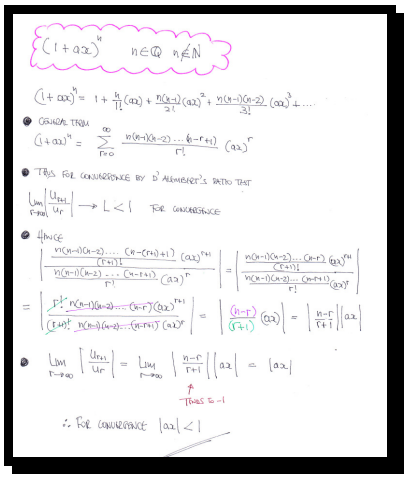
Consider the binomial infinite series expansion

$$(1+ax)^n,$$

where $a \in \mathbb{R}$, $n \in \mathbb{Q}$, $n \notin \mathbb{N}$.

Show that the series converges if $|ax| < 1$.

proof



Question 119 (*****)

The n^{th} term of a series is given recursively by

$$u_{n+1}=\frac{n}{2n+1}u_n,\ n\in\mathbb{N},\ \ u_1=2.$$

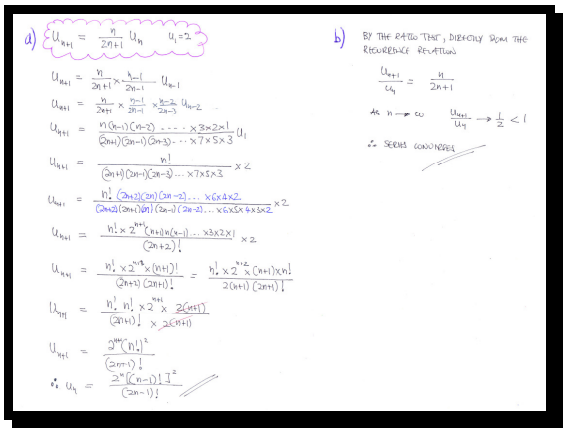
a) Show, by direct manipulation, that

$$u_n=\frac{2^n\times\left[(n-1)!\right]^2}{(2n-1)!}.$$

[*you may not use proof by induction in this part*]

b) Determine whether $\displaystyle\sum_{n}^{\infty}u_n$ converges or diverges.

converges

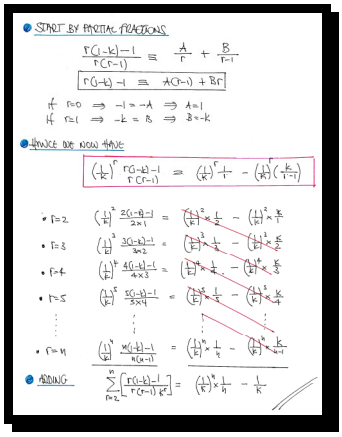


Question 120 (*****)

Determine, in terms of k and n, a simplified expression

$$\sum_{r=2}^{n}\left[\frac{r(1-k)-1}{r(r-1)k^r}\right].$$

$$\square, \quad \boxed{\frac{1}{n}\left(\frac{1}{k}\right)^n-\frac{1}{k}}$$



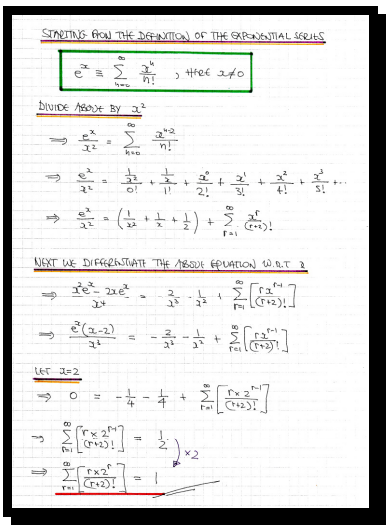
Question 121 (*****)

Use an appropriate method to sum the following series

$$\sum_{r=1}^{\infty}\frac{r\times 2^r}{(r+2)!}.$$

You may assume the series converges.

$\boxed{}$, $\boxed{1}$



Question 122 **(*****)**

The n^{th} term of a series is given recursively by

$$u_n = \frac{2n}{2n+1}u_{n-1}\,,\ n\in\mathbb{N}\,,\ \ u_0 = 1\,.$$

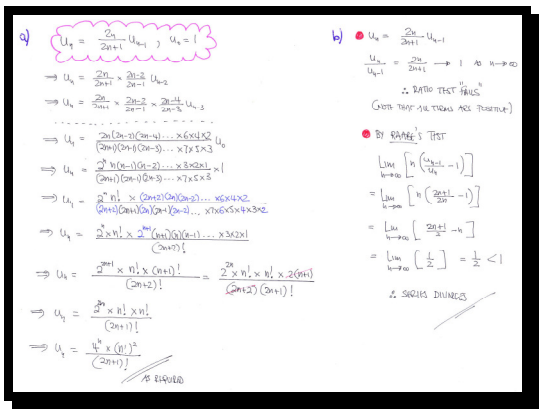
a) Show, by direct manipulation, that

$$u_n = \frac{4^n\times(n!)^2}{(2n+1)!}\,.$$

[*you may not use proof by induction in this part*]

b) Determine whether $\displaystyle\sum_n^{\infty} u_n$ converges or diverges.

diverges



Question 123 (*****)

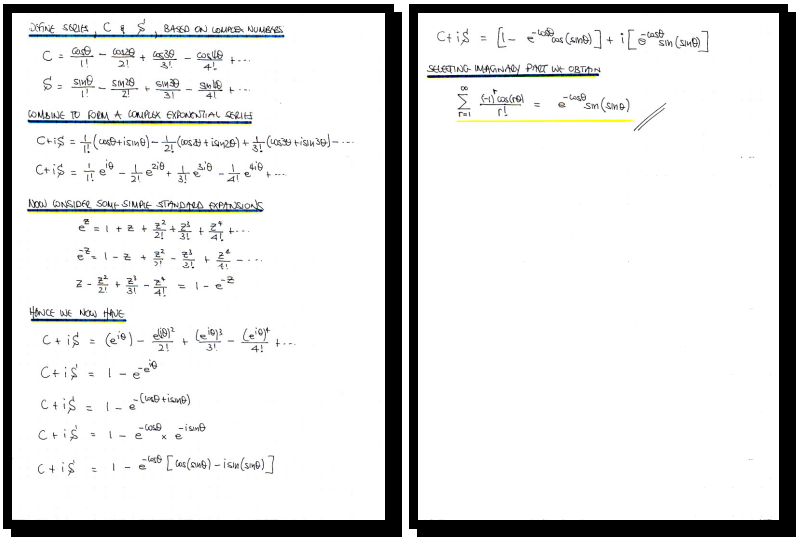
The following convergent series S is given below

$$S=\frac{\sin\theta}{1!}-\frac{\sin 2\theta}{2!}+\frac{\sin 3\theta}{3!}-\frac{\sin 4\theta}{4!}+\ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S=\mathrm{e}^{-\cos\theta}\sin\left(\sin\theta\right).$$

[ ], [proof]

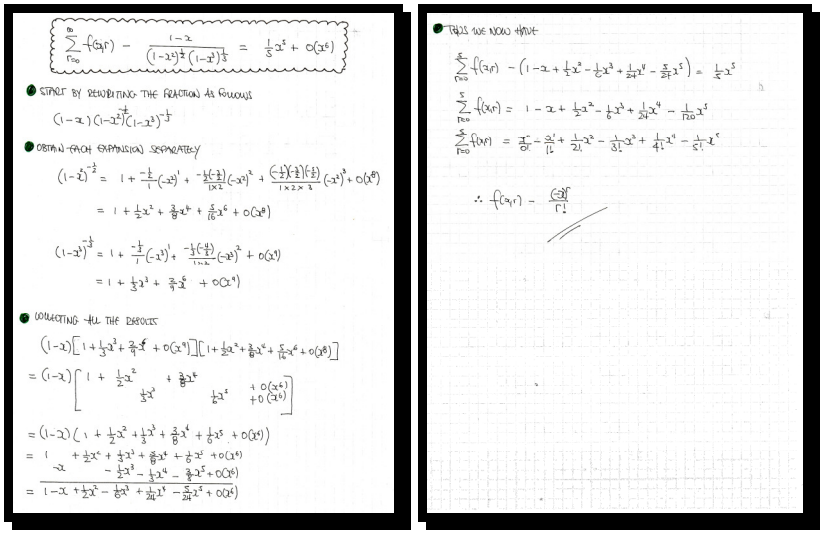


Question 124 (*****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x,r) \; - \; \frac{1-x}{\sqrt{1-x^2}\sqrt[3]{1-x^3}}, \; -1<x<1.$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x,r)$.

$\square$, $\boxed{f(x,r)=\dfrac{(-x)^r}{r!}}$

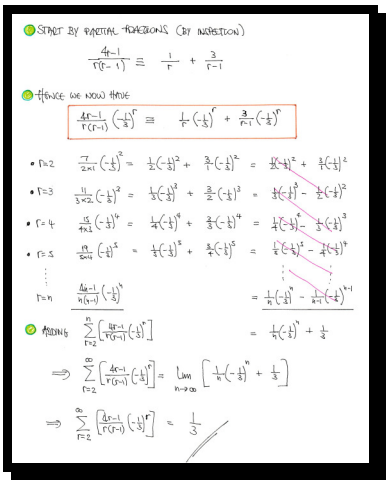


Question 125 (*****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^r\right].$$

$\square$, $\boxed{\frac{1}{3}}$

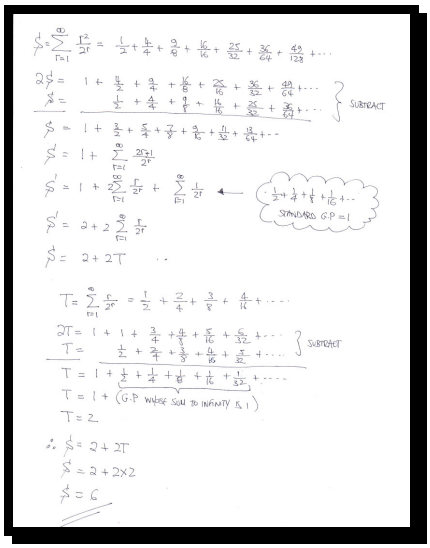


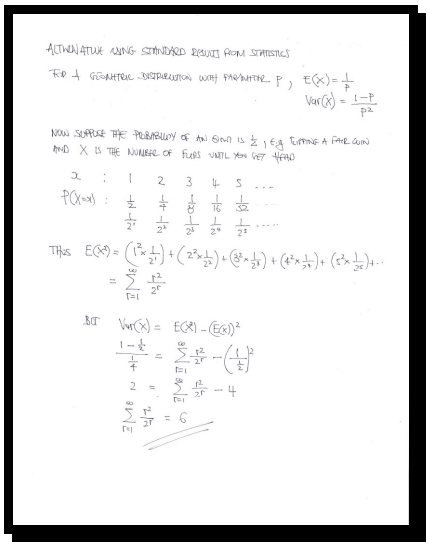
Question 126 (*****)

Show clearly that

$$\sum_{r=1}^{\infty}\frac{r^2}{2^r}=6\ .$$

proof



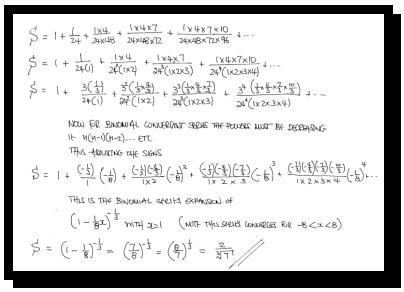


Question 127 (*****)

Show clearly that

$$1+\frac{1}{24}+\frac{1\cdot 4}{24\cdot 48}+\frac{1\cdot 4\cdot 7}{24\cdot 48\cdot 72}+\frac{1\cdot 4\cdot 7\cdot 10}{24\cdot 48\cdot 72\cdot 96}-\ldots=\frac{2}{\sqrt[3]{7}}\ .$$

, proof



Question 128 (*****)

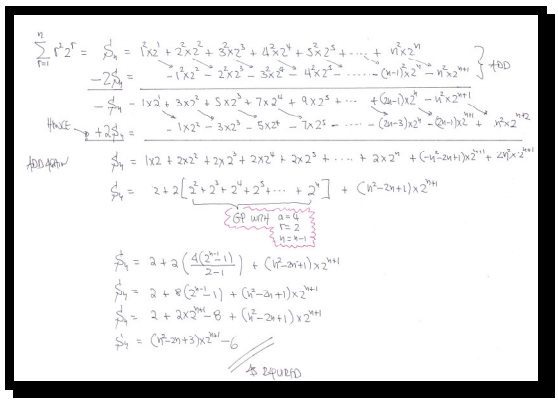
$$S_n = \sum_{r=1}^{n} \left(r^2 \times 2^r \right)$$

Use the standard techniques for the summation of a geometric series, to show that

$$S_n = \left(n^2 - 2n + 3 \right) \times 2^{n+1} - 6 \, .$$

[*You may not use proof by induction in this question.*]

proof



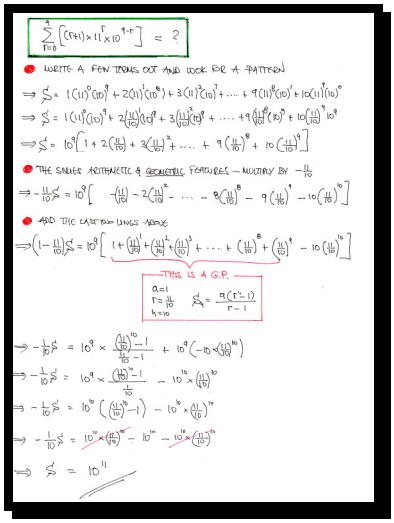
Question 129 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=0}^{9}\left[(r+1)\times 11^r\times 10^{9-r}\right].$$

You may leave the answer in index form.

$$\square\,,\qquad \sum_{r=0}^{9}\left[(r+1)\times 11^r\times 10^{9-r}\right]=10^{11}$$



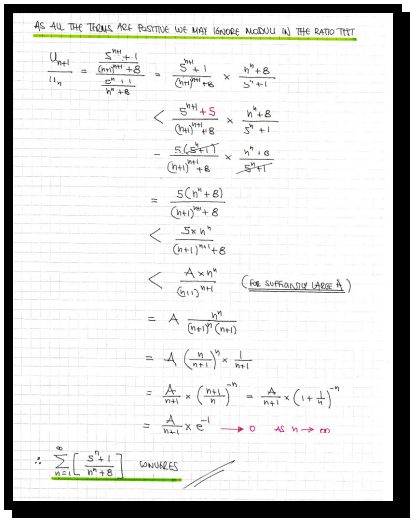
Question 130 (*****)

Use the ratio test to show that the following series converges

$$\sum_{n=1}^{\infty}\left[\frac{5^n+1}{n^n+8}\right].$$

You may assume without proof that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=\frac{1}{\mathrm{e}}$.

[ ], [proof]



Question 131 **(*****)**

$$f(x)=\frac{1}{\sqrt{1-x}},\ -1<x<1.$$

a) By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x)=\sum_{r=0}^{\infty}\binom{2r}{r}\left(\frac{1}{4}x\right)^r.$$

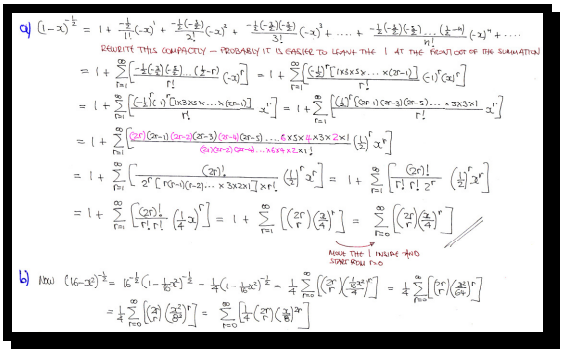
b) Find a similar expression for $\dfrac{1}{\sqrt{16-x^2}}$ and show further that

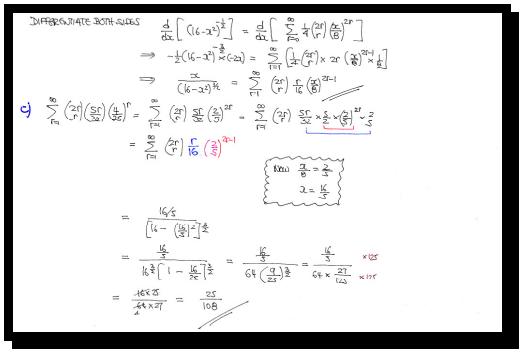
$$\frac{x}{\left(16-x^2\right)^{\frac{3}{2}}}=\sum_{r=1}^{\infty}\binom{2r}{r}\left(\frac{1}{16}r\right)\left(\frac{1}{8}x\right)^{2r-1}.$$

c) Determine the exact value of

$$\sum_{r=1}^{\infty}\binom{2r}{r}\left(\frac{5}{32}r\right)\left(\frac{4}{25}\right)^r.$$

$\square$, $\boxed{\dfrac{25}{108}}$





Question 132 (*****)

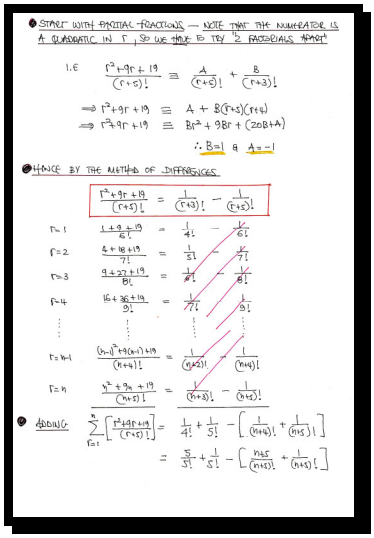
Determine, in terms of n, a simplified expression

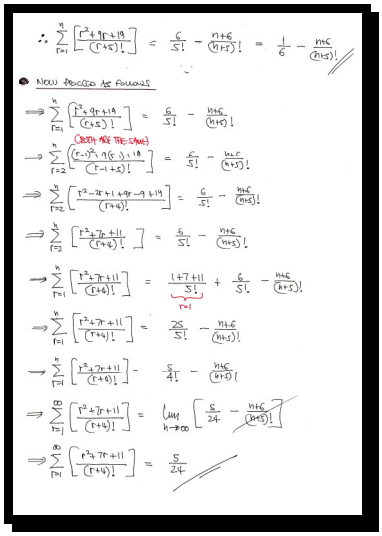
$$\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right].$$

$\square$, $\boxed{\displaystyle\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{1}{6}-\frac{n+5}{(n+5)!}}$, $\boxed{\displaystyle\sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{24}}$





Question 133 (*****)

The n^{th} term of a series is given recursively by

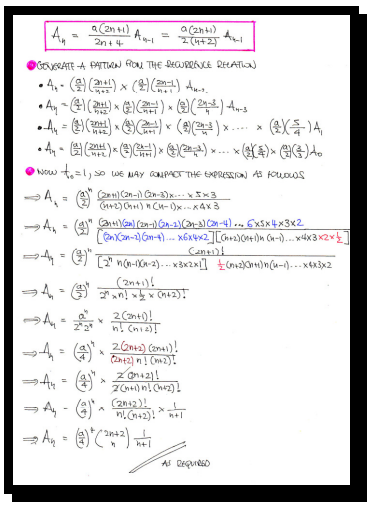
$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \ n \in \mathbb{N}, \ n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

, proof

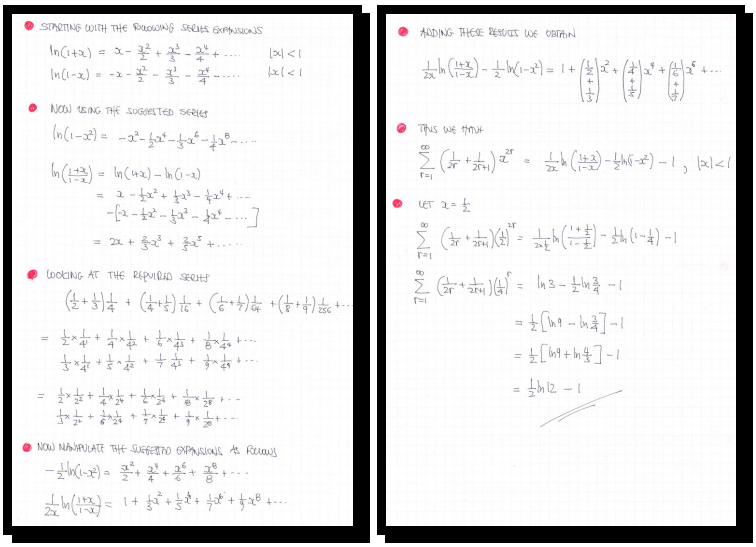


Question 134 (*****)

By considering the series expansions of $\ln\left(1-x^2\right)$ and $\ln\left(\dfrac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty}\left[\left(\frac{1}{2r}+\frac{1}{2r+1}\right)\left(\frac{1}{4}\right)^r\right].$$

$\boxed{\phantom{-1+\frac{1}{2}\ln 12}}$, $\boxed{-1+\frac{1}{2}\ln 12}$

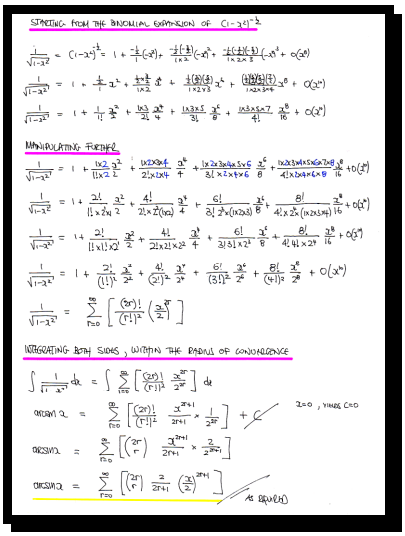


Question 135 (*****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right].$$

[ ], [proof]



Question 136 (*****)

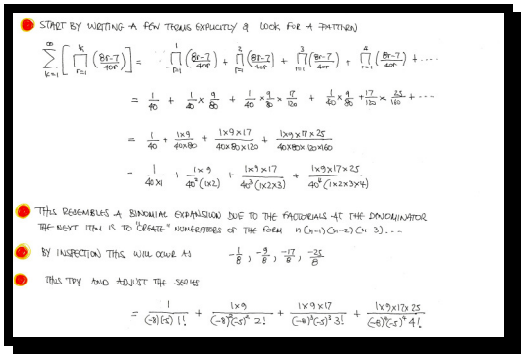
The product operator $\prod$, is defined as

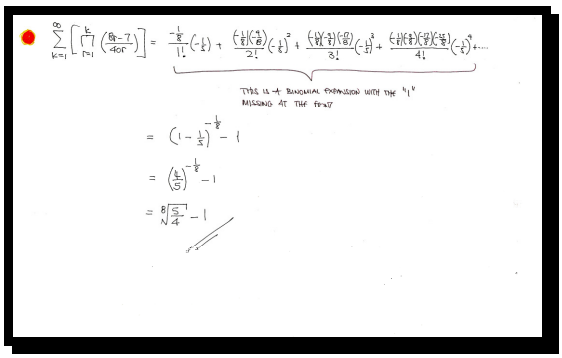
$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k \, .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

$\square$, $\sqrt[8]{\frac{5}{4}}-1$





Question 137 (*****)

A sequence $u_1,\ u_2,\ u_3,\ u_5,\ u_6,\ \ldots$ is generated by the recurrence relation

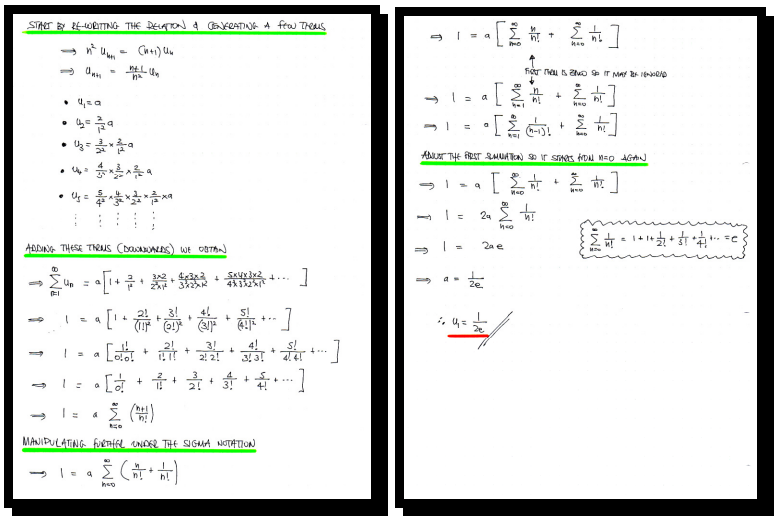
$$n^2 u_{n+1} = (n+1)u_n, \quad n = 1,\ 2,\ 3,\ 4,\ ...$$

It is further given that

$$\sum_{n=1}^{\infty} u_n = 1.$$

Find in exact form the value of u_1.

$\square$, $\boxed{u_1 = \dfrac{1}{2\mathrm{e}}}$

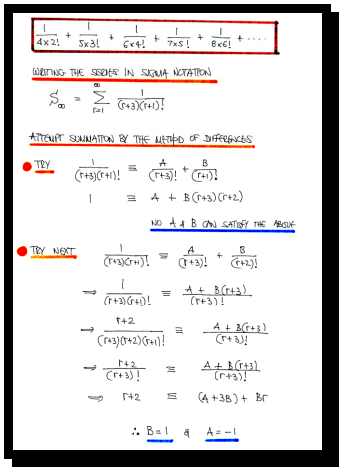


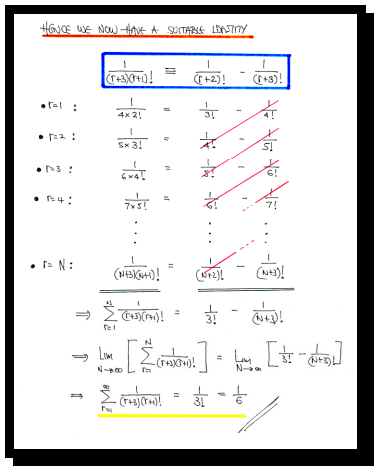
Question 138 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4\times 2!}+\frac{1}{5\times 3!}+\frac{1}{6\times 4!}+\frac{1}{7\times 5!}+\frac{1}{8\times 6!}+...$$

$\square$, $\boxed{\frac{1}{6}}$





Question 139 **(*****)**

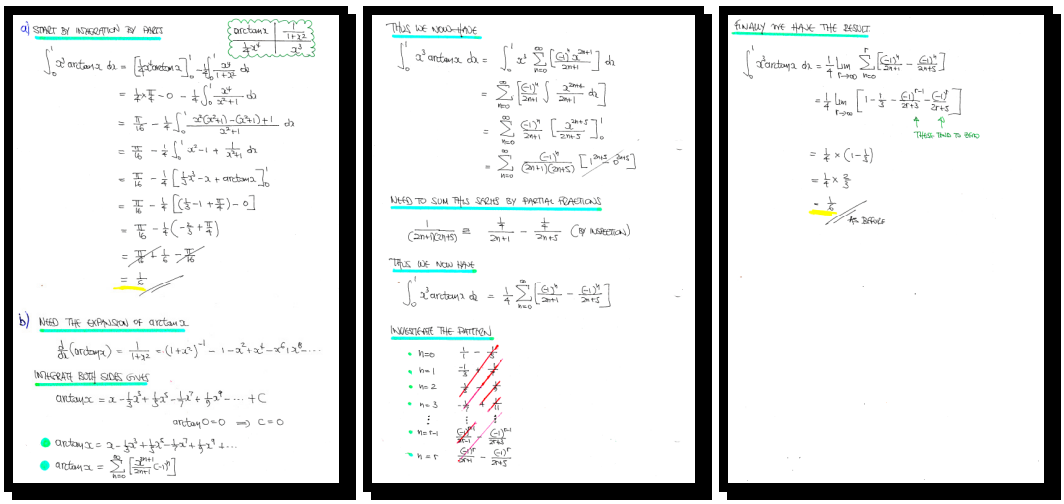
a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \ \ dx \, .$$

b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{6}}$

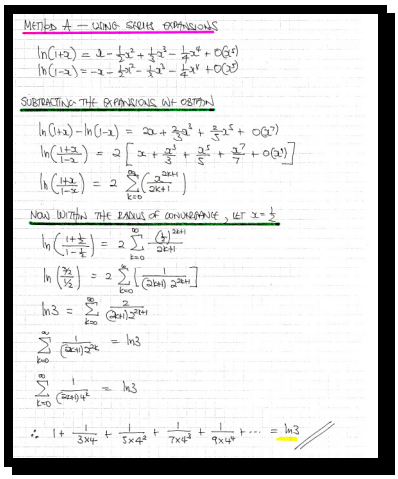


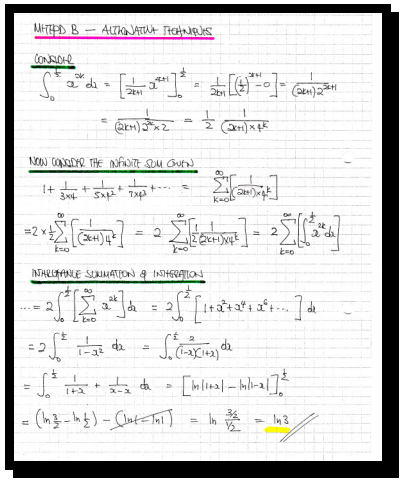
Question 140 (*****)

Find the sum to infinity of the following series.

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+...$$

$\boxed{}$, $\boxed{\ln 3}$



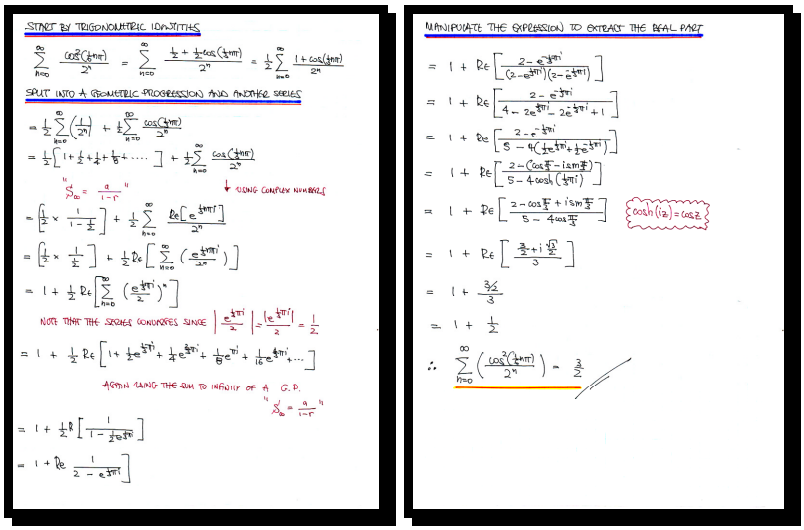


Question 141 (*****)

By showing a detailed method, sum the following series.

$$\sum_{n=0}^{\infty}\left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right].$$

$$\boxed{}, \quad \boxed{\frac{3}{2}}$$

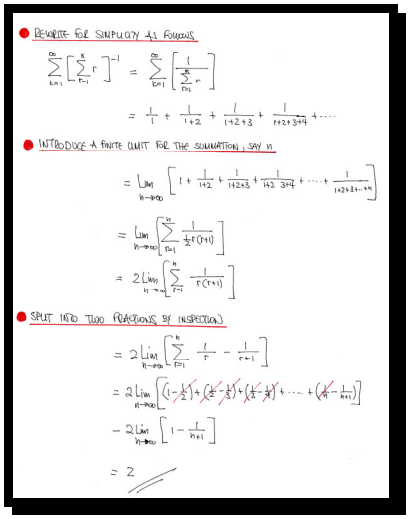


Question 142 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}.$$

$\square$, $\boxed{2}$



Question 143 (*****)

The first three terms of a series S are

$$S=7+9x+8x^2+...$$

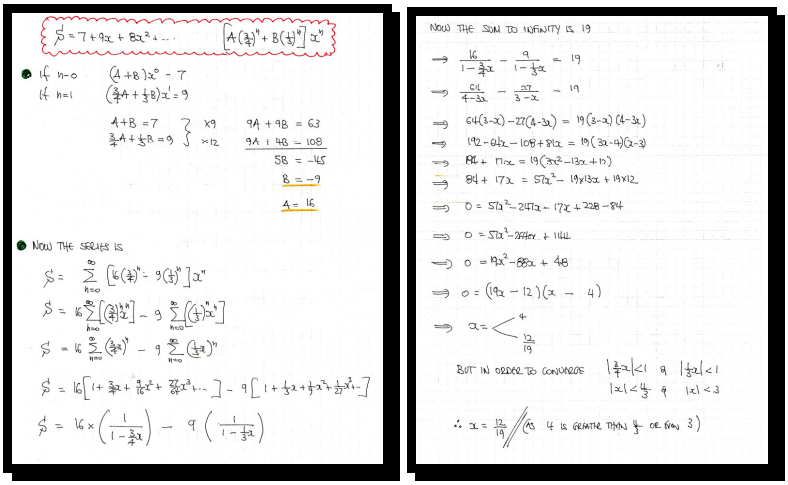
The n^{th} term of S is given by

$$A\left(\frac{3}{4}x\right)^n+B\left(\frac{1}{3}x\right)^n,$$

where A and B are non zero constants.

Given that the sum to infinity of S is 19, determine the value of x.

$\square$, $\boxed{x=\frac{12}{19}}$

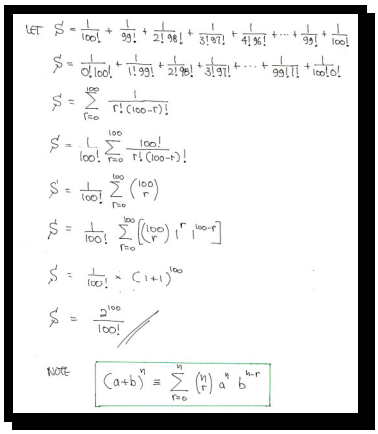


Question 144 (*****)

Find a simplified expression for the following sum

$$\frac{1}{100!}+\frac{1}{99!}+\frac{1}{2!98!}+\frac{1}{3!97!}+\frac{1}{4!96!}+\ldots+\frac{1}{2!98!}+\frac{1}{99!}+\frac{1}{100!}$$

$\square$, $\boxed{\dfrac{2^{100}}{100!}}$

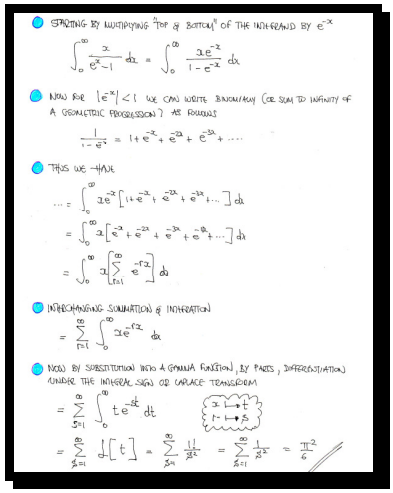


Question 145 (*****)

Show by detailed workings that

$$\int_0^{\infty} \frac{x}{\mathrm{e}^x-1}\,dx = \frac{\pi^2}{6}$$

proof



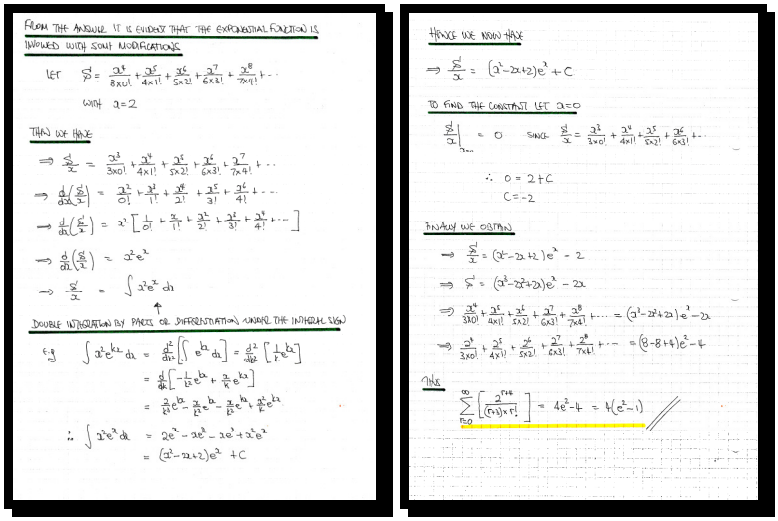
Question 146 (*****)

Consider the following convergent infinite series.

$$\sum_{r=0}^{\infty}\frac{2^{r+4}}{(r+3)r!}$$

Use appropriate techniques to show that the sum to infinity of the above series is $4\left(e^2-1\right)$

[ ], [proof]



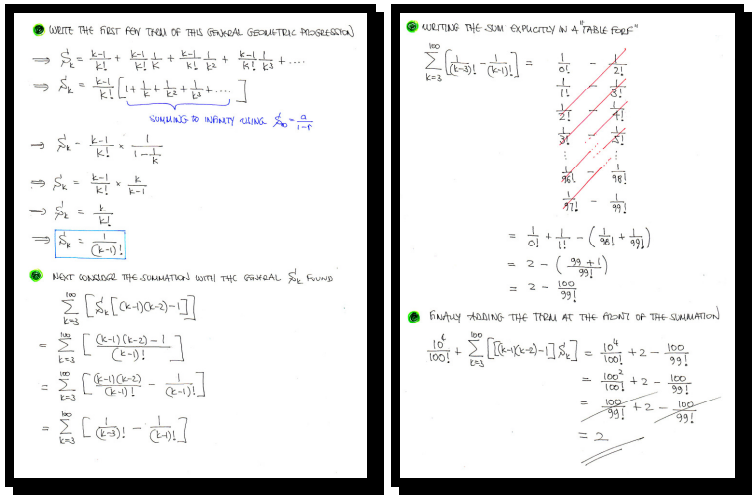
Question 147 (*****)

A family of infinite geometric series S_k, has first term $\dfrac{k-1}{k!}$ and common ratio $\dfrac{1}{k}$, where $k=3,4,5,6,...,99,100$.

Find the value of

$$\frac{10^4}{100!}+\sum_{k=3}^{100}\left[\left[(k-1)(k-2)-1\right]S_k\right].$$

$\square$, $\boxed{2}$



Question 148 (*****)

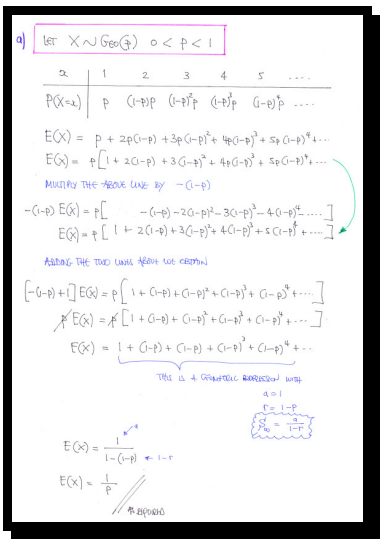
A discrete random variable X is geometrically distributed with parameter p.

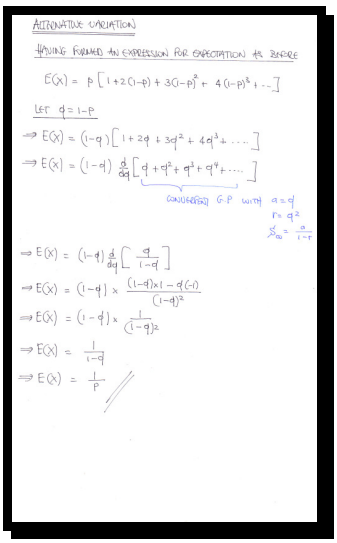
Show that …

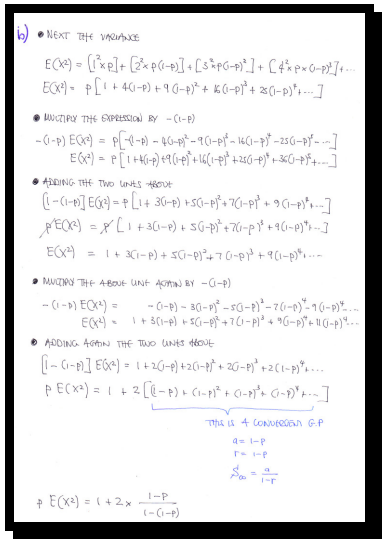
a) $\ldots \mathrm{E}(X)=\dfrac{1}{p}$.

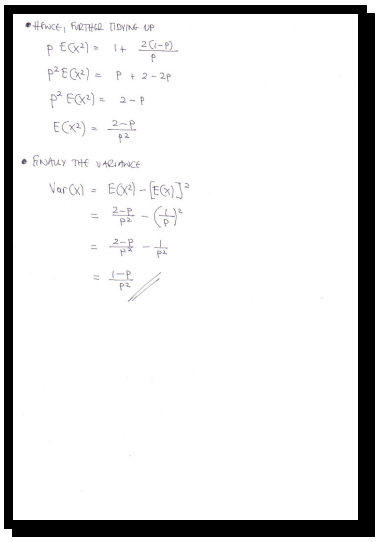
b) $\ldots \mathrm{E}(X)=\dfrac{1-p}{p^2}$.

[ ], [proof]







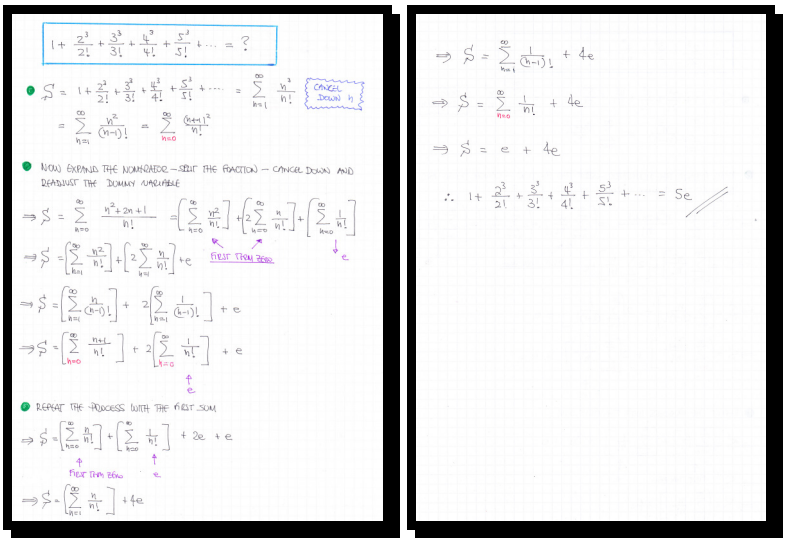


Question 149 (*****)

Find the sum to infinity of the following convergent series

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\frac{6^3}{6!}+\ldots.$$

5e



Question 150 (*****)

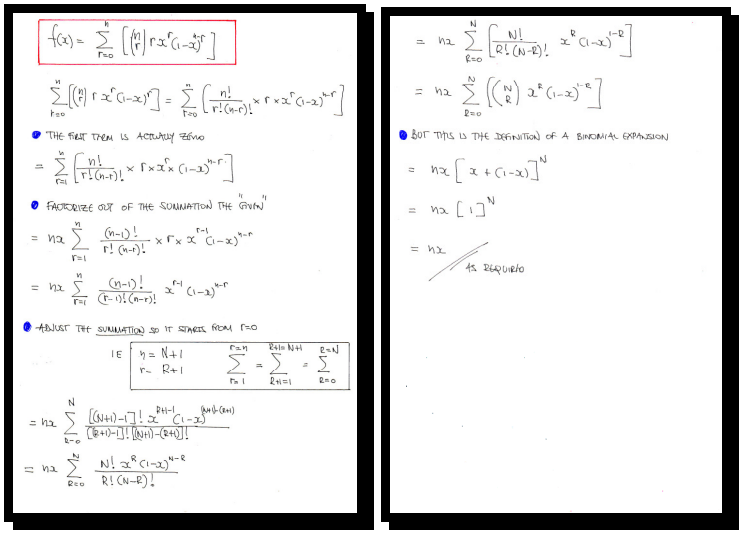
The function f is defined as

$$f(n,x) \equiv \sum_{r=0}^{n} \binom{n}{r} r\, x^r (1-x)^{n-r},$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$.

Show that $f(n,x) \equiv nx$.

, proof



Question 151 (*****)

The binomial probability distribution $X \sim \mathrm{B}(n, p)$ satisfies

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

where $r = 0,1,2,3,\ldots,n$ and $0 < p < 1$.

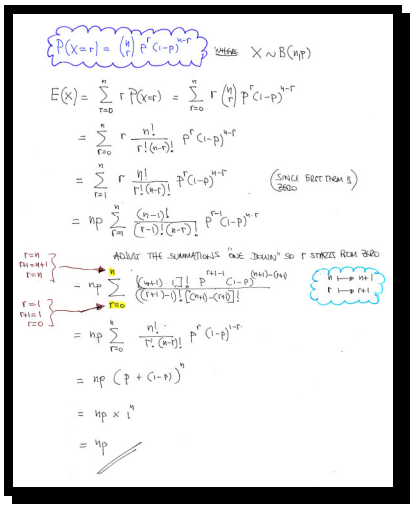
The expectation of X is defined as

$$\mathrm{E}(X) \equiv \sum_{r=0}^{n} \left[r\,\mathrm{P}(X = r) \right]$$

Show that

$$\mathrm{E}(X) = np .$$

, proof



Question 152 (*****)

The function f is defined in terms of the real constants, a, b and c, by

$$f(x)=\left(a+bx+cx^2\right)(1-x)^{-3}, \quad x\in\mathbb{R}, \quad |x|<1.$$

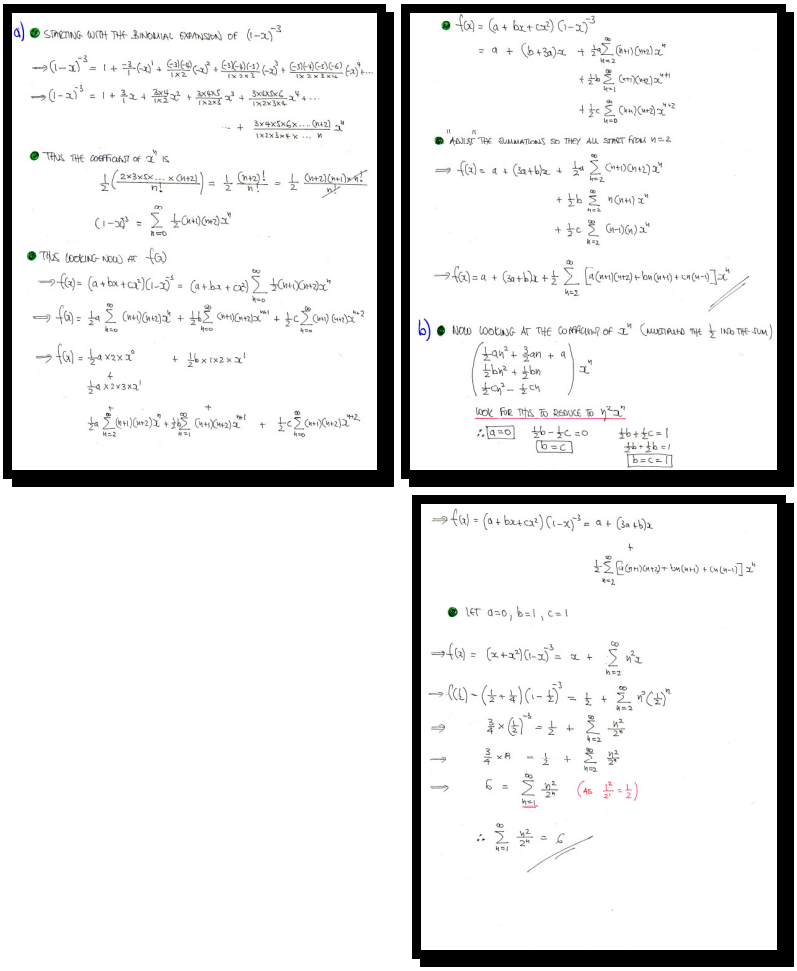
a) Show that

$$f(x)=a+(3a+b)x+\frac{1}{2}\sum_{n=2}^{\infty}\left[\left[a(n+1)(n+2)+bn(n+1)+cn(n-1)\right]x^n\right].$$

b) Use the expression of part **(a)** to deduce the value of

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}.$$

$\boxed{}$, 6



Question 153 (*****)

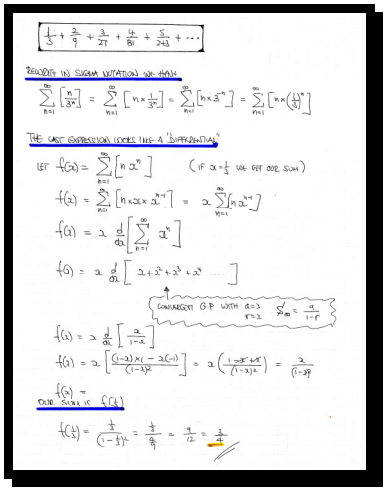
The function f is defined by

$$f(x) \equiv \sum_{n=1}^{\infty}\left[nx^n\right], \quad x\in\mathbb{R}, \quad |x|<1.$$

Use the above function to find the sum to infinity of the following series.

$$\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}+\ldots.$$

$\square$, $\boxed{\frac{3}{4}}$

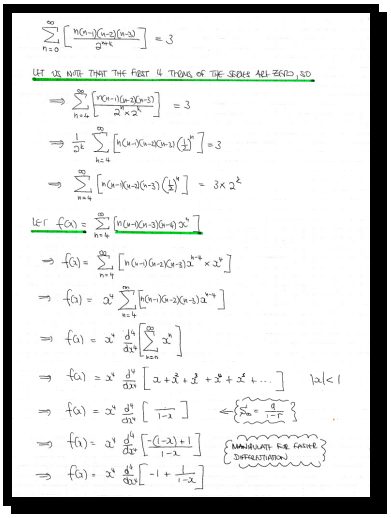


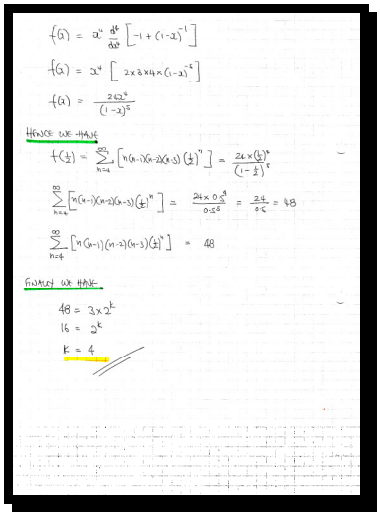
Question 154 (*****)

Find the value of $x \in \mathbb{R}$ in the following equation

$$\sum_{n=0}^{\infty}\left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}}\right] = 3 .$$

$\square$, $\boxed{k=4}$





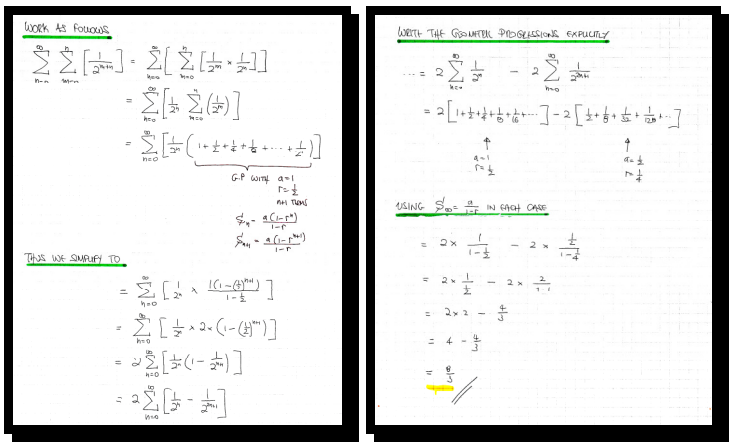
Question 155 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right].$$

Detailed workings must be shown.

$\square$, $\boxed{\frac{8}{3}}$



Question 156 (*****)

It is given that for $x \in \mathbb{R}$, $-\dfrac{1}{k} < x < \dfrac{1}{k}$, $k > 0$,

$$f(x,k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

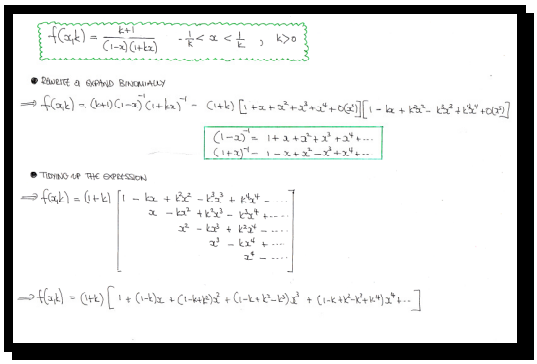
$$f(x,k) \equiv \sum_{r=0}^{\infty}\left[a_r x^r\right],$$

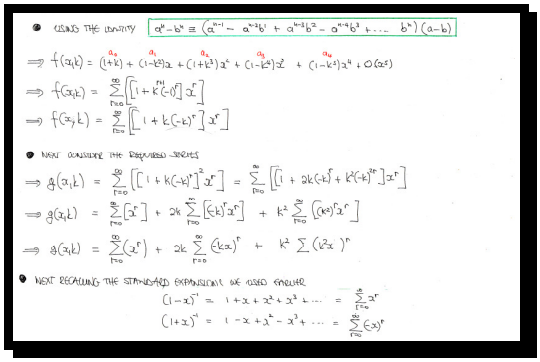
where a_r are functions of k, show that

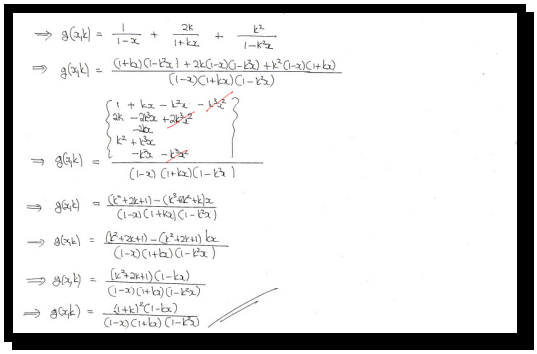
$$\sum_{r=0}^{\infty}\left[a_r{}^2 x^r\right] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)\left(1-k^2x\right)}.$$

You may assume that $\displaystyle\sum_{r=0}^{\infty}\left[a_r{}^2 x^r\right]$ converges.

, proof







Question 157 **(*****)**

It is given that

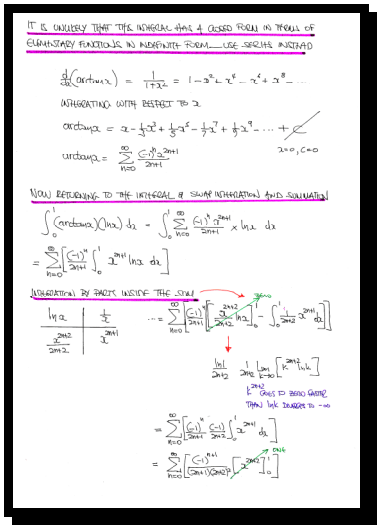
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots=\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2$

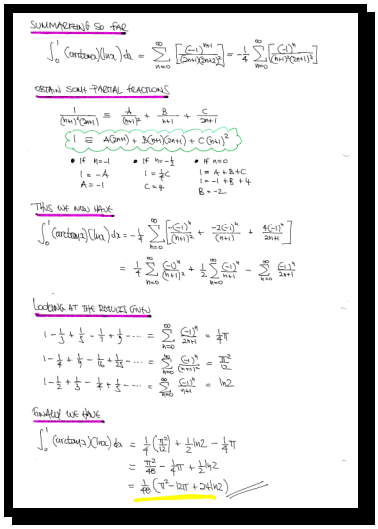
Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx\,.$$

[*you may assume that integration and summation commute*]

$\square$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$





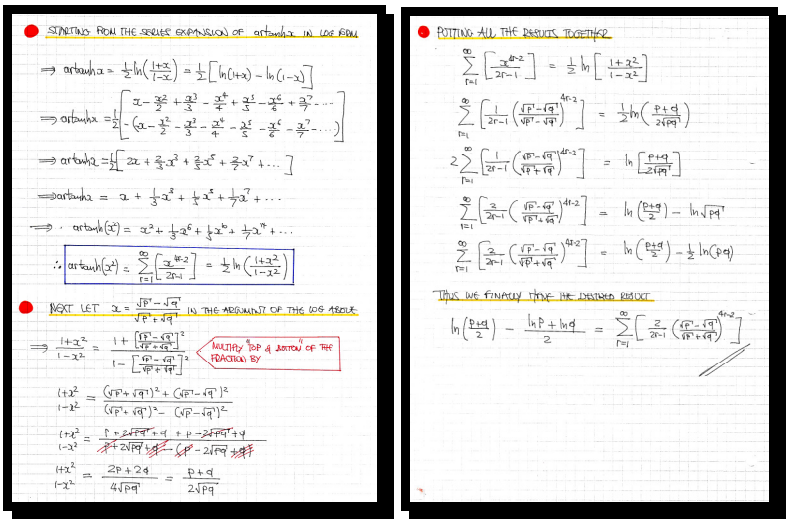
Question 158 (*****)

Given that p and q are positive, shown that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\operatorname{artanh}\left(x^2\right)$ useful in this question.

$\boxed{}$, proof



Question 159 **(*****)**

Show that

$$1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\frac{x^{12}}{12!}+\frac{x^{15}}{15!}+\ldots=\frac{1}{3}\left[\mathrm{e}^x+2\mathrm{e}^{\frac{1}{2}x}\cos\left(\frac{1}{2}\sqrt{3}x\right)\right].$$

You may find useful in this question the fact that if $z=\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}$ then $1+z+z^2=0$.

$\boxed{}$, proof

