RESIDUES and APPLICATIONS in SERIES SUMMATION

The Residue Theorem can often be used to sum various types of series.

The following results are valid under some restrictions on $f(z)$, which more often than not are satisfied when the series converges.

$$\sum_{r=-\infty}^{\infty} f(r)$$

use $\oint_{\Gamma_n} f(z)\pi\cot\pi z\, dz$, where Γ_n is the square with vertices at $\left(n+\frac{1}{2}\right)(\pm 1\pm \mathrm{i})$

$$\sum_{r=-\infty}^{\infty} (-1)^r f(r)$$

use $\oint_{\Gamma_n} f(z)\pi\operatorname{cosec}\pi z\, dz$, where Γ_n is the square with vertices at $\left(n+\frac{1}{2}\right)(\pm 1\pm \mathrm{i})$

$$\sum_{r=-\infty}^{\infty} f\left(\frac{2r+1}{2}\right)$$

use $\oint_{\Gamma_n} f(z)\pi\tan\pi z\, dz$, where Γ_n is the square with vertices at $n(\pm 1\pm \mathrm{i})$

$$\sum_{r=-\infty}^{\infty} (-1)^r f\left(\frac{2r+1}{2}\right)$$

use $\oint_{\Gamma_n} f(z)\pi\sec\pi z\, dz$, where Γ_n is the square with vertices at $n(\pm 1\pm \mathrm{i})$

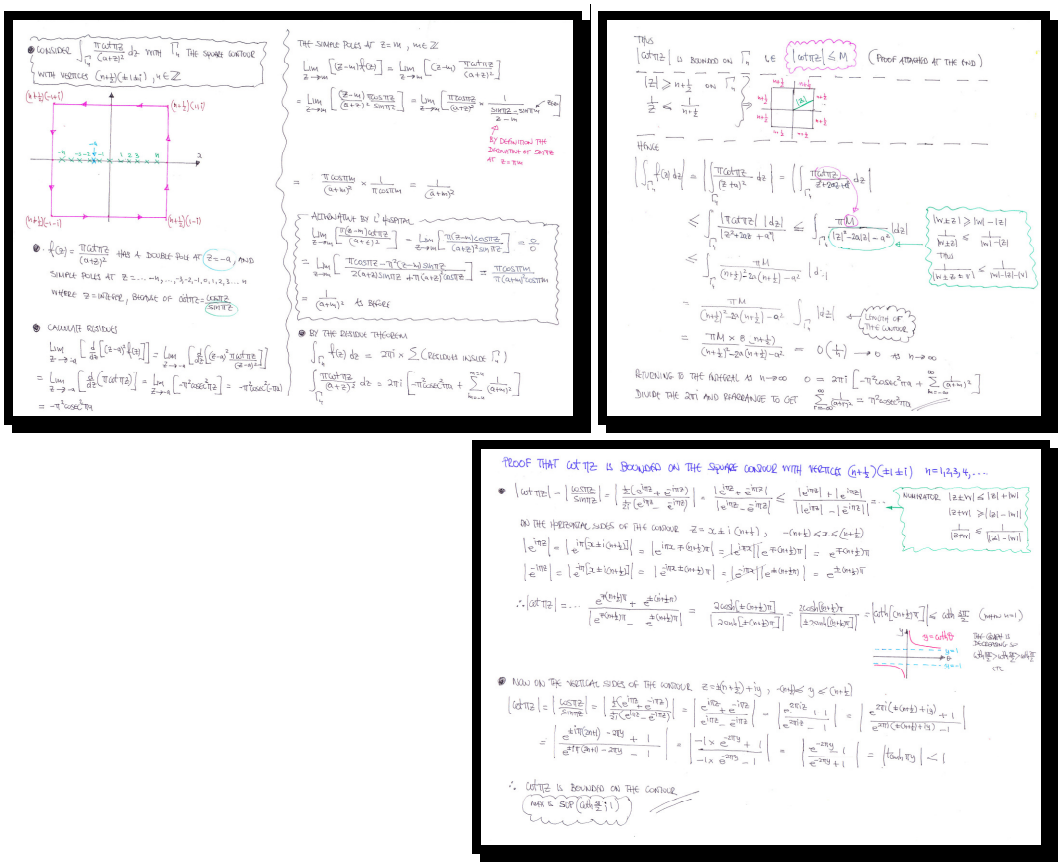
Question 1

$$f(z)=\frac{\pi\cot\pi z}{(a+z)^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty}\frac{1}{(a+r)^2}=\pi^2\operatorname{cosec}^2(\pi a),\ a\notin\mathbb{Z}.$$

proof



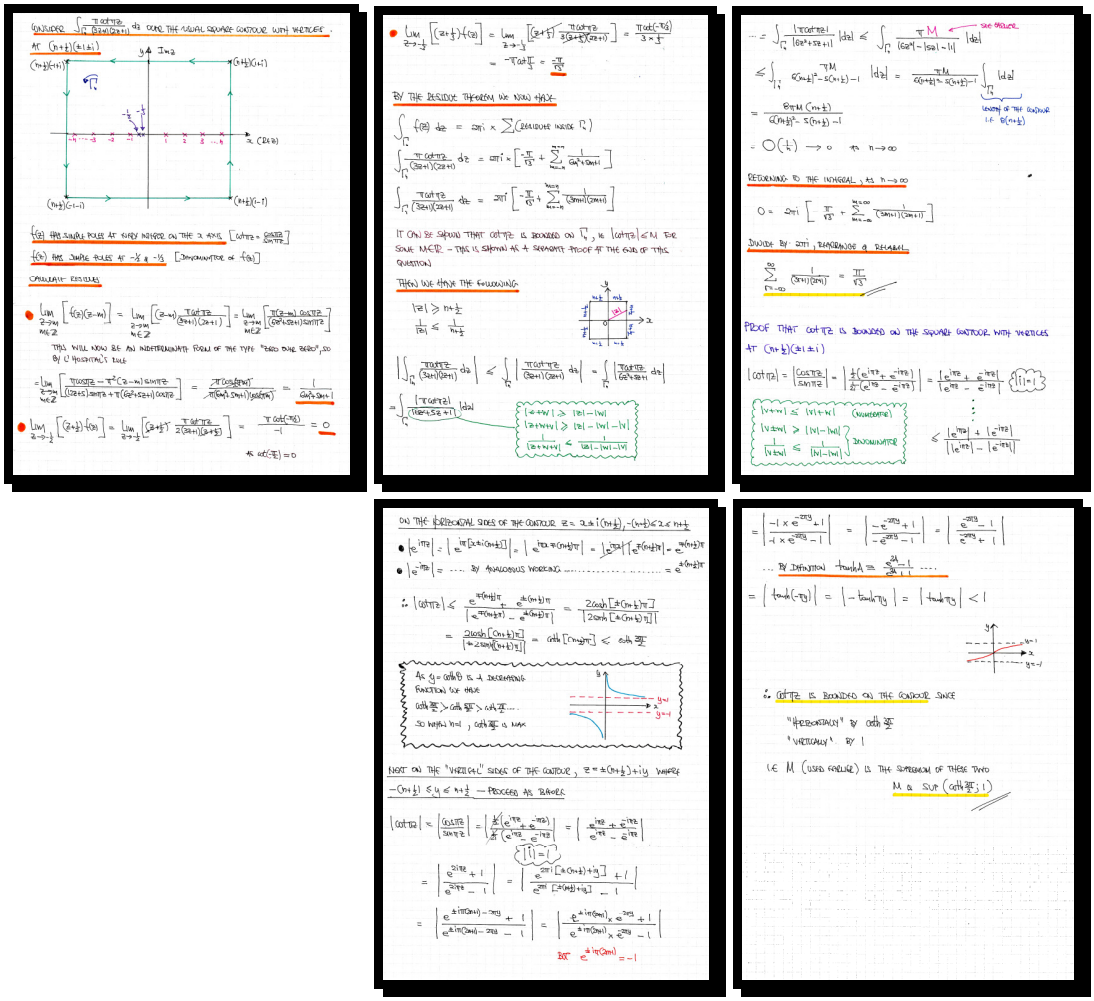
Question 2

$$f(z)=\frac{\pi\cot\pi z}{(3z+1)(2z+1)},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty}\frac{1}{(3r+1)(2r+1)}=\frac{\pi}{\sqrt{3}}.$$

[ ], [proof]



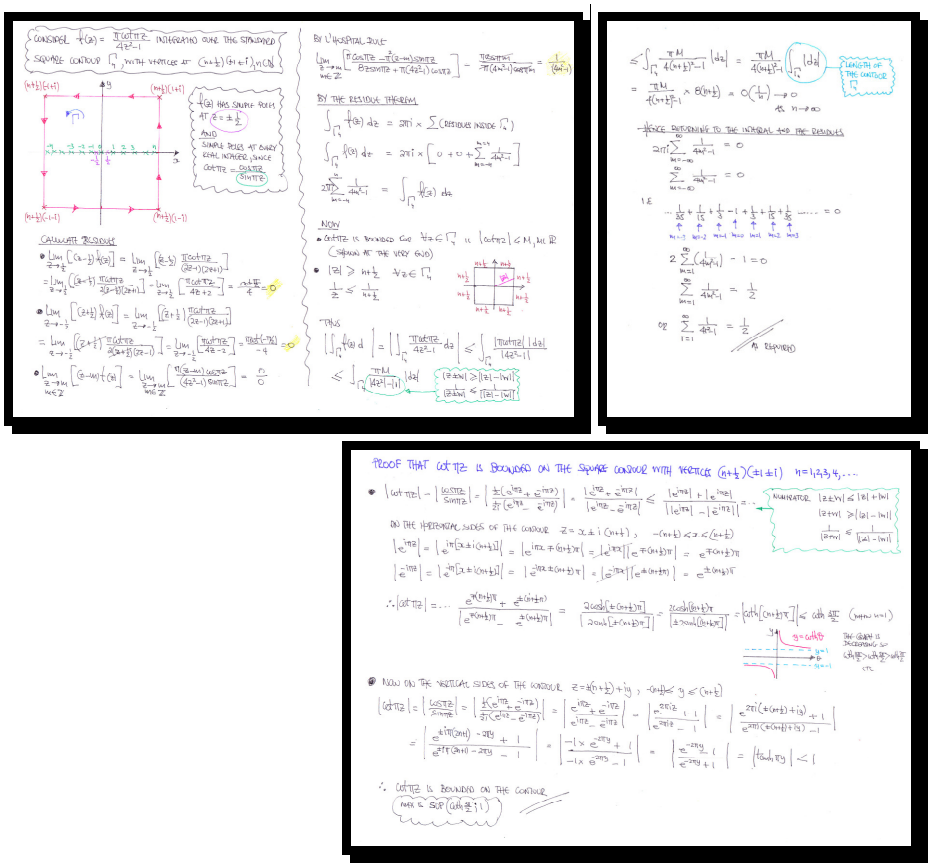
Question 3

$$f(z)=\frac{\pi\cot\pi z}{4z^2-1},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{1}{4r^2-1}=\frac{1}{2}.$$

proof



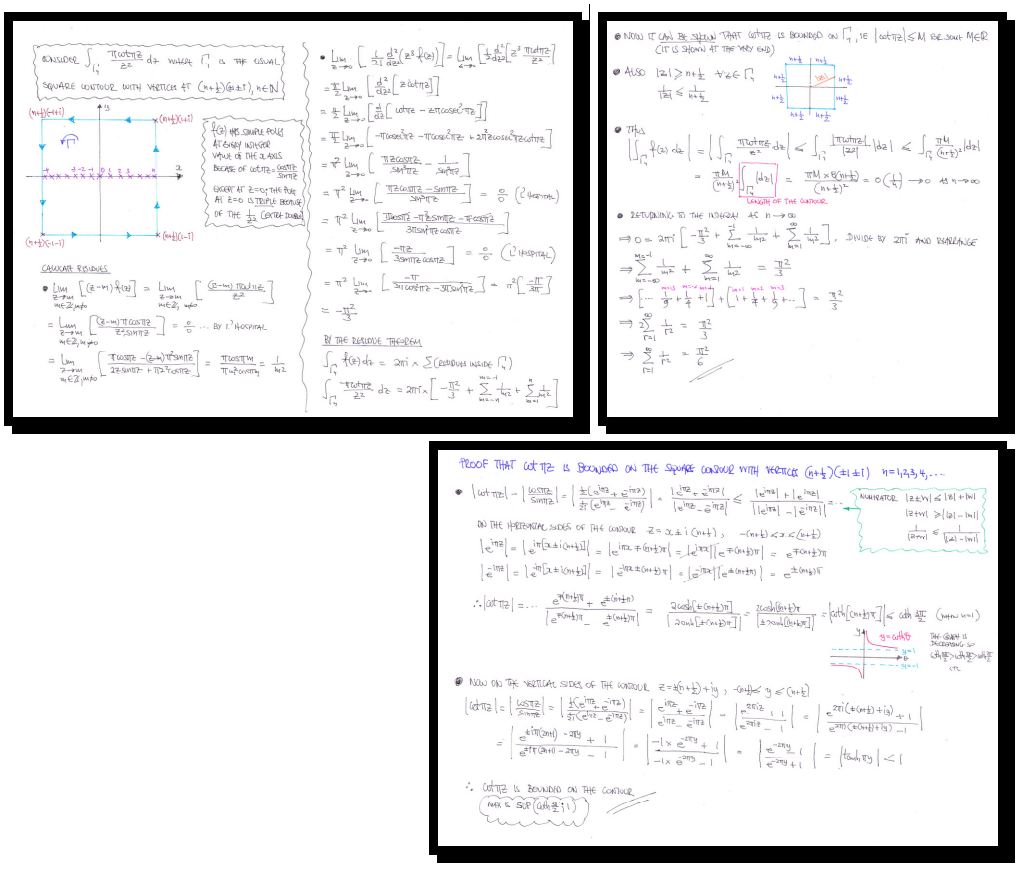
Question 4

$$f(z)=\frac{\pi\cot\pi z}{z^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}.$$

proof



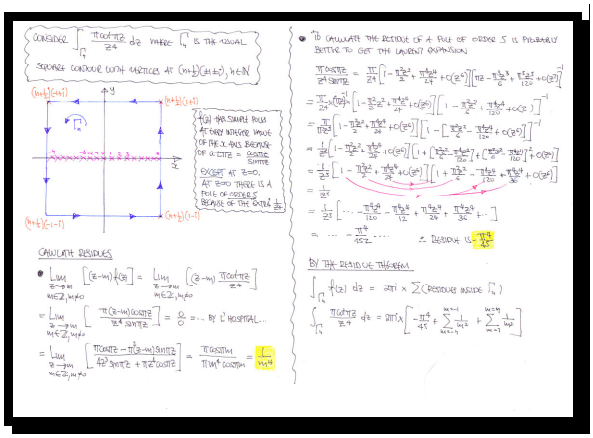
Question 5

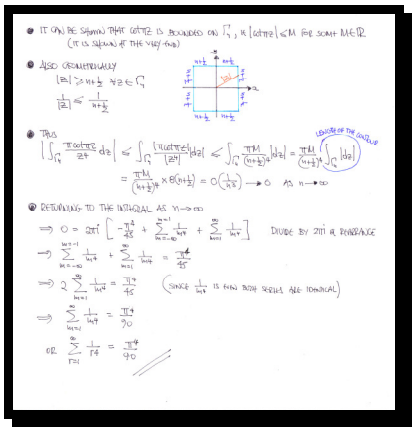
$$f(z)=\frac{\pi\cot\pi z}{z^4},\ z\in\mathbb{C}.$$

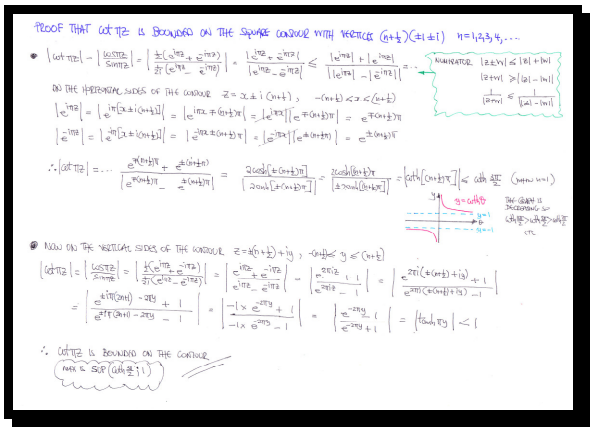
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{1}{r^4}=\frac{\pi^4}{90}.$$

proof







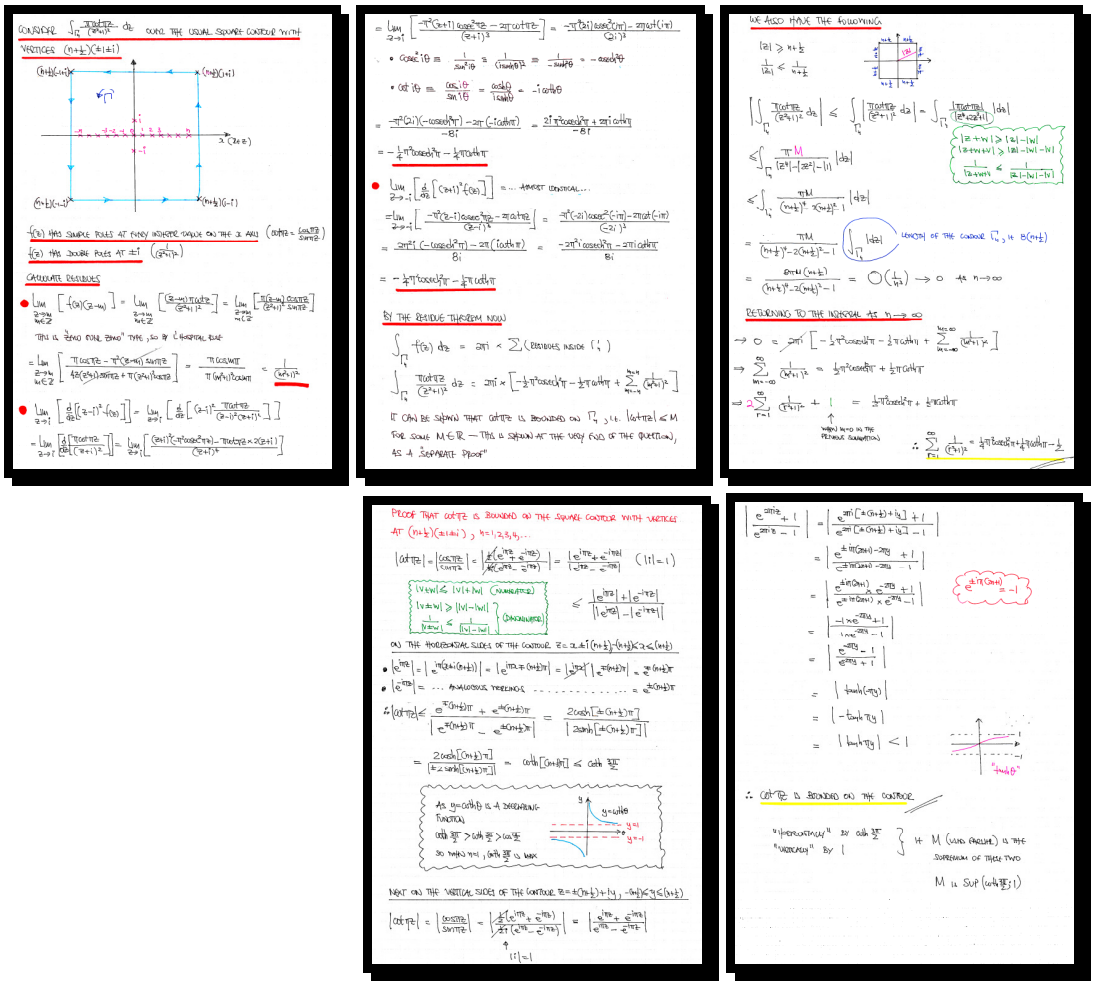
Question 6

$$f(z)=\frac{\pi\cot\pi z}{\left(z^2+1\right)^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{1}{\left(r^2+1\right)^2}=\frac{1}{4}\pi^2\operatorname{cosech}^2\pi+\frac{1}{4}\pi\coth\pi-\frac{1}{2}.$$

[ ], proof



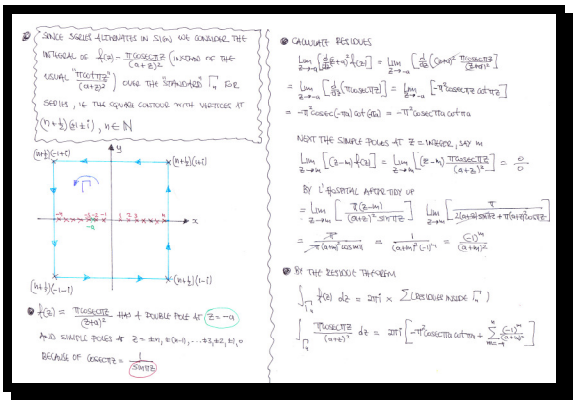
Question 7

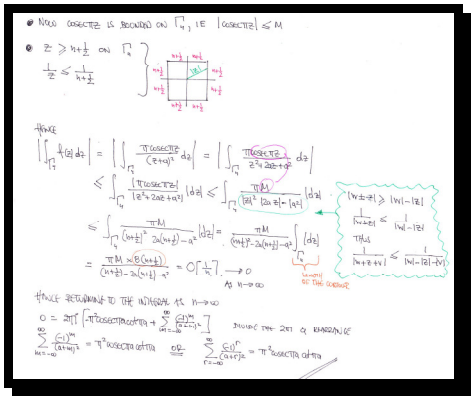
$$f(z)=\frac{\pi\operatorname{cosec}\pi z}{(a+z)^2},\ z\in\mathbb{C}.$$

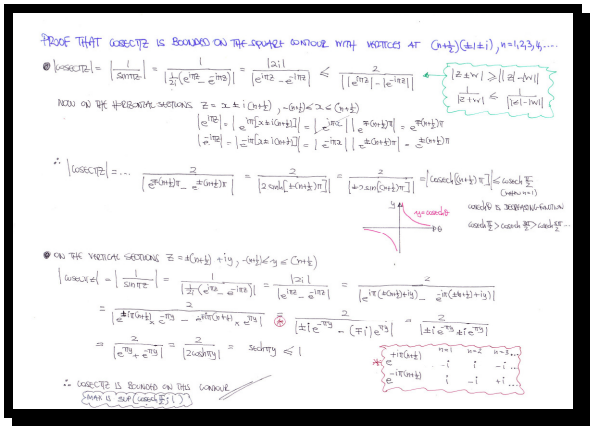
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(a+r)^2}=\pi^2\operatorname{cosec}(\pi a)\cot(\pi a),\ a\notin\mathbb{Z}.$$

proof







Question 8

$$f(z)=\frac{\pi\operatorname{cosec}\pi z}{(2z+1)(3z+1)},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{(2r+1)(3r+1)}=\frac{\pi}{3}\left(2\sqrt{3}-3\right).$$

[ ], [proof]



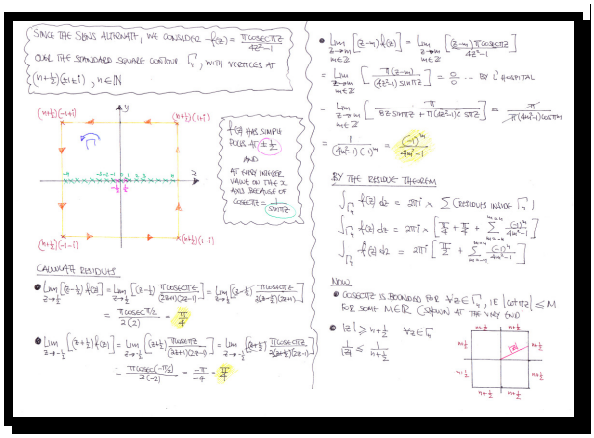
Question 9

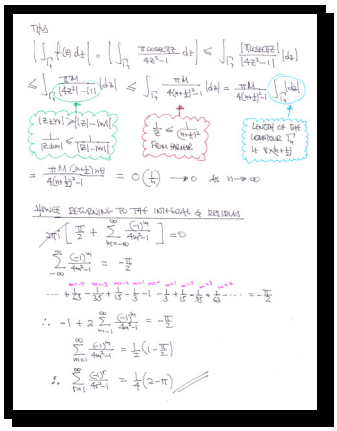
$$f(z)=\frac{\pi \operatorname{cosec} \pi z}{4z^2-1},\ z\in\mathbb{C}.$$

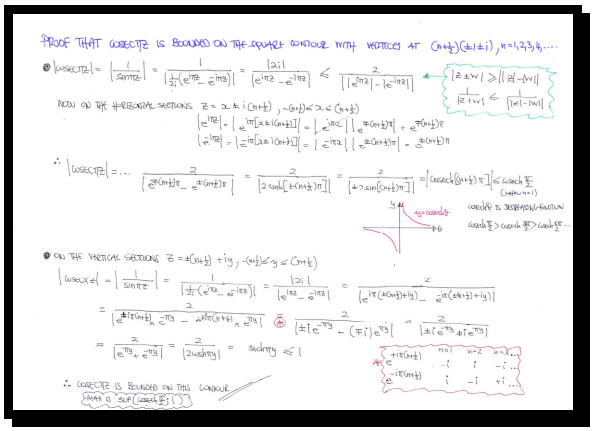
By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{(-1)^r}{4r^2-1}=\frac{1}{4}(2-\pi).$$

proof







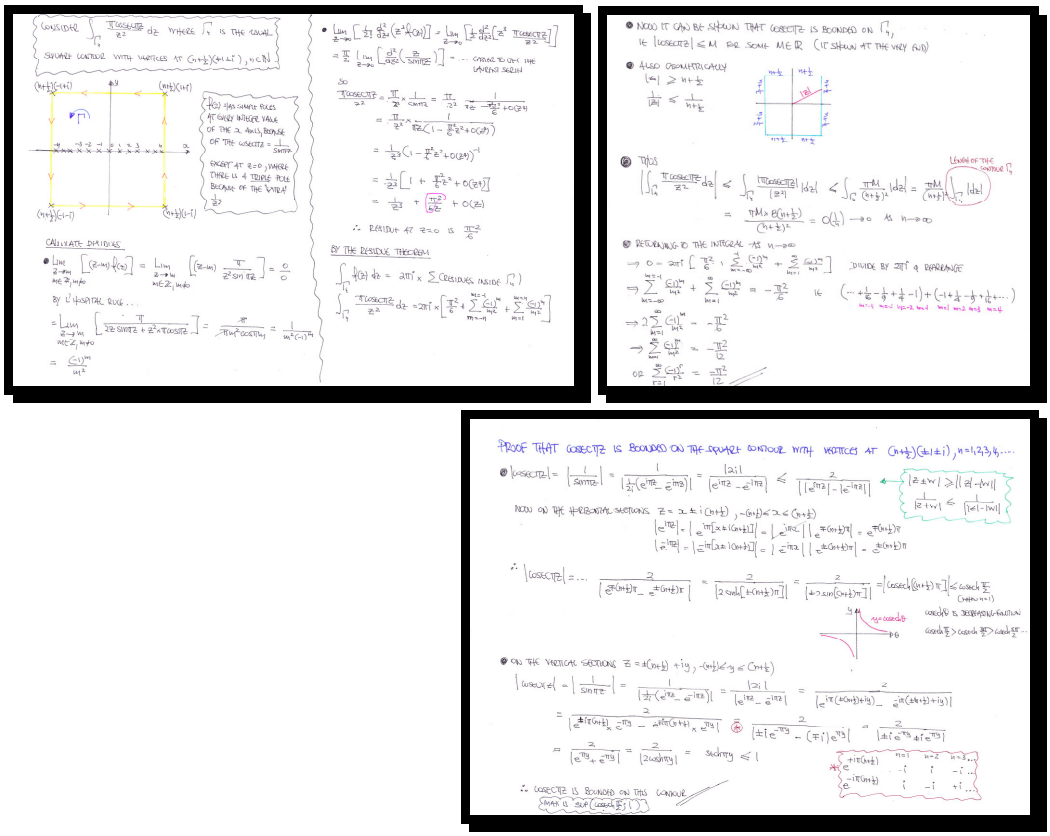
Question 10

$$f(z)=\frac{\pi \operatorname{cosec} \pi z}{z^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{(-1)^r}{r^2}=-\frac{1}{12}\pi^2.$$

proof



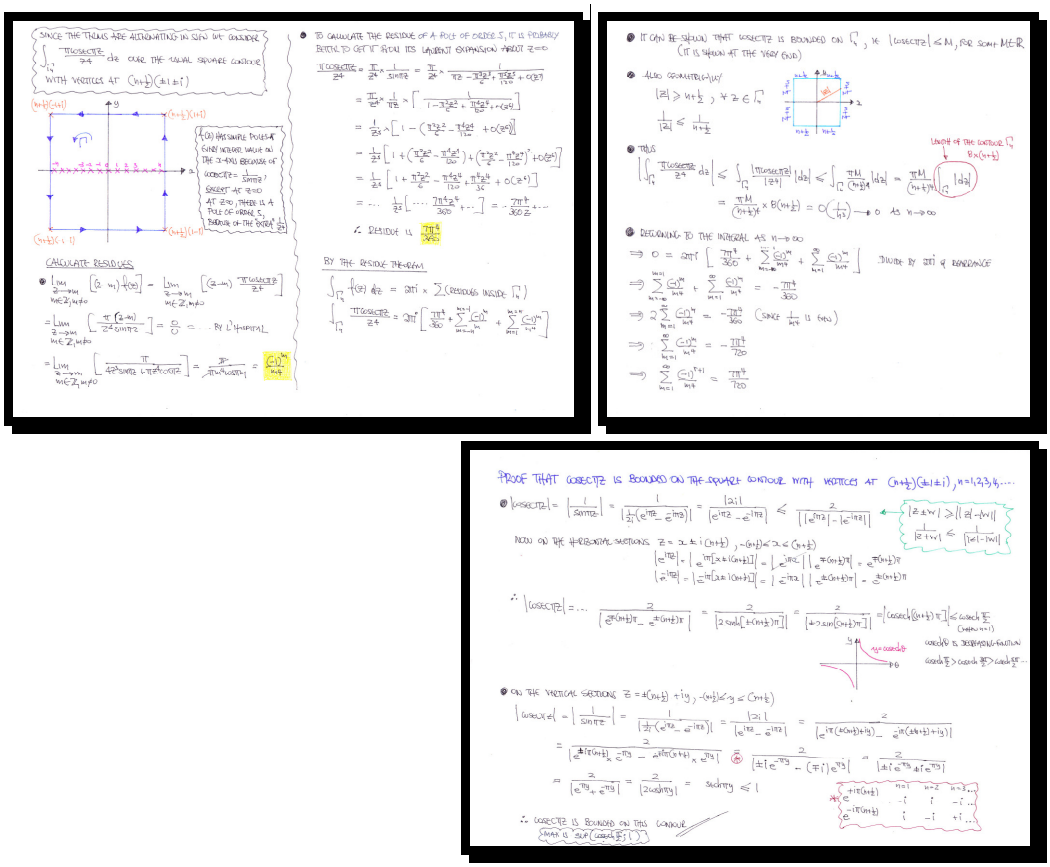
Question 11

$$f(z)=\frac{\pi \operatorname{cosec}\pi z}{z^4},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^4}=\frac{7\pi^4}{720}.$$

proof



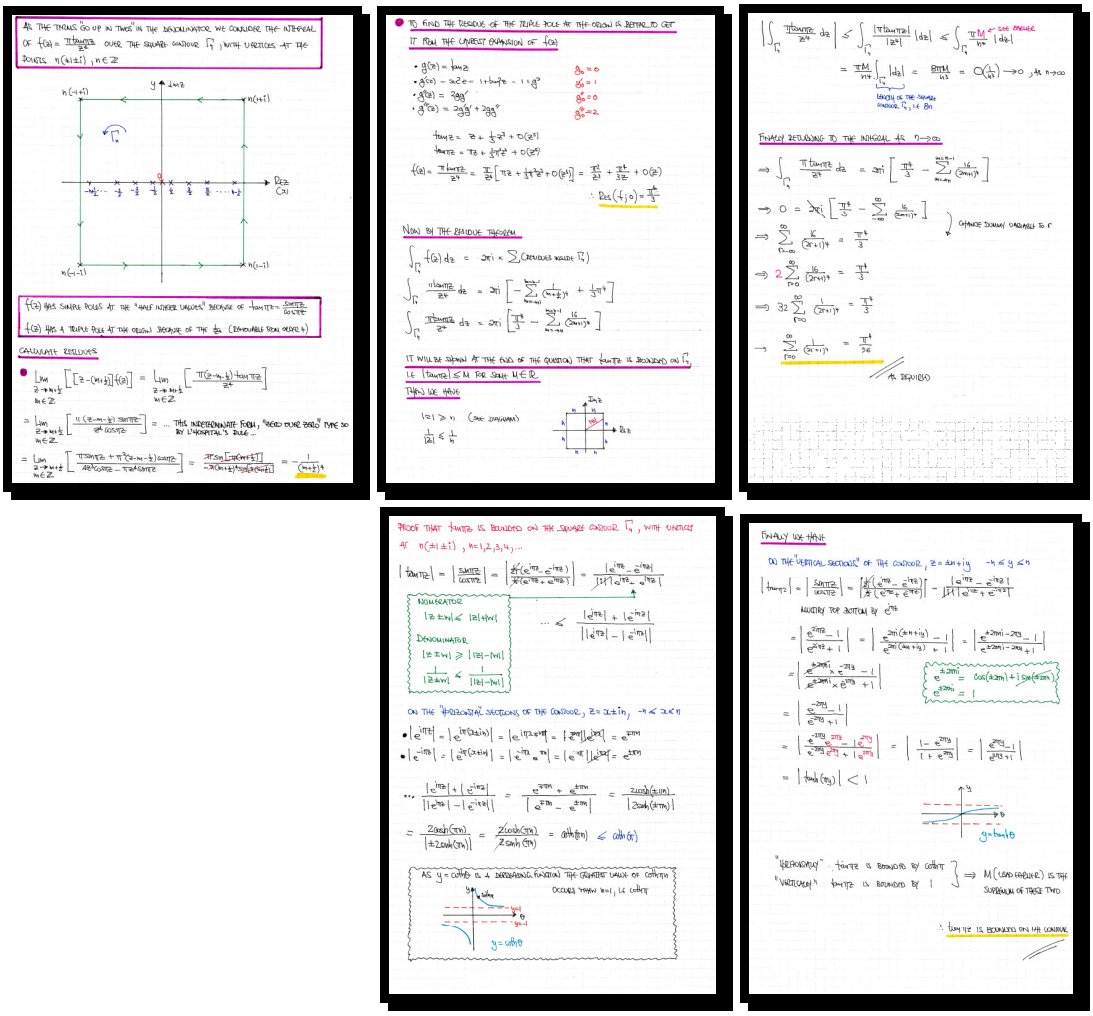
Question 12

$$f(z)=\frac{\pi\tan\pi z}{z^4},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=0}^{\infty}\frac{1}{(2r+1)^4}=\frac{\pi^4}{96}.$$

[ ], proof



Question 13

$$f(z)=\frac{\pi\sec\pi z}{z^3},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^3}=\frac{\pi^3}{32}.$$

[ ], proof

