RESIDUES and INTEGRATION

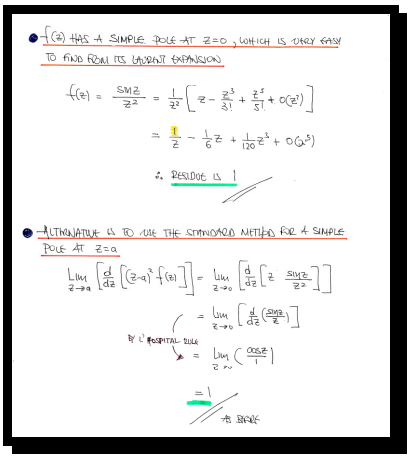
CALCULATIONS OF RESIDUES

Question 1

$$f(z)\equiv\frac{\sin z}{z^2},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=1}$

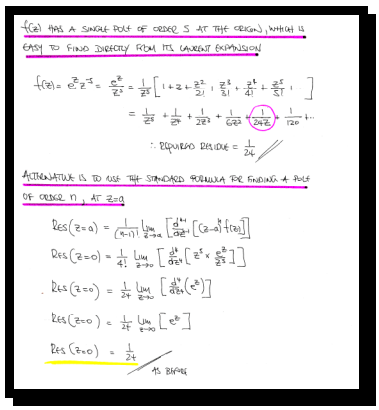


Question 2

$$f(z)\equiv e^z z^{-5},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=\frac{1}{24}}$

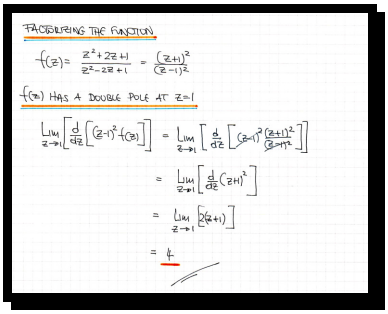


Question 3

$$f(z)\equiv\frac{z^2+2z+1}{z^2-2z+1},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=1)=4}$

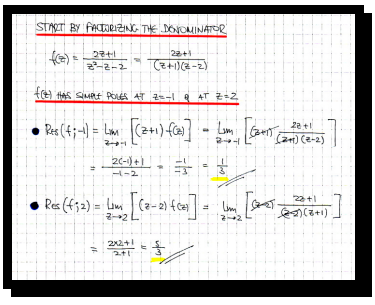


Question 4

$$f(z)\equiv\frac{2z+1}{z^2-z-2},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{}$, $\boxed{res(z=2)=\frac{5}{3}}$, $\boxed{res(z=-1)=\frac{1}{3}}$

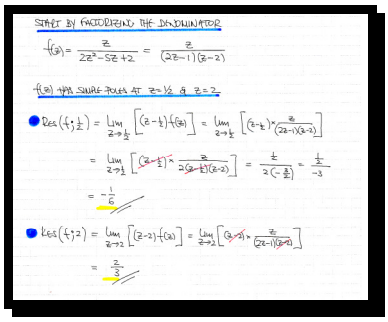


Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, \ z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\square$, $\boxed{res\left(z = \frac{1}{2}\right) = -\frac{1}{6}}$, $\boxed{res(z = 2) = \frac{2}{3}}$



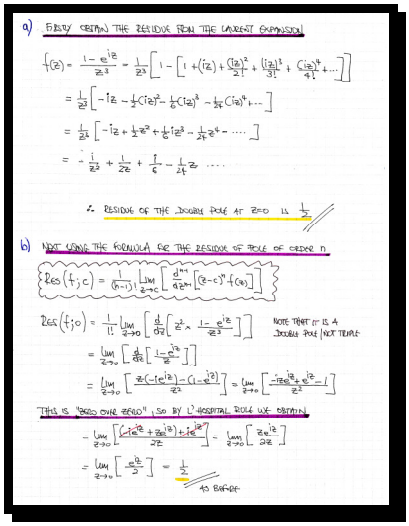
Question 6

$$f(z)\equiv\frac{1-\mathrm{e}^{\mathrm{i}z}}{z^3},\ z\in\mathbb{C}.$$

a) Find the first four terms in the Laurent expansion of $f(z)$ and hence state the residue of the pole of $f(z)$.

b) Determine the residue of the pole of $f(z)$ by an alternative method

$\square$, $\boxed{res(z=0)=\frac{1}{2}}$



Question 7

$$f(z)\equiv\frac{z^2+4}{z^3+2z^2+2z},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z=0)=2},\ \boxed{res(z=-1+\mathrm{i})=\frac{1}{2}(-1+3\mathrm{i})},\ \boxed{res(z=-1-\mathrm{i})=-\frac{1}{2}(1+3\mathrm{i})}$$

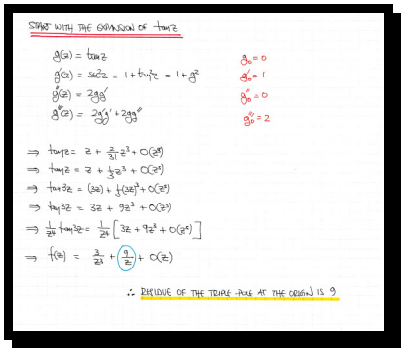


Question 8

$$f(z)\equiv\frac{\tan 3z}{z^4},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$$\boxed{},\ \boxed{res(z=0)=9}$$



Question 9

$$f(z)\equiv\frac{z^2-2z}{\left(z^2+4\right)(z+1)^2},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z=2\mathrm{i})=\frac{1}{25}(7+\mathrm{i})},\ \boxed{res(z=-2\mathrm{i})=\frac{1}{25}(7-\mathrm{i})},\ \boxed{res(z=-1)=-\frac{14}{25}}$$

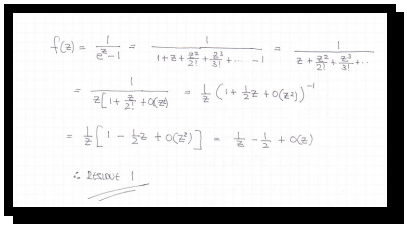


Question 10

$$f(z)\equiv\frac{1}{\mathrm{e}^z-1},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$, at the origin.

$$\boxed{res(z=0)=1}$$

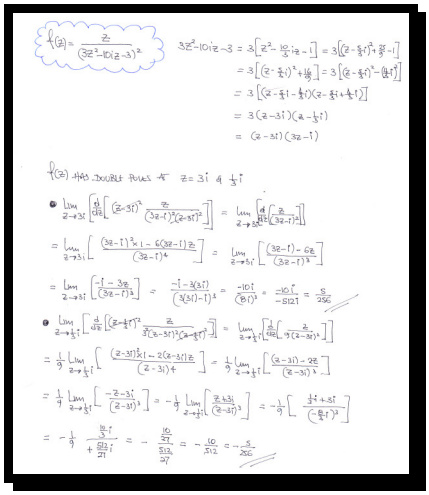


Question 11

$$f(z)\equiv\frac{z}{\left(3z^2-10\mathrm{i}\,z-3\right)^2},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res\left(z=3\mathrm{i}\right)=\frac{5}{256}},\ \boxed{res\left(z=\frac{1}{3}\mathrm{i}\right)=-\frac{5}{256}}$$

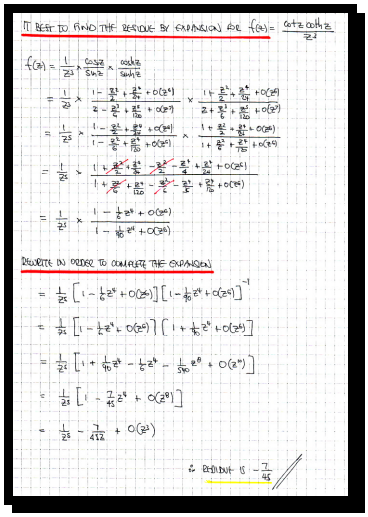


Question 12

$$f(z)\equiv\frac{\cot z\,\coth z}{z^3},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z=0$.

$\boxed{}$, $\boxed{res(z=0)=-\frac{7}{45}}$

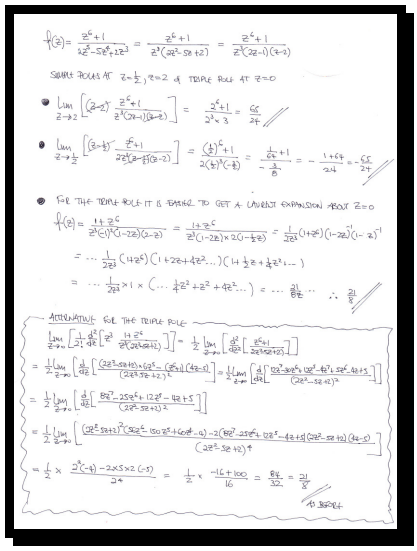


Question 13

$$f(z)\equiv\frac{z^6+1}{2z^5-5z^4+2z^3},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res\left(z=\frac{1}{2}\right)=-\frac{65}{24}},\ \boxed{res\left(z=2\right)=\frac{65}{24}},\ \boxed{res\left(z=0\right)=\frac{21}{8}}$$

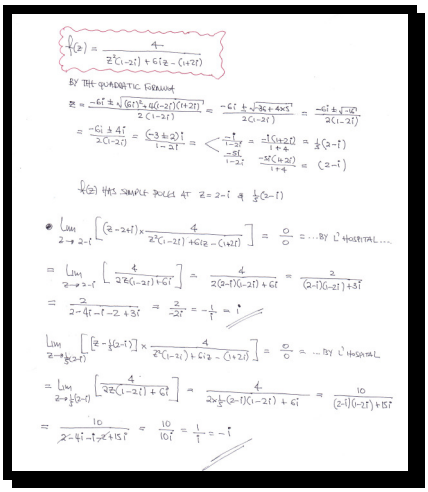


Question 14

$$f(z)\equiv\frac{4}{z^2(1-2\mathrm{i})+6z\,\mathrm{i}-(1+2\mathrm{i})},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res(z=2-\mathrm{i})=\mathrm{i}},\ \boxed{res\left(z=\frac{1}{5}(2-\mathrm{i})\right)=-\mathrm{i}}$$



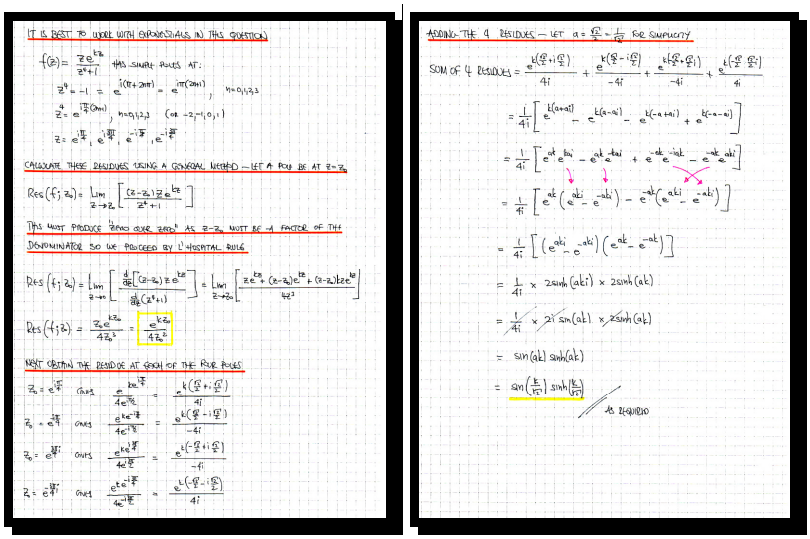
Question 15

$$f(z)\equiv\frac{z\,\mathrm{e}^{kz}}{z^4+1},\ z\in\mathbb{C},\ k\in\mathbb{R},\ k>0.$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right).$$

[ ], [proof]



REAL INTEGRALS

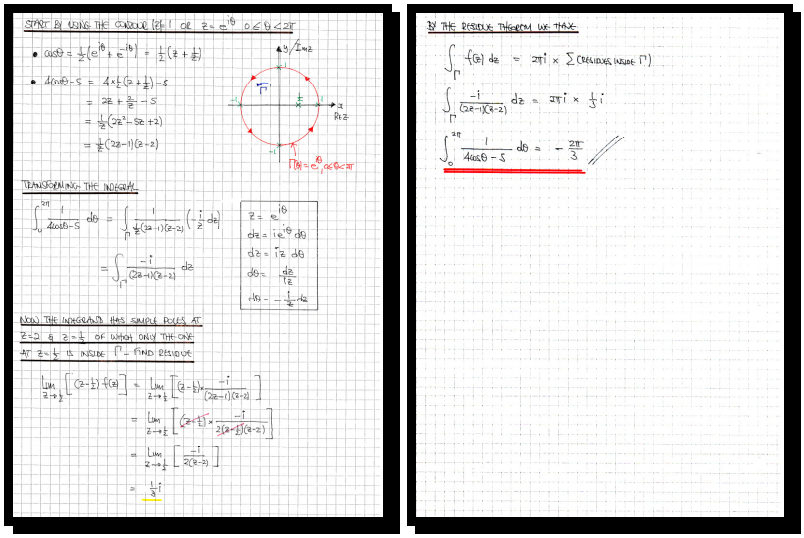
UNIT CIRCLE CONTOUR

Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\, d\theta .$$

V , [ ] , $\boxed{-\frac{2\pi}{3}}$

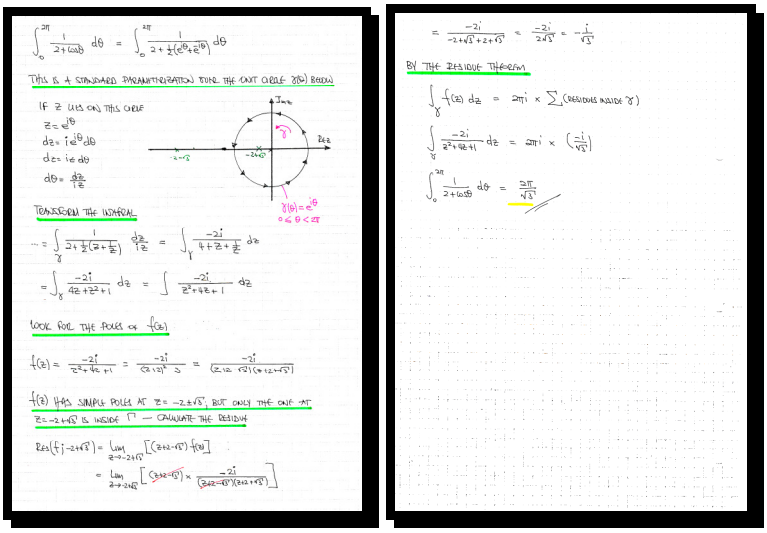


Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta\,.$$

$\square$, $\boxed{\dfrac{2\pi}{\sqrt{3}}}$

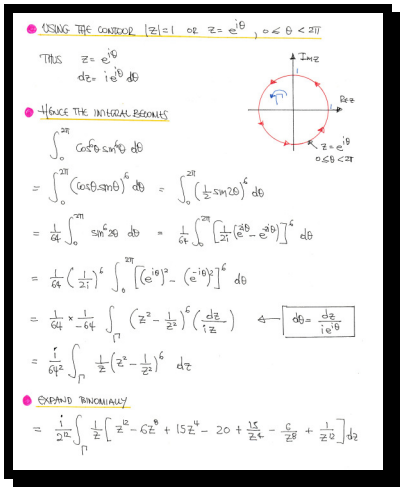


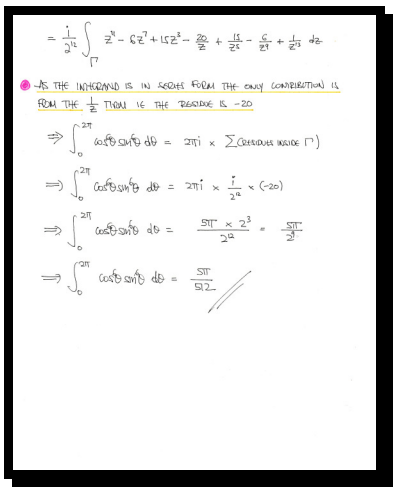
Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \cos^6\theta \sin^6\theta \ d\theta .$$

$\square$, $\boxed{\dfrac{5\pi}{512}}$



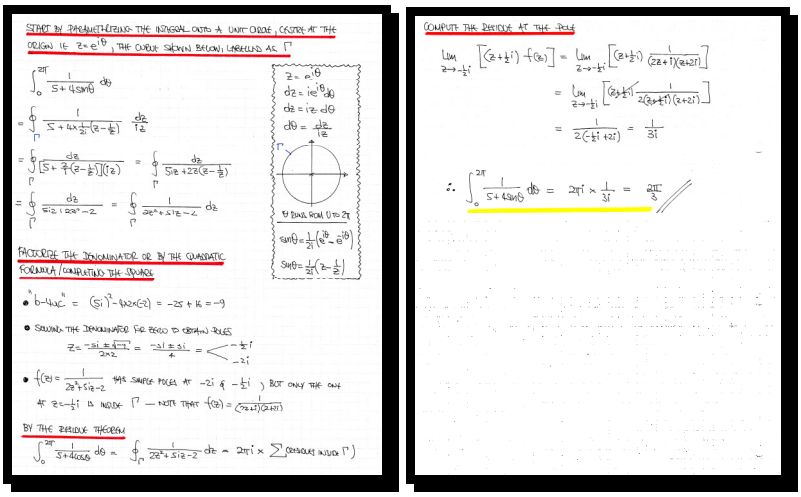


Question 4

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\, d\theta .$$

$\square$, $\boxed{\dfrac{2\pi}{3}}$

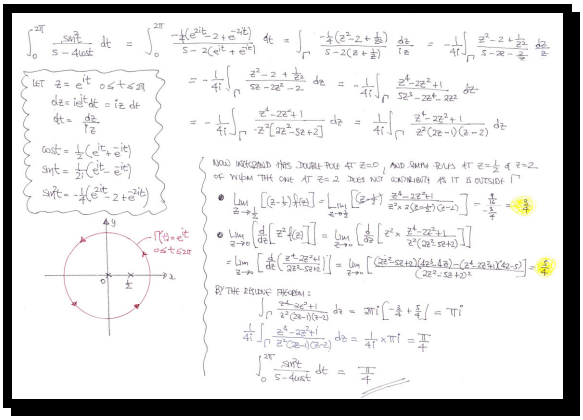


Question 5

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{\sin^2 t}{5-4\cos t}\, dt\,.$$

$$\boxed{\frac{\pi}{4}}$$

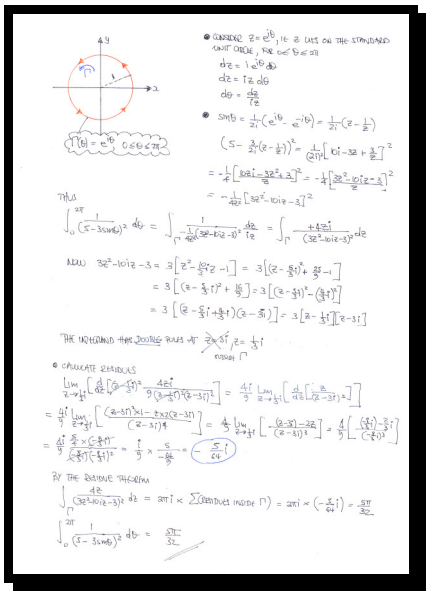


Question 6

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}\, d\theta .$$

$$\frac{5\pi}{32}$$

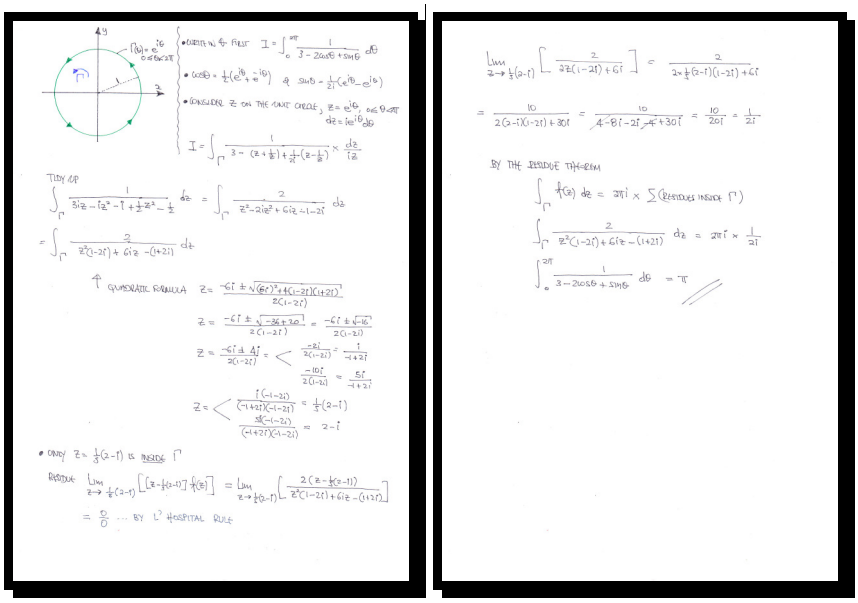


Question 7

$$I=\int_0^{2\pi}\frac{1}{3-2\cos x+\sin x}\,dx\,.$$

By integrating a suitable complex function over an appropriate contour find the exact value of I.

$\boxed{\pi}$

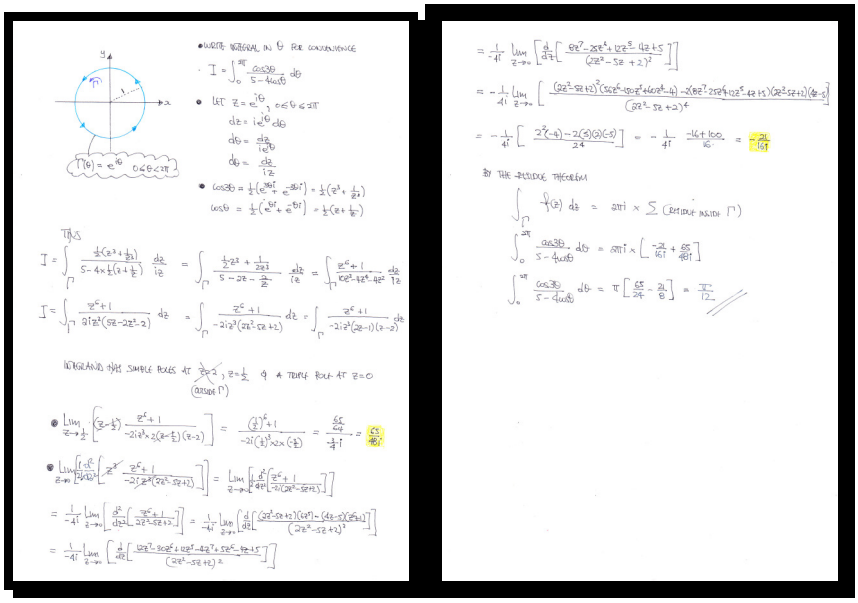


Question 8

$$I = \int_0^{2\pi} \frac{\cos 3x}{5-4\cos x}\, dx\,.$$

By integrating a suitable complex function over an appropriate contour find the exact value of I.

$$\frac{\pi}{12}$$



SEMI CIRCLE CONTOUR

Jordan's Lemma

Suppose that $f(z) \to 0$ uniformly, as $|z| \to \infty$, for $0 \le \arg z \le \pi$.

If $\alpha > 0$, then $\int_{\gamma_R} f(z) \mathrm{e}^{\mathrm{i}\alpha z}\, dz \to 0$ as $R \to \infty$, where $\gamma_R(\theta) = R\mathrm{e}^{\mathrm{i}\theta}$, for $0 \le \theta \le \pi$.

Proof

Given $\varepsilon > 0$ we may always pick R_0, so that if $R > R_0$, $|f(z)| < \varepsilon$, $\forall z \in \gamma_R$.

Thus

$$\left| \int_{\gamma_R} \mathrm{e}^{\mathrm{i}\alpha z} f(z)\, dz \right| = \left| \int_0^{\pi} \mathrm{e}^{\mathrm{i}\alpha R(\cos\theta + \mathrm{i}\sin\theta)} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\, d\theta \right| =$$

$$\left| \int_0^{\pi} \mathrm{e}^{\mathrm{i}\alpha R\cos\theta}\, \mathrm{e}^{-\alpha R\sin\theta} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\, d\theta \right| \le \int_0^{\pi} \left| \mathrm{e}^{\mathrm{i}\alpha R\cos\theta}\, \mathrm{e}^{-\alpha R\sin\theta} f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \right| d\theta =$$

$$\int_0^{\pi} \left| \mathrm{e}^{\mathrm{i}\alpha R\cos\theta} \right| \left| \mathrm{e}^{-\alpha R\sin\theta} \right| \left| f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \right| |\mathrm{i}| \left| \mathrm{e}^{\mathrm{i}\theta} \right| d\theta = \int_0^{\pi} \mathrm{e}^{-\alpha R\sin\theta} \left| f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) \right| d\theta \le$$

$$\varepsilon R \int_0^{\pi} \mathrm{e}^{-\alpha R\sin\theta}\, d\theta = 2\varepsilon R \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\alpha R\sin\theta}\, d\theta \qquad \left[\text{since } \sin\theta \text{ is even about } \frac{\pi}{2} \right]$$

Now by **Jordan's Inequality**

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1, \text{ if } 0 < \theta \le \frac{\pi}{2}$$

$$\sin\theta \ge \frac{2\theta}{\pi} \Rightarrow \mathrm{e}^{-\sin\theta} \le \mathrm{e}^{\frac{2}{\pi}\theta}, \text{ if } 0 < \theta \le \frac{\pi}{2}$$

Hence

$$2\varepsilon R \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\alpha R\sin\theta}\, d\theta \le 2\varepsilon R \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{2}{\pi}\alpha R\theta}\, d\theta = 2\varepsilon R \left[-\frac{\pi}{2\alpha R} \mathrm{e}^{-\frac{2}{\pi}\alpha R\theta} \right]_0^{\frac{\pi}{2}} =$$

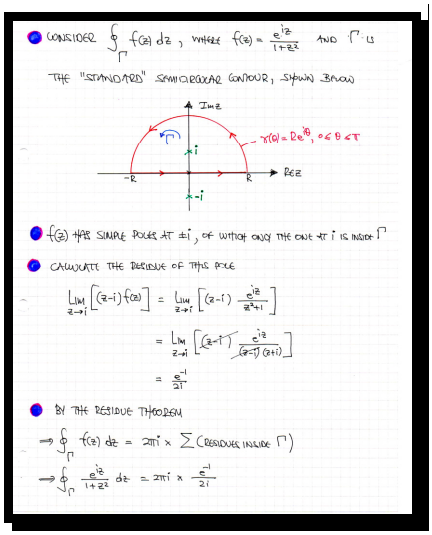
$$\frac{\varepsilon\pi}{\alpha} \left[\mathrm{e}^{-\frac{2}{\pi}\alpha R\theta} \right]_{\frac{\pi}{2}}^{0} = \frac{\varepsilon\pi}{\alpha} \left[1 - \mathrm{e}^{-\alpha R} \right] \to 0 \text{ since as } R \to \infty, \varepsilon \to 0 \ \square$$

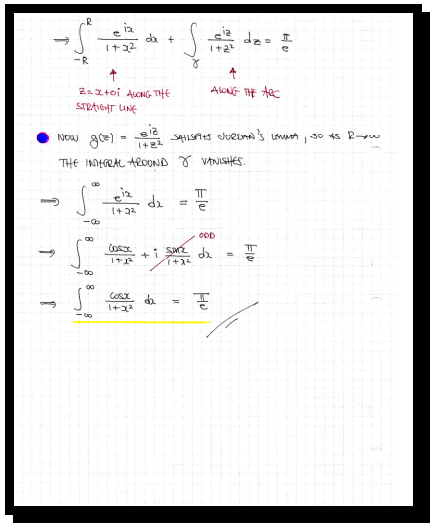
Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx\,.$$

V , [ ] , $\boxed{\frac{\pi}{e}}$



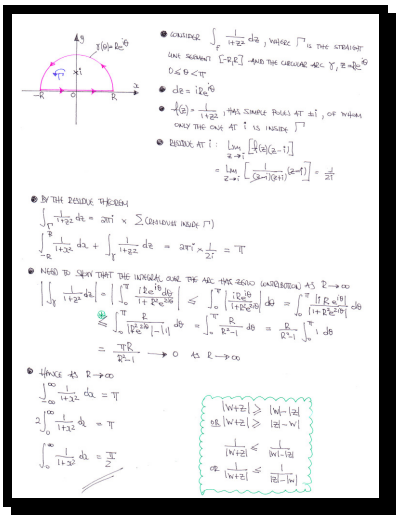


Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^2}\, dx\,.$$

$$\boxed{\frac{\pi}{2}}$$

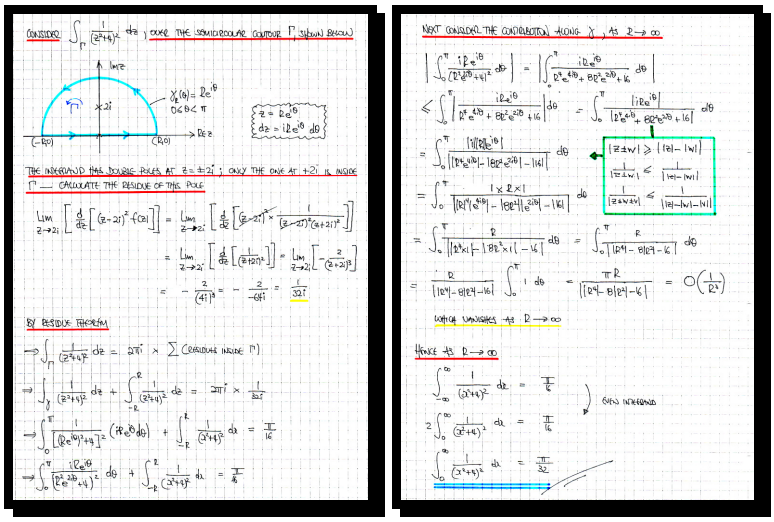


Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{\left(x^2+4\right)^2}\, dx\,.$$

$\square$, $\boxed{\dfrac{\pi}{32}}$

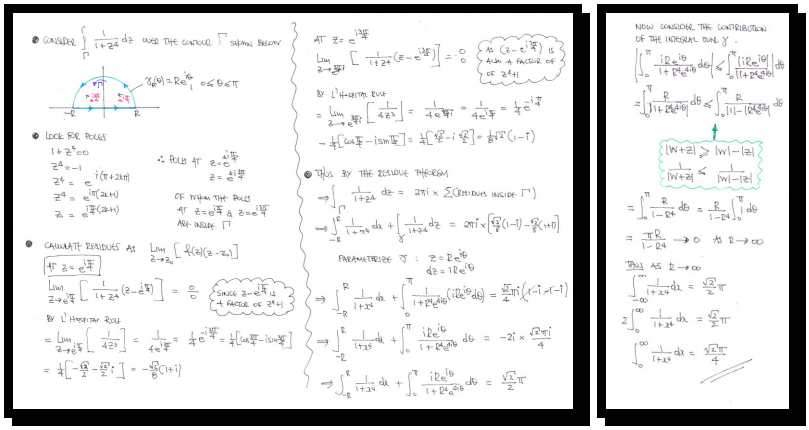


Question 4

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^4}\, dx\,.$$

$$\boxed{\frac{\pi\sqrt{2}}{4}}$$

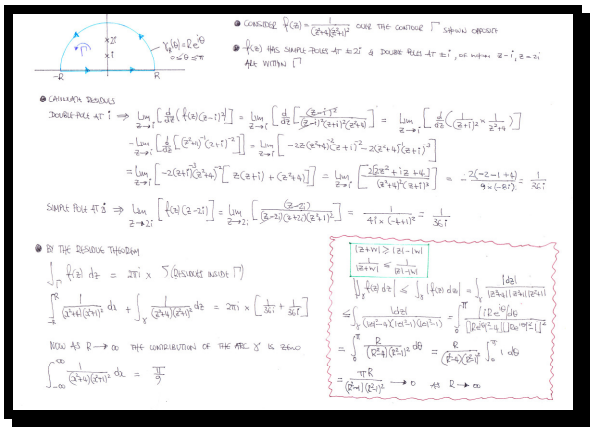


Question 5

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{\left(x^2+4\right)\left(x^2+1\right)^2}\, dx\,.$$

$$\boxed{\frac{\pi}{9}}$$

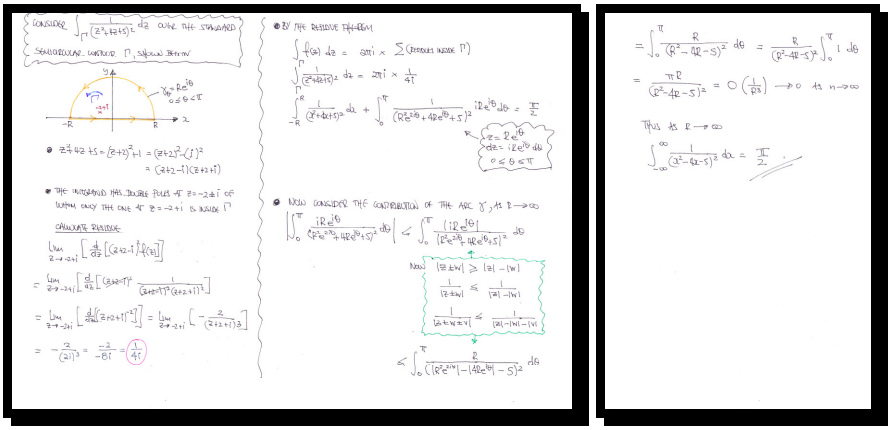


Question 6

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{\left(x^2+4x+5\right)^2}\, dx\,.$$

$$\boxed{\frac{\pi}{2}}$$

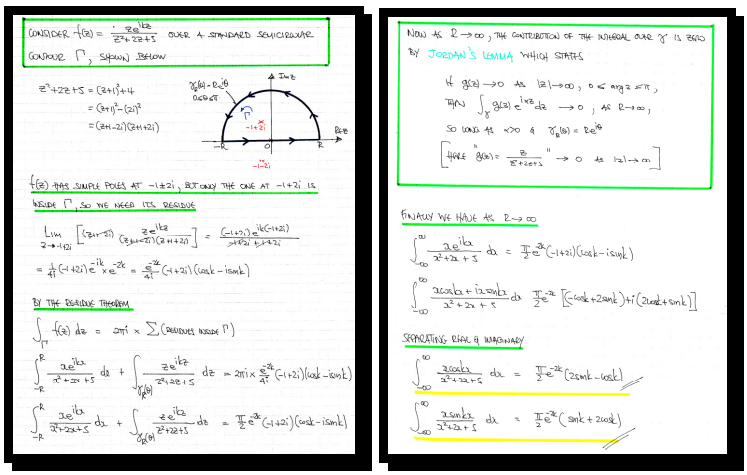


Question 7

Given that $k>0$ find the exact value of

$$\int_{-\infty}^{\infty} \frac{x\cos kx}{x^2+2x+5}\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{x\sin kx}{x^2+2x+5}\,dx\,.$$

$\boxed{}$, $\boxed{\frac{1}{2}\pi \mathrm{e}^{-2k}\left(2\sin k-\cos k\right)}$, $\boxed{\frac{1}{2}\pi \mathrm{e}^{-2k}\left(\sin k+2\cos k\right)}$



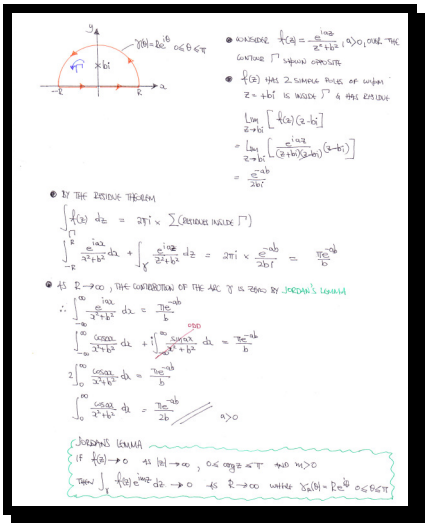
Question 8

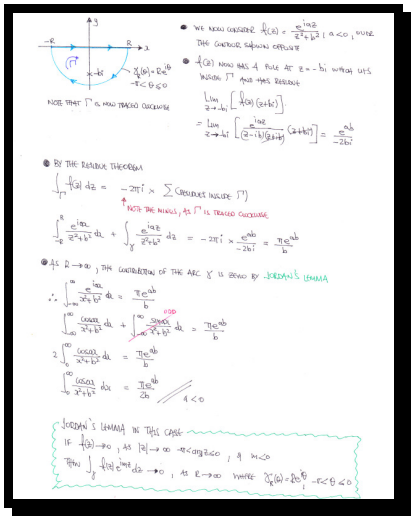
By integrating a suitable complex function over an appropriate contour find

a) $\ldots \displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\, dx\,,\ a>0\,.$

b) $\ldots \displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\, dx\,,\ a<0\,.$

$$\boxed{\frac{\pi \mathrm{e}^{-ab}}{2b},\ a>0}\ ,\ \boxed{\frac{\pi \mathrm{e}^{ab}}{2b},\ <0}$$



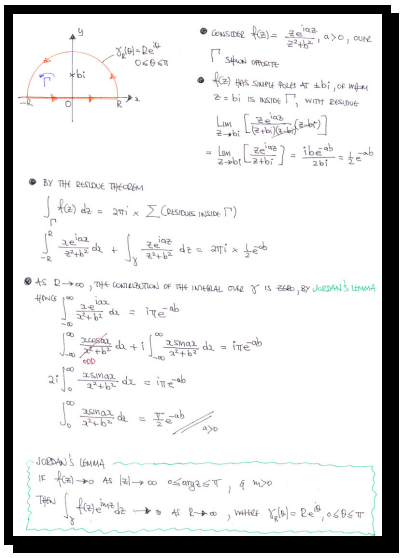


Question 9

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{x \sin ax}{x^2+b^2}\, dx\,,\ a>0\,.$$

$$\boxed{\frac{1}{2}\pi \mathrm{e}^{-ab}}$$

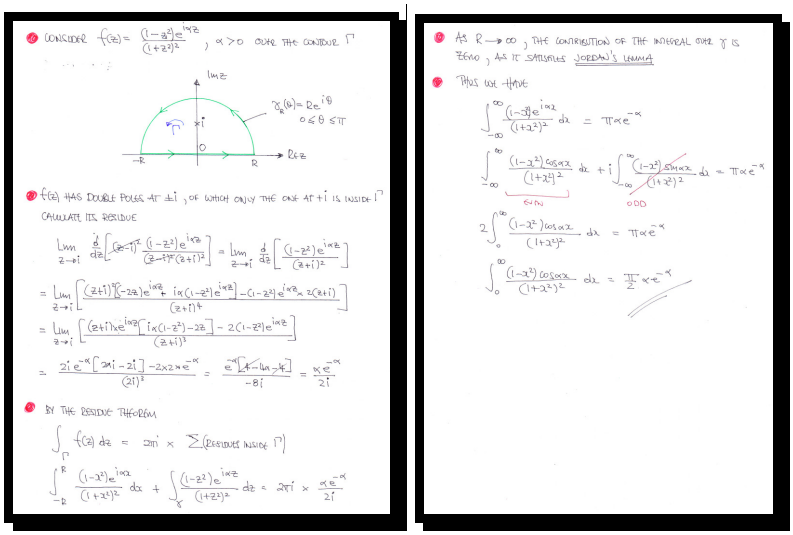


Question 10

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\left(1-x^2\right)\cos\alpha x}{\left(1+x^2\right)^2}\,dx\,,\ \alpha>0\,.$$

$$\boxed{\frac{1}{2}\pi\alpha\,\mathrm{e}^{-\alpha}}$$

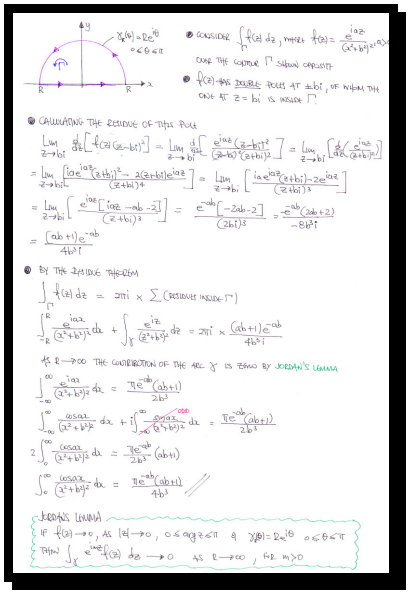


Question 11

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\cos ax}{\left(x^2+b^2\right)^2}\, dx\,,\ a>0\,.$$

$$\boxed{\frac{\pi \mathrm{e}^{-ab}\left(ab+1\right)}{4b^3}}$$

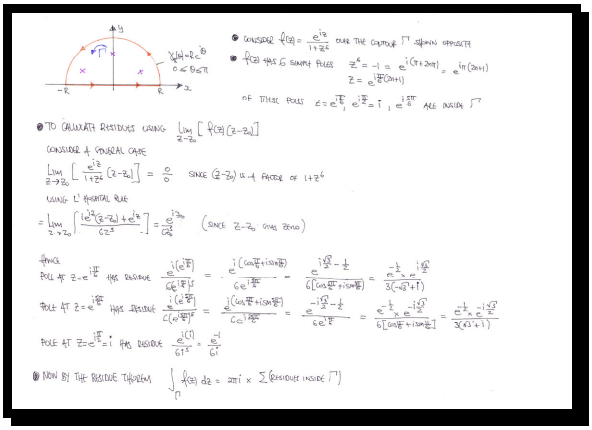


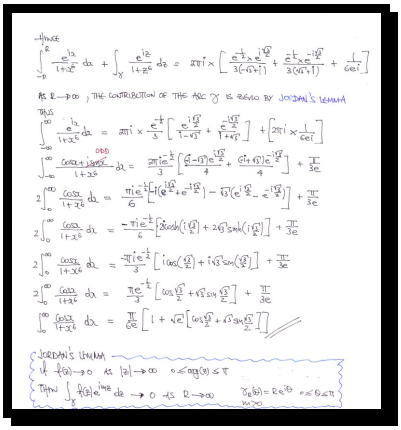
Question 12

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\cos x}{1+x^6}\ dx\,.$$

$$\boxed{\frac{\pi}{6\mathrm{e}}\left[1+\sqrt{\mathrm{e}}\left[\cos\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right]\right]}$$





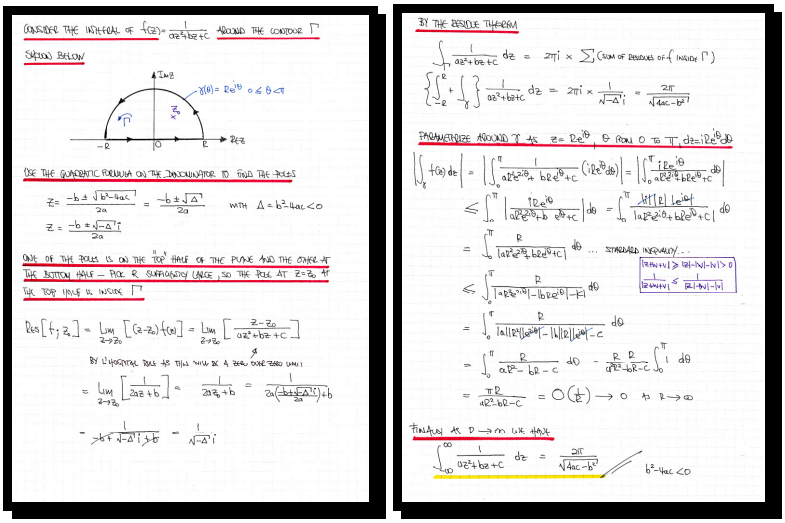
Question 13

By integrating a suitable complex function over an appropriate contour find an exact simplified value for

$$\int_{-\infty}^{\infty} \frac{1}{ax^2+bx+c}\,dx,$$

where a, b and c are real constants such that $a>0$ and $b^2-4ac<0$.

V, $\square$, $\dfrac{2\pi}{\sqrt{4ac-b^2}}$

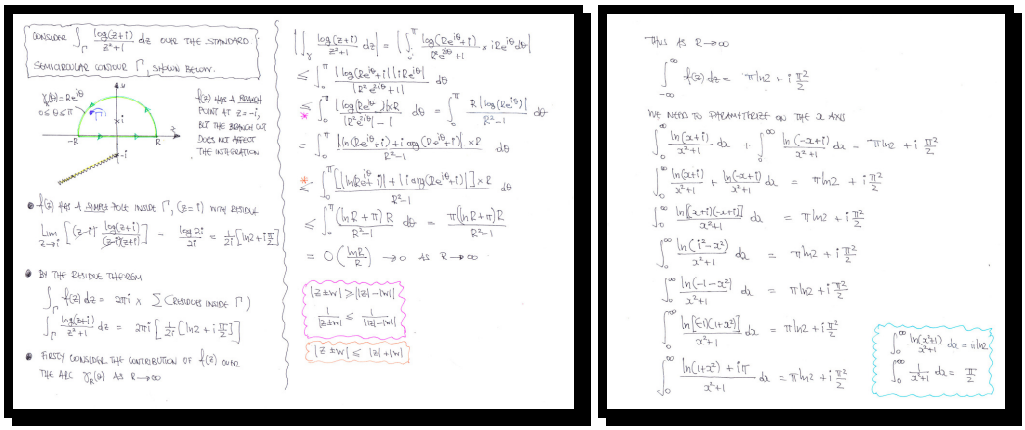


Question 14

$$I=\int_0^{\infty}\frac{\ln\left(x^2+1\right)}{x^2+1}\,dx\,.$$

By integrating $\dfrac{\ln(z+\mathrm{i})}{z^2+1}$ over a semicircular contour find the exact value of I.

$$\boxed{I=\pi\ln 2}$$



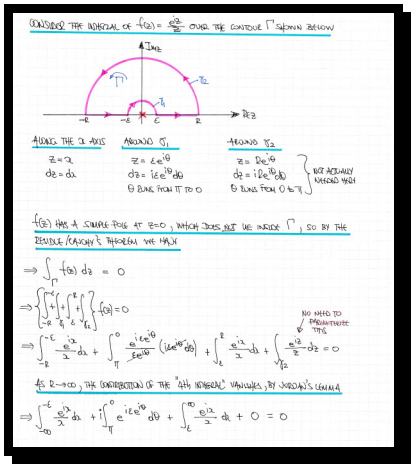
SEMI CIRCLE CONTOUR WITH HOLE

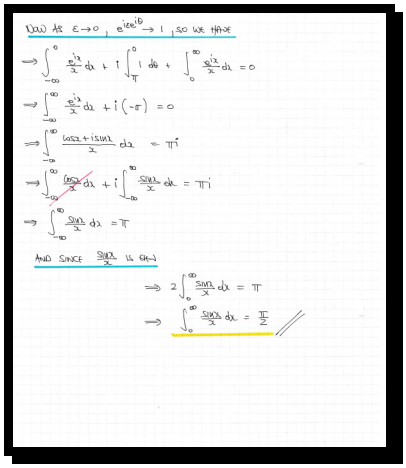
Question 1

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

V , [ ] , proof



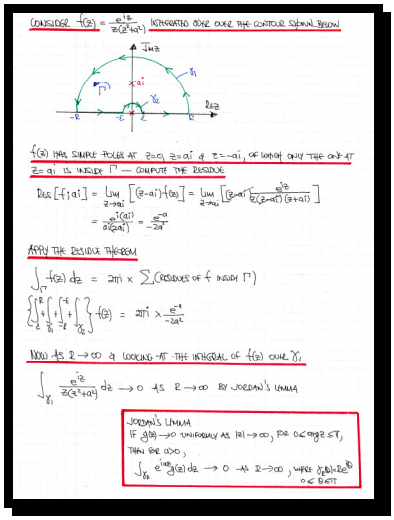


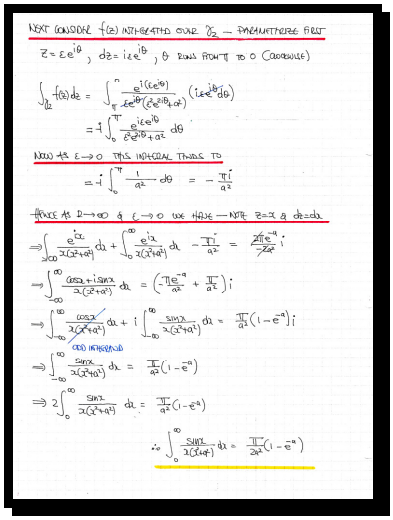
Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x\left(x^2+a^2\right)}\,dx=\frac{\pi}{2a^2}\left(1-\mathrm{e}^{-a}\right).$$

V , [ ] , proof



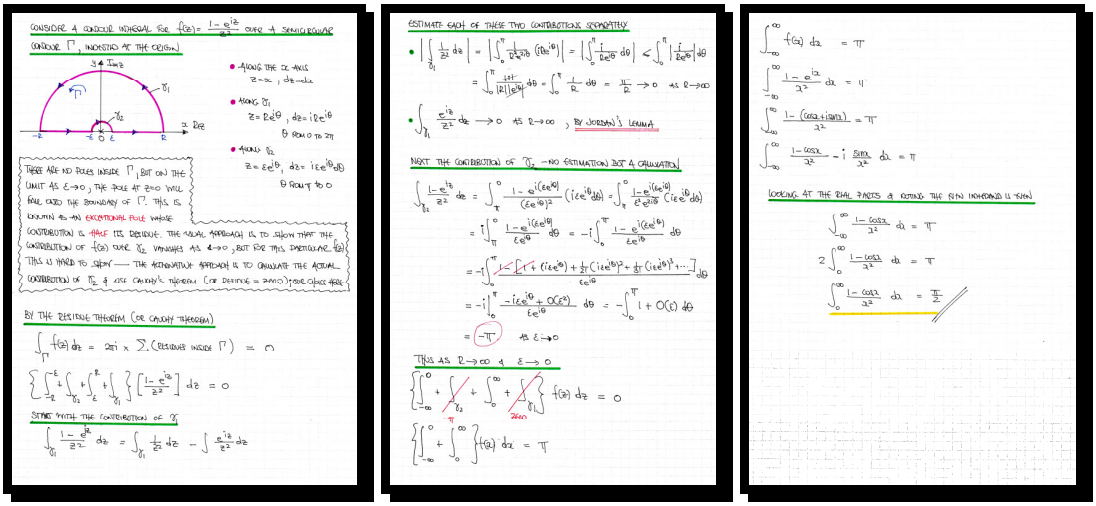


Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}.$$

[ ], [proof]



Question 4

$$\int_0^{\infty} \frac{\ln x}{1+x^4}\, dx\,.$$

a) Find the value of the above improper integral, by integrating

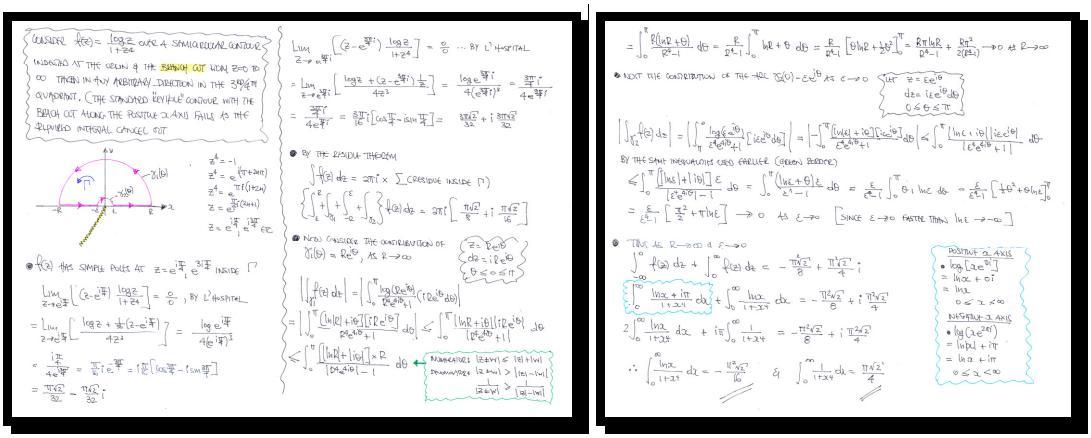
$$f(z)=\frac{\log z}{1+z^4},\ z\in\mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

b) State the value of

$$\int_0^{\infty} \frac{1}{1+x^4}\, dx\,.$$

$$-\frac{\pi^2\sqrt{2}}{16},\ \frac{\pi\sqrt{2}}{4}$$



Question 5

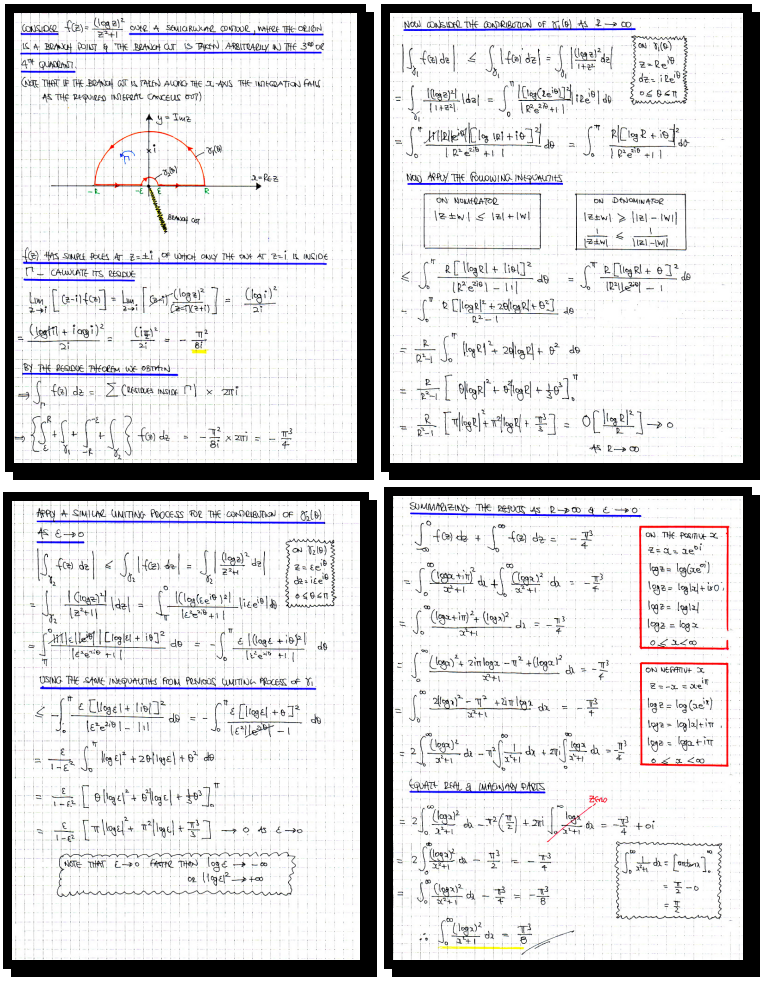
$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2}\, dx\,.$$

Find the value of the above improper integral, by integrating

$$f(z)=\frac{(\log z)^2}{1+z^2}\,,\ z\in\mathbb{C}\,,$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

$\square$, $\dfrac{\pi^3}{8}$



Question 6

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^4}\, dx\,.$$

a) Find the value of the above improper integral, by integrating

$$f(z)=\frac{(\log z)^2}{1+z^4},\ z\in\mathbb{C},$$

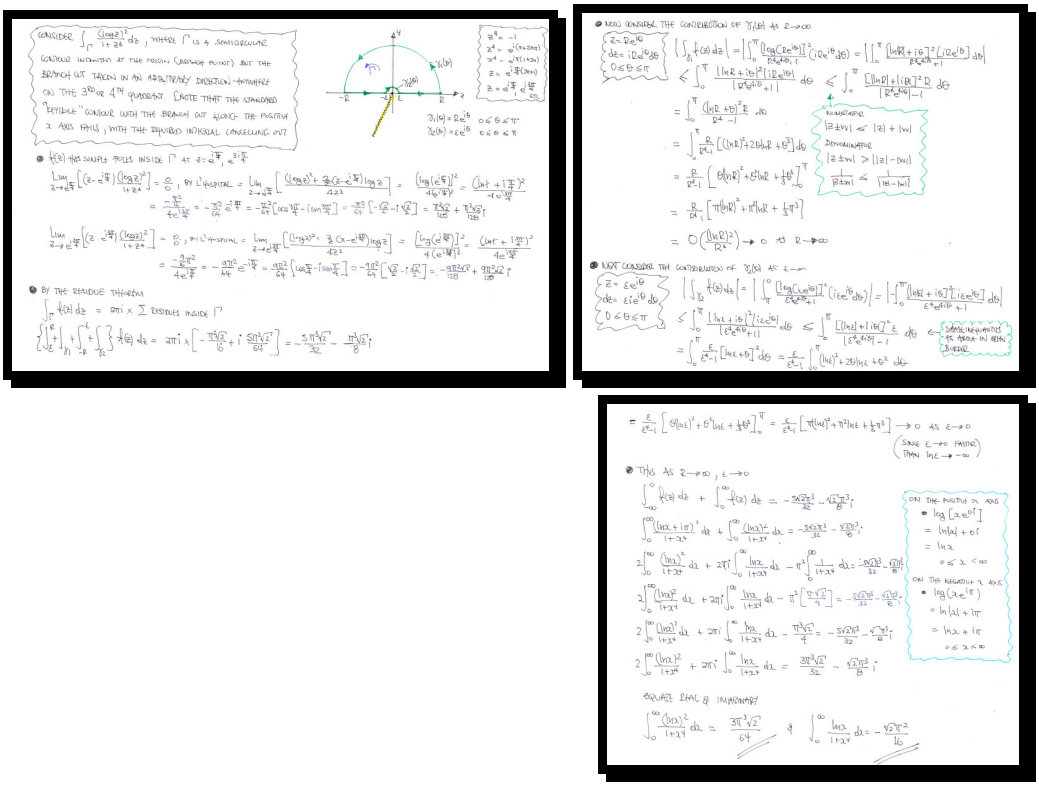
over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

[You may assume without proof that $\displaystyle\int_0^{\infty} \frac{1}{1+x^4}\, dx=\frac{\pi\sqrt{2}}{4}$]

b) State the value of

$$\int_0^{\infty} \frac{\ln x}{1+x^4}\, dx\,.$$

$$\boxed{\frac{3\pi^3\sqrt{2}}{64}},\ \boxed{-\frac{\pi^2\sqrt{2}}{16}}$$



KEYHOLE CONTOUR

(Branch Cuts)

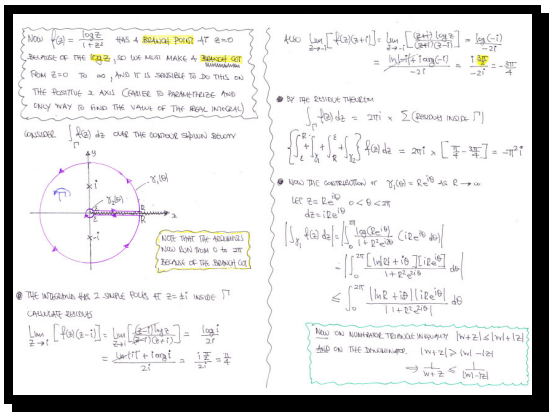
Question 1

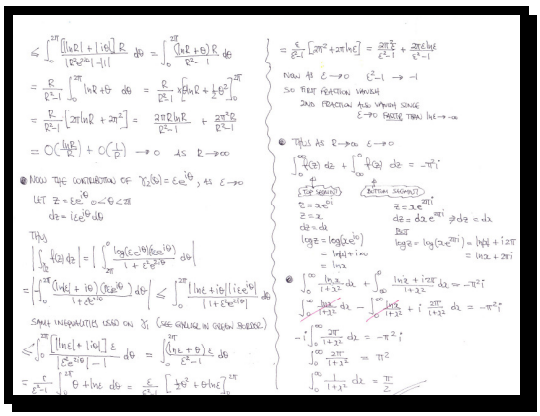
$$f(z)=\frac{\log z}{1+z^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_0^{\infty}\frac{1}{1+x^2}\,dx=\frac{\pi}{2}.$$

proof



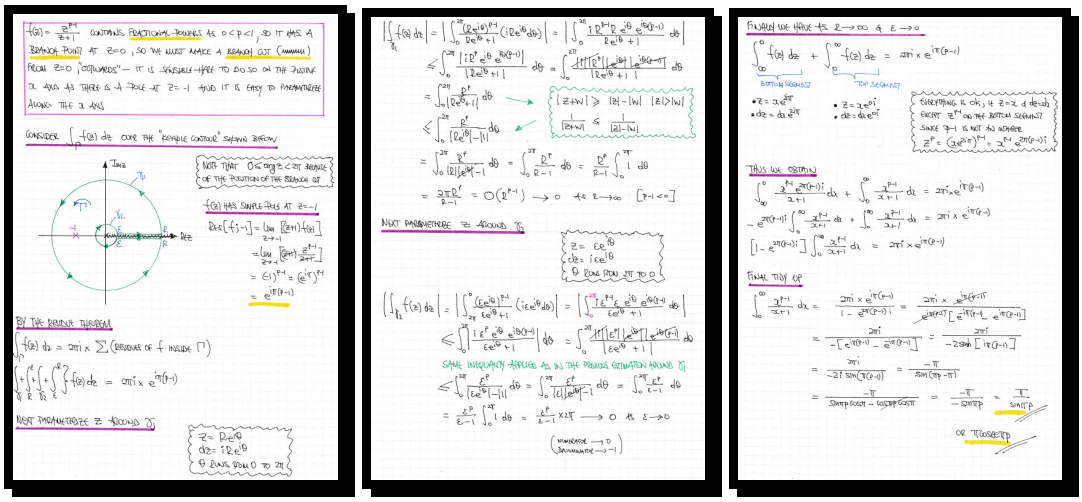


Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \pi \operatorname{cosec}\left(p\pi\right),\ 0<p<1.$$

V, proof

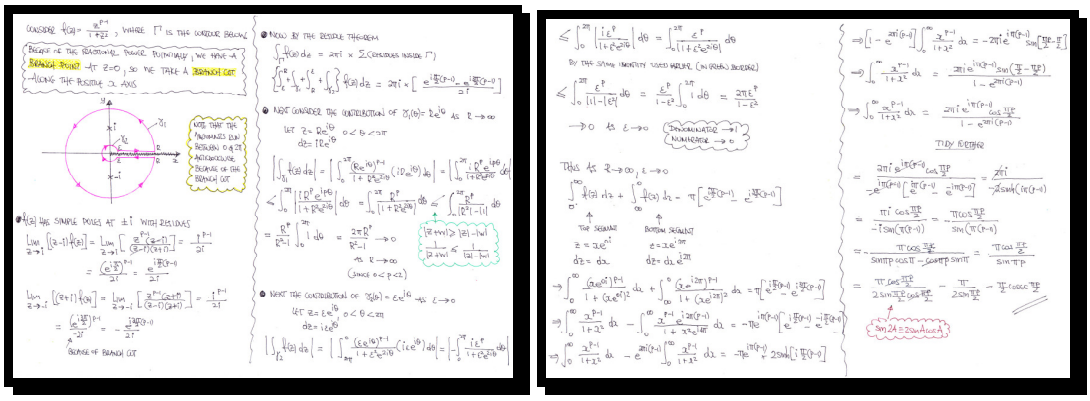


Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x^2}\, dx = \frac{\pi}{2}\operatorname{cosec}\left(\frac{p\pi}{2}\right), \quad 0<p<2\,.$$

proof



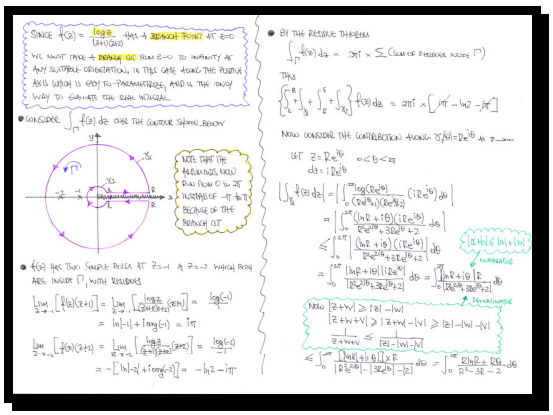
Question 4

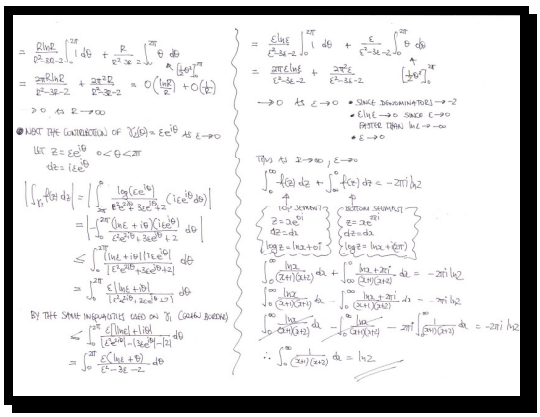
$$f(z)=\frac{\log z}{(z+1)(z+2)},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_0^{\infty}\frac{1}{(x+1)(x+2)}\,dx=\ln 2.$$

proof





Question 5

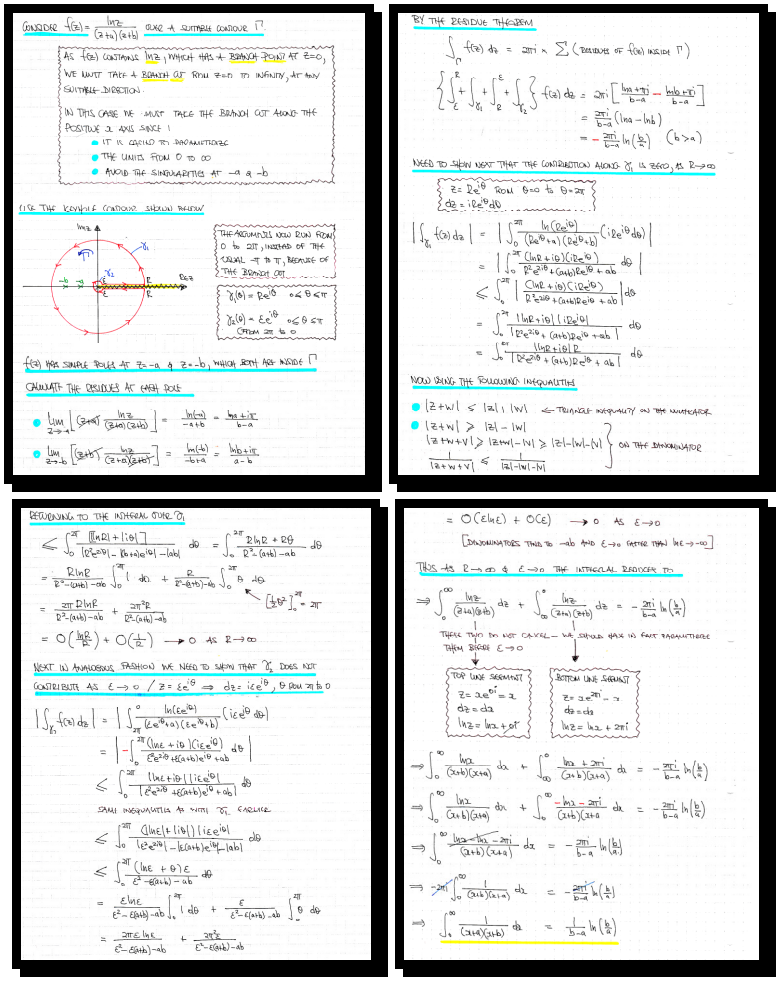
$$f(z)=\frac{\log z}{(z+a)(z+b)},\ z\in\mathbb{C},$$

where $a\in\mathbb{R}^+$, $b\in\mathbb{R}^+$ with $b>a$.

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_0^{\infty}\frac{1}{(x+a)(x+b)}\,dx=\frac{1}{b-a}\ln\left(\frac{b}{a}\right).$$

[ ], proof

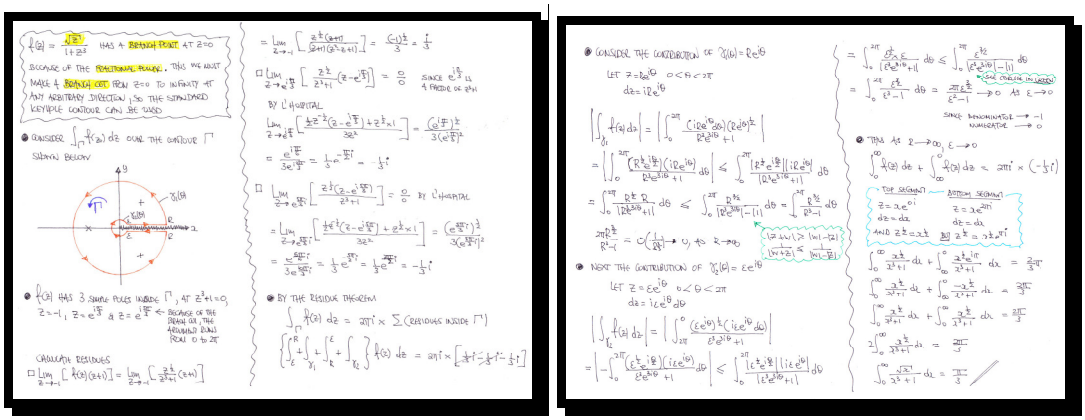


Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^3}\, dx = \frac{\pi}{3}.$$

proof



Question 7

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2}\, dx\,.$$

An attempt is made to find the value of the above improper integral, by integrating

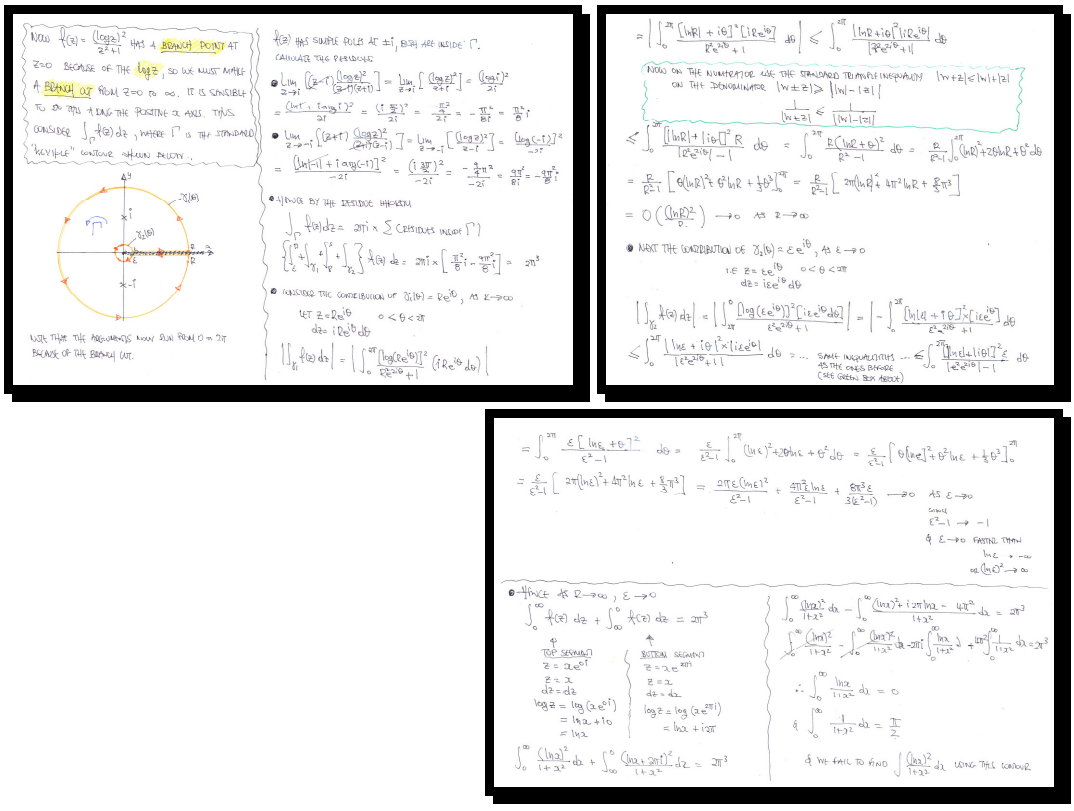
$$f(z)=\frac{(\ln z)^2}{1+z^2},\ z\in\mathbb{C},$$

over the standard "keyhole" contour with a branch cut taken on the positive x axis.

a) Show that such attempt fails.

b) Calculate the value of the two integrals that can be found during this attempt.

$$\boxed{\int_0^{\infty} \frac{1}{1+x^2}\, dx=\frac{\pi}{2}},\ \boxed{\int_0^{\infty} \frac{\ln x}{1+x^2}\, dx=0}$$



Question 8

Use a substitution followed by integration of a suitable complex function over an appropriate contour, to show that

$$\int_0^{\frac{1}{2}\pi} (\tan x)^{\alpha}\ dx = \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi\alpha\right), \quad -1<\alpha<1.$$

proof



SPECIAL CONTOURS

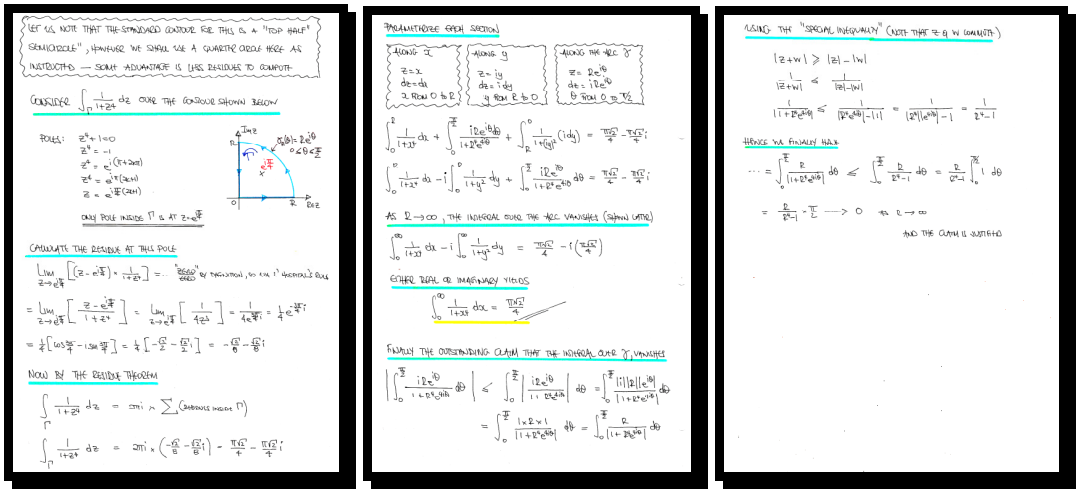
Question 1

Consider the contour Γ located in the first quadrant, defined as the boundary of a quarter circular sector of radius R, with centre at the origin O.

By integrating a suitable complex function over Γ show that

$$\int_0^{\infty} \frac{1}{1+x^4}\, dx = \frac{\pi\sqrt{2}}{4}.$$

$\square$, $\boxed{\dfrac{\pi\sqrt{2}}{4}}$



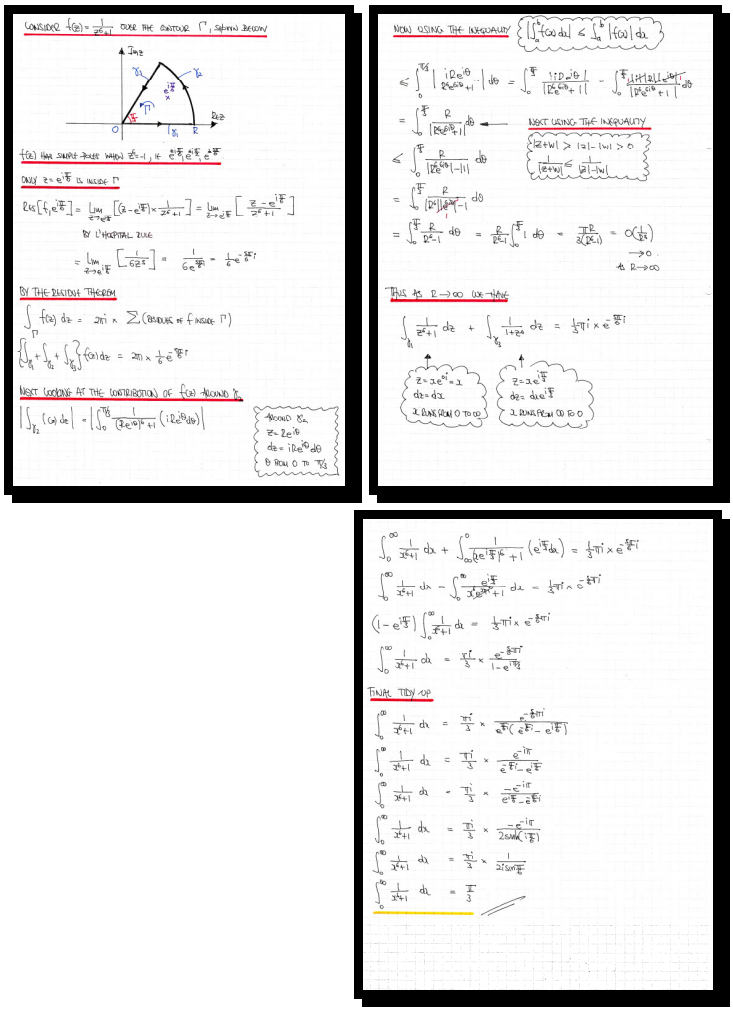
Question 2

By integrating a suitable complex function over a contour defined as the outline of a circular sector subtending an angle of $\frac{1}{3}\pi$ at the origin, find an exact value for

$$\int_0^{\infty} \frac{1}{1+x^6}\, dx\,.$$

No credit will be given for integration over alternative contours.

V, , $\frac{\pi}{3}$

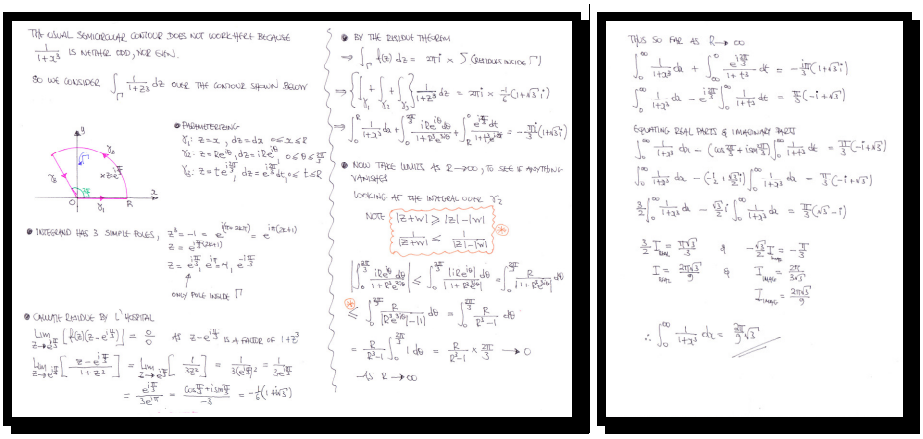


Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^3}\, dx\,.$$

$$\boxed{\frac{2\pi\sqrt{3}}{9}}$$

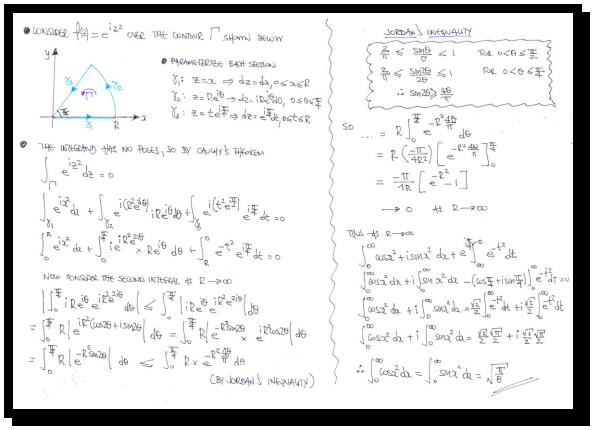


Question 4

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \cos\left(x^2\right)\ dx = \int_0^{\infty} \sin\left(x^2\right)\ dx = \sqrt{\frac{\pi}{8}}\,.$$

proof

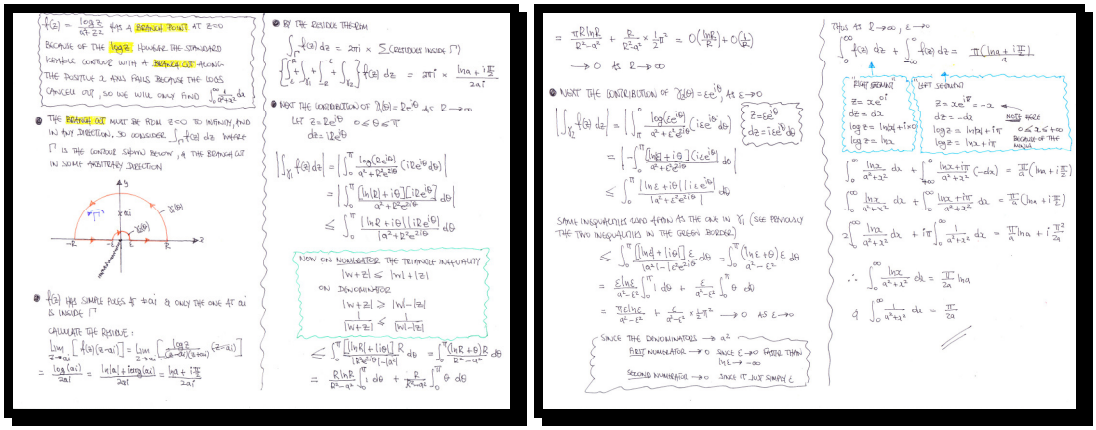


Question 5

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\ln x}{a^2+x^2}\, dx = \frac{\pi \ln a}{2a}.$$

proof

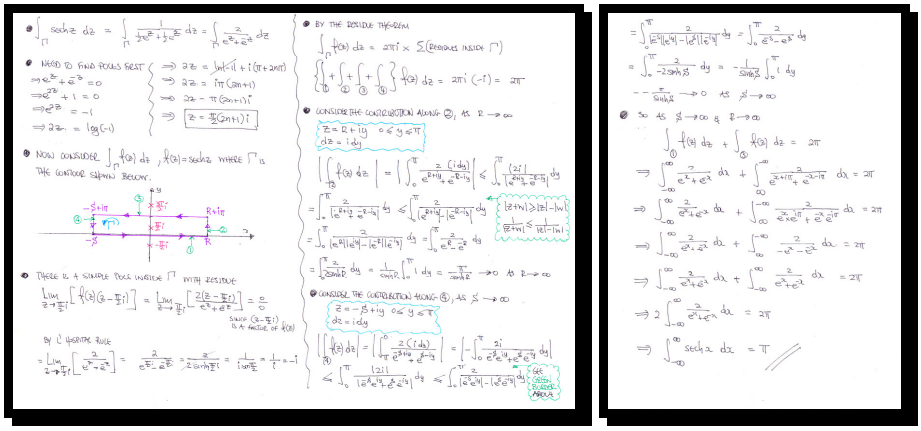


Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_{-\infty}^{\infty} \operatorname{sech} x \ \ dx = \pi \, .$$

proof



Question 7

It is required to evaluate the integral

$$\int_0^{\infty} \mathrm{e}^{-x^2} \cos x \ dx .$$

a) Show that the above integral can be written as

$$\frac{1}{2}\mathrm{e}^{-\frac{1}{4}} \int_{-\infty}^{\infty} \mathrm{e}^{-\left(x+\frac{1}{2}\mathrm{i}\right)^2} \ dx$$

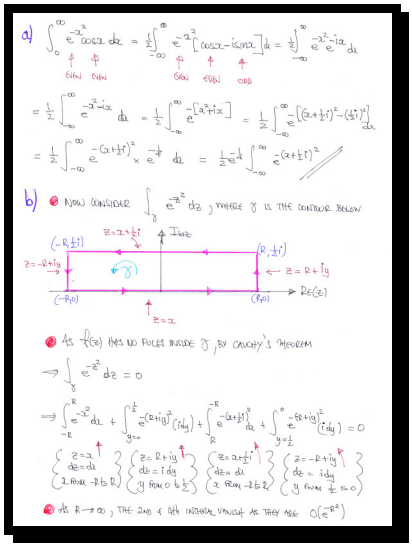
b) By integrating the complex function $f(z)=\mathrm{e}^{-z^2}$, over a rectangular contour with vertices at $(-R,0)$, $(R,0)$, $\left(R,\frac{1}{2}\mathrm{i}\right)$ and $\left(-R,\frac{1}{2}\mathrm{i}\right)$, show that

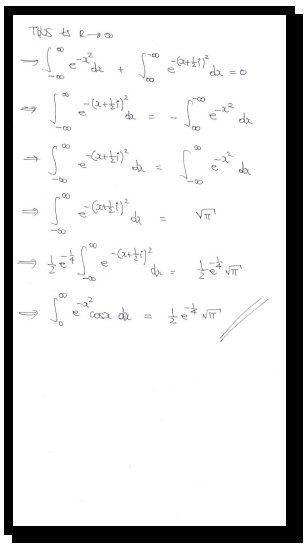
$$\int_0^{\infty} \mathrm{e}^{-x^2} \cos x \ dx = \frac{1}{2}\,\mathrm{e}^{-\frac{1}{4}}\sqrt{\pi} .$$

You may assume without proof that

$$\int_0^{\infty} \mathrm{e}^{-x^2} \ dx = \frac{1}{2}\sqrt{\pi} .$$

proof





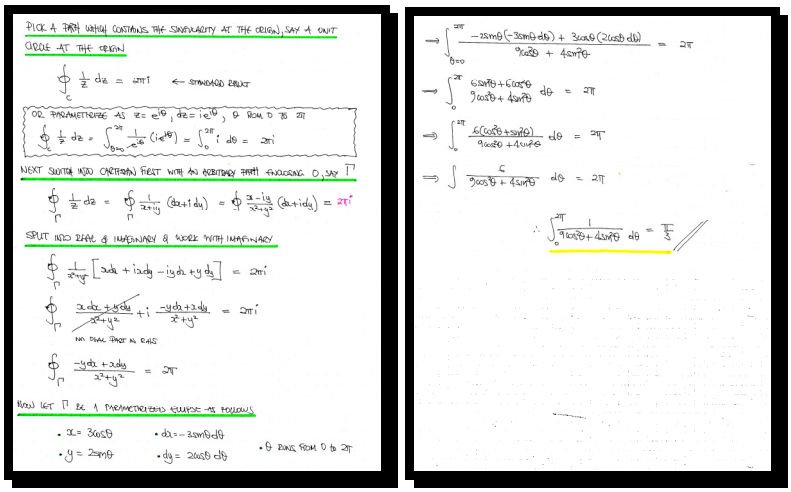
Question 8

$$f(z) \equiv \frac{1}{z},\ z \in \mathbb{C},\ z \neq 0.$$

By considering the integral of $f(z)$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{9\cos^2\theta + 4\sin^2\theta}\, d\theta = \frac{\pi}{3}.$$

[ ], [proof]



Question 9

The complex number $z = c + a\cos\theta + \mathrm{i}b\sin\theta$, $0 \le \theta < 2\pi$, traces a closed contour C, where a, b and c are positive real numbers with $a > c$.

By considering

$$\oint_C \frac{1}{z}\, dz,$$

show that

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + (b\sin\theta)^2}\, d\theta = \frac{2\pi}{b}.$$

proof

