LAPLACE'S EQUATION

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$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=0, \quad \Phi=\Phi\left(x, y\right)$$

Two Dimensional in Cartesian

Question 1

The temperature distribution, $T(x,y)$, in a rectangular plate for which $0 \le x \le 2$ and $0 \le y \le 1$, satisfies Laplace's Equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0.$$

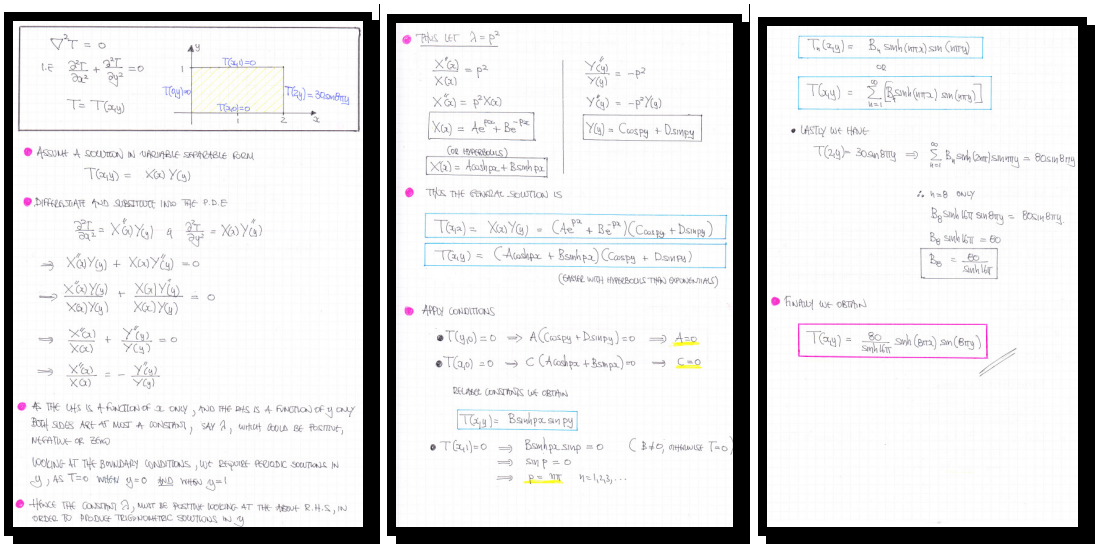
The edges of the rectangular plate are maintained at the following temperatures

- $T(x,0)=0, \quad x\in[0,2]$
- $T(x,1)=0, \quad x\in[0,2]$
- $T(0,y)=0, \quad y\in[0,1]$
- $T(2,y)=30\sin(8\pi y), \quad y\in[0,1].$

Determine the temperature distribution of the plate

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{T(x,y)=\frac{80\sinh(8\pi x)\sin(8\pi y)}{\sinh 16\pi}}$$



Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

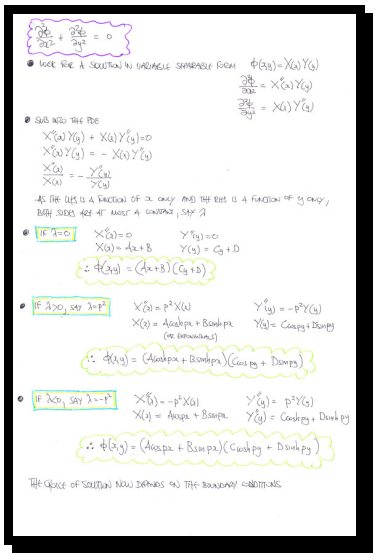
where $0 \le x \le \pi$ and $y > 0$.

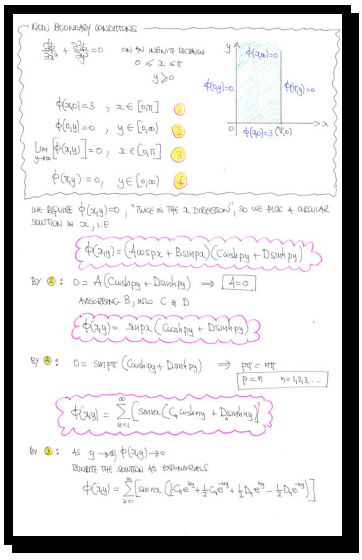
Determine an expression for $\varphi(x, y)$, given further that

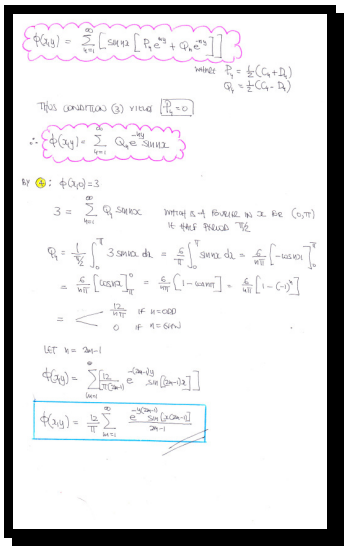
- $\varphi(x, 0) = 3, \quad x \in [0, \pi]$
- $\varphi(0, y) = 0, \quad y \in [0, \infty)$
- $\lim_{y \to \infty} \left[\varphi(x, y) \right] = 0, \quad x \in [0, \pi]$
- $\varphi(\pi, y) = 0, \quad y \in [0, \infty)$.

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{\varphi(x, y) = \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{\exp\left[-y(2n-1)\right] \sin\left[x(2n-1)\right]}{(2n-1)^3}}$$







Question 3

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

where $0 \le x \le a$ and $0 \le y \le b$, with a and b positive constants.

Determine an expression for $\varphi(x, y)$, given further that

- $\varphi(x,0) = 0, \quad x \in [0,a]$
- $\varphi(x,b) = 0, \quad x \in [0,a]$
- $\varphi(0, y) = 0, \quad y \in [0,b]$
- $\varphi(a, y) = by - y^2, \quad y \in [0,b]$.

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{\varphi(x, y) = \frac{8b^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sinh\left[\frac{\pi x}{b}(2n-1)\right] \sin\left[\frac{\pi y}{b}(2n-1)\right]}{(2n-1)^3 \sinh\left[\frac{\pi a}{b}(2n-1)\right]}}$$

[solution overleaf]



Question 4

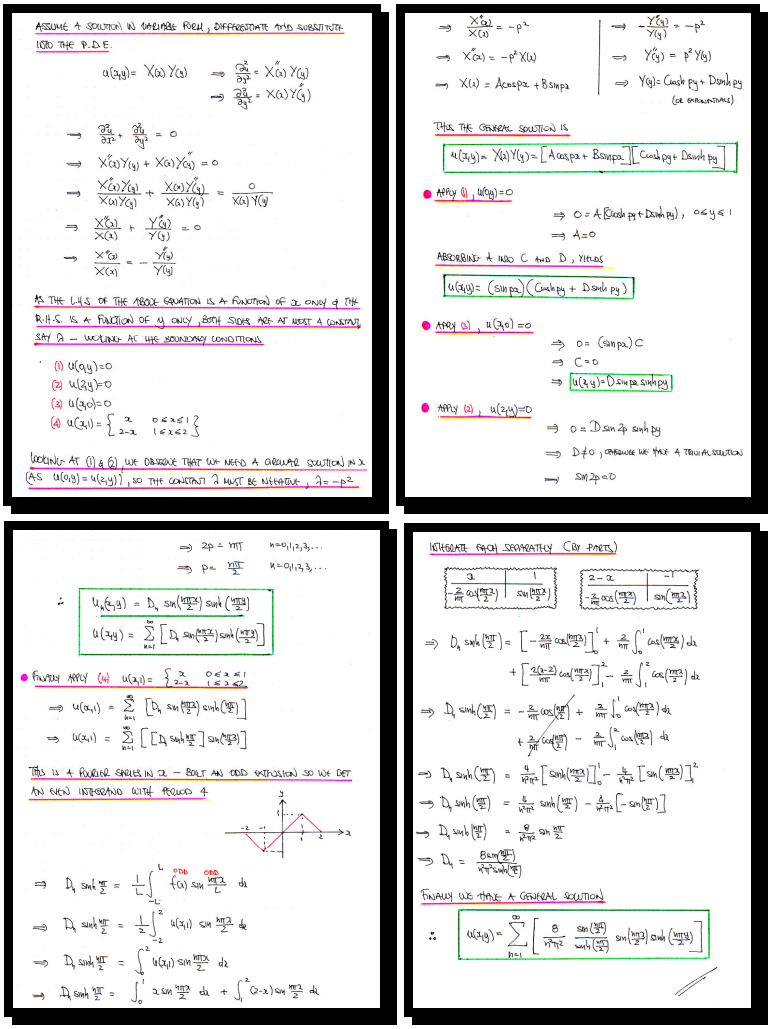
The function $u = u(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le 2, \ 0 \le y \le 1.$$

Determine an expression for $u = u(x, y)$, given further that

$$u(0, y) = 0, \quad u(2, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases}$$

$$\boxed{}, \quad \boxed{u(x, y) = \sum_{n=1}^{\infty} \left[\frac{8 \sin\left(\frac{1}{2} n\pi\right)}{n^2 \pi^2 \sinh\left(\frac{1}{2} n\pi\right)} \sin\left(\frac{1}{2} n\pi x\right) \sinh\left(\frac{1}{2} n\pi y\right) \right]}$$



Question 5

A square plate of unit length has three of its sides kept at temperature $0\ ^\circ\text{C}$, while the fourth side is kept at temperature $50\ ^\circ\text{C}$.

In a steady state the temperature, $\theta(x,y)$ satisfies Laplace's Equation

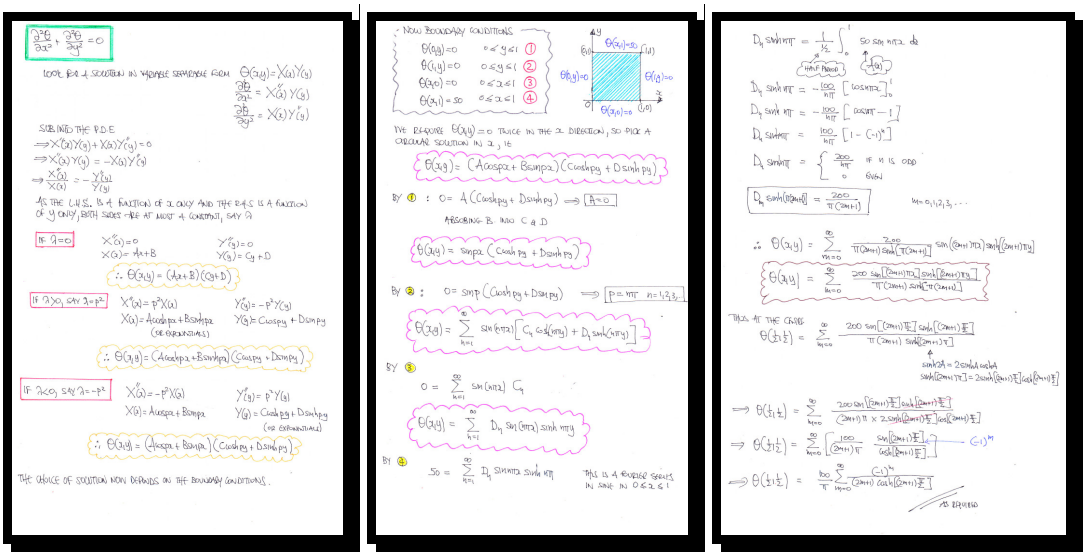
$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0,$$

Solve the equation and hence show that

$$\theta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{100}{\pi}\sum_{n=0}^{\infty}\left\{\frac{(-1)^n}{(2n+1)\cosh\left[(2n+1)\dfrac{\pi}{2}\right]}\right\}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

proof



Question 6

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in a rectangular region where $0 \le x \le 3$ and $0 \le y \le 2$.

It is further given that

- $\varphi(x,0) = 0, \quad x \in [0,3]$
- $\varphi(x,2) = 0, \quad x \in [0,3]$
- $\varphi(0,y) = 0, \quad y \in [0,2]$
- $\varphi(3,y) = \begin{cases} y & y \in [0,1] \\ 2-y & y \in (1,2] \end{cases}$.

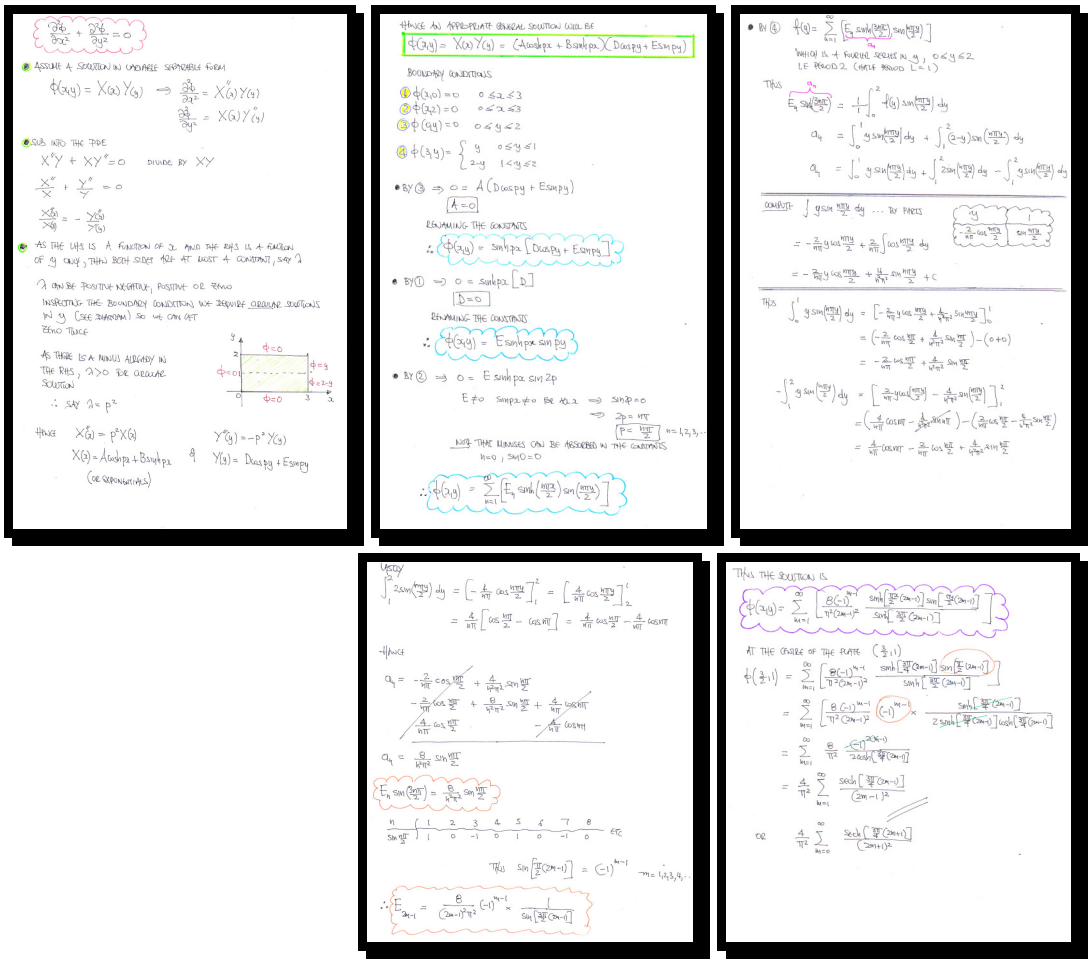
Solve the equation and hence show that

$$\varphi\left(\frac{3}{2},1\right) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\operatorname{sech}\left[\frac{3\pi}{4}(2n-1)\right]}{(2n-1)^2}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

proof

[solution overleaf]



Question 7

The temperature $\theta = \theta(x, y)$ for a **steady** two-dimensional heat flow in the semi-infinite region for which $y \geq 0$ satisfies Laplace's Equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0,$$

subject to the boundary conditions

- $\frac{\partial \theta}{\partial x}(0, y) = 0, \quad y \in [0, \infty)$
- $\frac{\partial \theta}{\partial x}(L, y) = 0, \quad y \in [L, \infty)$
- $\lim_{y \to \infty} \left[\theta(x, y) \right] \leq |M|, \ M \in \mathbb{R}, \quad x \in (-\infty, \infty), \ y \in [x, \infty)$
- $\theta(x, 0) = \frac{x(L - x)}{L^2}, \quad x \in (-\infty, \infty), \quad f(-x) = f(x), \quad f(x) = f(x + 2L)$

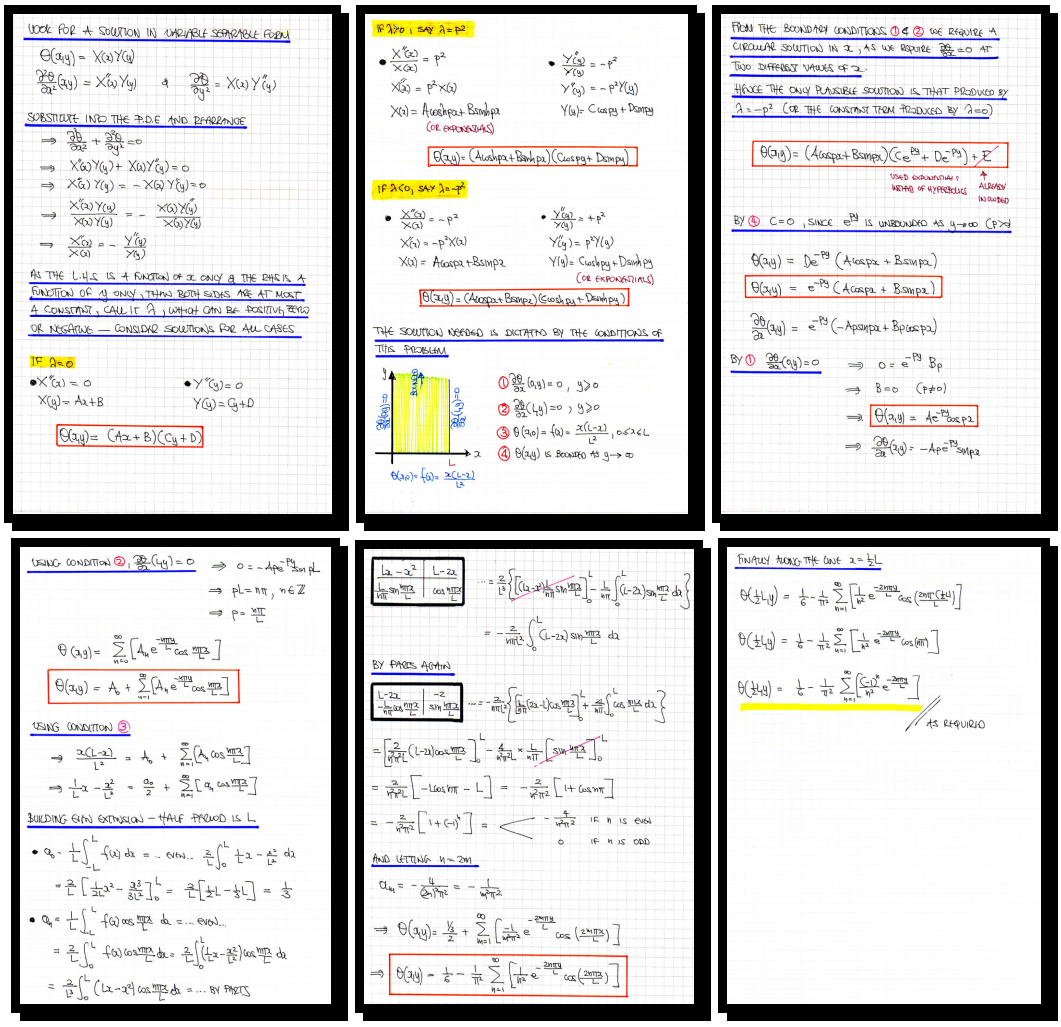
Solve the equation and hence show that

$$\theta\left(\frac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \exp\left[-\frac{2n\pi y}{L}\right]}{n^2}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

, proof

[solution overleaf]



LOOK FOR A SOLUTION IN VARIABLE SEPARABLE FORM

$\theta(x,y) = X(x)Y(y)$

$\frac{\partial^2\theta}{\partial x^2}(x,y) = X''(x)Y(y)$ & $\frac{\partial^2\theta}{\partial y^2} = X(x)Y''(y)$

SUBSTITUTE INTO THE P.D.E AND REARRANGE

$\Rightarrow \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$

$\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$

$\Rightarrow X''(x)Y(y) = -X(x)Y''(y) = 0$

$\Rightarrow \frac{X''(x)Y(y)}{X(x)Y(y)} = -\frac{X(x)Y''(y)}{X(x)Y(y)}$

$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$

AS THE L.H.S IS A FUNCTION OF x ONLY & THE RHS IS A FUNCTION OF y ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, CALL IT λ, WHICH CAN BE POSITIVE, ZERO OR NEGATIVE — CONSIDER SOLUTIONS FOR ALL CASES

IF $\lambda=0$

- $X''(x) = 0$, $X(x) = Ax+B$
- $Y''(y) = 0$, $Y(y) = Cy+D$

$\theta(x,y) = (Ax+B)(Cy+D)$

IF $\lambda>0$, SAY $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$, $X''(x) = p^2X(x)$, $X(x) = Ae^{px} + Be^{-px}$ (OR EXPONENTIALS)
- $\frac{Y''(y)}{Y(y)} = -p^2$, $Y''(y) = -p^2Y(y)$, $Y(y) = C\cos py + D\sin py$

$\theta(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

IF $\lambda<0$, SAY $\lambda = -p^2$

- $\frac{X''(x)}{X(x)} = -p^2$, $X''(x) = -p^2X(x)$, $X(x) = Ae^{px} + Be^{-px}$ (OR EXPONENTIALS)
- $\frac{Y''(y)}{Y(y)} = +p^2$, $Y''(y) = p^2Y(y)$, $Y(y) = Ce^{py} + De^{-py}$ (OR EXPONENTIALS)

$\theta(x,y) = (Ae^{px} + Be^{-px})(Ce^{py} + De^{-py})$

THE SOLUTION NEEDED IS DICTATED BY THE CONDITIONS OF THIS PROBLEM

① $\frac{\partial\theta}{\partial x}(0,y) = 0$, $y \geqslant 0$

② $\frac{\partial\theta}{\partial x}(L,y) = 0$, $y \geqslant 0$

③ $\theta(x,0) = f(x) = \frac{x(L-x)}{L^2}$, $0 \leqslant x \leqslant L$

④ $\theta(x,y)$ is BOUNDED AS $y \to \infty$

FROM THE BOUNDARY CONDITIONS ① & ② WE REQUIRE A CIRCULAR SOLUTION IN x, AS WE REQUIRE $\frac{\partial\theta}{\partial x} = 0$ AT TWO DIFFERENT VALUES OF x.

HENCE THE ONLY PLAUSIBLE SOLUTION IS THAT PRODUCED BY $\lambda = -p^2$ (OR THE CONSTANT TERM PRODUCED BY $\lambda = 0$)

$\theta(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) + E$

USED EXPONENTIAL METHOD OF HYPERBOLIC — ALREADY INCLUDED

BY ④ $C = 0$, SINCE e^{py} IS UNBOUNDED AS $y \to \infty$ ($p>0$)

$\theta(x,y) = De^{-py}(A\cos px + B\sin px)$

$\theta(x,y) = e^{-py}(A\cos px + B\sin px)$

$\frac{\partial\theta}{\partial x}(x,y) = e^{-py}(-Ap\sin px + Bp\cos px)$

BY ① $\frac{\partial\theta}{\partial x}(0,y) = 0$

$\Rightarrow 0 = e^{-py}Bp$

$\Rightarrow B = 0 \quad (p \neq 0)$

$\Rightarrow \theta(x,y) = Ae^{-py}\cos px$

$\Rightarrow \frac{\partial\theta}{\partial x}(x,y) = -Ape^{-py}\sin px$

USING CONDITION ②, $\frac{\partial\theta}{\partial x}(L,y) = 0$

$\Rightarrow 0 = -Ape^{-py}\sin pL$

$\Rightarrow pL = n\pi$, $n \in \mathbb{Z}$

$\Rightarrow p = \frac{n\pi}{L}$

$\theta(x,y) = \sum_{n=0}^{\infty}\left[A_n e^{-\frac{n\pi}{L}y}\cos\frac{n\pi x}{L}\right]$

$\theta(x,y) = A_0 + \sum_{n=1}^{\infty}\left[A_n e^{-\frac{n\pi}{L}y}\cos\frac{n\pi x}{L}\right]$

USING CONDITION ③

$\Rightarrow \frac{x(L-x)}{L^2} = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi x}{L}\right]$

$\Rightarrow \frac{1}{L}x - \frac{x^2}{L^2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}\right]$

BUILDING EVEN EXTENSION — HALF PERIOD IS L

- $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \ldots$ EVEN $\ldots \frac{2}{L}\int_0^L \frac{1}{L}x - \frac{x^2}{L^2}\,dx$
  $= \frac{2}{L}\left[\frac{1}{2L}x^2 - \frac{2}{3L^2}x^3\right]_0^L = \frac{2}{L}\left[\frac{1}{2}L - \frac{1}{3}L\right] = \frac{1}{3}$
- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \ldots$ EVEN $\ldots$
  $= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L\left(\frac{1}{L}x - \frac{x^2}{L^2}\right)\cos\frac{n\pi x}{L}\,dx$
  $= \frac{2}{L^2}\int_0^L (Lx - x^2)\cos\frac{n\pi x}{L}\,dx = \ldots$ BY PARTS

$Lx - x^2$	$L - 2x$
$\frac{L}{n\pi}\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$

$\ldots = \frac{2}{L^2}\left\{\left[(Lx-x^2)\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L - \frac{L}{n\pi}\int_0^L (L-2x)\sin\frac{n\pi x}{L}\,dx\right\}$

$= -\frac{2}{n\pi L}\int_0^L (L-2x)\sin\frac{n\pi x}{L}\,dx$

BY PARTS AGAIN

$L-2x$	-2
$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$

$\ldots = -\frac{2}{n\pi L}\left\{\left[\frac{L}{n\pi}(Lx-x^2)\cos\frac{n\pi x}{L}\right]_0^L + \frac{2L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right\}$

$= \left[\frac{2}{n^2\pi^2}(L-2x)\cos\frac{n\pi x}{L}\right]_0^L - \frac{4}{n^2\pi^2}\times\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L$

$= \frac{2}{n^2\pi^2}\left[-L\cos n\pi - L\right] = -\frac{2}{n^2\pi^2}\left[1+\cos n\pi\right]$

$= -\frac{2}{n^2\pi^2}\left[1+(-1)^n\right] = \begin{cases} -\frac{4}{n^2\pi^2} & \text{IF } n \text{ IS EVEN} \\ 0 & \text{IF } n \text{ IS ODD} \end{cases}$

AND LETTING $n = 2m$

$a_{2m} = -\frac{4}{(2m)^2\pi^2} = -\frac{1}{m^2\pi^2}$

$\Rightarrow \theta(x,y) = \frac{1}{2} + \sum_{m=1}^{\infty}\left[-\frac{1}{m^2\pi^2}e^{-\frac{2m\pi}{L}y}\cos\left(\frac{2m\pi x}{L}\right)\right]$

$\Rightarrow \theta(x,y) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi}{L}y}\cos\left(\frac{2n\pi x}{L}\right)\right]$

FINALLY ALONG THE LINE $x = \frac{1}{2}L$

$\theta\left(\frac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi}{L}y}\cos\left(\frac{2n\pi}{L}\cdot\frac{1}{2}L\right)\right]$

$\theta\left(\frac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi}{L}y}\cos(n\pi)\right]$

$\theta\left(\frac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}e^{-\frac{2n\pi}{L}y}\right]$

AS REQUIRED

LAPLACE'S EQUATION

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}=0, \quad \Phi=\Phi\left(r,\theta\right)$$

Two Dimensional in Plane Polars

Question 1

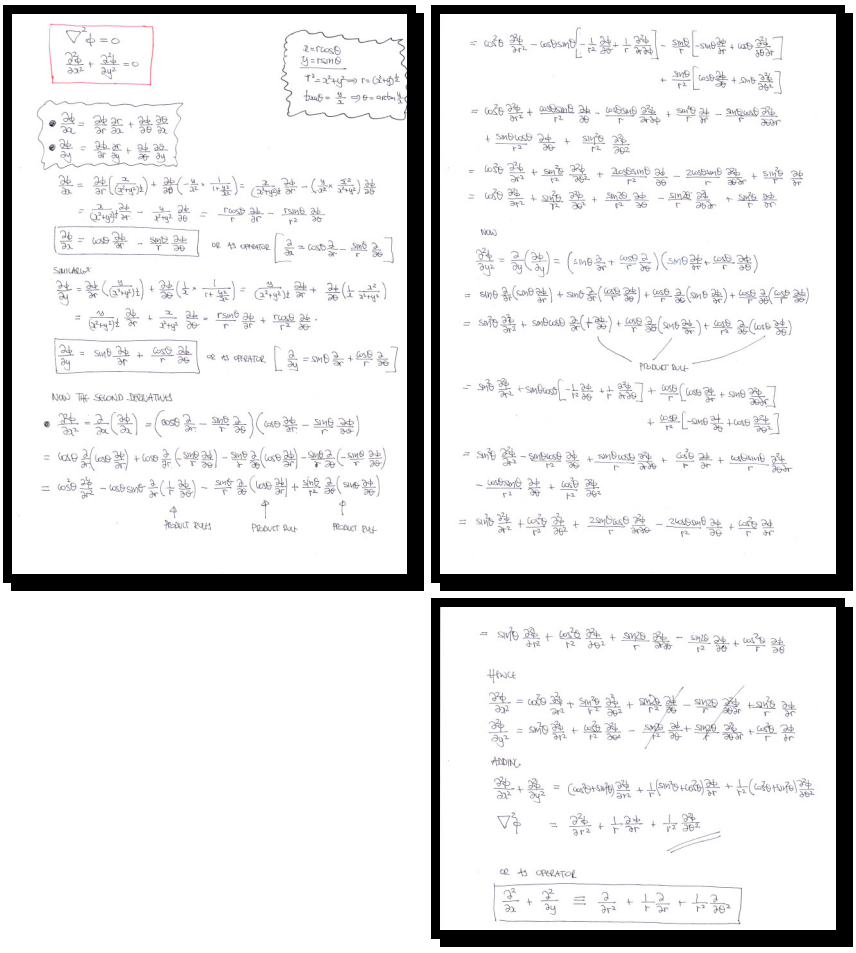
$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0\,.$$

The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2}+\frac{1}{r}\frac{\partial \varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}=0\,.$$

proof



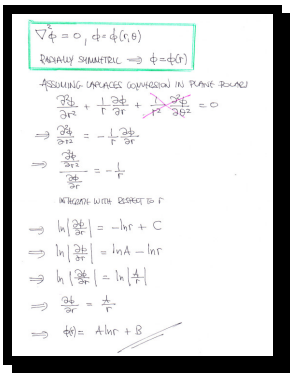
Question 2

In a two dimensional universe the gravitational Φ would satisfy the two dimensional Laplace's equation

$$\nabla^2\Phi = 0.$$

Find a general circularly symmetric solution for Φ.

$$\boxed{\Phi(r) = A\ln r + B}$$



Question 3

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}=0\,.$$

a) Derive the general solution of the above equation, in variable separable form.

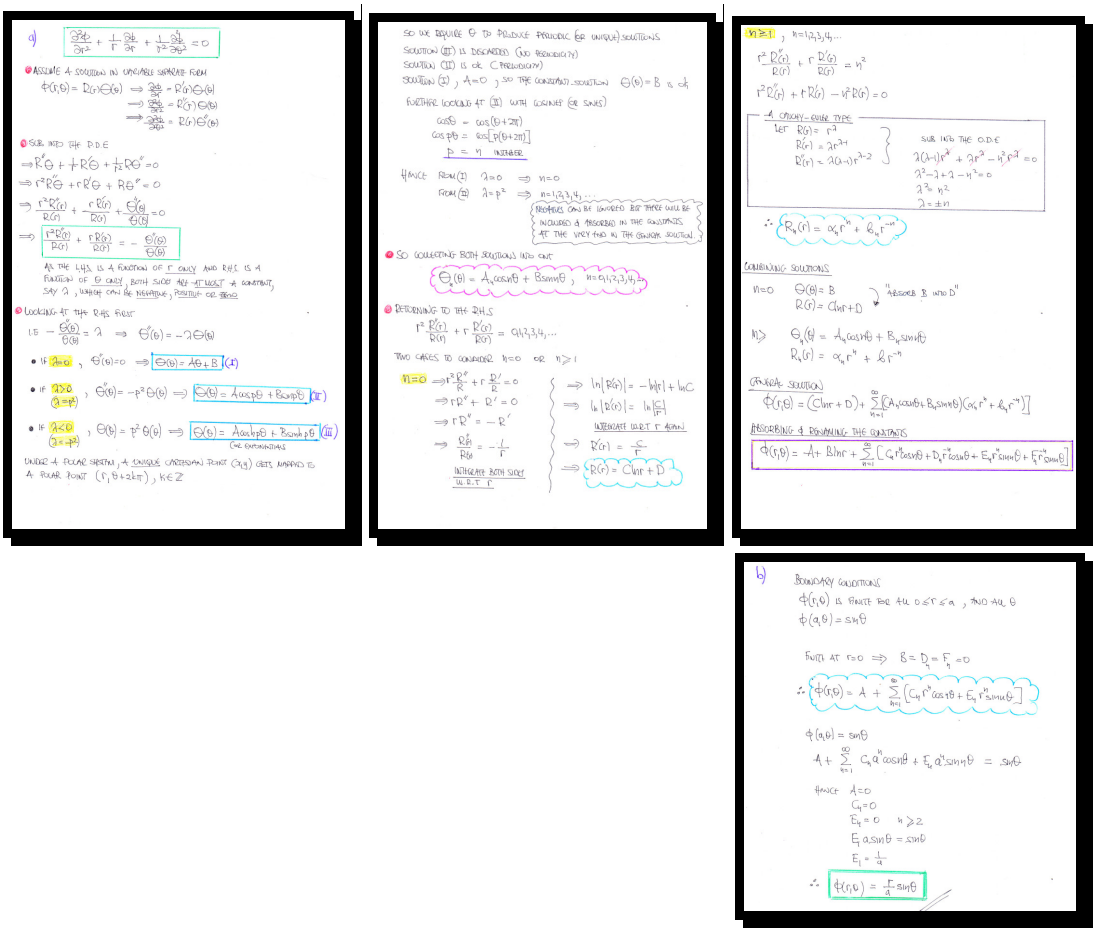
b) Hence find a specific solution subject to the conditions

i. $\Phi(0,\theta)=0$

ii. $\Phi(r,\theta)$ is finite for $0\le r\le a$ and for all θ.

$$\boxed{\Phi(r,\theta)=A+B\ln r+\sum_{n=1}^{\infty}\left[\left(C_n r^n+D_n r^{-n}\right)\cos n\theta+\left(E_n r^n+F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi(r,\theta)=\frac{r}{a}\sin\theta}$$



Question 4

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}=0\,.$$

a) Derive the general solution of the above equation, in variable separable form.

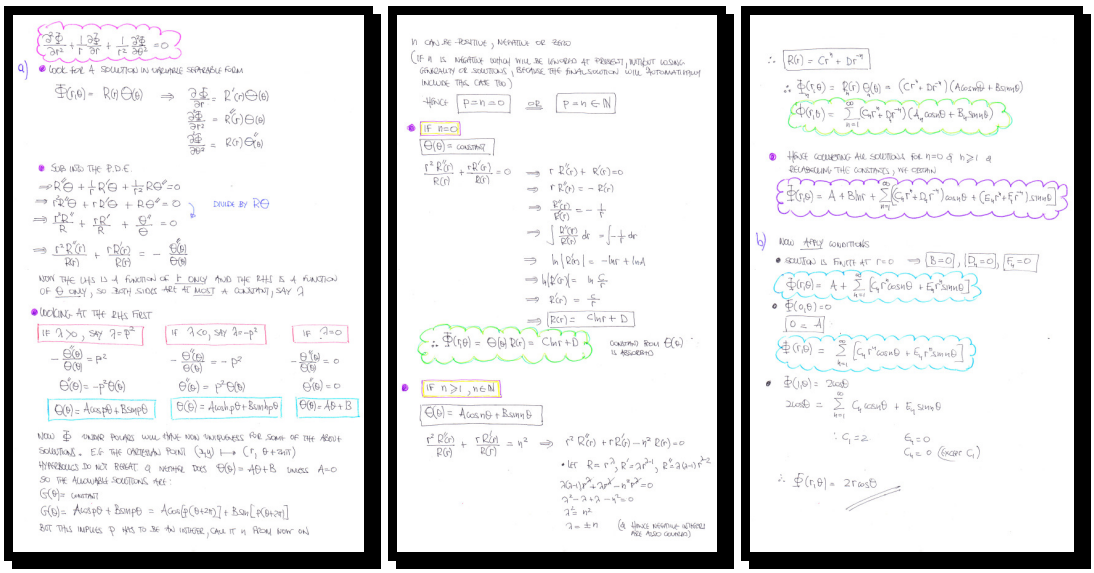
b) Hence find a specific solution subject to the conditions

i. $\Phi(0,\theta)=0$

ii. $\Phi(1,\theta)=2\cos\theta\,.$

$$\boxed{\Phi(r,\theta)=A+B\ln r+\sum_{n=1}^{\infty}\left[\left(C_n r^n+D_n r^{-n}\right)\cos n\theta+\left(E_n r^n+F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi(r,\theta)=2r\cos\theta}$$



Question 5

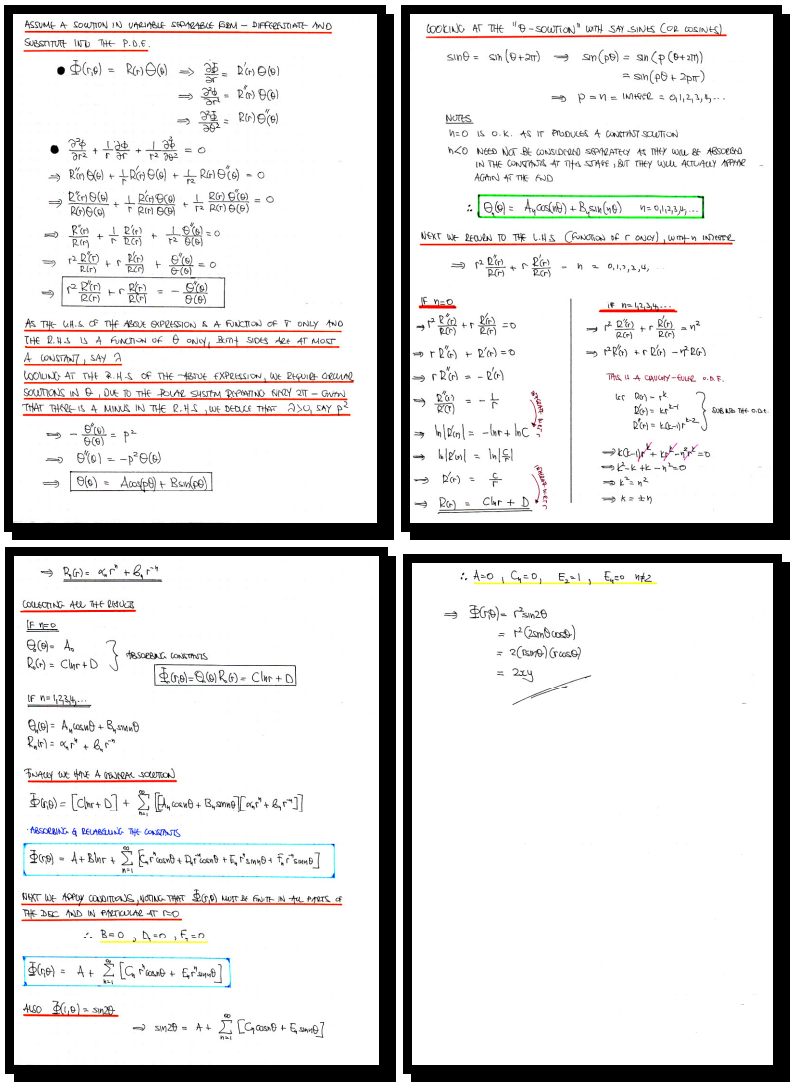
The steady state temperature distribution $\Phi = \Phi(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

Given further that $\Phi(1,\theta) = \sin 2\theta$, determine a simplified expression for $\Phi(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$\boxed{}$, $\boxed{\Phi(r,\theta) = r^2 \sin 2\theta}$



Question 6

The steady state temperature distribution $\Phi = \Phi(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

Given further that

$$2\Phi(1,\theta) + \frac{\partial \Phi}{\partial r}(1,\theta) = 100 - 2\cos 2\theta ,$$

determine a simplified expression for $\Phi(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{\Phi(r,\theta) = 50 - \frac{1}{2}r^2\cos 2\theta}$$



Question 7

The steady state temperature distribution $u = u(r,\theta)$ in a circular thin metal disc of radius a, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

The top half of the circumference of the disc, $0\le\theta\le\pi$, is maintained at $100\,^{\circ}\text{C}$, and the bottom half of the circumference of the disc, $\pi<\theta<2\pi$, is maintained at $0\,^{\circ}\text{C}$.

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$u(r,\theta)=50+\frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin n\theta}{n}\left(\frac{r}{a}\right)^n\right]$$



Question 8

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0 , \ 1 < r < 2$$

$$\nabla^2\Phi_2 = 0 , \ r \geq 2 .$$

It is further given that

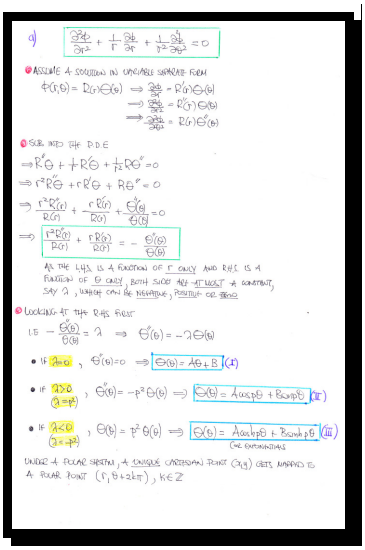
$$\Phi_1(1,\theta) = 0 , \quad \Phi_1(2,\theta) = 1 , \quad \Phi_2(2,\theta) = 1 \quad \text{and} \quad \lim_{r\to\infty}\left[\Phi_2(r,\theta) - r\cos\theta\right] = 1 .$$

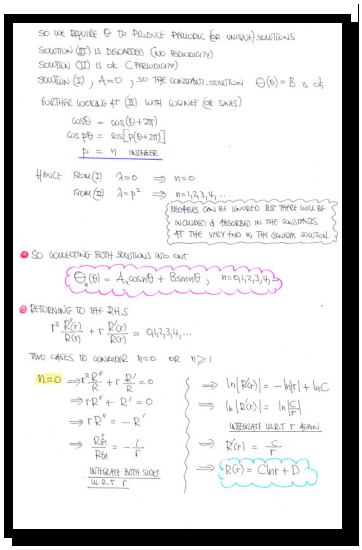
b) Determine expressions for Φ_1 and Φ_2.

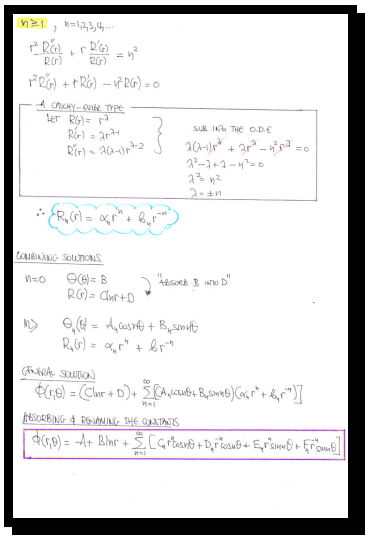
$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

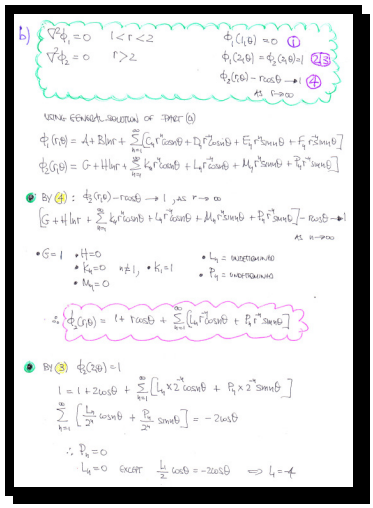
$$\boxed{\Phi_1(r,\theta) = \frac{\ln r}{\ln 2}} , \quad \boxed{\Phi_2(r,\theta) = 1 + r\cos\theta - \frac{4}{r}\cos\theta}$$

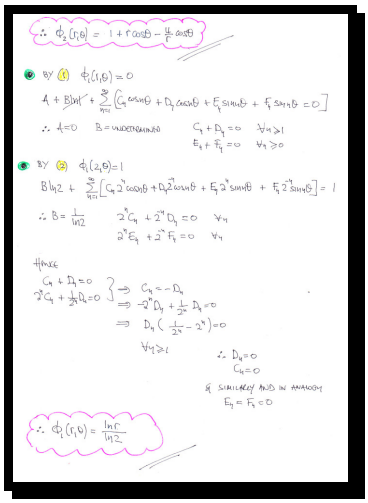
[solution overleaf]











Question 9

The steady state temperature distribution $u = u(r,\theta)$ in a thin metal disc in the shape of a circular sector of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

It is further given that $u(r,0)=u\left(r,\frac{1}{2}\pi\right)=0$ and $\frac{\partial u}{\partial r}(\theta,1)=\theta$.

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta)=\sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}r^{2k}\sin(2k\theta)}{2k^2}\right]}$$



Question 10

The steady state temperature distribution $u=u(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

It is further given that $u(1,\theta)=\begin{cases}\theta & -\frac{1}{2}\pi<\theta<\frac{1}{2}\pi\\ 0 & \text{otherwise}\end{cases}$

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\left[\left[\frac{1}{n}\cos\left(\frac{1}{2}n\pi\right)+\frac{2}{\pi n^2}\sin\left(\frac{1}{2}n\pi\right)\right]r^n\sin n\theta\right]}$$



Question 11

The steady state temperature distribution $u=u(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\,.$$

It is further given that $u(1,\theta)=\begin{cases}\pi-\theta & 0\le\theta\le\pi\\ 0 & \text{otherwise}\end{cases}$

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$u(r,\theta)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{r^n}{n}\left[\sin n\theta+\frac{1-(-1)^n}{\pi n}\cos n\theta\right]\right]$$



Question 12

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0 , \ 0 < r < 1$$

$$\nabla^2\Phi_2 = 0 , \ r \geq 1 .$$

It is further given that

- $\left|\Phi_1(0,\theta)\right| \leq M$, $M \in \mathbb{R}$.
- $\lim\limits_{r\to\infty}\left[\Phi_2(r,\theta) - r\cos\theta\right] = 0$.
- $\Phi_1(1,\theta) = \Phi_2(1,\theta)$.
- $\frac{\partial\Phi_1}{\partial r}(1,\theta) + 3\frac{\partial\Phi_2}{\partial r}(1,\theta) = 0$.

b) Determine expressions for Φ_1 and Φ_2 .

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_1(r,\theta) = 3r\cos\theta} , \ \boxed{\Phi_2(r,\theta) = r\cos\theta + \frac{2}{r}\cos\theta}$$

[solution overleaf]



Question 13

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0 , \ r > 1$$

$$\nabla^2\Phi_2 = 0 , \ 0 < r < 1 .$$

It is further given that

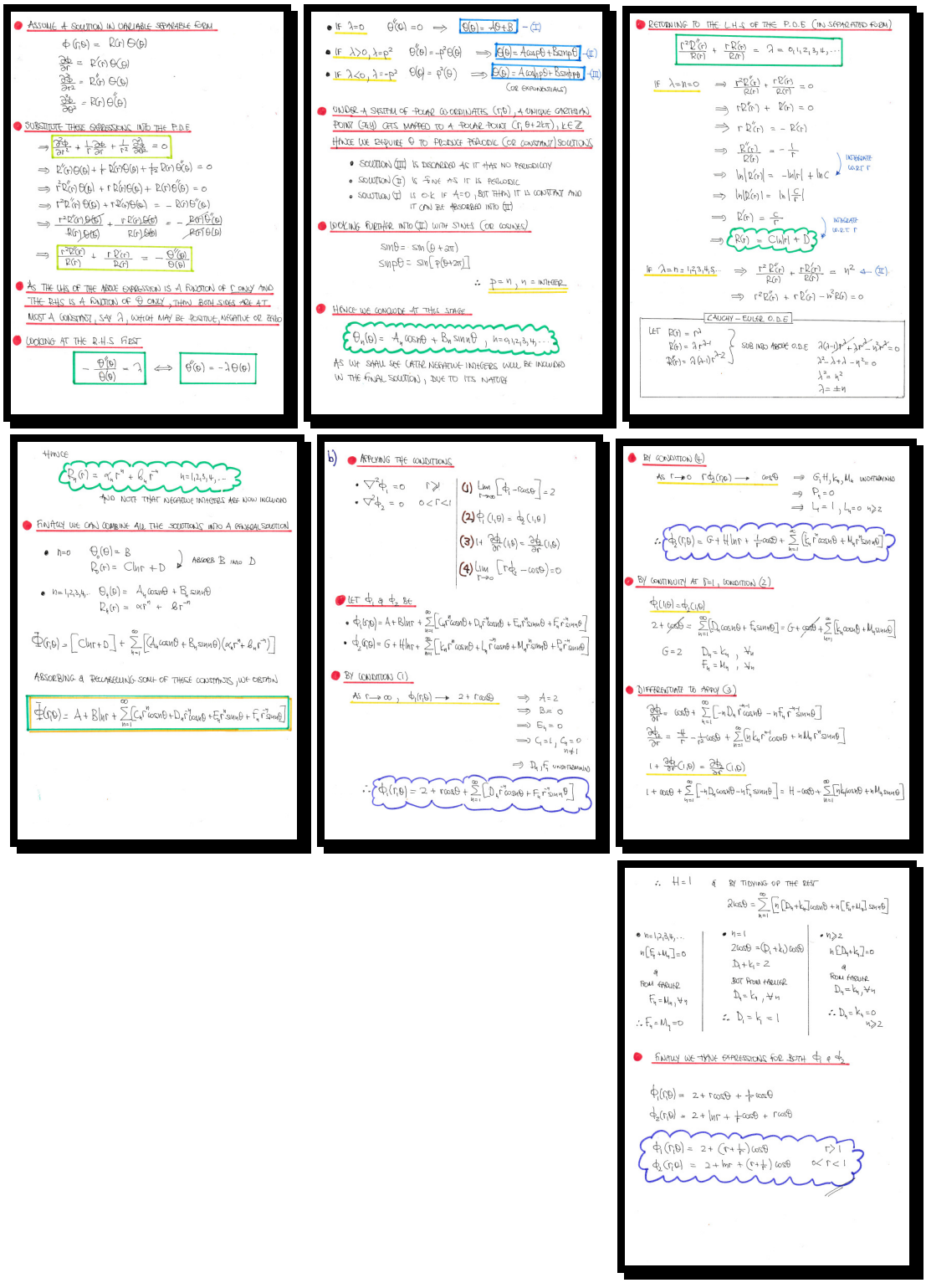
- $\lim\limits_{r\to\infty}\left[\Phi_1(r,\theta) - r\cos\theta\right] = 2 .$
- $\Phi_1(1,\theta) = \Phi_2(1,\theta) .$
- $1 + \dfrac{\partial\Phi_1}{\partial r}(1,\theta) = \dfrac{\partial\Phi_2}{\partial r}(1,\theta) .$
- $\lim\limits_{r\to 0}\left[r\Phi_2(r,\theta) - \cos\theta\right] = 0 .$

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$\boxed{}$, $\boxed{\Phi_1(r,\theta) = 2 + \left(r + \frac{1}{r}\right)\cos\theta}$, $\boxed{\Phi_2(r,\theta) = 2 + \ln r + \left(r + \frac{1}{r}\right)\cos\theta}$

[solution overleaf]


ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM
SUBSTITUTE THESE EXPRESSIONS INTO THE P.D.E
AS THE LHS OF THE ABOVE EXPRESSION IS A FUNCTION OF r ONLY AND THE RHS IS A FUNCTION OF θ ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH MAY BE POSITIVE, NEGATIVE OR ZERO
LOOKING AT THE R.H.S FIRST
IF λ=0
IF λ>0, λ=p²
IF λ<0, λ=-p²
(OR EXPONENTIALS)
UNDER A SYSTEM OF POLAR CO-ORDINATES (r,θ), A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT (r, θ+2kπ), k∈Z
HENCE WE REQUIRE Θ TO PRODUCE PERIODIC (OR CONSTANT) SOLUTIONS
SOLUTION (III) IS DISCARDED AS IT HAS NO PERIODICITY
SOLUTION (II) IS FINE AS IT IS PERIODIC
SOLUTION (I) IS O.K IF A=0, BUT THEN IT IS CONSTANT AND IT CAN BE ABSORBED INTO (II)
LOOKING FURTHER INTO (II) WITH STATES (OR GOGGLES)
HENCE WE CONCLUDE AT THIS STAGE
AS WE SHALL SEE LATER NEGATIVE INTEGERS WILL BE INCLUDED IN THE FINAL SOLUTION, DUE TO ITS NATURE
RETURNING TO THE L.H.S OF THE P.D.E (IN SEPARATED FORM)
INTEGRATE W.R.T r
INTEGRATE W.R.T r
CAUCHY – EULER O.D.E
SUB INTO ABOVE O.D.E
HENCE
AND NOTE THAT NEGATIVE INTEGERS ARE NOW INCLUDED
FINALLY WE CAN COMBINE ALL THE SOLUTIONS INTO A GENERAL SOLUTION
ABSORBING & RELABELLING SOME OF THESE CONSTANTS, WE OBTAIN
APPLYING THE CONDITIONS
LET Φ₁ & Φ₂ BE
BY CONDITION (1)
BY CONDITION (4)
BY CONDITION (2)
DIFFERENTIATE TO APPLY (3)
∴ H=1 & BY TIDYING UP THE REST
FINALLY WE HAVE EXPRESSIONS FOR BOTH Φ₁ & Φ₂

Question 14

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_0 = \Phi_0(r,\theta)$ and $\Phi_1 = \Phi_1(r,\theta)$ satisfy

$$\nabla^2\Phi_0 = 0 , \ r > 1$$

$$\nabla^2\Phi_1 = 0 , \ 0 \le r < 1 .$$

It is further given that

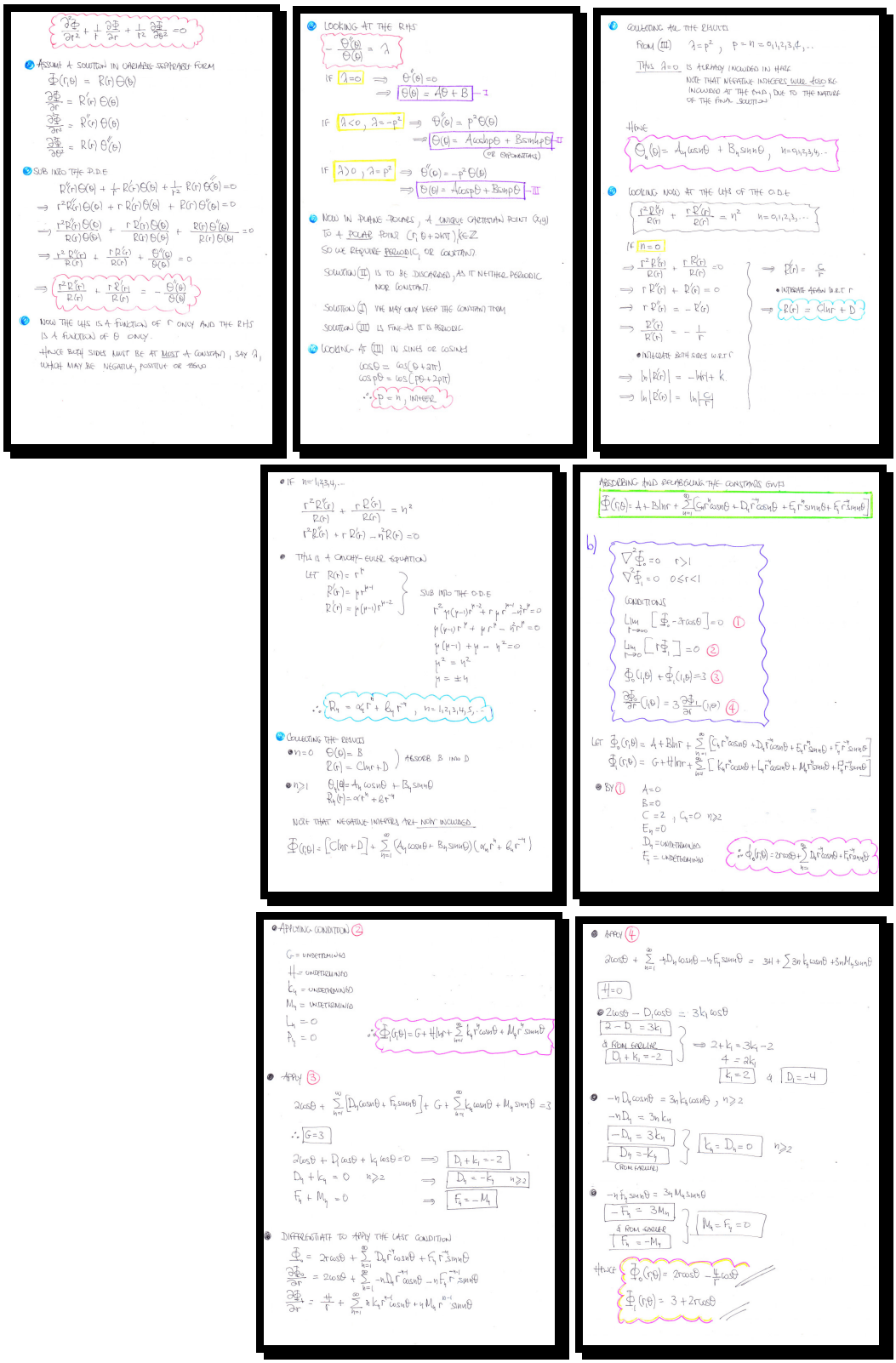
- $\lim\limits_{r\to\infty}\left[\Phi_0(r,\theta) - 2r\cos\theta\right] = 0 .$
- $\lim\limits_{r\to 0}\left[r\Phi_1(r,\theta)\right] = 0$
- $\Phi_0(1,\theta) + \Phi_1(1,\theta) = 3 .$
- $\dfrac{\partial\Phi_0}{\partial r}(1,\theta) = 3\dfrac{\partial\Phi_1}{\partial r}(1,\theta) .$
- .

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_0(r,\theta) = \left(2r - \frac{4}{r}\right)\cos\theta}, \ \boxed{\Phi_1(r,\theta) = 3 + 2r\cos\theta}$$

[solution overleaf]



Question 15

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_0 = \Phi_0(r,\theta)$ and $\Phi_1 = \Phi_1(r,\theta)$ satisfy

$$\nabla^2\Phi_0 = 0 , \ r > 1$$

$$\nabla^2\Phi_1 = 0 , \ 0 \le r \le 1 .$$

It is further given that

- $\lim\limits_{r\to\infty}\left[\Phi_0(r,\theta) - r\cos\theta\right] = 0 .$
- $\Phi_1(0,\theta) \le M , \ M \in \mathbb{R} .$
- $2\Phi_0(1,\theta) + \Phi_1(1,\theta) = 2\pi .$
- $\dfrac{\partial\Phi_0}{\partial r}(1,\theta) = \dfrac{\partial\Phi_1}{\partial r}(1,\theta) .$

b) Determine expressions for Φ_0 and Φ_1.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_0(r,\theta) = \left(r - \frac{3}{r}\right)\cos\theta} , \ \boxed{\Phi_1(r,\theta) = 2\pi + 4r\cos\theta}$$

[solution overleaf]



Question 16

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$, $\Phi_2 = \Phi_2(r,\theta)$ and $\Phi_3 = \Phi_3(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0 , \ r > 2$$

$$\nabla^2\Phi_2 = 0 , \ 1 < r < 2$$

$$\nabla^2\Phi_3 = 0 , \ 0 < r < 1 .$$

It is further given that

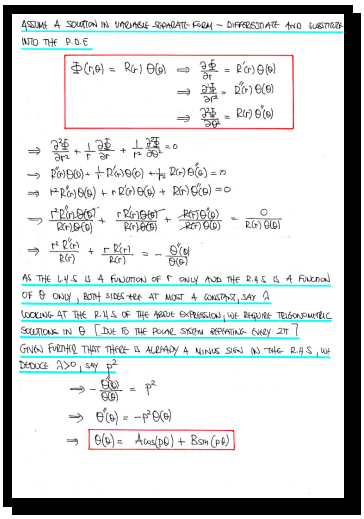
- $\lim\limits_{r\to\infty}\left[\Phi_1(r,\theta) - r\cos\theta\right] = 0 .$
- $\dfrac{\partial\Phi_1}{\partial r}(2,\theta) = \dfrac{\partial\Phi_2}{\partial r}(2,\theta) = 2\cos\theta .$
- $\Phi_2(1,\theta) = \Phi_3(1,\theta) = \cos\theta .$
- $\lim\limits_{r\to 0}\left[r\dfrac{\partial\Phi_3}{\partial r}(r,\theta)\right] = 1 .$

b) Determine expressions for Φ_1, Φ_2 and Φ_3.

$\square$, $\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]}$,

$\boxed{\Phi_1(r,\theta) = r\cos\theta - \frac{4}{r}\cos\theta}$, $\boxed{\Phi_2(r,\theta) = \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta}$, $\boxed{\Phi_3(r,\theta) = \ln r + r\cos\theta}$

[solution overleaf]



ASSUME A SOLUTION IN VARIABLE SEPARATE FORM – DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\Phi(r,\theta) = R(r)\,\Theta(\theta) \quad \Rightarrow \quad \frac{\partial \Phi}{\partial r} = R'(r)\,\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \Phi}{\partial r^2} = R''(r)\,\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\,\Theta''(\theta)$$

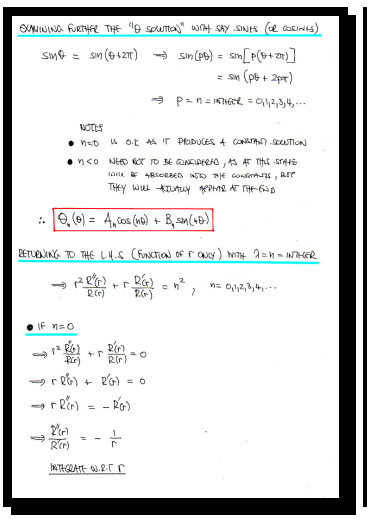
$$\Rightarrow \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$
$$\rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$\Rightarrow r^2 R''(r)\Theta(\theta) + r R'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$
$$\Rightarrow \frac{r^2 R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{r R'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = \frac{0}{R(r)\Theta(\theta)}$$
$$\Rightarrow \frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

AS THE L.H.S IS A FUNCTION OF r ONLY AND THE R.H.S IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

LOOKING AT THE R.H.S OF THE ABOVE EXPRESSION, WE REQUIRE TRIGONOMETRIC SOLUTIONS IN θ [DUE TO THE POLAR SYSTEM REPEATING EVERY 2π]

GIVEN FURTHER THAT THERE IS ALREADY A MINUS SIGN IN THE R.H.S, WE DEDUCE $\lambda > 0$, SAY p^2

$$\Rightarrow -\frac{\Theta''(\theta)}{\Theta(\theta)} = p^2$$
$$\Rightarrow \Theta''(\theta) = -p^2\Theta(\theta)$$
$$\Rightarrow \Theta(\theta) = A\cos(p\theta) + B\sin(p\theta)$$



EXAMINING FURTHER THE "θ SOLUTION" WITH SAY SINES (OR COSINES)

$$\sin\theta = \sin(\theta + 2\pi) \rightarrow \sin(p\theta) = \sin[p(\theta + 2\pi)]$$
$$= \sin(p\theta + 2p\pi)$$
$$\Rightarrow p = n = \text{INTEGER} = 0, 1, 2, 3, 4, \ldots$$

NOTES

- $n = 0$ IS O.K AS IT PRODUCES A CONSTANT SOLUTION
- $n < 0$ NEED NOT TO BE CONSIDERED, AS AT THIS STAGE WOULD BE ABSORBED INTO THE CONSTANTS, BUT THEY WILL ACTUALLY APPEAR AT THE END

$$\therefore \Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$$

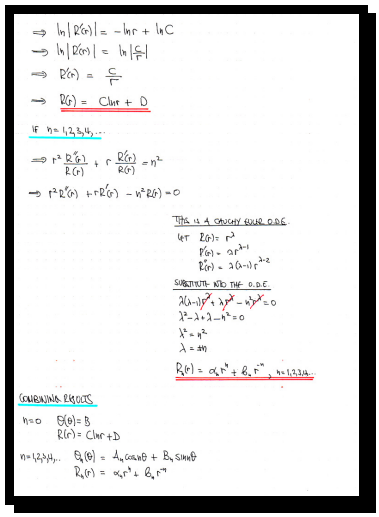
RETURNING TO THE L.H.S (FUNCTION OF r ONLY) WITH $\lambda = n =$ INTEGER

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2, \quad n = 0, 1, 2, 3, 4, \ldots$$

- IF $n = 0$

$$\Rightarrow r^2 R''(r) + r R'(r) = 0$$
$$\Rightarrow r R''(r) + R'(r) = 0$$
$$\Rightarrow r R'(r) = -R'(r)$$
$$\Rightarrow \frac{R'(r)}{R(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r



$$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$
$$\Rightarrow R'(r) = \frac{C}{r}$$
$$\Rightarrow R(r) = C\ln r + D$$

IF $n = 1, 2, 3, 4, \ldots$

$$\Rightarrow r^2 R''(r) + r\frac{R'(r)}{R(r)} = n^2$$
$$\Rightarrow r^2 R''(r) + r R'(r) - n^2 R(r) = 0$$

THIS IS A CAUCHY EULER O.D.E

LET $R(r) = r^\lambda$
$R'(r) = \lambda r^{\lambda-1}$
$R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

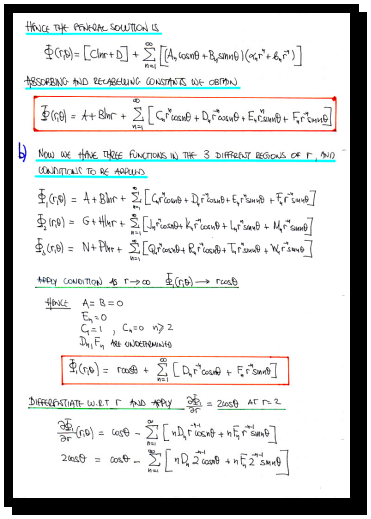
SUBSTITUTE INTO THE O.D.E

$$\lambda(\lambda-1)r^\lambda + \lambda r^\lambda - n^2 r^\lambda = 0$$
$$\lambda^2 - \lambda + \lambda - n^2 = 0$$
$$\lambda^2 = n^2$$
$$\lambda = \pm n$$
$$R(r) = \alpha_n r^n + \beta_n r^{-n}, \quad n = 1, 2, 3, 4, \ldots$$

COMBINING RESULTS

$n = 0$ $\quad \Theta(\theta) = B$
$R(r) = C\ln r + D$

$n = 1, 2, 3, 4, \ldots$ $\quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$
$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$



HENCE THE GENERAL SOLUTION IS

$$\Phi(r,\theta) = (C\ln r + D) + \sum_{n=1}^{\infty}\left[(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING AND RELABELLING CONSTANTS WE OBTAIN

$$\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$$

NOW WE HAVE THREE FUNCTIONS IN THE 3 DIFFERENT REGIONS OF r, AND CONDITIONS TO BE APPLIED

$$\Phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$$
$$\Phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta + K_n r^{-n}\cos n\theta + L_n r^n\sin n\theta + M_n r^{-n}\sin n\theta\right]$$
$$\Phi_3(r,\theta) = N + P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$

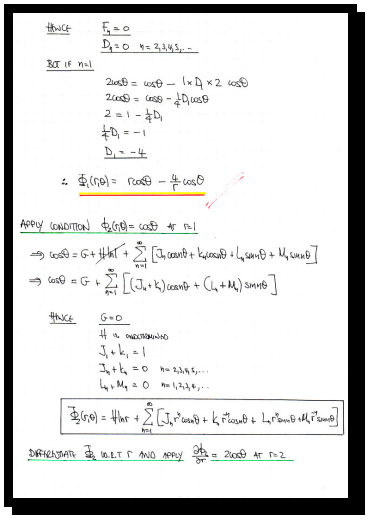
APPLY CONDITION AS $r \to \infty$ $\quad \Phi_1(r,\theta) \longrightarrow r\cos\theta$

HENCE $\quad A = B = 0$
$E_n = 0$
$C_n = 1$; $C_n = 0$ $\forall n \geq 2$
D_n, F_n ARE UNDETERMINED

$$\Phi_1(r,\theta) = r\cos\theta + \sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta\right]$$

DIFFERENTIATE W.R.T r AND APPLY $\frac{\partial \Phi_1}{\partial r} = 2\cos\theta$ AT $r = 2$

$$\frac{\partial \Phi_1}{\partial r}(r,\theta) = \cos\theta - \sum_{n=1}^{\infty}\left[n D_n r^{-n-1}\cos n\theta + n E_n r^{-n-1}\sin n\theta\right]$$
$$2\cos\theta = \cos\theta - \sum_{n=1}^{\infty}\left[n D_n 2^{-n-1}\cos n\theta + n E_n 2^{-n-1}\sin n\theta\right]$$



HENCE $\quad F_n = 0$
$D_n = 0 \quad n = 2, 3, 4, 5, \ldots$

BUT IF $n = 1$

$$2\cos\theta = \cos\theta - 1\times D_1 \times 2\cos\theta$$
$$2\cos\theta = \cos\theta - \tfrac{1}{2}D_1\cos\theta$$
$$2 = 1 - \tfrac{1}{2}D_1$$
$$\tfrac{1}{2}D_1 = -1$$
$$D_1 = -4$$

$$\therefore \Phi_1(r,\theta) = r\cos\theta - \frac{4}{r}\cos\theta$$

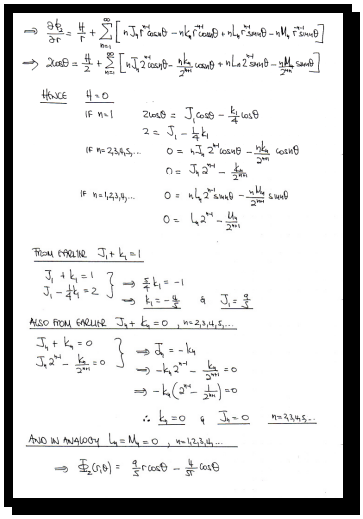
APPLY CONDITION $\Phi_2(r,\theta) = \cos\theta$ AT $r = 1$

$$\Rightarrow \cos\theta = G + H\ln 1 + \sum_{n=1}^{\infty}\left[J_n\cos n\theta + K_n\cos n\theta + L_n\sin n\theta + M_n\sin n\theta\right]$$
$$\Rightarrow \cos\theta = G + \sum_{n=1}^{\infty}\left[(J_n + K_n)\cos n\theta + (L_n + M_n)\sin n\theta\right]$$

HENCE $\quad G = 0$
H IS UNDETERMINED
$J_1 + K_1 = 1$
$J_n + K_n = 0 \quad n = 2, 3, 4, 5, \ldots$
$L_n + M_n = 0 \quad n = 1, 2, 3, 4, \ldots$

$$\Phi_2(r,\theta) = H\ln r + \sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta + K_n r^{-n}\cos n\theta + L_n r^n\sin n\theta + M_n r^{-n}\sin n\theta\right]$$

DIFFERENTIATE Φ_2 W.R.T r AND APPLY $\frac{\partial \Phi_2}{\partial r} = 2\cos\theta$ AT $r = 2$



$$\Rightarrow \frac{\partial \Phi_2}{\partial r} = \frac{H}{r} + \sum_{n=1}^{\infty}\left[n J_n r^{n-1}\cos n\theta - n K_n r^{-n-1}\cos n\theta + n L_n r^{n-1}\sin n\theta - n M_n r^{-n-1}\sin n\theta\right]$$
$$\Rightarrow 2\cos\theta = \frac{H}{2} + \sum_{n=1}^{\infty}\left[n J_n 2^{n-1}\cos n\theta - \frac{n K_n}{2^{n+1}}\cos n\theta + n L_n 2^{n-1}\sin n\theta - \frac{n M_n}{2^{n+1}}\sin n\theta\right]$$

HENCE $\quad H = 0$

IF $n = 1$ $\quad 2\cos\theta = J_1\cos\theta - \frac{K_1}{4}\cos\theta$
$2 = J_1 - \frac{1}{4}K_1$

IF $n = 2, 3, 4, 5, \ldots$ $\quad 0 = n J_n 2^{n-1}\cos n\theta - \frac{n K_n}{2^{n+1}}\cos n\theta$
$0 = J_n 2^{n-1} - \frac{K_n}{2^{n+1}}$

IF $n = 1, 2, 3, 4, \ldots$ $\quad 0 = n L_n 2^{n-1}\sin n\theta - \frac{n M_n}{2^{n+1}}\sin n\theta$
$0 = L_n 2^{n-1} - \frac{M_n}{2^{n+1}}$

FROM EARLIER $J_1 + K_1 = 1$

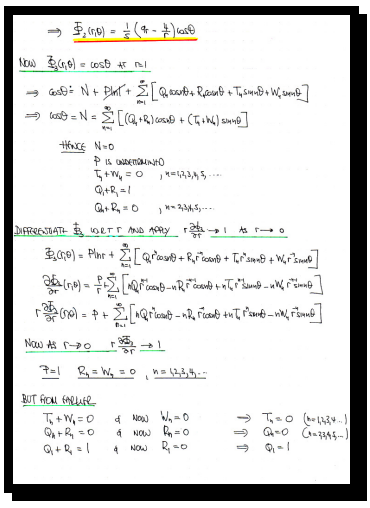
$$\left.\begin{aligned} J_1 + K_1 &= 1 \\ J_1 - \tfrac{1}{4}K_1 &= 2 \end{aligned}\right\} \Rightarrow \tfrac{5}{4}K_1 = -1 \Rightarrow K_1 = -\tfrac{4}{5} \text{ & } J_1 = \tfrac{9}{5}$$

ALSO FROM EARLIER $J_n + K_n = 0$, $n = 2, 3, 4, 5, \ldots$

$$\left.\begin{aligned} J_n + K_n &= 0 \\ J_n 2^{n-1} - \frac{K_n}{2^{n+1}} &= 0 \end{aligned}\right\} \Rightarrow J_n = -K_n$$
$$\Rightarrow -K_n 2^{n-1} - \frac{K_n}{2^{n+1}} = 0$$
$$\Rightarrow -K_n\left(2^{n-1} - \frac{1}{2^{n+1}}\right) = 0$$
$$\therefore K_n = 0 \text{ & } J_n = 0 \qquad n = 2, 3, 4, 5, \ldots$$

AND IN ANALOGY $L_n = M_n = 0$, $n = 1, 2, 3, 4, \ldots$

$$\Rightarrow \Phi_2(r,\theta) = \frac{9}{5}r\cos\theta - \frac{4}{5r}\cos\theta$$



$$\Rightarrow \Phi_3(r,\theta) = \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta$$

NOW $\Phi_3(r,\theta) = \cos\theta$ AT $r = 1$

$$\Rightarrow \cos\theta = N + P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$
$$\Rightarrow \cos\theta = N + \sum_{n=1}^{\infty}\left[(Q_n + R_n)\cos n\theta + (T_n + W_n)\sin n\theta\right]$$

HENCE $\quad N = 0$
P IS UNDETERMINED
$T_n + W_n = 0 \quad , n = 1, 2, 3, 4, 5, \ldots$
$Q_1 + R_1 = 1$
$Q_n + R_n = 0 \quad , n = 2, 3, 4, 5, \ldots$

DIFFERENTIATE Φ_3 W.R.T r AND APPLY $r\frac{\partial \Phi_3}{\partial r} \to 1$ AS $r \to 0$

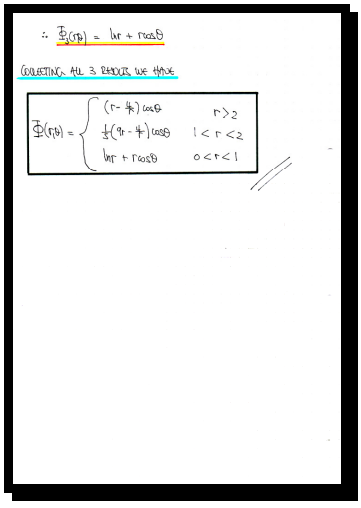
$$\Phi_3(r,\theta) = P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$
$$\frac{\partial \Phi_3}{\partial r}(r,\theta) = \frac{P}{r} + \sum_{n=1}^{\infty}\left[n Q_n r^{n-1}\cos n\theta - n R_n r^{-n-1}\cos n\theta + n T_n r^{n-1}\sin n\theta - n W_n r^{-n-1}\sin n\theta\right]$$
$$r\frac{\partial \Phi_3}{\partial r}(r,\theta) = P + \sum_{n=1}^{\infty}\left[n Q_n r^n\cos n\theta - n R_n r^{-n}\cos n\theta + n T_n r^n\sin n\theta - n W_n r^{-n}\sin n\theta\right]$$

NOW AS $r \to 0$ $\quad r\frac{\partial \Phi_3}{\partial r} \to 1$

$P = 1$ $\quad R_n = W_n = 0$, $n = 1, 2, 3, 4, \ldots$

BUT FROM EARLIER

$T_n + W_n = 0$ & NOW $W_n = 0$ $\Rightarrow T_n = 0$ $(n = 1, 2, 3, 4 \ldots)$
$Q_n + R_n = 0$ & NOW $R_n = 0$ $\Rightarrow Q_n = 0$ $(n = 2, 3, 4, 5 \ldots)$
$Q_1 + R_1 = 1$ & NOW $R_1 = 0$ $\Rightarrow Q_1 = 1$



$$\therefore \Phi_3(r,\theta) = \ln r + r\cos\theta$$

COLLECTING ALL 3 RESULTS WE HAVE

$$\Phi(r,\theta) = \begin{cases} \left(r - \frac{4}{r}\right)\cos\theta & r > 2 \\ \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta & 1 < r < 2 \\ \ln r + r\cos\theta & 0 < r < 1 \end{cases}$$

LAPLACE'S EQUATION

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}+\frac{\partial^2 \Phi}{\partial z^2}=0, \quad \Phi=\Phi\left(r,\theta,z\right)$$

Three Dimensional in Cylindrical Polars

Question 1

The potential function $V = V(r, \theta, z)$ satisfies Laplace's equation in cylindrical polar coordinates, shown below.

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 .$$

Use separation of variables to show that the radial part of the general solution of Laplace's equation in cylindrical polar coordinates, satisfies Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 , \quad n = 0, 1, 2, 3, \ldots$$

proof

