HEAT EQUATION

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$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad \theta = \theta(x,t)$$

One Dimensional

Question 1

A thin rod of length 2 m has temperature $z = 20$ °C throughout its length.

At time $t = 0$, the temperature z is suddenly dropped to $z = 0$ °C at both its ends at $x = 0$, and at $x = 2$.

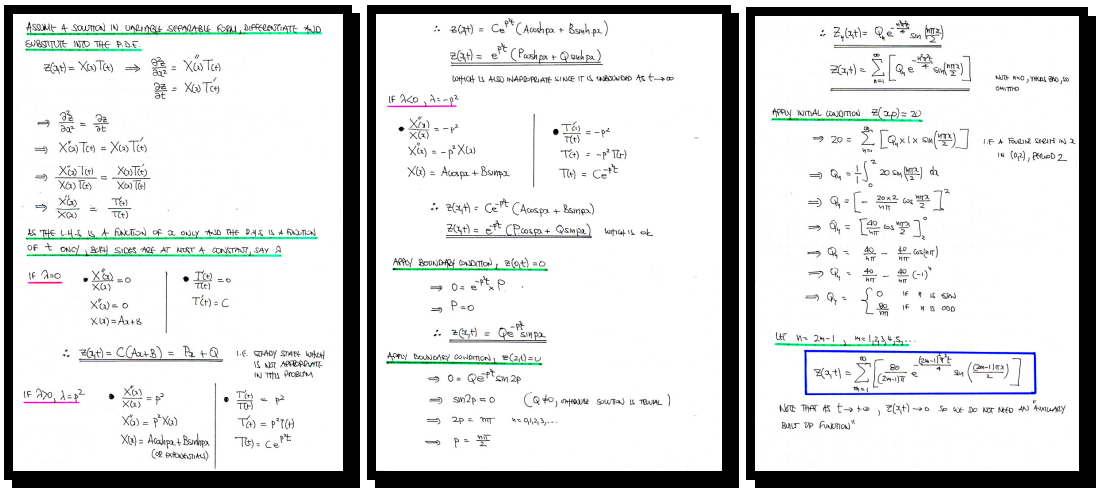
The temperature distribution along the rod $z(x,t)$, satisfies the standard heat equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}, \quad 0 \le x \le 2, \quad t \ge 0.$$

Assuming the rod is insulated along its length, determine an expression for $z(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{}, \quad \boxed{z(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{80}{\pi(2n-1)} \exp\left[-\frac{\pi^2 (2n-1)^2 t}{4} \right] \sin\left[\frac{(2n-1)\pi x}{2} \right] \right\}}$$



Question 2

At time $t<0$, a long thin rod of length l has temperature distribution $\theta(x)$ given by

$$\theta(x)=xl-x^2 .$$

At time $t=0$ the temperature is suddenly dropped to 0 °C at both ends of the rod, and maintained at 0 °C for $t\geq 0$.

The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t},\quad 0\leq x\leq l,\quad t\geq 0,$$

where α is a positive constant.

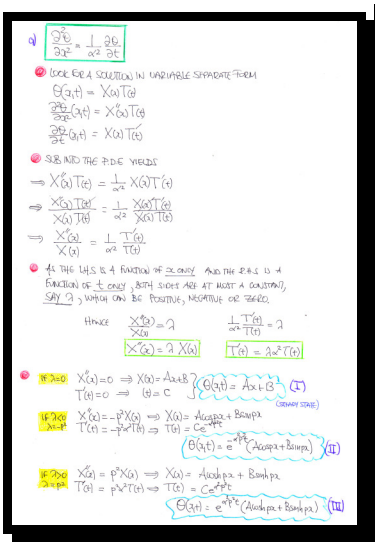
Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$ and hence show that

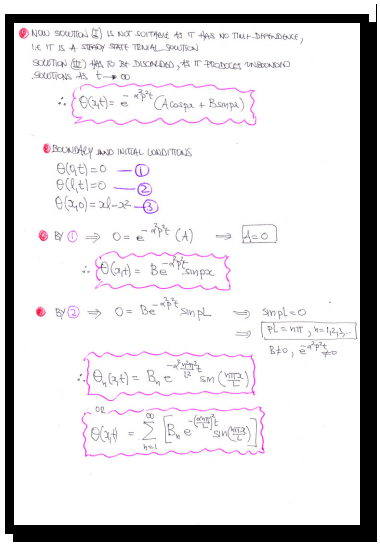
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32}.$$

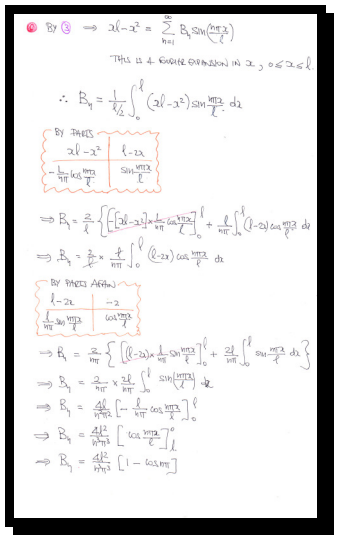
[*You must derive the standard solution of the heat equation in variable separate form*]

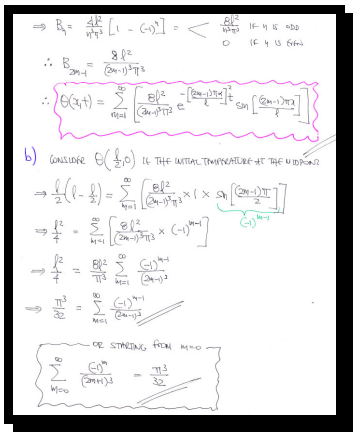
$$\boxed{\theta(x,t)=\sum_{n=1}^{\infty}\left\{\frac{8l^2}{(2n-1)^3\pi^3}\exp\left[-\frac{\alpha^2\pi^2(2n-1)^2t}{l^2}\right]\sin\left[\frac{(2n-1)\pi x}{l}\right]\right\}}$$

[solution overleaf]









Question 3

The temperature distribution $\theta(x,t)$ along a thin bar of length 2 m satisfies the partial differential equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{9}\frac{\partial\theta}{\partial t},\quad 0\le x\le 2,\quad t\ge 0.$$

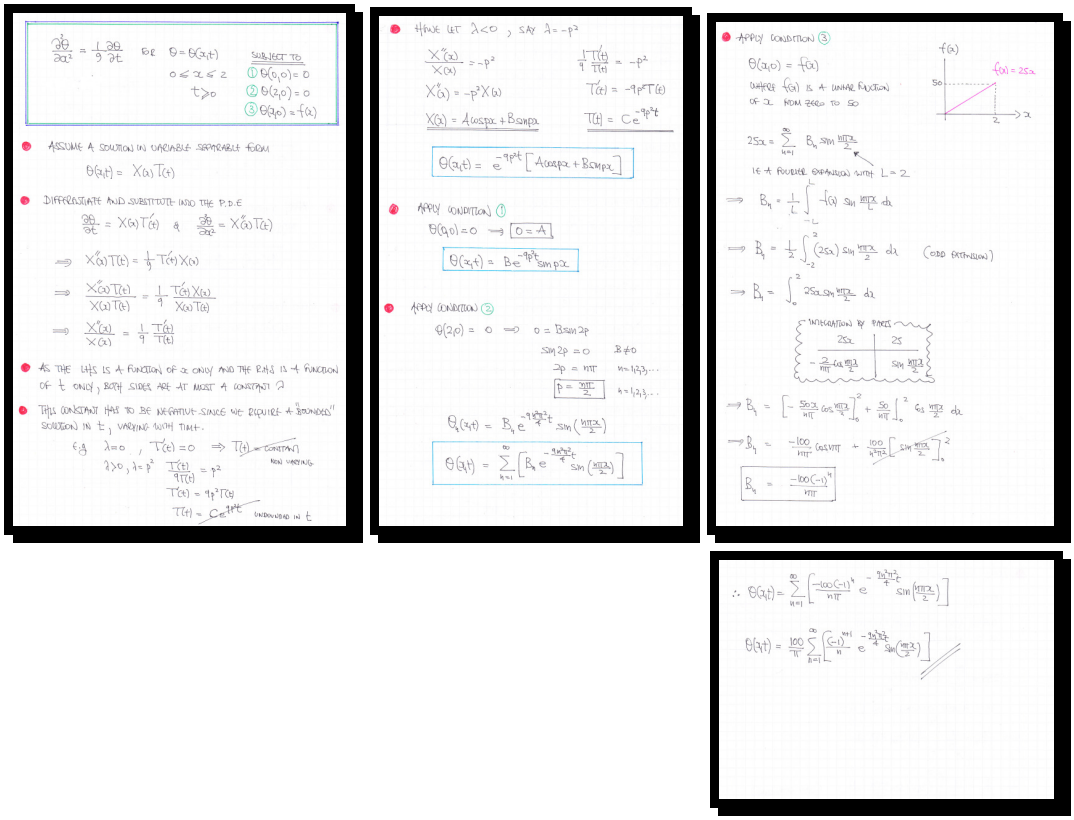
Initially the bar has a linear temperature distribution, with temperature 0 °C at one end of the bar where $x=0$ m, and temperature 50 °C at the other end where $x=2$ m.

At time $t=0$ the temperature is suddenly dropped to 0 °C at both ends of the rod, and maintained at 0 °C for $t\ge 0$.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$ and hence show that

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\theta(x,t)=\frac{100}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n}\exp\left[-\frac{9n^2\pi^2t}{4}\right]\sin\left[\frac{n\pi x}{2}\right]\right\}$$



Question 4

The temperature $\Theta(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 \Theta}{\partial x^2} = 4\frac{\partial \Theta}{\partial t},$$

where x is a spatial coordinate and t is time, with $t \geq 0$.

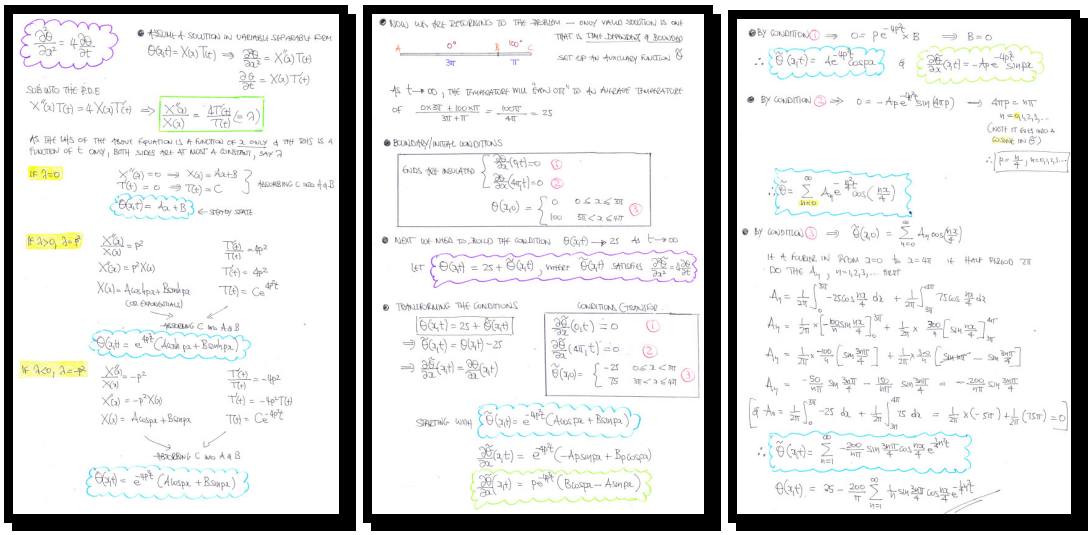
For $t < 0$, two thin rods, of lengths 3π and π, have temperatures $0\,^{\circ}\text{C}$ and $100\,^{\circ}\text{C}$, respectively. At time $t = 0$ the two rods are joined end to end into a single rod of length 4π.

The rods are made of the same material, have perfect thermal contact and are insulated along their length.

Determine an expression for $\Theta(x,t)$, $t \geq 0$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{\Theta(x,t) = 25 - \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\mathrm{e}^{-\frac{1}{4}n^2 t}\sin\left(\frac{3}{4}n\pi\right)\cos\left(\frac{1}{4}nx\right)\right]}$$



Question 5

Solve the heat equation for $u = u(x,t)$

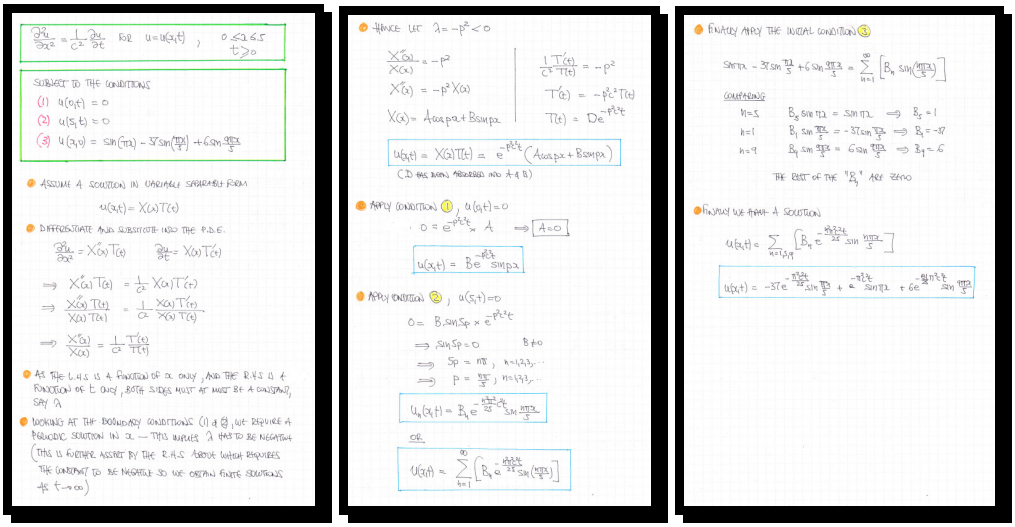
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}, \quad 0 \le x \le 5, \quad t \ge 0,$$

subject to the conditions

$$u(0,t) = 0, \qquad u(5,t) = 0 \qquad \text{and} \qquad u(x,0) = \sin \pi x - 37 \sin\left(\frac{1}{5}\pi x\right) + 6\sin\left(\frac{9}{5}\pi x\right).$$

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{u(x,t) = -37\,\mathrm{e}^{-\frac{1}{25}\pi^2 c^2 t}\sin\left(\frac{1}{5}\pi x\right) + \mathrm{e}^{-\pi^2 c^2 t}\sin(\pi x) + 6\,\mathrm{e}^{-\frac{81}{25}\pi^2 c^2 t}\sin\left(\frac{9}{5}\pi x\right)}$$



Question 6

A long thin rod of length L has temperature $\theta = 0$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_1 at both ends of the rod, at $x = 0$ and at $x = L$.

Both ends of the rod are maintained at temperature T_1 for $t \geq 0$.

The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

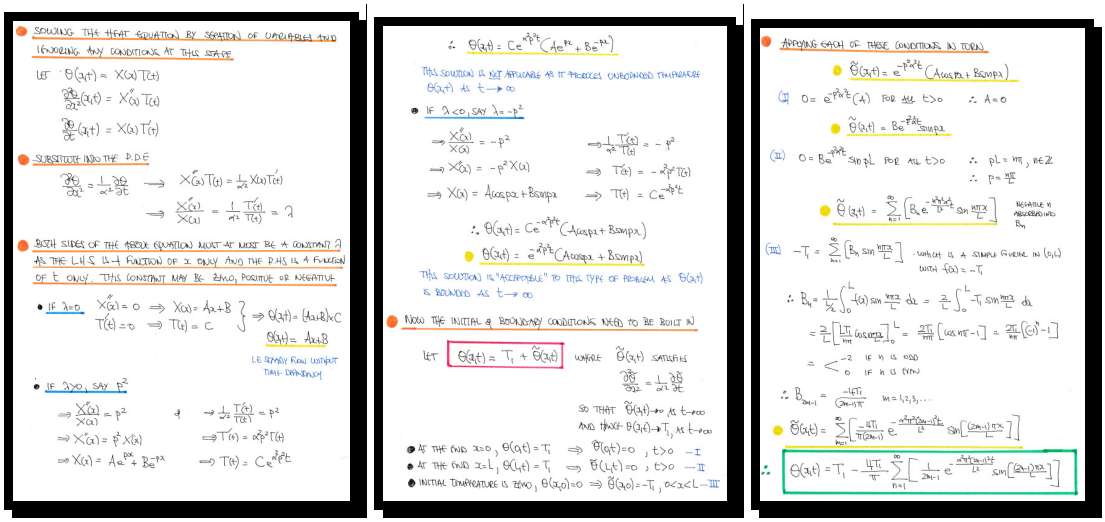
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\theta(x,t) = T_1\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)}\exp\left[-\frac{\alpha^2\pi^2(2n-1)^2 t}{L^2}\right]\sin\left[\frac{(2n-1)\pi x}{L}\right]\right\}\right],$$



Question 7

A long thin rod of length L has temperature $\theta = T_1$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_2 at one of its ends at $x = 0$, and is maintained at T_2 for $t \geq 0$.

The temperature distribution along the rod $\theta(x,t)$, satisfies the standard heat equation

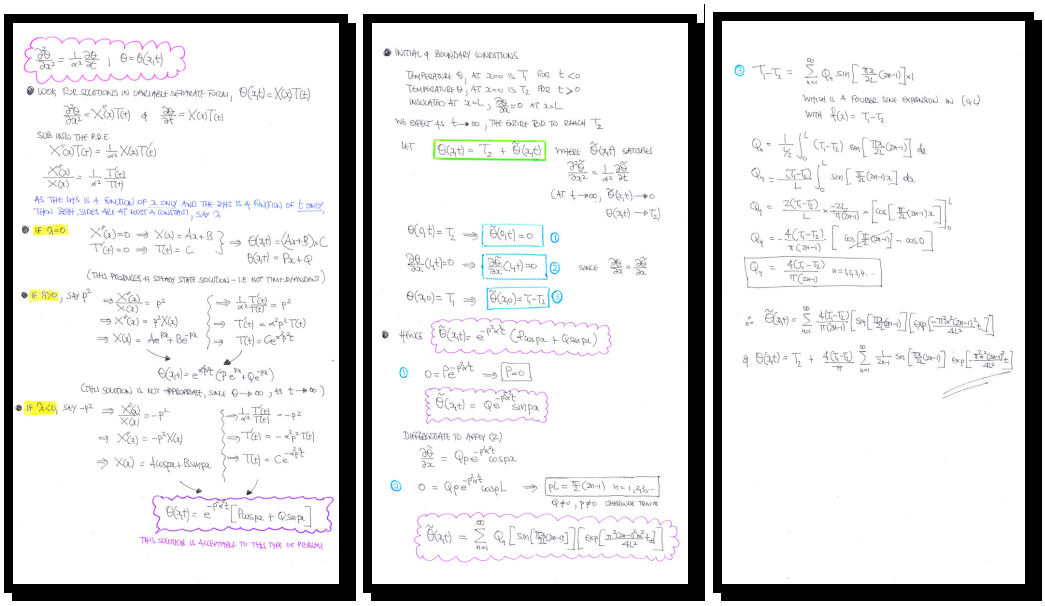
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\theta(x,t) = T_2 + \frac{4(T_1 - T2)}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)}\exp\left[-\frac{\alpha^2\pi^2(2n-1)^2 t}{4L^2}\right]\sin\left[\frac{(2n-1)\pi x}{2L}\right]\right\},$$



Question 8

The temperature $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{9}\frac{\partial u}{\partial t}\,,\ t\geq 0\,,\ 0\leq x\leq 2$$

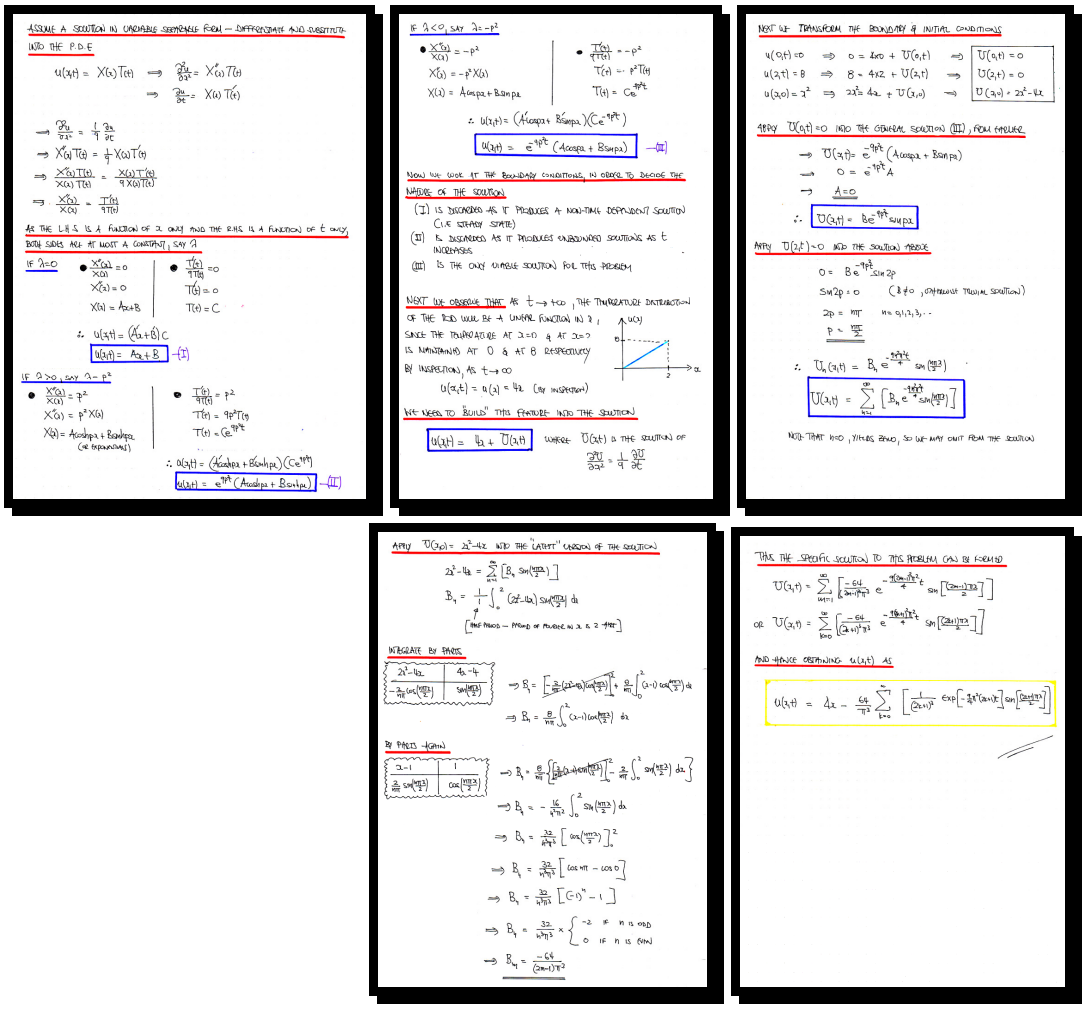
where x is a spatial coordinate and t is time.

It is further given that

$$u(0,t)=0\,,\qquad u(2,t)=8\,,\qquad u(x,0)=2x^2$$

Determine an expression for $u(x,t)$.

$\square$, $u(x,t)=4x-\displaystyle\sum_{k=0}^{\infty}\left[\left[\frac{64}{(2k+1)^3\pi^3}\right]\exp\left[-\frac{9(2k+1)^2t}{4}\right]\sin\left[\frac{(2k+1)\pi x}{2}\right]\right]$



Question 9

The temperature $u(x,t)$ in a thin rod of length π satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial u}{\partial t},\quad 0\le x\le \pi,\quad t\ge 0,$$

where c is a positive constant.

The initial temperature distribution of the rod is

$$u(x,0)=\frac{1}{2}\cos(4x),\ 0\le x\le \pi.$$

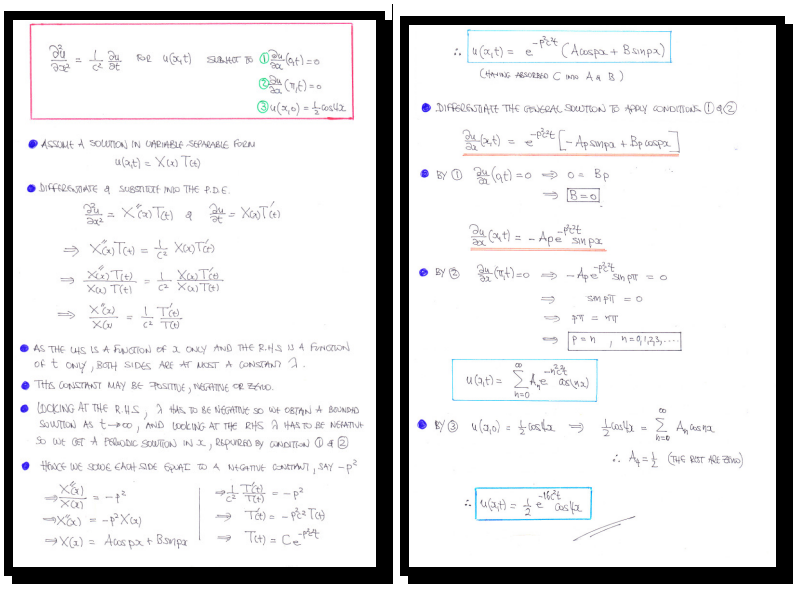
For $t\ge 0$, heat is allowed to flow freely along the rod, with the rod including its endpoints insulated.

Show that

$$u(x,t)=\frac{1}{2}\mathrm{e}^{-16c^2t}\cos 4x.$$

[*You must derive the standard solution of the heat equation in variable separate form*]

proof



Question 10

The temperature $\theta(x,t)$ in a thin rod of length L satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t},\ \ 0\le x\le L,\quad t\ge 0.$$

The initial temperature distribution is

$$\theta(x,0)=\begin{cases}\dfrac{\theta_0 x}{L} & 0\le x<\dfrac{L}{2}\\[2ex] 0 & \dfrac{L}{2}<x\le L\end{cases}$$

where θ_0 is a constant.

The endpoints of the rod are maintained at zero temperature for $t\ge 0$.

a) Assuming the rod is insulated along its length, find an expression for $\theta(x,t)$.

b) By considering the initial temperature at $x=\dfrac{L}{2}$, show that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.$$

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{\theta(x,t)=\frac{\theta_0}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\left[2\sin\left(\frac{n\pi}{2}\right)-n\pi\cos\left(\frac{n\pi}{2}\right)\right]\exp\left[-\frac{n^2\pi^2 t}{L^2}\right]\sin\left[\frac{n\pi x}{L}\right]\right\}}$$

[solution overleaf]



Question 11

The temperature $\theta(x,t)$ in a long thin rod of length L satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t},\ \ 0\le x\le L,\quad t\ge 0,$$

where α is a positive constant.

The initial temperature distribution of the rod is

$$\theta(x,0)=\sin\left(\frac{\pi x}{L}\right),\ 0\le x\le L.$$

For $t\ge 0$, heat is allowed to flow freely with the endpoints of the rod insulated.

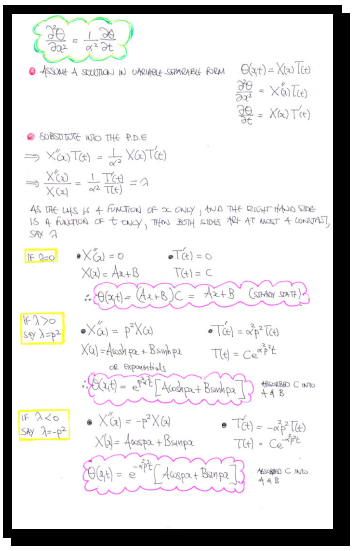
Show that

$$\theta(x,t)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\exp\left(-\frac{4\alpha^2n^2\pi^2t}{L^2}\right)\cos\left(\frac{2m\pi x}{L}\right)}{4n^2-1}\right].$$

[*You must derive the standard solution of the heat equation in variable separate form*]

proof

[solution overleaf]



$\dfrac{\partial^2\theta}{\partial x^2}=\dfrac{1}{\alpha^2}\dfrac{\partial\theta}{\partial t}$

ASSUME A SOLUTION IN VARIABLE-SEPARABLE FORM $\quad \theta(x,t)=X(x)T(t)$

$\dfrac{\partial^2\theta}{\partial x^2}=X''(x)T(t)$

$\dfrac{\partial\theta}{\partial t}=X(x)T'(t)$

SUBSTITUTE INTO THE P.D.E

$\Rightarrow X''(x)T(t)=\dfrac{1}{\alpha^2}X(x)T'(t)$

$\Rightarrow \dfrac{X''(x)}{X(x)}=\dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)}=\lambda$

AS THE LHS IS A FUNCTION OF x ONLY, AND THE RIGHT HAND SIDE IS A FUNCTION OF t ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda=0$

$X''(x)=0 \qquad T'(t)=0$

$X(x)=Ax+B \qquad T(t)=C$

$\therefore \theta(x,t)=(Ax+B)C=Px+B$ (SUMMARY SOLN)

IF $\lambda>0$ SAY $\lambda=p^2$

$X''(x)=p^2X(x) \qquad T'(t)=\alpha^2p^2T(t)$

$X(x)=A\cosh px+B\sinh px \qquad T(t)=Ce^{\alpha^2p^2t}$

OR EXPONENTIALS

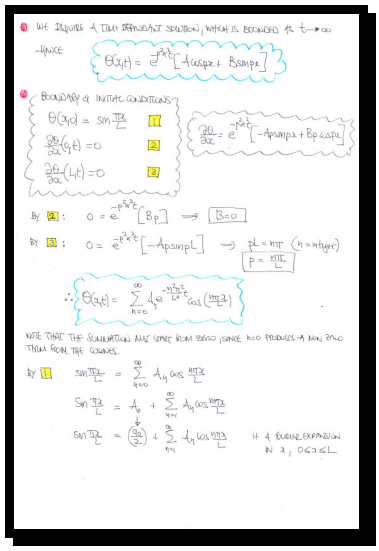
$\therefore \theta(x,t)=e^{\alpha^2p^2t}\left[A\cosh px+B\sinh px\right]$ ABSORBED C INTO A & B

IF $\lambda<0$ SAY $\lambda=-p^2$

$X''(x)=-p^2X(x) \qquad T'(t)=-\alpha^2p^2T(t)$

$X(x)=A\cos px+B\sin px \qquad T(t)=Ce^{-\alpha^2p^2t}$

$\theta(x,t)=e^{-\alpha^2p^2t}\left[A\cos px+B\sin px\right]$ ABSORBED C INTO A & B



WE IGNORE A TIME DEPENDANT SOLUTION, WHICH IS BOUNDED AS $t\to\infty$

HENCE $\quad \theta(x,t)=e^{-\alpha^2p^2t}\left[A\cos px+B\sin px\right]$

BOUNDARY & INITIAL CONDITIONS

$\theta(x,0)=\sin\dfrac{\pi x}{L}$ [1]

$\dfrac{\partial\theta}{\partial x}(0,t)=0$ [2]

$\dfrac{\partial\theta}{\partial x}(L,t)=0$ [3]

$\dfrac{\partial\theta}{\partial x}=e^{-\alpha^2p^2t}\left[-Ap\sin px+Bp\cos px\right]$

BY [2]: $\quad 0=e^{-\alpha^2p^2t}\left[Bp\right] \Rightarrow B=0$

BY [3]: $\quad 0=e^{-\alpha^2p^2t}\left[-Ap\sin pL\right] \Rightarrow pL=n\pi \quad (n=\text{integer})$

$P=\dfrac{n\pi}{L}$

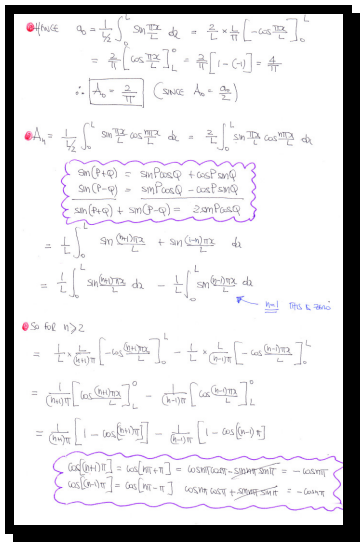
$\therefore \theta(x,t)=\sum_{n=0}^{\infty}A_ne^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\cos\left(\dfrac{n\pi x}{L}\right)$

NOTE THAT THE SOLUTIONS MAY START FROM ZERO (SINCE $n=0$ PRODUCES A NON ZERO TERM FROM THE COSINES.

BY [1]: $\quad \sin\dfrac{\pi x}{L}=\sum_{n=0}^{\infty}A_n\cos\dfrac{n\pi x}{L}$

$\sin\dfrac{\pi x}{L}=A_0+\sum_{n=1}^{\infty}A_n\cos\dfrac{n\pi x}{L}$

$\sin\dfrac{\pi x}{L}=\left(\dfrac{2A_0}{2}\right)+\sum_{n=1}^{\infty}A_n\cos\dfrac{n\pi x}{L}$ IS A FOURIER EXPANSION IN x, $0\le x\le L$



HENCE $\quad A_0=\dfrac{1}{\frac{1}{2}L}\int_0^L\sin\dfrac{\pi x}{L}\,dx=\dfrac{2}{L}\times\dfrac{L}{\pi}\left[-\cos\dfrac{\pi x}{L}\right]_0^L$

$=\dfrac{2}{\pi}\left[\cos\dfrac{\pi x}{L}\right]_L^0=\dfrac{2}{\pi}\left[1-(-1)\right]=\dfrac{4}{\pi}$

$\therefore \boxed{A_0=\dfrac{2}{\pi}} \quad (\text{SINCE } A_0=\dfrac{a_0}{2})$

$A_n=\dfrac{1}{L}\int_0^L\sin\dfrac{\pi x}{L}\cos\dfrac{n\pi x}{L}\,dx=\dfrac{2}{L}\int_0^L\sin\dfrac{\pi x}{L}\cos\dfrac{n\pi x}{L}\,dx$

$\sin(P+Q)=\sin P\cos Q+\cos P\sin Q$

$\sin(P-Q)=\sin P\cos Q-\cos P\sin Q$

$\sin(P+Q)+\sin(P-Q)=2\sin P\cos Q$

$=\dfrac{1}{L}\int_0^L\sin\dfrac{(n+1)\pi x}{L}+\sin\dfrac{(1-n)\pi x}{L}\,dx$

$=\dfrac{1}{L}\int_0^L\sin\dfrac{(n+1)\pi x}{L}\,dx-\dfrac{1}{L}\int_0^L\sin\dfrac{(n-1)\pi x}{L}\,dx$

$n=1$ THIS IS ZERO

SO FOR $n\ge2$

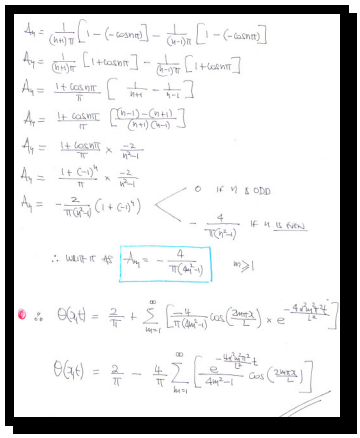
$=\dfrac{1}{L}\times\dfrac{L}{(n+1)\pi}\left[-\cos\dfrac{(n+1)\pi x}{L}\right]_0^L-\dfrac{1}{L}\times\dfrac{L}{(n-1)\pi}\left[-\cos\dfrac{(n-1)\pi x}{L}\right]_0^L$

$=\dfrac{1}{(n+1)\pi}\left[\cos\dfrac{(n+1)\pi x}{L}\right]_L^0-\dfrac{1}{(n-1)\pi}\left[\cos\dfrac{(n-1)\pi x}{L}\right]_L^0$

$=\dfrac{1}{(n+1)\pi}\left[1-\cos(n+1)\pi\right]-\dfrac{1}{(n-1)\pi}\left[1-\cos(n-1)\pi\right]$

$\cos\left[(n+1)\pi\right]=\cos\left[n\pi+\pi\right]=\cos n\pi\cos\pi-\sin n\pi\sin\pi=-\cos n\pi$

$\cos\left[(n-1)\pi\right]=\cos\left[n\pi-\pi\right]=\cos n\pi\cos\pi+\sin n\pi\sin\pi=-\cos n\pi$



$A_n=\dfrac{1}{(n+1)\pi}\left[1-(-\cos n\pi)\right]-\dfrac{1}{(n-1)\pi}\left[1-(-\cos n\pi)\right]$

$A_n=\dfrac{1}{(n+1)\pi}\left[1+\cos n\pi\right]-\dfrac{1}{(n-1)\pi}\left[1+\cos n\pi\right]$

$A_n=\dfrac{1+\cos n\pi}{\pi}\left[\dfrac{1}{n+1}-\dfrac{1}{n-1}\right]$

$A_n=\dfrac{1+\cos n\pi}{\pi}\left[\dfrac{(n-1)-(n+1)}{(n+1)(n-1)}\right]$

$A_n=\dfrac{1+\cos n\pi}{\pi}\times\dfrac{-2}{n^2-1}$

$A_n=\dfrac{1+(-1)^n}{\pi}\times\dfrac{-2}{n^2-1}$

$A_n=-\dfrac{2}{\pi(n^2-1)}\left(1+(-1)^n\right)$

0 IF n IS ODD

$-\dfrac{4}{\pi(n^2-1)}$ IF n IS EVEN

$\therefore$ WRITE IT AS $\boxed{A_n=-\dfrac{4}{\pi(4n^2-1)}} \quad n\ge1$

$\therefore \theta(x,t)=\dfrac{2}{\pi}+\sum_{n=1}^{\infty}\left[\dfrac{-4}{\pi(4n^2-1)}\cos\left(\dfrac{2n\pi x}{L}\right)\times e^{-\frac{4n^2\pi^2\alpha^2}{L^2}t}\right]$

$\theta(x,t)=\dfrac{2}{\pi}-\dfrac{4}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{e^{-\frac{4n^2\pi^2\alpha^2}{L^2}t}}{4n^2-1}\cos\left(\dfrac{2n\pi x}{L}\right)\right]$

Question 12

A long thin rod AB, of length L, has constant temperature $\theta = 0$ throughout its length. Another long thin rod CD, also of length L, has constant temperature $\theta = 100$ throughout its length.

At time $t = 0$ the temperature the ends B and C are brought into full contact, while the ends A and D are maintained at respective temperatures $\theta = 0$ and $\theta = 100$.

Show that, for $t \geqslant 0$, the temperature $\theta(t)$ of the point where the two rods are joined satisfies

$$50 - \frac{100}{\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \exp\left[-\frac{\alpha^2 \pi^2 (2n+1)^2 t}{4L^2} \right] \right].$$

You may assume

- $ABCD$ is a straight line
- The rods is insulated along their lengths
- The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

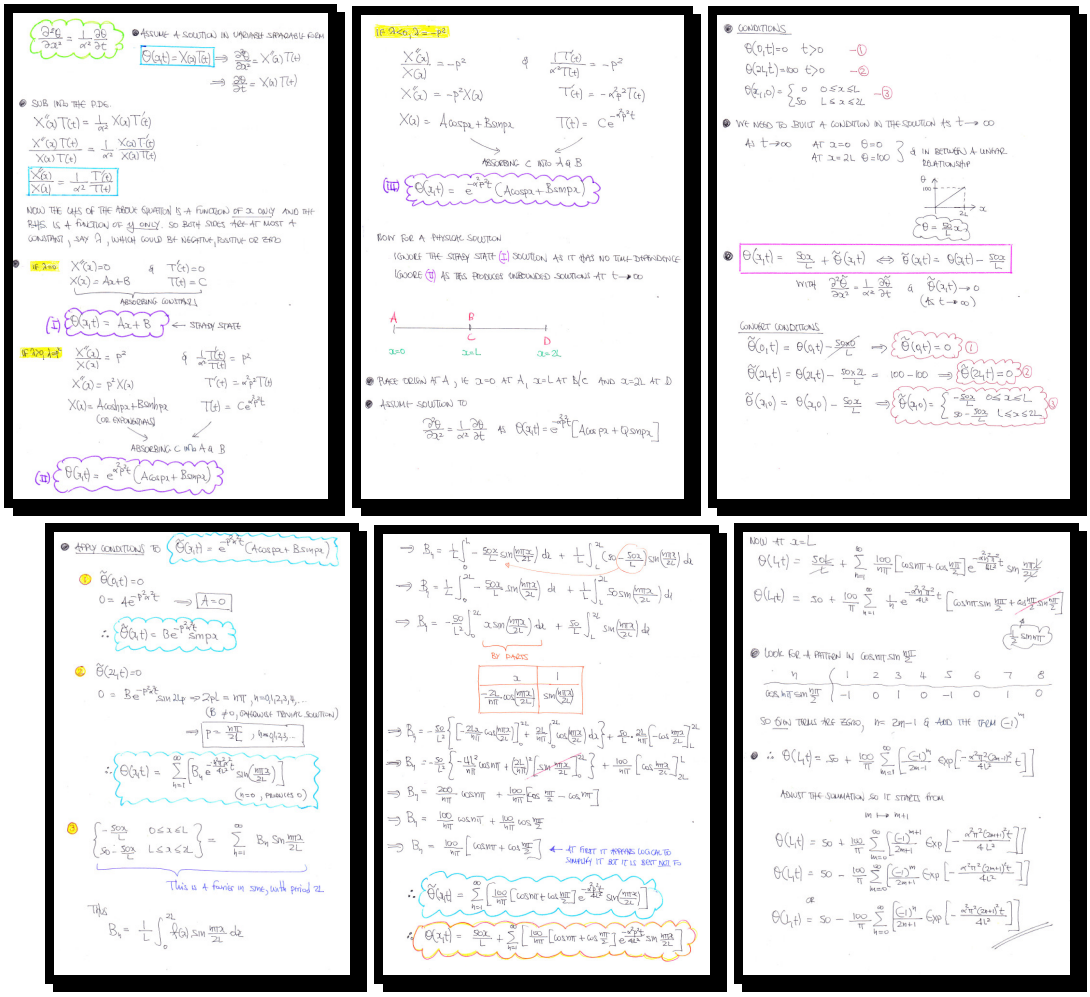
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leqslant x \leqslant L, \quad t \geqslant 0,$$

where α is a positive constant.

[*You must derive the standard solution of the heat equation in variable separate form*]

proof

[solution overleaf]



Question 13

The one dimensional heat equation is given by

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \le x \le L, \quad t \ge 0,$$

where α is a positive constant, known as the thermal diffusivity.

a) Obtain a general solution of the above equation by trying a solution in variable separable form.

A long thin rod AB, of length $2L$, has its endpoints, at $x = 0$ and $x = 2L$, maintained at constant temperature $\theta = 0$ and its midpoint is maintained at temperature $\theta = 100$, until a steady temperature distribution $\Theta(x)$ is reached throughout its length.

b) Show that

$$\Theta(x) = \begin{cases} \dfrac{100}{L}x & 0 \le x < L \\ \dfrac{100}{L}(2L - x) & L < x \le 2L \end{cases}$$

c) Prove that

$$\int_L^{2L} \Theta(x) \sin\left(\frac{n\pi x}{2L}\right) dx = (-1)^{n+1} \int_0^L \Theta(x) \sin\left(\frac{n\pi x}{2L}\right) dx$$

At $t = 0$, the heat source which was maintaining the midpoint of the rod at $\theta = 100$ is removed, but its endpoints are still maintained at $\theta = 0$. The rod is insulated throughout its length and allowed to cool.

d) Show that for $t \ge 0$, the temperature $\theta(t)$ of the midpoint of the rod satisfies

$$\frac{800}{\pi^2} \sum_{n=0}^{\infty} \left[\frac{1}{(2n+1)^2} \exp\left[-\frac{\alpha^2 \pi^2 (2n+1)^2 t}{4L^2} \right] \right].$$

proof

[solution overleaf]



HEAT EQUATION

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=\frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad \theta=\theta\left(x,y,t\right)$$

Two Dimensional

Question 1

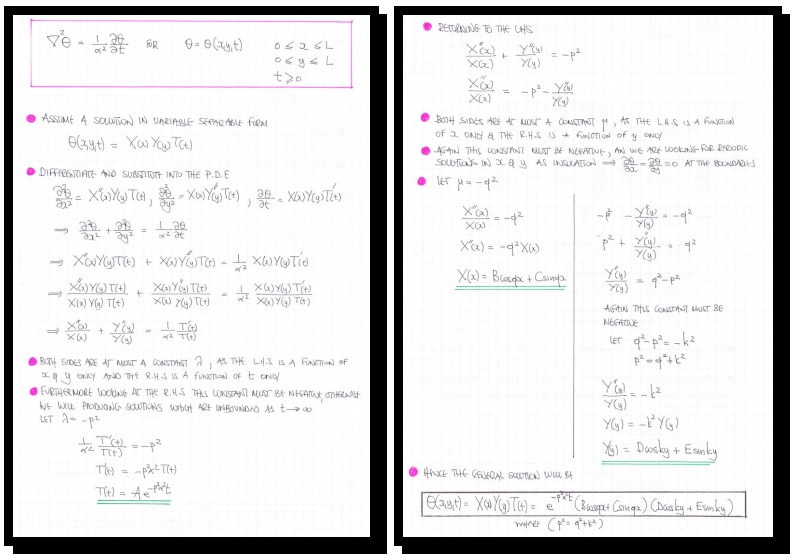
The temperature distribution, $\theta(x, y, t)$, on a square plate satisfies the equation

$$\nabla^2\theta = \frac{1}{\alpha^2}\frac{\partial z}{\partial t}, \quad 0 \le x \le L, \quad 0 \le y \le L, \quad t \ge 0.$$

Find a general solution for $\theta(x, y, t)$, which is periodic in x and in y.

Define any constants used.

$$\boxed{\theta(x, y, t) = \mathrm{e}^{-p^2\alpha^2 t}\left[A\cos qx + B\sin qx\right]\left[C\cos ky + D\sin ky\right], \quad p^2 = q^2 + k^2}$$



HEAT EQUATION

Miscellaneous Questions

Question 1

The smooth function $u = u(x,t)$ satisfies the diffusion equation

$$\frac{\partial^2 u}{dx^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t},$$

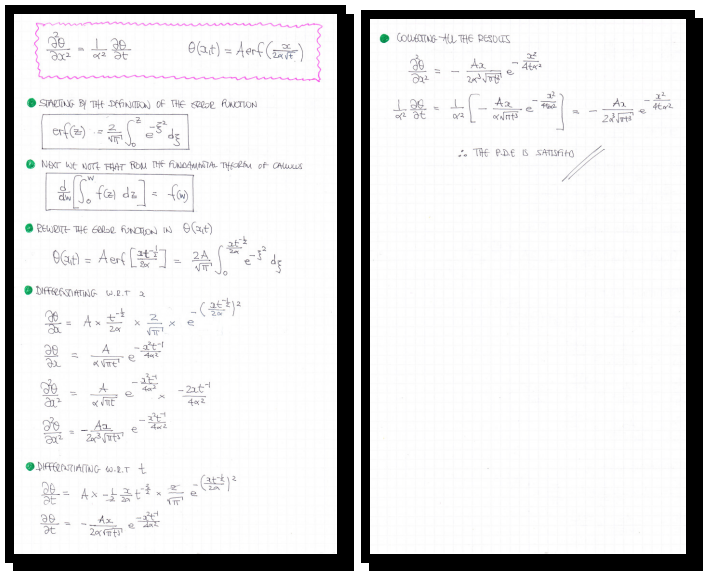
where α is a positive constant.

Show by differentiation that $u(x,t) = A\,\text{erf}\left(\dfrac{x}{2\alpha\sqrt{t}}\right)$, where A is a non zero constant, satisfies the diffusion equation.

You may assume that

- $\text{erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \text{e}^{-\xi}\ d\xi$.
- $\dfrac{d}{dw}\left[\displaystyle\int_0^w f(z)\ dz\right] = f(w)$.

proof



Question 2

The function $u = u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{dx^2} - 2u = 0,$$

subject to the conditions

$$u(0,t) = 0, \quad u(1,t) = 0 \quad \text{and} \quad u(x,0) = 1.$$

Use the substitution $u(x,t) = \mathrm{e}^{kt} w(x,t)$, with a suitable value for the constant k, to find a simplified expression for $u(x,t)$.

$$\boxed{u(x,t) = \frac{2\mathrm{e}^{2t}}{\pi} \sum_{n=1}^{\infty} \left[\frac{\exp\left[-(2n-1)^2 \pi^2 t\right] \sin\left[(2n-1)\pi x\right]}{2n-1} \right]}$$

