PARTIAL DIFFERENTIAL EQUATIONS

(by integral transformations)

Created by T. Madas

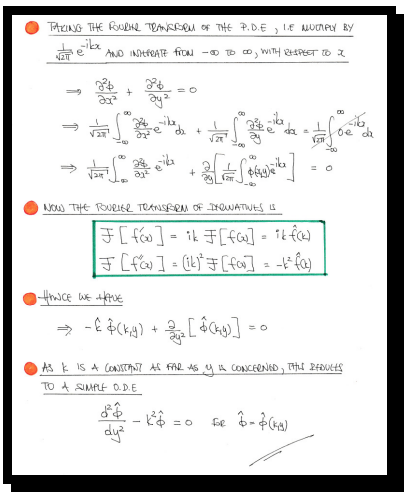
Question 1

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation for $\hat{\varphi}(k, y)$, where $\hat{\varphi}(k, y)$ is the Fourier transform of $\varphi(x, y)$ with respect to x.

$$\boxed{}, \quad \boxed{\frac{d^2\hat{\varphi}}{dx^2} - k^2\hat{\varphi} = 0}$$



Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

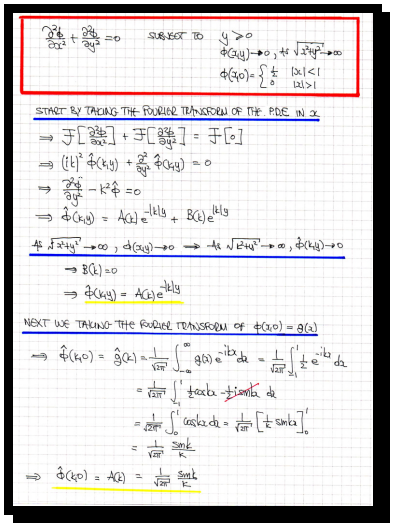
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

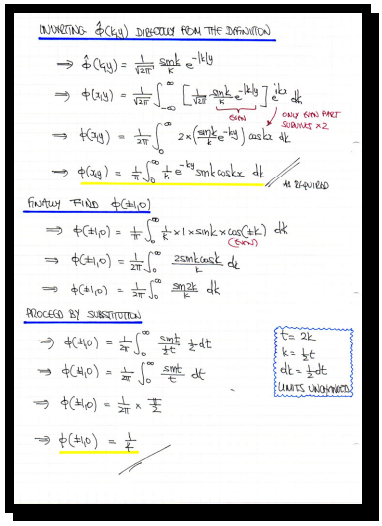
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} \mathrm{e}^{-ky} \sin k \cos kx \ dk,$$

and hence deduce the value of $\varphi(\pm 1, 0)$.

$\square$, $\boxed{\varphi(\pm 1, 0) = \frac{1}{4}}$





Question 3

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

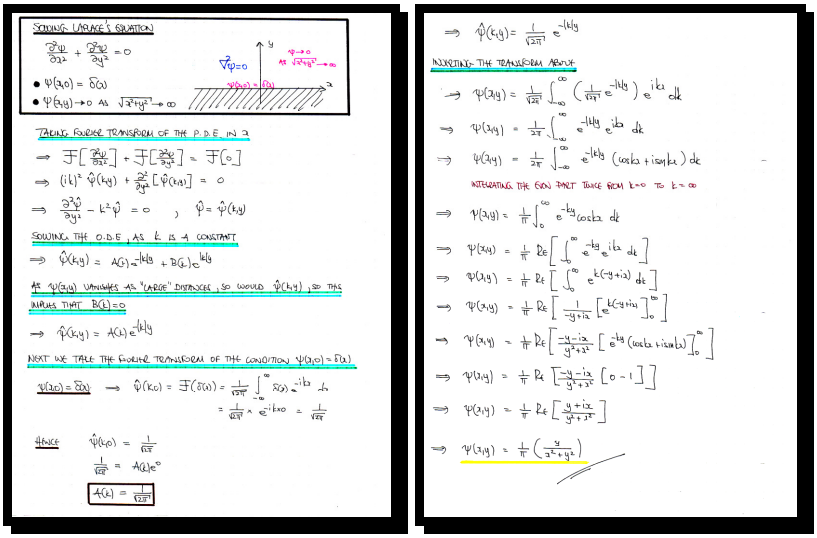
It is further given that

- $\psi(x, 0) = \delta(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{1}{\pi}\left(\frac{y}{x^2 + y^2}\right).$$

, proof



Question 4

The function $u = u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} = 0 .$$

It is further given that

- $u(x,0) = \delta(x)$
- $u(x,t) \to 0$ as $|x| \to \infty$

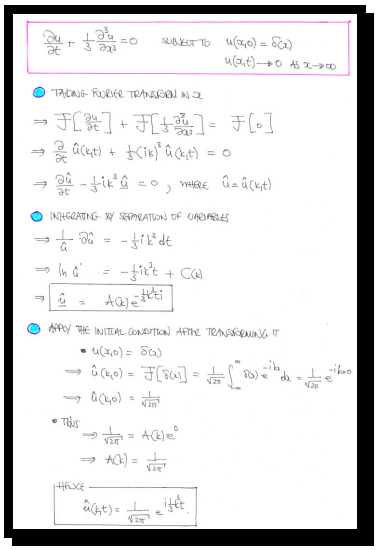
Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

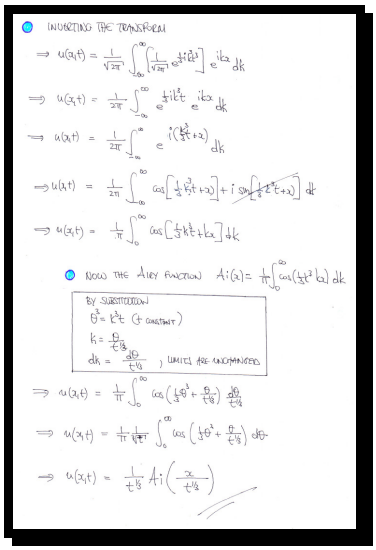
$$u(x,t) = \frac{1}{t^{\frac{1}{3}}}\text{Ai}\left(\frac{x}{t^{\frac{1}{3}}}\right),$$

where the $\text{Ai}(x)$ is the Airy function, defined as

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left[\frac{1}{3}k^3 + kx\right]\, dk .$$

proof





Question 5

The function $\Phi = \Phi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

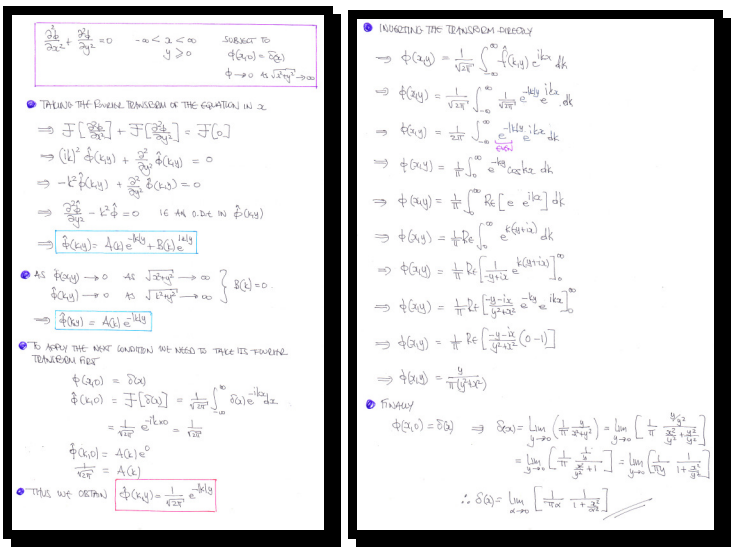
It is further given that

- $\Phi(x,0) = \delta(x)$
- $\Phi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to find the solution of the above partial differential equation and hence show that

$$\delta(x) = \lim_{\alpha \to 0} \left[\frac{1}{\pi\alpha} \left(1 + \frac{y^2}{\alpha^2} \right)^{-1} \right].$$

proof



Question 6

The function $u = u(t, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y\,,\ t \geq 0\,,\ y > 0\,,$$

subject to the following conditions

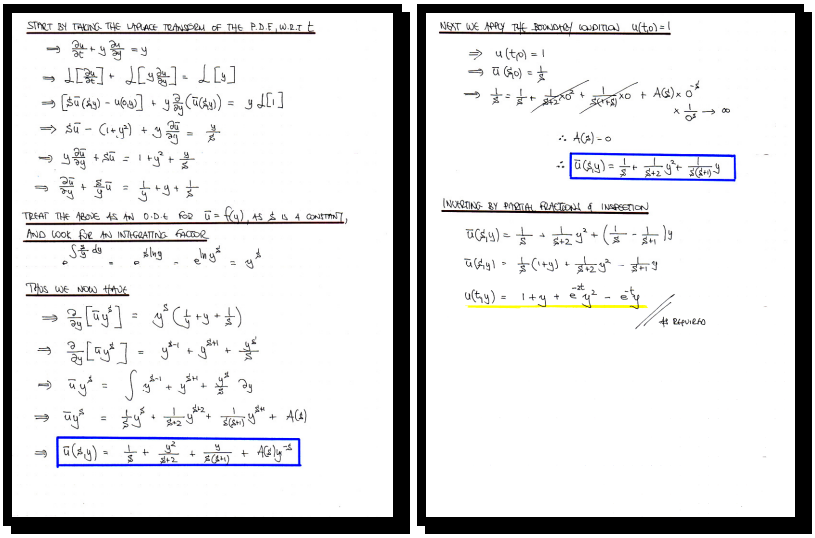
i. $u(0, y) = 1 + y^2\,,\ y > 0$

ii. $u(t, 0) = 1\,,\ t \geq 0$

Use Laplace transforms in t to show that

$$u(t, y) = 1 + y - y\,\mathrm{e}^{-t} + y^2\,\mathrm{e}^{-2t}\,.$$

[ ], proof



Question 7

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

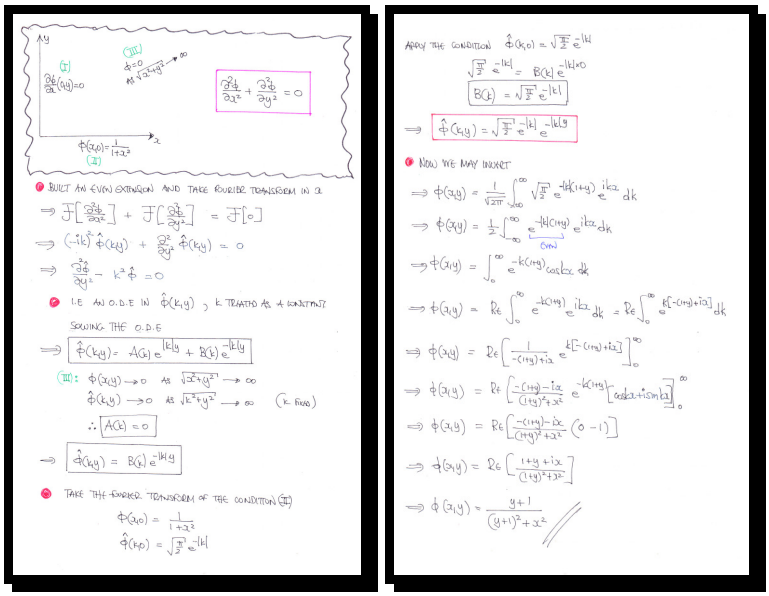
It is further given that

- $\varphi(x,0) = \dfrac{1}{1+x^2}$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\dfrac{\partial}{\partial x}\left[\varphi(x,0)\right] = 0$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y+1}{x^2 + (y+1)^2}.$$

proof



Question 8

The function $z = z(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial z}{\partial t} + z, \quad x \geq 0, \quad t \geq 0,$$

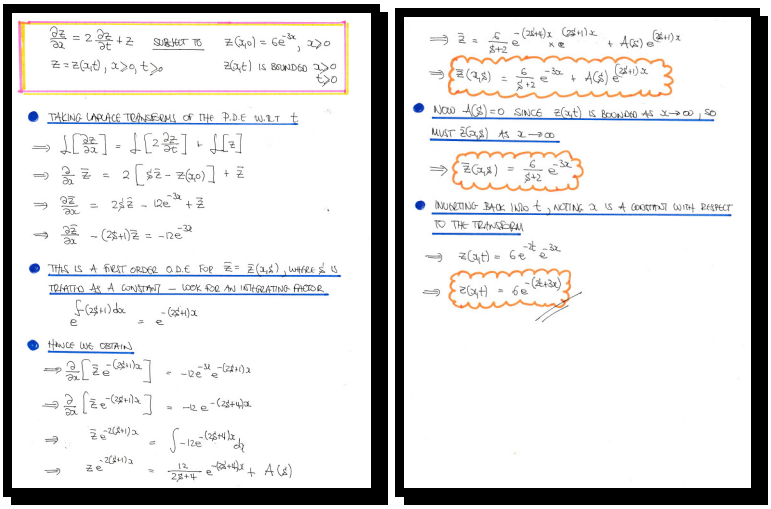
subject to the following conditions

i. $z(x,0) = 6\mathrm{e}^{-3x}$, $x > 0$.

ii. $z(x,t)$, is bounded for all $x \geq 0$ and $t \geq 0$.

Find the solution of partial differential equation by using Laplace transforms.

$\square$, $\boxed{z(x,t) = 6\mathrm{e}^{-(3x+2t)}}$



Question 9

$$\theta(x)=8\sin(2\pi x),\ 0\le x\le 1$$

The above equation represents the temperature distribution $\theta\,^{\circ}\mathrm{C}$, maintained along the 1 m length of a thin rod.

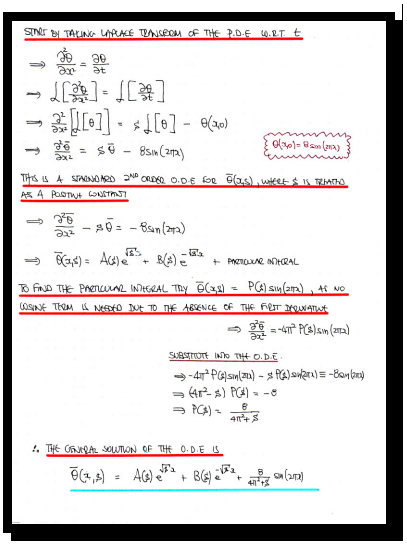
At time $t=0$, the temperature θ is suddenly dropped to $\theta=0\,^{\circ}\mathrm{C}$ at both the ends of the rod at $x=0$, and at $x=1$, and the source which was previously maintaining the temperature distribution is removed.

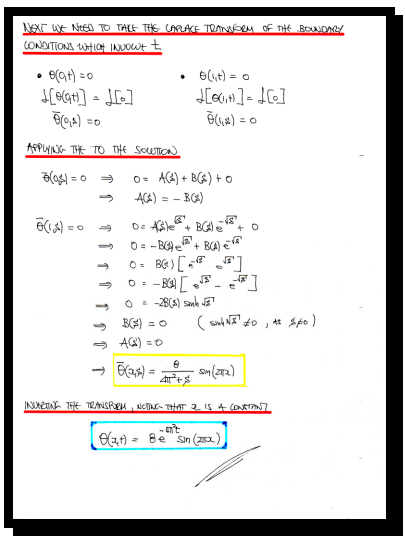
The new temperature distribution along the rod $\theta(x,t)$, satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t},\quad 0\le x\le 1,\quad t\ge 0.$$

Use Laplace transforms to determine an expression for $\theta(x,t)$.

$\square$, $\boxed{\theta(x,t)=8\mathrm{e}^{-4\pi^2 t}\sin(2\pi x)}$





Question 10

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that

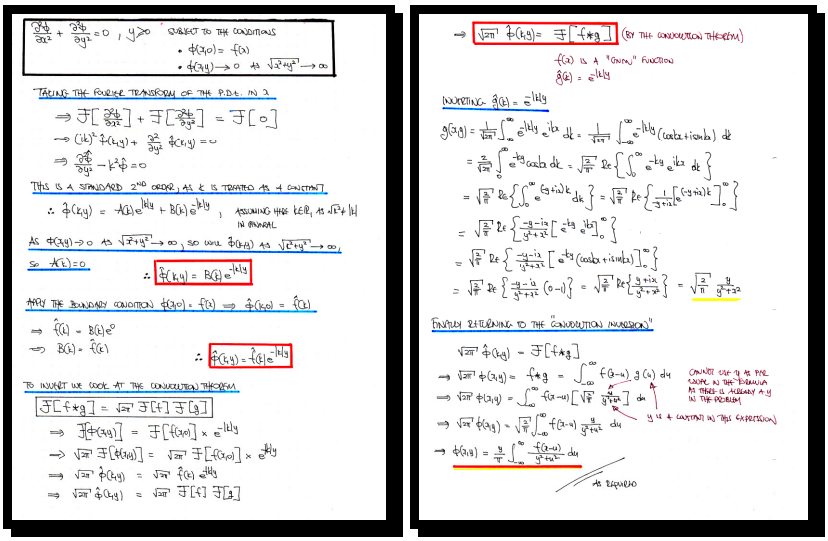
$$\varphi(x,0) = f(x)$$

$$\varphi(x, y) \to 0 \text{ as } \sqrt{x^2 + y^2} \to \infty$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2 + y^2} \, du .$$

, proof



Question 11

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that for a given function $f = f(x)$

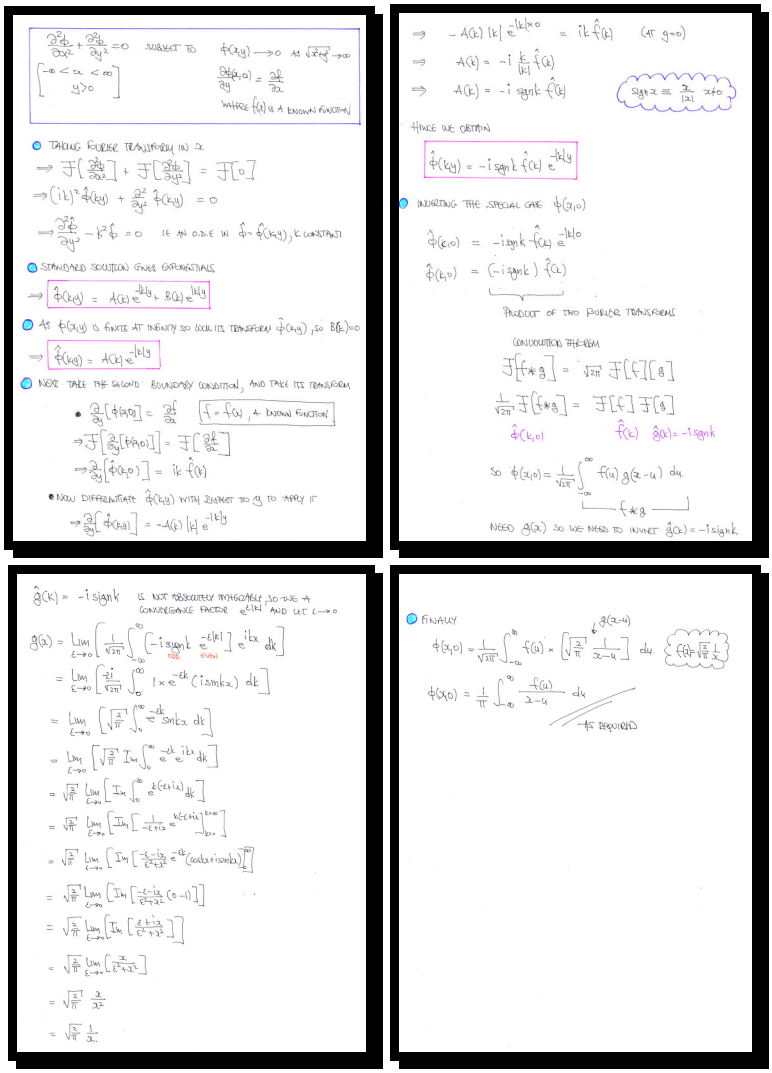
- $\dfrac{\partial}{\partial y}\left[\varphi(x,0)\right] = \dfrac{\partial}{\partial x}\left[f(x)\right]$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{x-u}\, du .$$

proof

[**solution overleaf**]



Question 12

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad -\infty < x < \infty, \ y \geq 0.$$

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \mathrm{H}(x)$, the Heaviside function.

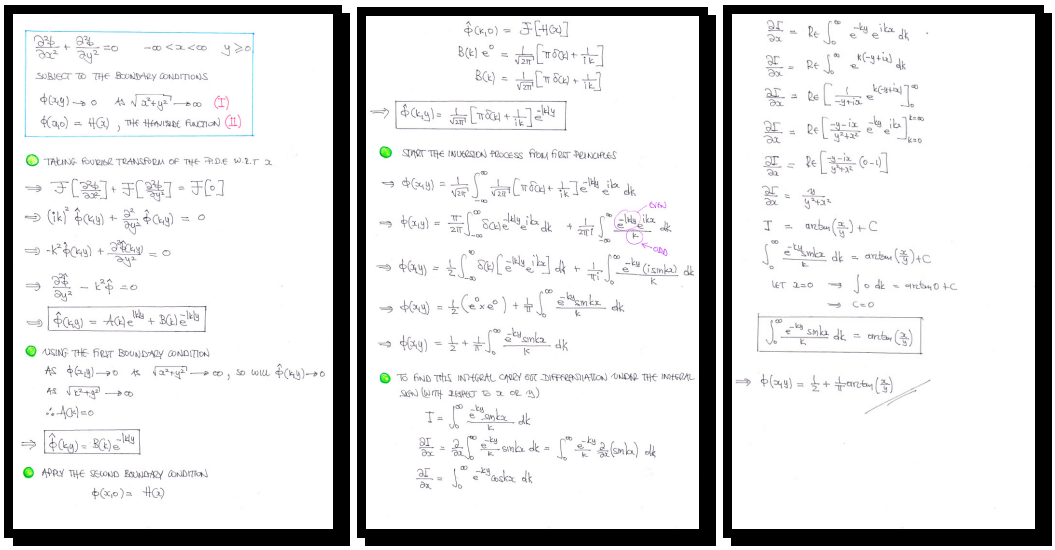
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right).$$

You may assume that

$$\mathcal{F}[\mathrm{H}(x)] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{\mathrm{i}k}\right].$$

proof



Question 13

The temperature $\theta(x,t)$ in a semi-infinite thin rod satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t},\quad x\geq 0,\quad t\geq 0.$$

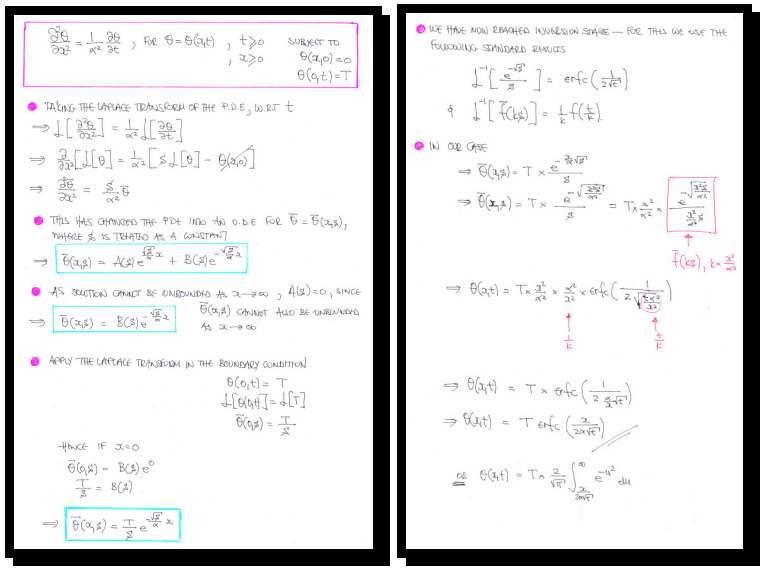
The initial temperature of the rod is 0 °C, and for $t>0$ the endpoint at $x=0$ is maintained at T °C.

Assuming the rod is insulated along its length, use Laplace transforms to find an expression for $\theta(x,t)$.

You may assume that

- $\mathcal{L}^{-1}\left[\dfrac{\mathrm{e}^{-\sqrt{s}}}{s}\right]=\operatorname{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$
- $\mathcal{L}^{-1}\left[\overline{f}(k\,s)\right]=\dfrac{1}{k}f\left(\dfrac{t}{k}\right)$, where k is a constant.

$$\theta(x,t)=\frac{2T}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty}\mathrm{e}^{-u^2}\,du\quad=\quad T\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$



Question 14

The function $u=u(x,y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,,\quad -\infty<x<\infty\,,\quad 0<y<1\,.$$

It is further given that

- $u(x,0)=0$
- $u(x,1)=f(x)$

  where $f(-x)=f(x)$ and $f(x)\to 0$ as $x\to\infty$

a) Use Fourier transforms to show that

$$u(x,y)=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\hat{f}(k)\cos kx\sinh ky}{\sinh k}\,dk\,,\quad \hat{f}(k)=\mathcal{F}\left[f(x)\right].$$

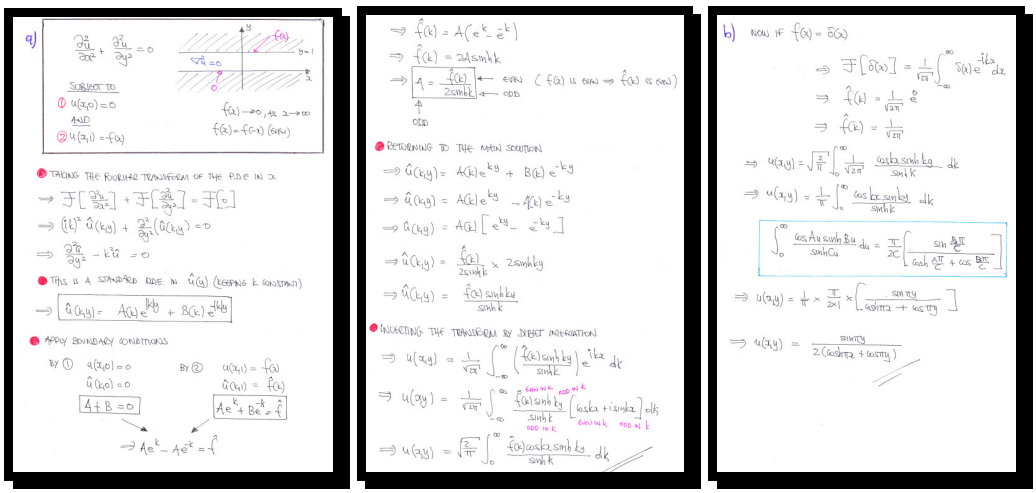
b) Given that $f(x)=\delta(x)$ show further that

$$u(x,y)=\frac{\sin\pi y}{2\left[\cosh\pi x+\cos\pi y\right]}.$$

You may assume without proof

$$\int_0^{\infty}\frac{\cos Au\sinh Bu}{\sinh Cu}\,du=\frac{\pi}{2C}\left[\frac{\sin(B\pi/C)}{\cosh(A\pi/C)+\cos(B\pi/C)}\right],\ 0\le B<C\,.$$

proof



Question 15

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 ,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\psi(x,0) = f(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

c) Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2 + y^2} \, du .$$

d) Evaluate the above integral for …

i. … $f(x) = 1$.

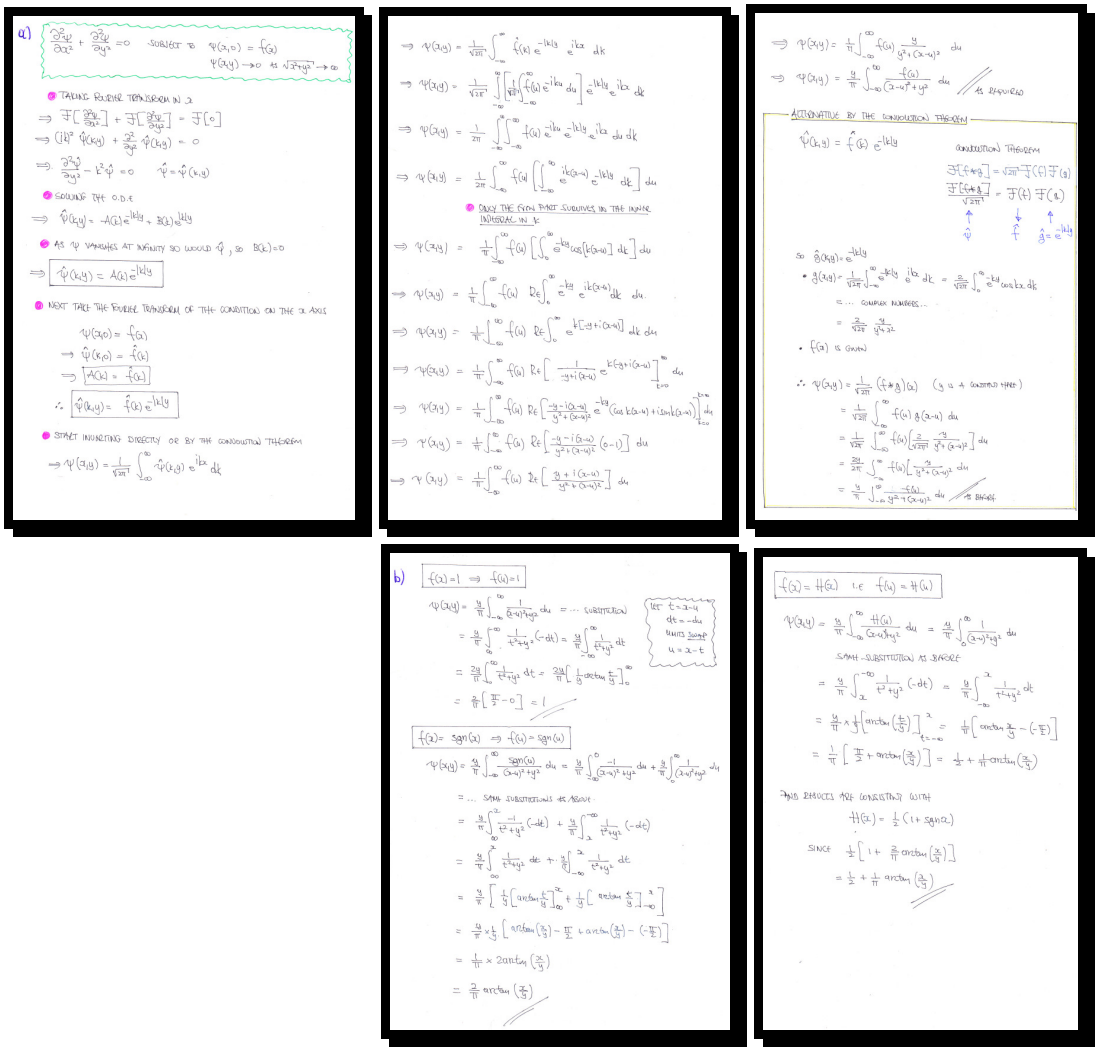
ii. … $f(x) = \operatorname{sgn} x$

iii. … $f(x) = \mathrm{H}(x)$

commenting further whether these answers are consistent.

$$\boxed{\psi(x, y) = 1} , \quad \boxed{\psi(x, y) = \frac{2}{\pi} \arctan\left(\frac{x}{y}\right)} , \quad \boxed{\psi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right)}$$

[solution overleaf]



Question 16

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

It is further given that

- $u(0, y) = 0$
- $u(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $u(x, 0) = f(x)$, $f(0) = 0$, $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to show that

$$u(x, y) = \frac{y}{\pi} \int_0^{\infty} f(w) \left[\frac{1}{y^2 + (x - w)^2} - \frac{1}{y^2 + (x + w)^2} \right] dw .$$

proof

[solution overleaf]



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

SUBJECT TO

① $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$

② $u(0,y)=0$

③ $u(x,0)=f(x)$, $f(0)=0$, $f(x)\to 0$ AS $x\to\infty$

$\nabla u=0$

A) ALTHOUGH THE REGION IS NOT SYMMETRICAL IN x (OR IN y), EXTEND $u(x,y)$ & $f(x)$ IN THE NEGATIVE OF INFLECTION, SO BOTH ARE ODD

B) TAKE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right]=\mathcal{F}[0]$$

$$\Rightarrow (ik)^2\hat{u}(k,y)+\frac{\partial^2}{\partial y^2}\hat{u}(k,y)=0$$

$$\Rightarrow \frac{\partial^2 \hat{u}}{\partial y^2}-k^2\hat{u}=0$$

$$\Rightarrow \boxed{\hat{u}(k,y)=A(k)e^{|k|y}+B(k)e^{-|k|y}}$$

C) APPLY BOUNDARY CONDITION ①

IF $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$, THEN $\hat{u}(k,y)\to 0$ AS $\sqrt{k^2+y^2}\to\infty$

$\therefore A(k)=0$

$$\Rightarrow \boxed{\hat{u}(k,y)=B(k)e^{-|k|y}}$$

D) APPLY BOUNDARY CONDITION ③

$\Rightarrow u(x,0)=f(x)$

$\Rightarrow \hat{u}(k,0)=\hat{f}(k)$

$\Rightarrow \hat{f}(k)=B(k)e^{0}$

$\Rightarrow B(k)=\hat{f}(k)$

$\therefore \boxed{\hat{u}(k,y)=\hat{f}(k)e^{-|k|y}}$

E) AS WE DO NOT KNOW $f(x)$ EXPLICITLY, WE START THE INVERSION FROM FIRST PRINCIPLES

$$\Rightarrow u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,y)e^{ikx}\,dk$$

- AS $u(x,y)$ IS ODD (WE BUILT ODD EXTENSION), $\hat{u}(k,y)$ WILL ALSO BE ODD

$$\Rightarrow u(x,y)=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\hat{u}(k,y)i\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}\hat{f}(k)e^{-|k|y}\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}e^{-|k|y}\sin kx\left[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(w)e^{-ikw}\,dw}_{\hat{f}(k)}\right]dk$$

- AS f IS "BUILT" ODD, ONLY THE ODD (SINE PART) SURVIVES

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}e^{-|k|y}\sin kx\left[-i\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=-\frac{2}{\pi}(i)^2\int_{0}^{\infty}e^{-|k|y}\sin kx\left[\int_{0}^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}e^{-|k|y}f(w)\sin kx\sin kw\,dw\,dk$$

F) REVERSING THE ORDER OF INTEGRATION NOTING THAT THE LIMITS ARE UNDERSTOOD ("BOX" REGION FROM 0 TO ∞)

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-|k|y}\sin kx\sin kw\,dk\right]dw$$

NEED TO DERIVE AN IDENTITY

$$\left.\begin{aligned}\cos(kx+kw)&\equiv\cos kx\cos kw-\sin kx\sin kw\\ \cos(kx-kw)&\equiv\cos kx\cos kw+\sin kx\sin kw\end{aligned}\right\}$$

$$\cos(kx-kw)-\cos(kx+kw)\equiv 2\sin kx\sin kw$$

$$\Rightarrow u(x,y)=\frac{1}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-|k|y}\cos[k(x-w)]-e^{-|k|y}\cos[k(x+w)]\,dk\right]dw$$

G) LOOKING AT EACH OF THE "IMAGE" INTEGRALS

$$\int_{k=0}^{\infty}e^{-|k|y}\cos[k(x-w)]\,dk=\mathrm{Re}\int_{k=0}^{\infty}e^{-|k|y}e^{ik(x-w)}\,dk$$

$$=\mathrm{Re}\int_{k=0}^{\infty}e^{k[-y+i(x-w)]}\,dk=\mathrm{Re}\left[\frac{1}{-y+i(x-w)}e^{k(-y+i(x-w))}\right]_{k=0}^{\infty}$$

$$=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}e^{-|k|y}e^{ik(x-w)}\right]_{k=0}^{\infty}=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}(0-1)\right]$$

$$=\frac{y}{y^2+(x-w)^2}$$

SIMILARLY THE OTHER INTEGRAL EXCEPT THAT WE HAVE $(x+w)$ INSTEAD OF $(x-w)$

H) $$u(x,y)=\frac{1}{\pi}\int_{0}^{\infty}f(w)\left[\frac{y}{y^2+(x-w)^2}-\frac{y}{y^2+(x+w)^2}\right]dw$$

$$u(x,y)=\frac{y}{\pi}\int_{0}^{\infty}f(w)\left[\frac{1}{y^2+(x-w)^2}-\frac{1}{y^2+(x+w)^2}\right]dw$$

Question 17

The function $\theta = \theta(x,t)$ satisfies the heat equation in one spatial dimension,

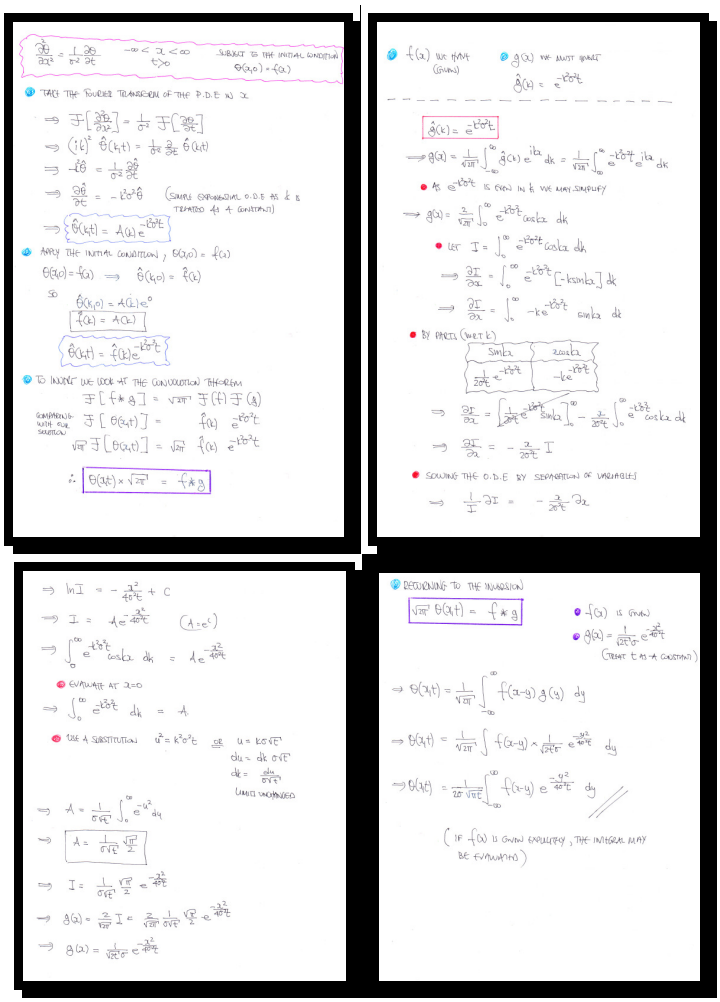
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial \theta}{\partial t}, \quad -\infty < x < \infty, \ t \geq 0,$$

where σ is a positive constant.

Given further that $\theta(x,0) = f(x)$, use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-u) \exp\left(\frac{u^2}{4t\sigma^2}\right) du .$$

proof



Question 18

The function $T=T(x,t)$ satisfies the heat equation in one spatial dimension,

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\sigma}\frac{\partial\theta}{\partial t},\quad x\geq 0,\ t\geq 0,$$

where σ is a positive constant.

It is further given that

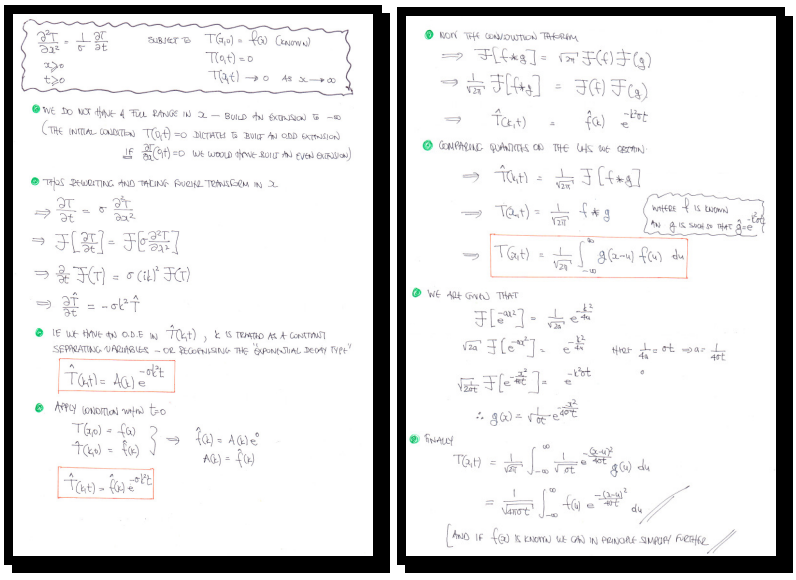
- $T(x,0)=f(x)$
- $T(0,t)=0$
- $T(x,t)\to 0$ as $x\to\infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$T(x,t)=\frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty} f(u)\exp\left[\frac{(x-u)^2}{4t\sigma}\right]du\,.$$

You may assume that $\mathcal{F}\left[\mathrm{e}^{ax^2}\right]=\frac{1}{\sqrt{2a}}\mathrm{e}^{\frac{k^2}{4a}}$.

proof



Question 19

The one dimensional heat equation for the temperature, $T(x,t)$, satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}, \ t \geq 0,$$

where t is the time, x is a spatial dimension and σ is a positive constant.

The temperature $T(x,t)$ is subject to the following conditions.

i. $\lim_{x\to\infty}\left[T(x,t)\right] = 0$

ii. $T(0,t) = 1$

iii. $T(x,0) = 0$

a) Use Laplace transforms to show that

$$\mathcal{L}\left[T(x,t)\right] = \overline{T}(x,s) = \frac{1}{s}\exp\left[-\sqrt{\frac{s}{\sigma}}\, x\right].$$

b) Use contour integration on the Laplace transformed temperature gradient $\frac{\partial}{\partial x}\left[\overline{T}(x,s)\right]$ to show further that

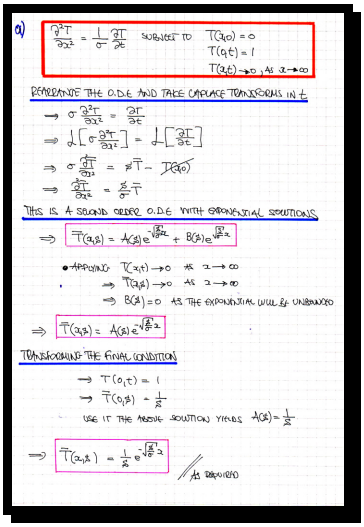
$$T(x,t) = 1 - \operatorname{erf}\left[\frac{x}{\sqrt{4\sigma t}}\right].$$

You may assume without proof that

- $\displaystyle\int_0^\infty \mathrm{e}^{-ax^2}\cos kx \ dx = \sqrt{\frac{\pi}{4a}}\exp\left[-\frac{k^2}{4a}\right]$
- $\displaystyle\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\xi^2} \ d\xi$

[ ], [proof]

[solution overleaf]



a) $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}$ SUBJECT TO $T(x,0) = 0$, $T(0,t) = 1$, $T(x,t) \to 0$, AS $x \to \infty$

REARRANGE THE O.D.E AND TAKE LAPLACE TRANSFORMS IN t

$\Rightarrow \sigma\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

$\Rightarrow \mathcal{L}\left[\sigma\frac{\partial^2 T}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial T}{\partial t}\right]$

$\Rightarrow \sigma\frac{\partial^2 \bar{T}}{\partial x^2} = s\bar{T} - T(x,0)$

$\Rightarrow \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{s}{\sigma}\bar{T}$

THIS IS A SECOND ORDER O.D.E WITH EXPONENTIAL SOLUTIONS

$\Rightarrow \bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x} + B(s)e^{\sqrt{\frac{s}{\sigma}}x}$

- APPLYING $T(x,t) \to 0$ AS $x \to \infty$
  $\Rightarrow \bar{T}(x,s) \to 0$ AS $x \to \infty$
  $\Rightarrow B(s) = 0$ AS THE EXPONENTIAL WILL BE UNBOUNDED

$\Rightarrow \bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x}$

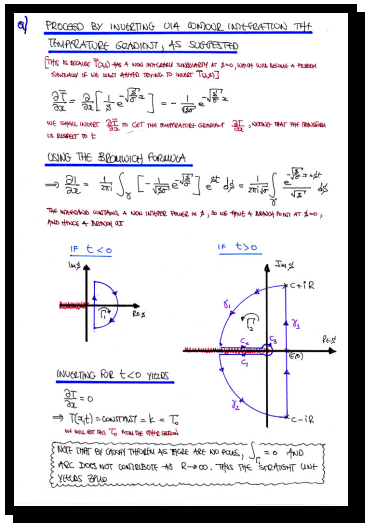
TRANSFORMING THE FINAL CONDITION

$\Rightarrow T(0,t) = 1$

$\Rightarrow \bar{T}(0,s) = \frac{1}{s}$

USE IT THE ABOVE SOLUTION YIELDS $A(s) = \frac{1}{s}$

$\Rightarrow \bar{T}(x,s) = \frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}$ // AS REQUIRED



a) PROCEED BY INVERTING VIA CONTOUR INTEGRATION THE TEMPERATURE GRADIENT, AS SUGGESTED

THIS IS BECAUSE $\bar{T}(0,s)$ HAS A NON INTEGRABLE DISCONTINUITY AT $s=0$, WHICH WILL BECAUSE A PROBLEM SUBSEQUENTLY IF WE WILL ATTEMPT TO INVERT $\bar{T}(x,s)$

$\frac{\partial \bar{T}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}\right] = -\frac{1}{\sqrt{\sigma s}}e^{-\sqrt{\frac{s}{\sigma}}x}$

WE SHALL INVERT $\frac{\partial \bar{T}}{\partial x}$ TO GET THE DISTRIBUTED GRADIENT $\frac{\partial T}{\partial x}$, NOTING THAT THE PROBLEM IS DISTANT TO t

USING THE BROMWICH FORMULA

$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i}\int_\gamma \left[-\frac{1}{\sqrt{\sigma s}}e^{-\sqrt{\frac{s}{\sigma}}x}\right]e^{st}\,ds = \frac{1}{2\pi i\sqrt{\sigma}}\int_\gamma \frac{e^{st-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s}}\,ds$

THE INTEGRAND CONTAINS A NON INTEGRABLE BRANCH IN s, SO WE HAVE A BRANCH POINT AT $s=0$, AND HENCE A BRANCH CUT

IF $t<0$

IF $t>0$

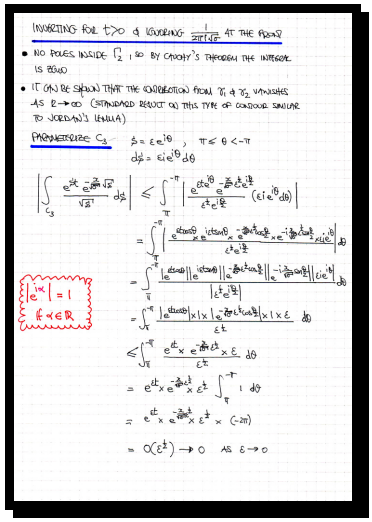
INVERTING FOR $t<0$ YIELDS

$\frac{\partial T}{\partial x} = 0$

$\Rightarrow T(x,t) = \text{CONSTANT} = k = T_0$

WE WILL USE THIS T_0 FROM THE FIRST SECTION

NOTE THAT BY CAUCHY THEOREM AS THERE ARE NO POLES, $\int_\Gamma = 0$ AND ARC DOES NOT CONTRIBUTE AS $R \to \infty$. THUS THE "EXPECTED" UNIT VELOCITY HOLDS



INVERTING FOR $t>0$ & LOOKING $\frac{1}{2\pi i\sqrt{\sigma}}$ AT THE FRONT

- NO POLES INSIDE Γ_2, SO BY CAUCHY'S THEOREM THE INTEGRAL IS ZERO
- IT CAN BE SHOWN THAT THE CONTRIBUTION FROM γ_1 & γ_2 VANISHES AS $R \to \infty$ (STANDARD RESULT ON THIS TYPE OF CONTOUR SIMILAR TO JORDAN'S LEMMA)

PARAMETRIZE C_3: $s = \varepsilon e^{i\theta}$, $\pi \le \theta \le -\pi$, $ds = \varepsilon i e^{i\theta}\,d\theta$

$$\left|\int_{C_3}\frac{e^{st}e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s}}\,ds\right| \le \int_\pi^{-\pi}\left|\frac{e^{\varepsilon t\cos\theta}e^{-i\varepsilon t\sin\theta}}{\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}\left(\varepsilon i e^{i\theta}\right)\right|d\theta$$

$$= \int_\pi^{-\pi}\left|e^{\varepsilon t\cos\theta}\right|\left|e^{-i\varepsilon t\sin\theta}\right|\times\left|e^{-\sqrt{\frac{\varepsilon}{\sigma}}x\cos\frac{\theta}{2}}\right|\times\left|e^{-i\sqrt{\frac{\varepsilon}{\sigma}}x\sin\frac{\theta}{2}}\right|\left|\varepsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}\right|d\theta$$

$$= \int_\pi^{-\pi}\left|e^{\varepsilon t\cos\theta}\right|\left|e^{-\sqrt{\frac{\varepsilon}{\sigma}}x\cos\frac{\theta}{2}}\right|\times\left|e^{-i\sqrt{\frac{\varepsilon}{\sigma}}x\sin\frac{\theta}{2}}\right|\times\left|\varepsilon^{\frac{1}{2}}\right|\left|e^{\frac{i\theta}{2}}\right|d\theta$$

$$= \int_\pi^{-\pi}\left|e^{\varepsilon t\cos\theta}\right|\times\left|e^{-\sqrt{\frac{\varepsilon}{\sigma}}x\cos\frac{\theta}{2}}\right|\times\left|e^{-i\sqrt{\frac{\varepsilon}{\sigma}}x\sin\frac{\theta}{2}}\right|\times\varepsilon^{\frac{1}{2}}\,d\theta$$

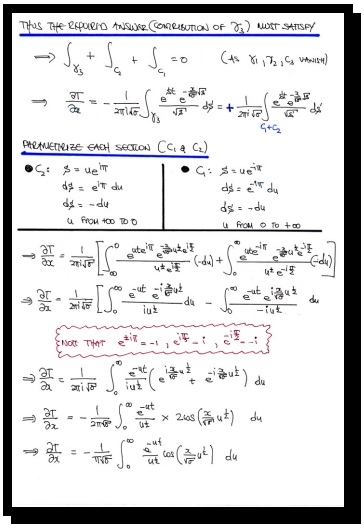
$$\le \int_\pi^{-\pi}\frac{e^{\varepsilon t}\times e^{\sqrt{\frac{\varepsilon}{\sigma}}x}\times\varepsilon^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$= e^{\varepsilon t}\times e^{\sqrt{\frac{\varepsilon}{\sigma}}x}\times\varepsilon^{\frac{1}{2}}\int_\pi^{-\pi}1\,d\theta$$

$$= e^{\varepsilon t}\times e^{\sqrt{\frac{\varepsilon}{\sigma}}x}\times\varepsilon^{\frac{1}{2}}\times(-2\pi)$$

$$= O\left(\varepsilon^{\frac{1}{2}}\right) \to 0 \quad \text{AS } \varepsilon \to 0$$

$|e^{iA}| = 1$ IF $A \in \mathbb{R}$



THUS THE REQUIRED INVERSE (CONTRIBUTION OF γ_3) MUST SATISFY

$\Rightarrow \int_{\gamma_3} + \int_{C_2} + \int_{C_1} = 0$ (AS γ_1, γ_2, C_3 VANISH)

$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi i\sqrt{\sigma}}\int_{\gamma_3}\frac{e^{st}e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s}}\,ds = +\frac{1}{2\pi i\sqrt{\sigma}}\left(\int_{C_1+C_2}\frac{e^{st}e^{-\sqrt{\frac{s}{\sigma}}x}}{\sqrt{s}}\,ds\right)$

PARAMETRIZE EACH SECTION (C_1 & C_2)

- C_2: $s = ue^{i\pi}$, $ds = e^{i\pi}\,du$, $ds = -du$, u FROM ∞ TO 0
- C_1: $s = ue^{-i\pi}$, $ds = e^{-i\pi}\,du$, $ds = -du$, u FROM 0 TO $+\infty$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_\infty^0\frac{e^{ute^{i\pi}}e^{-\sqrt{\frac{ue^{i\pi}}{\sigma}}x}}{\sqrt{ue^{i\pi}}}(-du) + \int_0^\infty\frac{e^{ute^{-i\pi}}e^{-\sqrt{\frac{ue^{-i\pi}}{\sigma}}x}}{\sqrt{ue^{-i\pi}}}(-du)\right]$$

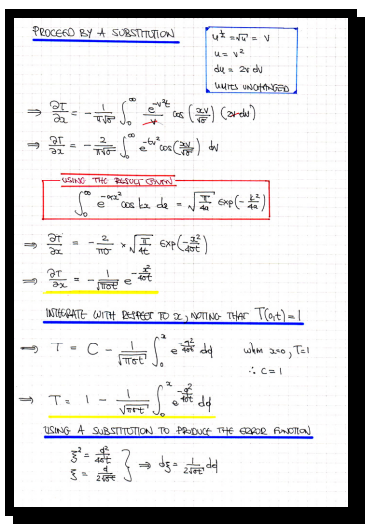
$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\left[\int_0^\infty\frac{e^{-ut}e^{-i\sqrt{\frac{u}{\sigma}}x}}{iu^{\frac{1}{2}}}du - \int_0^\infty\frac{e^{-ut}e^{i\sqrt{\frac{u}{\sigma}}x}}{-iu^{\frac{1}{2}}}du\right]$$

NOTE THAT $e^{\frac{i\pi}{2}} = i$, $e^{\frac{-i\pi}{2}} = -i$, $\frac{1}{i} = -i$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}}\int_0^\infty\frac{e^{-ut}}{iu^{\frac{1}{2}}}\left(e^{i\sqrt{\frac{u}{\sigma}}x} + e^{-i\sqrt{\frac{u}{\sigma}}x}\right)du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi\sqrt{\sigma}}\int_0^\infty\frac{e^{-ut}}{u^{\frac{1}{2}}}\times 2\cos\left(\sqrt{\frac{u}{\sigma}}x\right)du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_0^\infty\frac{e^{-ut}}{u^{\frac{1}{2}}}\cos\left(\sqrt{\frac{u}{\sigma}}x\right)du$$



PROCEED BY A SUBSTITUTION

$u^{\frac{1}{2}} = \sqrt{v} = v$, $u = v^2$, $du = 2v\,dv$, LIMITS UNCHANGED

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}}\int_0^\infty\frac{e^{-v^2t}}{v}\cos\left(\frac{xv}{\sqrt{\sigma}}\right)(2v\,dv)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}}\int_0^\infty e^{-tv^2}\cos\left(\frac{xv}{\sqrt{\sigma}}\right)dv$$

USING THE RESULT GIVEN

$$\int_0^\infty e^{-\alpha x^2}\cos bx\,dx = \sqrt{\frac{\pi}{4\alpha}}\exp\left(-\frac{b^2}{4\alpha}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sigma}\times\sqrt{\frac{\pi}{4t}}\exp\left(-\frac{x^2}{4\sigma t}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\sqrt{\pi\sigma t}}e^{-\frac{x^2}{4\sigma t}}$$

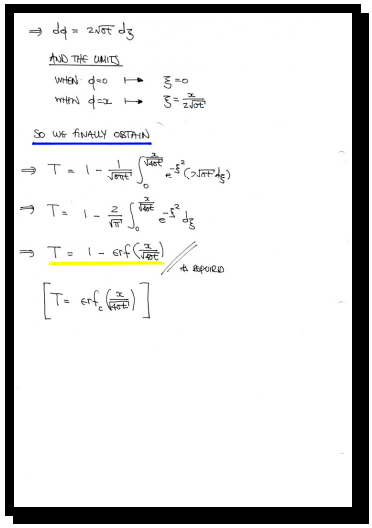
INTEGRATE WITH RESPECT TO x, NOTING THAT $T(0,t) = 1$

$$\Rightarrow T = C - \frac{1}{\sqrt{\pi\sigma t}}\int_0^x e^{-\frac{x^2}{4\sigma t}}\,dx \qquad \text{WHEN } x=0,\ T=1 \ \therefore C = 1$$

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\pi\sigma t}}\int_0^x e^{-\frac{x^2}{4\sigma t}}\,dx$$

USING A SUBSTITUTION TO PRODUCE THE ERROR FUNCTION

$\left.\begin{aligned} s^2 &= \frac{x^2}{4\sigma t} \\ s &= \frac{x}{2\sqrt{\sigma t}} \end{aligned}\right\} \Rightarrow ds = \frac{1}{2\sqrt{\sigma t}}\,dx$



$\Rightarrow dq = 2\sqrt{\sigma t}\,ds$

AND THE LIMITS

WHEN $q=0 \mapsto s=0$

WHEN $q=x \mapsto s = \frac{x}{2\sqrt{\sigma t}}$

SO WE FINALLY OBTAIN

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\pi\sigma t}}\int_0^{\frac{x}{2\sqrt{\sigma t}}}e^{-s^2}(2\sqrt{\sigma t}\,ds)$$

$$\Rightarrow T = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\sigma t}}}e^{-s^2}\,ds$$

$$\Rightarrow T = 1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\sigma t}}\right)$$ // AS REQUIRED

$$\left[T = \operatorname{erfc}\left(\frac{x}{2\sqrt{\sigma t}}\right)\right]$$