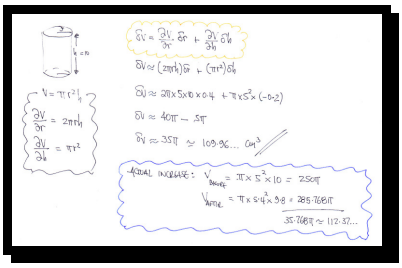
PARTIAL DIFFERENTIATION INTRODUCTION

Question 1 (**)

A right circular cylinder has radius 5 cm and height 10 cm.

Use a differential approximation to find an approximate increase in the volume of this cylinder if the radius increases by 0.4 cm and its height decreases by 0.2 cm.

$$\boxed{35\pi \approx 109.96...\text{cm}^3}$$

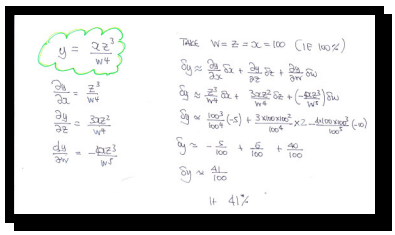


Question 2 (**)

$$y=\frac{xz^3}{w^4},\ w\neq 0.$$

Determine an approximate percentage increase in y, if x decreases by 5%, z increases by 2% and w decreases by 10%.

$$\boxed{\approx 41\%}$$



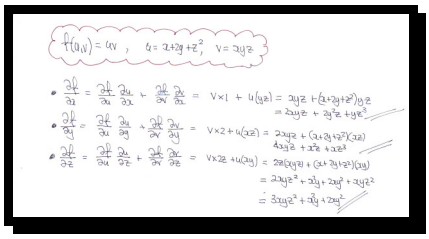
Question 3 (**)

The function f depends on u and v so that

$$f\left[u(x,y,z),v(x,y,z)\right]=uv\,,\quad u=x+2y+z^2\quad\text{and}\quad v=xyz\,.$$

Find simplified expressions for $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, in terms of x, y and z.

$$\boxed{\frac{\partial f}{\partial x}=2xyz+2y^2z+yz^3}\,,\quad\boxed{\frac{\partial f}{\partial y}=4xyz+x^2z+xz^3}\,,\quad\boxed{\frac{\partial f}{\partial z}=3xyz^2+x^2y+2xy^2}$$



Question 4 (**)

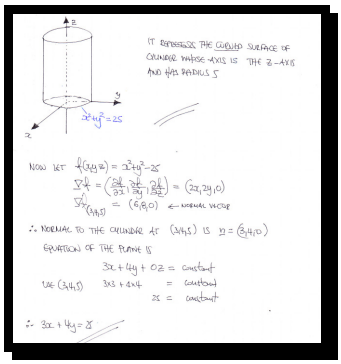
A surface S is defined by the Cartesian equation

$$x^2+y^2=25\,.$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3,4,5)$.

$$\boxed{3x+4y=25}$$



Question 5 (**)

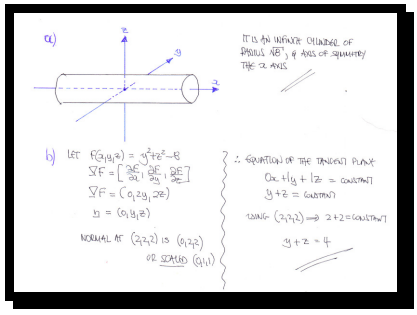
A surface S is defined by the Cartesian equation

$$y^2+z^2=8\,.$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2,2,2)$.

$$\boxed{y+z=4}$$



Question 6 (**)

The function φ depends on u, v and w so that

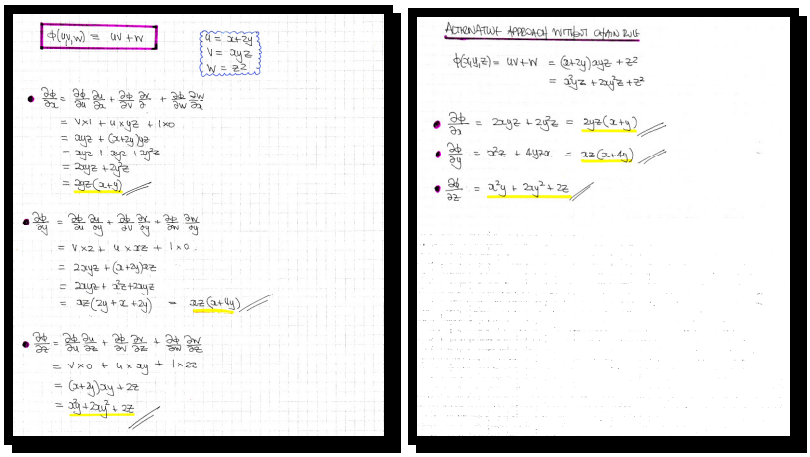
$$\varphi\big[u(x,y,z),v(x,y,z),w(x,y,z)\big]=uv+w.$$

It is further given that

$$u=x+2y,\qquad v=xyz\quad\text{and}\quad w=z^2.$$

By using **the chain rule for partial differentiation** find simplified expressions for $\dfrac{\partial\varphi}{\partial x}$, $\dfrac{\partial\varphi}{\partial y}$ and $\dfrac{\partial\varphi}{\partial z}$, in terms of x, y and z.

$\square$, $\boxed{\dfrac{\partial\varphi}{\partial x}=2yz(x+y)}$, $\boxed{\dfrac{\partial\varphi}{\partial y}=xz(x+4y)}$, $\boxed{\dfrac{\partial\varphi}{\partial z}=x^2y+2xy^2+2z}$

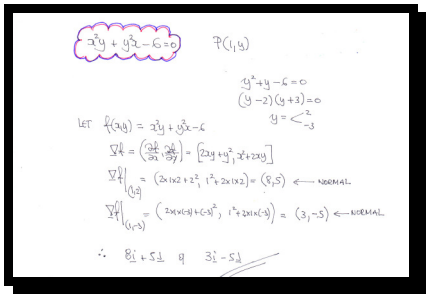


Question 7 (**)

The point $P(1, y)$ lies on the contour with equation $x^2y + y^2x - 6 = 0$.

Determine the possible normal vectors at P

$$\boxed{8\mathbf{i} + 5\mathbf{j}}, \quad \boxed{3\mathbf{i} - 5\mathbf{j}}$$

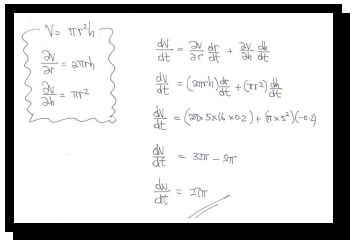


Question 8 (**)

The radius of a right circular cylinder is increasing at the constant rate of 0.2 cms^{-1} and its height is decreasing at the constant rate of 0.2 cms^{-1}.

Determine the rate at which the volume of this cylinder is increasing when the radius is 5 cm and its height is 16 cm.

$$\boxed{27\pi \approx 84.82...\text{cm}^3\text{s}^{-1}}$$



Question 9 (**)

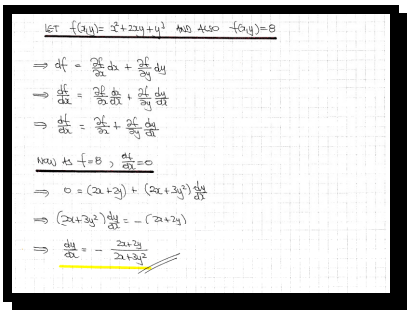
A curve has implicit equation

$$x^2+2xy+y^3=8\,.$$

Use partial differentiation to find an expression for $\dfrac{dy}{dx}$.

No credit will be given for obtaining the answer with alternative methods

$\square$, $\boxed{\dfrac{dy}{dx}=-\dfrac{2x+2y}{2x+3y^2}}$



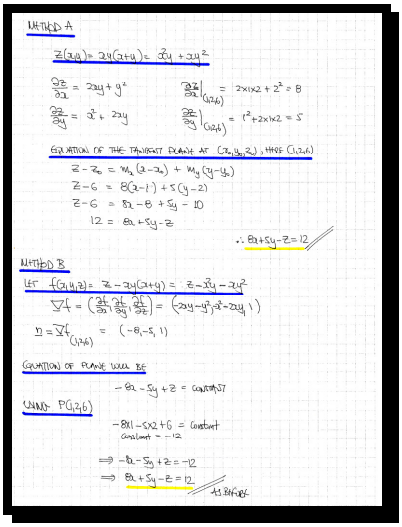
Question 10 (**)

A surface S is defined by the Cartesian equation

$$z = xy(x+y).$$

Find an equation of the tangent plane on S at the point $(1,2,6)$.

$\boxed{}$, $\boxed{8x+5y-z=12}$



Question 11 (**)

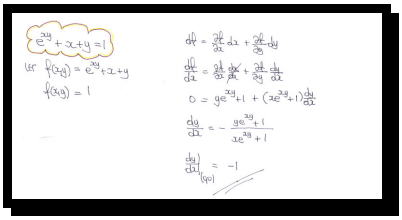
A curve has implicit equation

$$e^{xy}+x+y=1\,.$$

Use partial differentiation to find the value of $\dfrac{dy}{dx}$ at $(0,0)$.

No credit will be given for obtaining the answer with alternative methods

$$\left.\frac{dy}{dx}\right|_{(0,0)}=-1$$



Question 12 (+)**

The function f is defined as

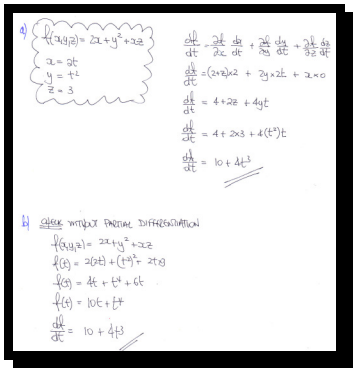
$$f(x,y,z) \equiv 2x + y^2 + xz ,$$

where $x = 2t$, $y = t^2$ and $z = 3$.

a) Use partial differentiation to find an expression for $\dfrac{df}{dt}$, in terms of t.

b) Verify the answer obtained in part **(a)** by a method **not** involving partial differentiation.

$$\boxed{\frac{df}{dt} = 4t^3 + 10}$$



Question 13 (+)**

The function φ is defined as

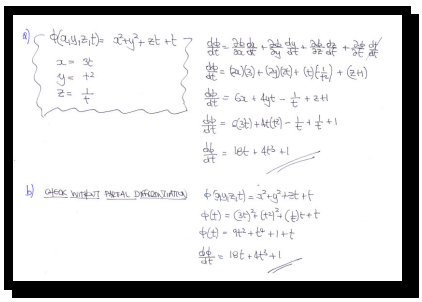
$$\varphi(x,y,z)\equiv x^2+y^2+tz+t\ ,\ \ t\neq 0\,,$$

where $x=3t$, $y=t^2$ and $z=\dfrac{1}{t}$.

a) Use partial differentiation to find an expression for $\dfrac{d\varphi}{dt}$, in terms of t.

b) Verify the answer obtained in part **(a)** by a method not involving partial differentiation.

$$\boxed{\frac{d\varphi}{dt}=4t^3+18t+1}$$



Question 14 (+)**

Plane Cartesian coordinates $(x.y)$ are related to plane polar coordinates (r,θ) by the transformation equations

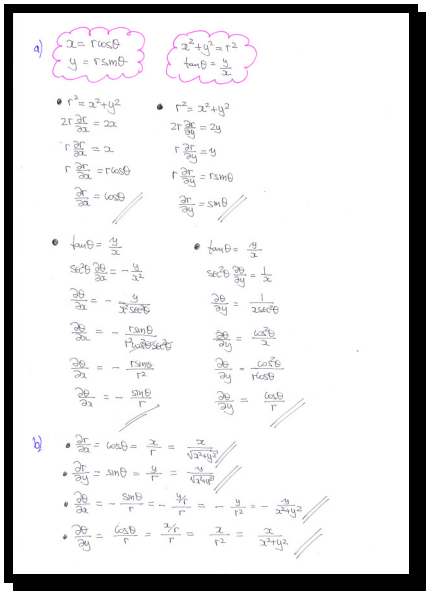
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,.$$

a) Find simplified expressions for $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial r}{\partial y}$, $\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial \theta}{\partial y}$, in terms of r and θ.

b) Deduce simplified expressions for $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial r}{\partial y}$, $\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial \theta}{\partial y}$, in terms of x and y.

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}\,,\quad \boxed{\frac{\partial r}{\partial y} = \sin\theta}\,,\quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}}\,,\quad \boxed{\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}}\,,\quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}}\,,\quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2}}\,,\quad \boxed{\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}}$$



Question 15 (+)**

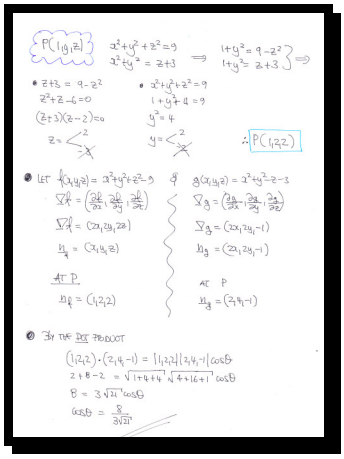
The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3 .$$

At the point P, the two surfaces intersect each other at an angle θ.

Given further that P lies in the first octant, determine the exact value of $\cos\theta$.

$$\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$



Question 16 (+)**

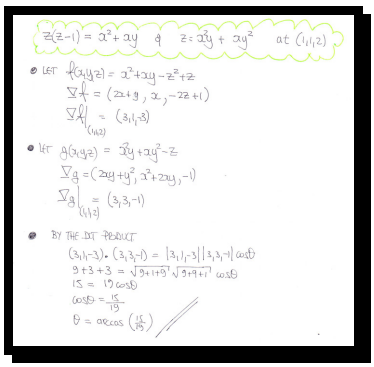
The point $P(1,1,2)$ lies on both surfaces with Cartesian equations

$$z(z-1)=x^2+xy \quad \text{and} \quad z=x^2y+xy^2 .$$

At the point P, the two surfaces intersect each other at an angle θ.

Determine the exact value of θ.

$$\boxed{\theta=\arccos\left(\frac{15}{19}\right)}$$



Question 17 (***)

The point $P(-1,1,3)$ lies on both surfaces with Cartesian equations

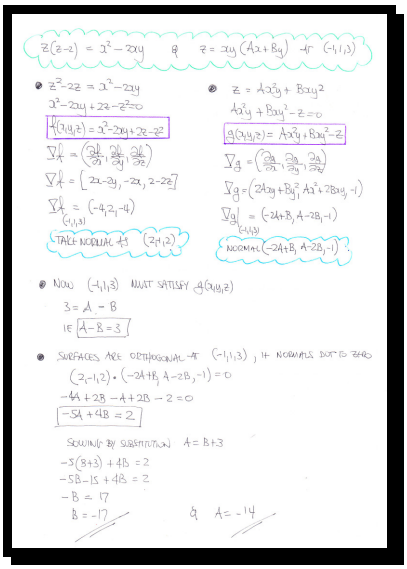
$$z(z-2)=x^2-2xy \quad \text{and} \quad z=xy(Ax+By),$$

where A and B are non zero constants.

The two surfaces intersect each other orthogonally at the point P.

Determine the value of A and the value of B.

$\boxed{A=-14}$, $\boxed{B=-17}$



Question 18 (***)

The function f depends on u, v and t so that

$$f\left\{u\left[x(t),y(t),z(t)\right],v\left[x(t),y(t),z(t)\right],t\right\}=u^2+v+2t\,.$$

It is further given that

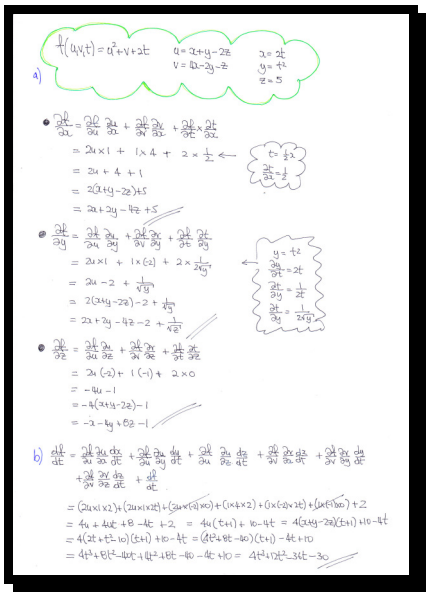
$$u=x+y-2z\,,\quad v=4x-2y-z\qquad\text{and}\qquad x=2t\,,\ y=t^2\,,\ z=5\,.$$

a) Find simplified expressions for $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, in terms of x, y and z.

b) Determine an expression for $\dfrac{df}{dt}$, in terms of t.

$$\boxed{\frac{\partial f}{\partial x}=2x+2y-4z+5}\,,\quad \boxed{\frac{\partial f}{\partial y}=2x+2y-4z+\frac{1}{\sqrt{z}}-2}\,,\quad \boxed{\frac{\partial f}{\partial z}=-x-4y+8z-1}\,,$$

$$\boxed{\frac{df}{dt}=4t^3+12t^2-36t-30}$$



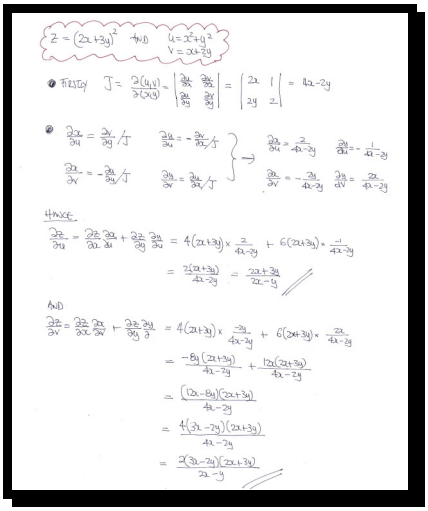
Question 19 (***)

The function z depends on u and v so that

$$z=(2x+3y)^2,\quad u=x^2+y^2 \quad \text{and} \quad v=x+2y.$$

Find simplified expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$, in terms of x and y.

$$\boxed{\frac{\partial z}{\partial u}=\frac{2x+3y}{2x-y}},\quad \boxed{\frac{\partial z}{\partial v}=\frac{2(3x-2y)(2x+3y)}{2x-y}}$$



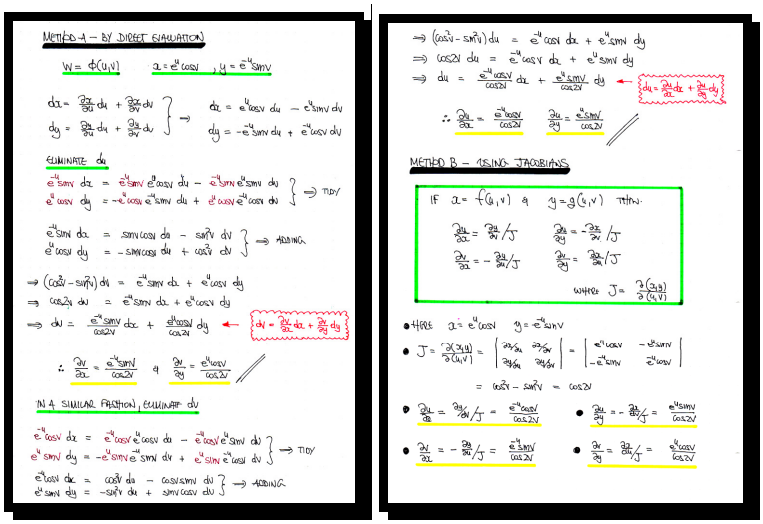
Question 20 (***)

The function $w=\varphi\left[u(x,y),v(x,y)\right]$ satisfies

$$x=\mathrm{e}^{u}\cos v \quad \text{and} \quad y=\mathrm{e}^{-u}\sin v\,.$$

Determine simplified expressions for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$, in terms of u and v.

$\square$, $\boxed{\dfrac{\partial u}{\partial x}=\dfrac{\mathrm{e}^{-u}\cos v}{\cos 2v}}$, $\boxed{\dfrac{\partial u}{\partial y}=\dfrac{\mathrm{e}^{u}\sin v}{\cos 2v}}$, $\boxed{\dfrac{\partial v}{\partial x}=\dfrac{\mathrm{e}^{-u}\sin v}{\cos 2v}}$, $\boxed{\dfrac{\partial v}{\partial y}=\dfrac{\mathrm{e}^{u}\cos v}{\cos 2v}}$



Question 21 (***)

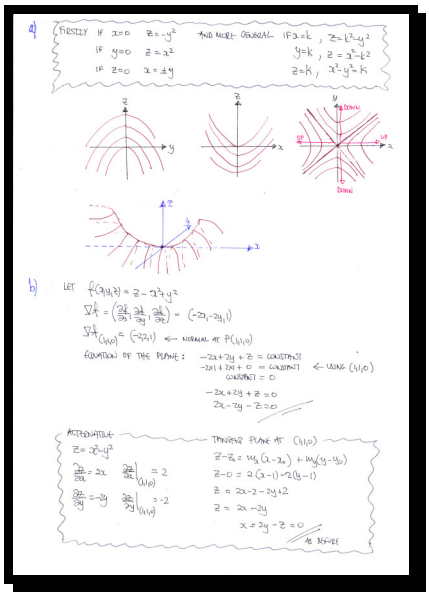
A surface S has Cartesian equation

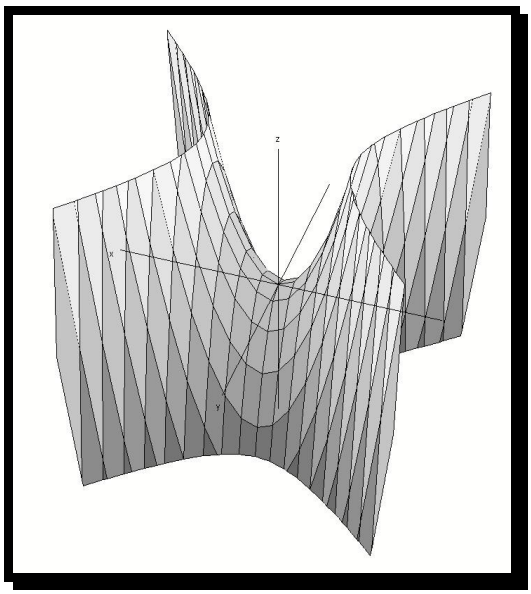
$$z = x^2 - y^2 .$$

a) Sketch profiles of S parallel to the y-z plane, parallel to the x-z plane, and parallel to the x-y plane.

b) Find an equation of the tangent plane on S, at the point $P(1,1,0)$.

$$\boxed{2x - 2y - z = 0}$$





Question 22 (***)

A surface S is given parametrically by

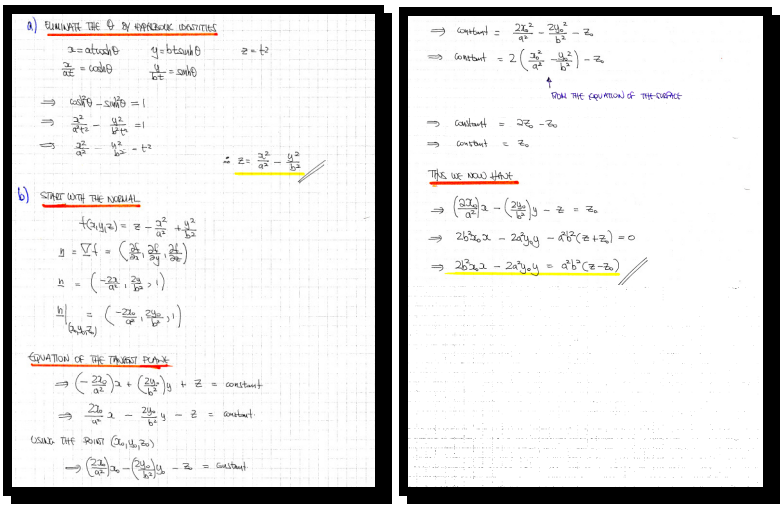
$$x = at\cosh\theta\,,\ x = bt\sinh\theta\,,\ z = t^2\,,$$

where t and θ are real parameters, and a and b are non zero constants .

a) Find a Cartesian equation for S .

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0).

$$\boxed{}\,,\quad \boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,,\quad \boxed{2b^2 x\, x_0 - 2a^2 y\, y_0 = a^2 b^2 (z + z_0)}$$



Question 23 (***)

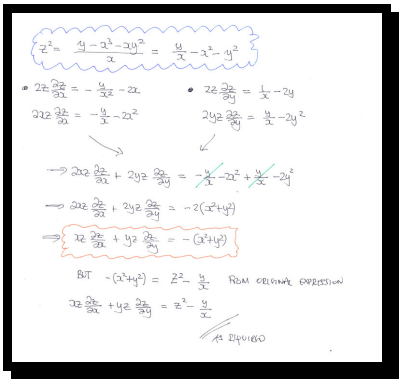
The function z depends on x and y so that

$$z^2(x,y)=\frac{y-x^3-xy^2}{x},\ x\neq 0.$$

Show that

$$xz\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}=z^2-\frac{y}{x}.$$

proof



Question 24 (***)

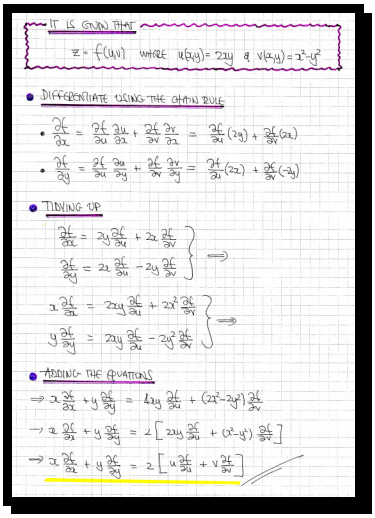
The function f depends on u and v where

$$u = 2xy \qquad \text{and} \qquad v = x^2 - y^2 .$$

Assuming $x \neq y$, $x \neq 0$ and $y \neq 0$, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left[u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} \right].$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 25 (***)

The functions F and G satisfy

$$G(x,y)\equiv F\left[u(x,y),\ v(x,y)\right],$$

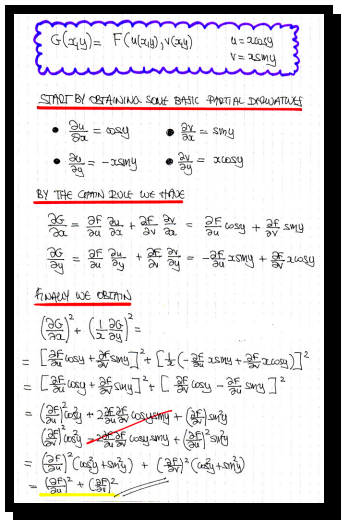
where u and v satisfy the following transformation equations.

$$u=x\cos y\,,\quad v=x\sin y\,.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial x}\right]^2+\left[\frac{1}{x}\frac{\partial G}{\partial y}\right]^2=\left[\frac{\partial F}{\partial u}\right]^2+\left[\frac{\partial F}{\partial v}\right]^2.$$

, proof



Question 26 (*+)**

The function f is defined as

$$f(x,y,z)\equiv x^3-75x+3z(y-1)^2+z^3.$$

The point Q lies on f.

The derivatives at Q in the directions $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}-\mathbf{k}$, are equal.

a) Show that Q must lie on the surface of a sphere S.

The point $P(1,3,a)$ lies on S.

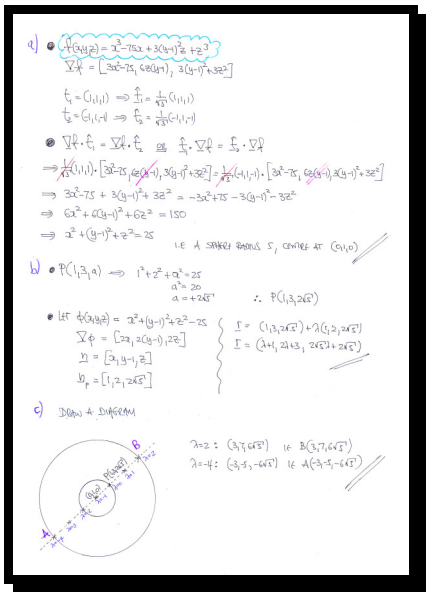
b) Find a vector equation of the normal line to S at P.

A sphere T is concentric to S and has radius three times as large as that of S.

The normal line to S at P intersects the surface of T at the points A and B.

c) Determine the coordinates of A and B.

$$\boxed{(x,y,z)=\left[\lambda+1,2\lambda+3,2(\lambda+1)\sqrt{5}\right]},\quad \boxed{A\left(3,7,6\sqrt{5}\right)\quad B\left(-3,-5,-6\sqrt{5}\right)}$$



Question 27 **(***+)**

The function z depends on x and y so that

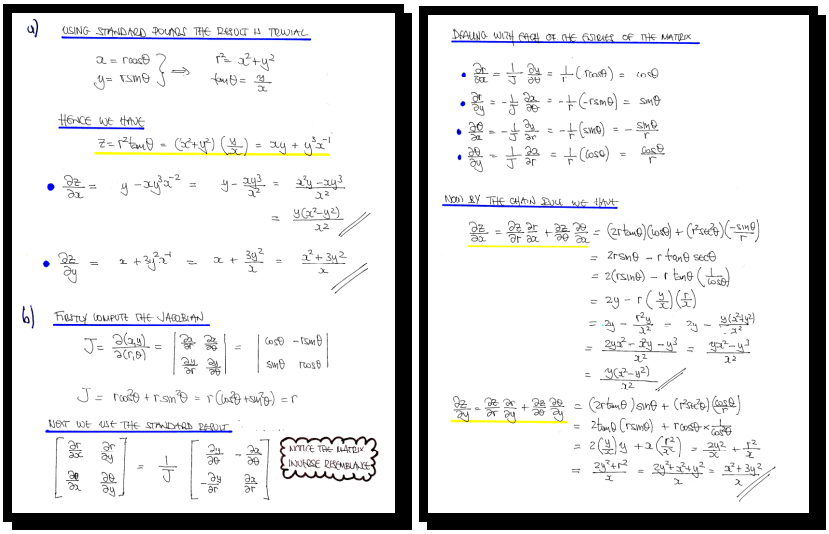
$$z = r^2 \tan\theta\,, \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,.$$

a) Express r and θ in terms of x and y and hence determine expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, in terms of x and y.

Give each of the answers as a single simplified fraction

b) Verify the answer to part **(a)** by implicit differentiation using Jacobians

$$\square\,, \quad \boxed{\frac{\partial z}{\partial x} = \frac{y\left(x^2 - y^2\right)}{x^2}}\,, \quad \boxed{\frac{\partial z}{\partial y} = \frac{x^2 + 3y^2}{x}}$$



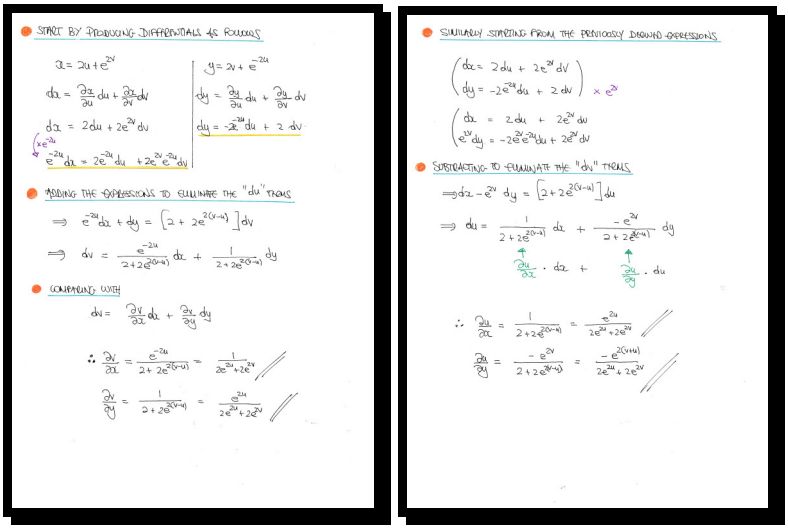
Question 28 (*+)**

The function φ depends on u and v so that

$$x=2u+\mathrm{e}^{2v} \qquad \text{and} \qquad y=2v+\mathrm{e}^{-2u}.$$

Without using standard results involving Jacobians, determine simplified expressions for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$, in terms of u and v.

$\square$, $\boxed{\dfrac{\partial u}{\partial x}=\dfrac{1}{2+2\mathrm{e}^{2(v-u)}}}$, $\boxed{\dfrac{\partial u}{\partial y}=-\dfrac{\mathrm{e}^{2v}}{2+2\mathrm{e}^{2(v-u)}}}$, $\boxed{\dfrac{\partial v}{\partial x}=\dfrac{\mathrm{e}^{-2u}}{2+2\mathrm{e}^{2(v-u)}}}$, $\boxed{\dfrac{\partial v}{\partial y}=\dfrac{1}{2+2\mathrm{e}^{2(v-u)}}}$



Question 29 (*+)**

A hill is modelled by the equation

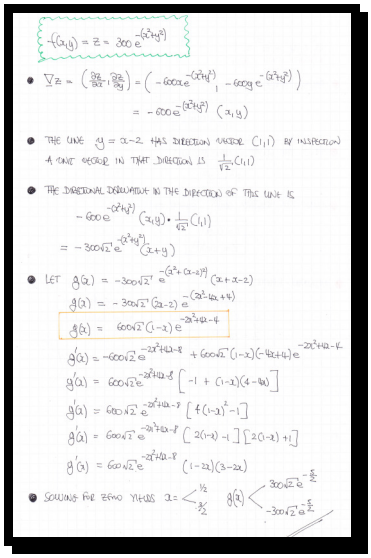
$$f(x,y)=300\,\mathrm{e}^{-\left(x^2+y^2\right)},\ x\in\mathbb{R},\ y\in\mathbb{R}.$$

A railway runs along the straight line with equation

$$y=x-2.$$

Determine the steepest slope that the train needs to climb.

$$\boxed{\pm\,300\sqrt{2}\,\mathrm{e}^{-\frac{5}{2}}}$$



Question 30 (*+)**

The functions F and G satisfy

$$G(u,v)\equiv F\Big[x(u,v),\ y(u,v)\Big],$$

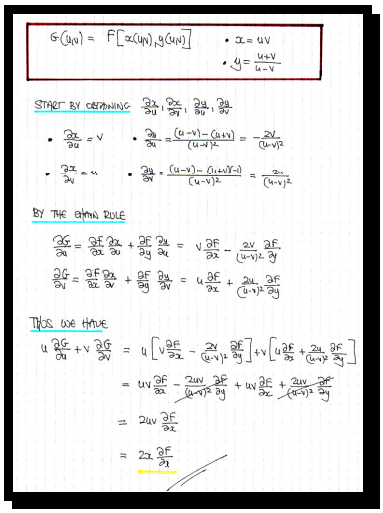
where x and y satisfy the following transformation equations.

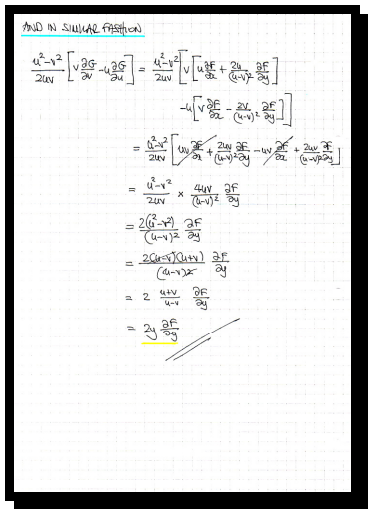
$$x=uv\,,\qquad y=\frac{u+v}{u-v}\,.$$

Use the chain rule for partial derivatives to show that

$$u\frac{\partial G}{\partial u}+v\frac{\partial G}{\partial v}=2x\frac{\partial F}{\partial x}\qquad\text{and}\qquad\frac{u^2-v^2}{2uv}\left[v\frac{\partial G}{\partial v}-u\frac{\partial G}{\partial u}\right]=2y\frac{\partial F}{\partial y}\ .$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 31 (****)

The function z depends on x and y so that

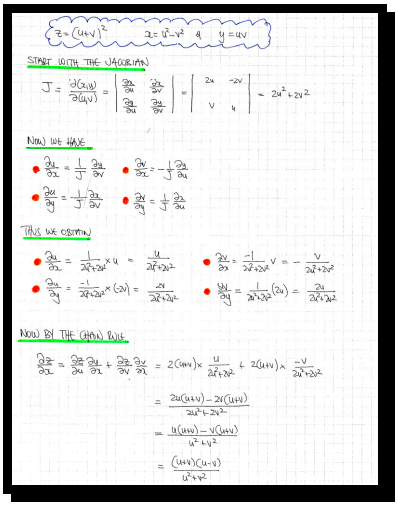
$$z=(u+v)^2,\quad x=u^2-v^2 \quad \text{and} \quad y=uv.$$

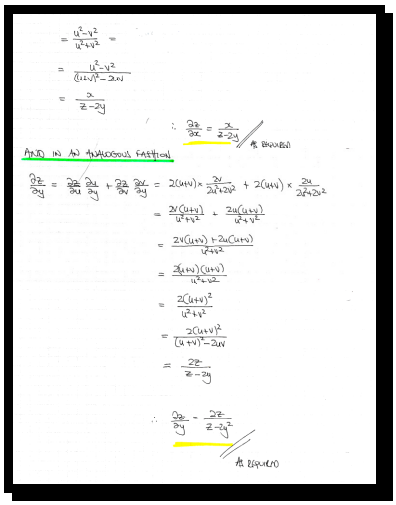
Show clearly that …

i. … $\dfrac{\partial z}{\partial x}=\dfrac{x}{z-2y}$.

ii. … $\dfrac{\partial z}{\partial y}=\dfrac{2z}{z-2y}$.

[ ], [proof]





Question 32 (****)

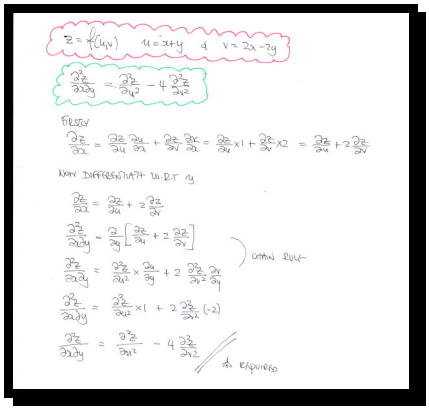
The function z depends on x and y so that

$$z=f\left(u,v\right),\quad u=x+y\quad\text{and}\quad v=2x-2y\,.$$

Show clearly that

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial u^2}-4\frac{\partial^2 z}{\partial v^2}\,.$$

proof



Question 33 (****)

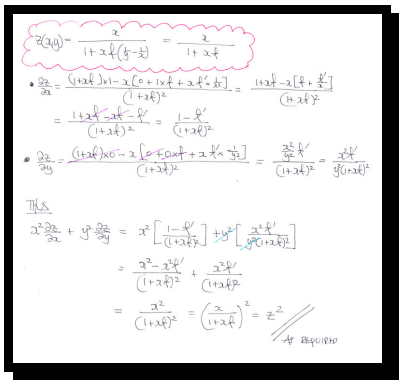
The function z depends on x and y so that

$$z=\frac{x}{1+x\,f} \qquad \text{where} \quad f=f\left(\frac{1}{y}-\frac{1}{x}\right).$$

Show clearly that

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=z^2.$$

proof



Question 34 (****)

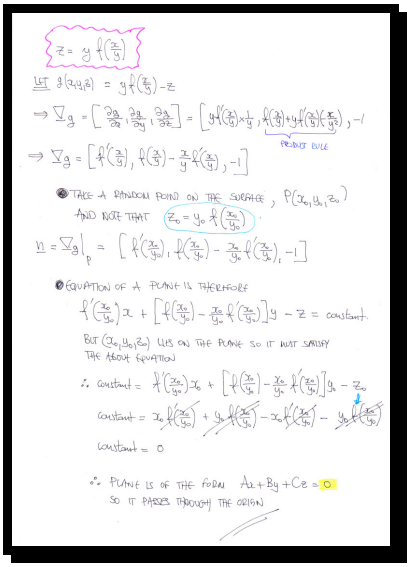
The surface S has equation

$$z = y\, f\left(\frac{x}{y}\right),$$

where the function $f\left(\frac{x}{y}\right)$ is differentiable.

Show that the tangent plane at any point on S passes through the origin O

proof



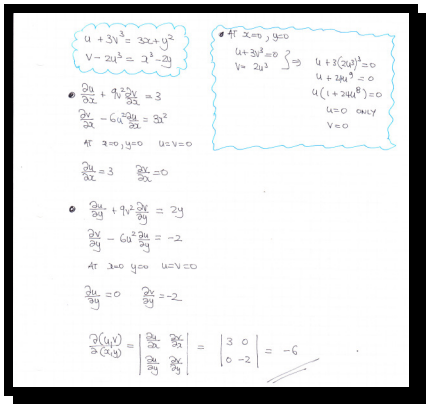
Question 35 (****)

The functions $u=u(x,y)$ and $v=v(x,y)$ satisfy

$$u+3v^3=3x+y^2 \quad \text{and} \quad v-2u^3=x^3-2y\,.$$

Determine the value of $\dfrac{\partial(u,v)}{\partial(x,y)}$ at $(x,y)=(0,0)$.

$$\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{(0,0)}=-6$$



Question 36 (****)

A surface S has equation $f(x,y,z)=0$, where

$$f(x,y,z)=x^2+2xy-4x+2y^2+2yz-8y-z^2+4z\,.$$

a) Show that there is no point on S where the normal to S is parallel to the z axis and hence state the geometric significance of this result with reference to the stationary points of S.

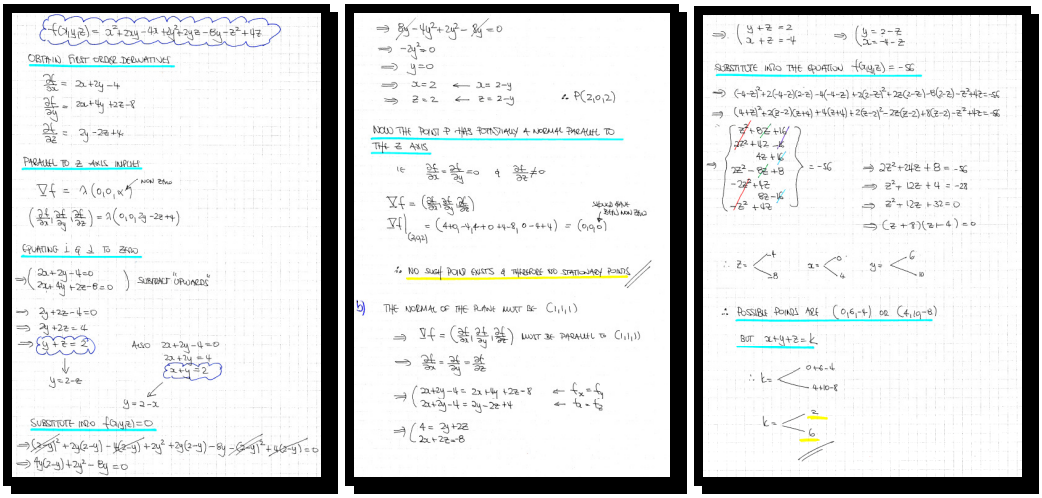
S is translated to give a new surface T with equation

$$f(x,y,z)=-56\,.$$

The plane with equation $x+y+z=k$, where k is a constant, is a tangent plane to T.

b) Determine the two possible values of k.

$\boxed{}$, $\boxed{k=2\ \cup\ k=6}$



Question 37 (****)

A surface S has equation $f(x,y,z)=0$, where

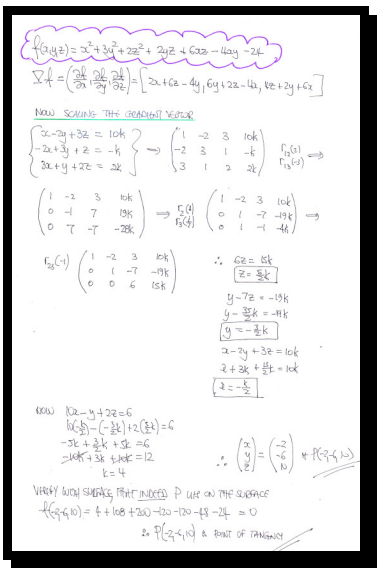
$$f(x,y,z)=x^2+3y^2+2z^2+2yz+6xz-4xy-24.$$

Show that the plane with equation

$$10x-y+2z=6,$$

is a tangent plane to S, and find the coordinates of the point of tangency.

$$\boxed{(-2,-6,10)}$$



Question 38 (****)

It is given that g is a twice differentiable function of one variable, with domain all real numbers.

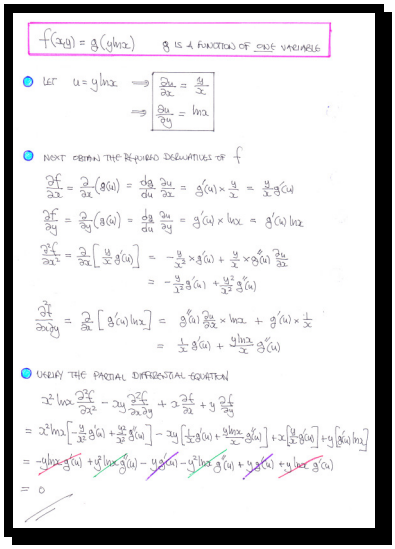
It is further given that for $x>0$

$$f(x,y)=g(y\ln x).$$

Show that

$$x^2\ln x\frac{\partial^2 f}{\partial x^2}-xy\frac{\partial^2 f}{\partial x\partial y}+x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0.$$

proof



Question 39 (****)

The function w depends on x and y so that

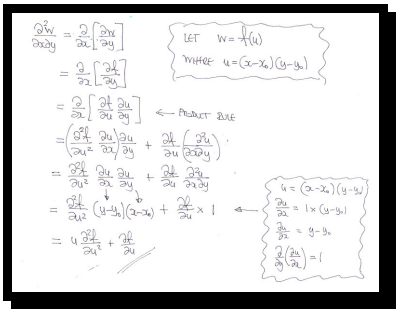
$$w = f(u), \quad \text{and} \quad u = (x - x_0)(y - y_0),$$

where x_0 and y_0 are constants.

Show clearly that

$$\frac{\partial^2 w}{\partial x \partial y} = u\frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u}.$$

proof



Question 40 (****)

The functions f and G satisfy

$$G(r,\theta,\varphi)\equiv f\left[x(r,\theta,\varphi),\ y(r,\theta,\varphi),\ z(r,\theta,\varphi)\right],$$

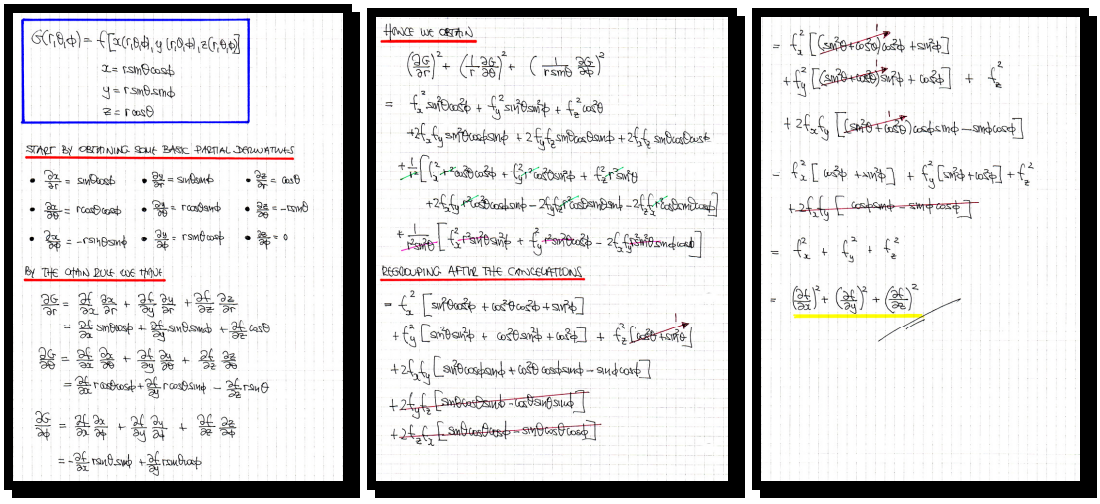
where x, y and z satisfy the standard Spherical Polar Coordinates transformation relationships

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial r}\right]^2+\left[\frac{1}{r}\frac{\partial G}{\partial\theta}\right]^2+\left[\frac{1}{r\sin\theta}\frac{\partial G}{\partial\varphi}\right]^2=\left[\frac{\partial f}{\partial x}\right]^2+\left[\frac{\partial f}{\partial y}\right]^2+\left[\frac{\partial f}{\partial z}\right]^2.$$

[ ], [proof]



Question 41 (**+)**

It is given that

$$z(x,y)=f(u,v),$$

so that

$$u=x^3+y^3 \quad \text{and} \quad v=\frac{y}{x}.$$

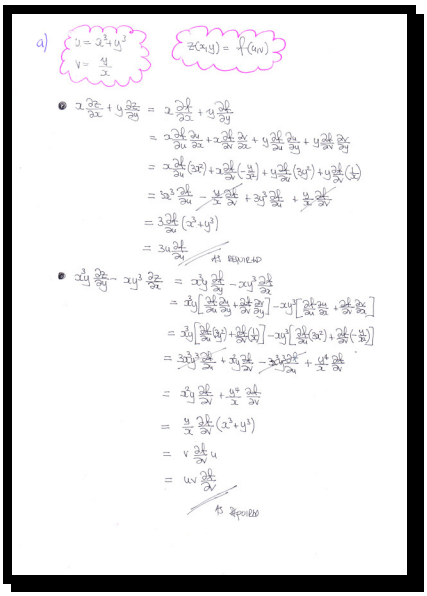
a) Use the chain rule to show that

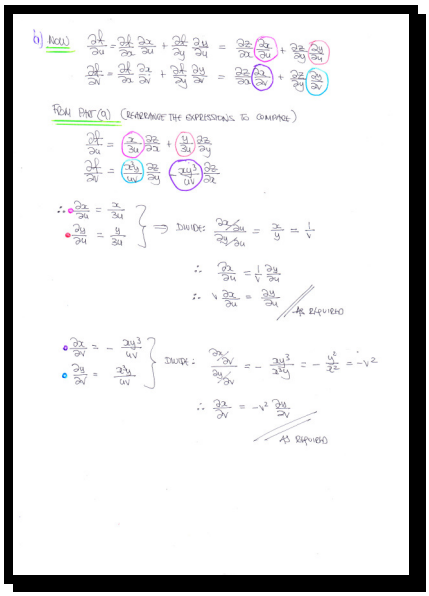
$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3u\frac{\partial f}{\partial u} \qquad \text{and} \qquad yx^3\frac{\partial z}{\partial y}-xy^3\frac{\partial z}{\partial x}=uv\frac{\partial f}{\partial v}.$$

b) Hence show further that

$$v\frac{\partial x}{\partial u}=\frac{\partial y}{\partial u} \qquad \text{and} \qquad \frac{\partial x}{\partial v}=-v^2\frac{\partial y}{\partial v}.$$

proof





Question 42 (**+)**

It is given that f and g are differentiable functions of one variable, with domain all real numbers.

It is further given that for $x>0$

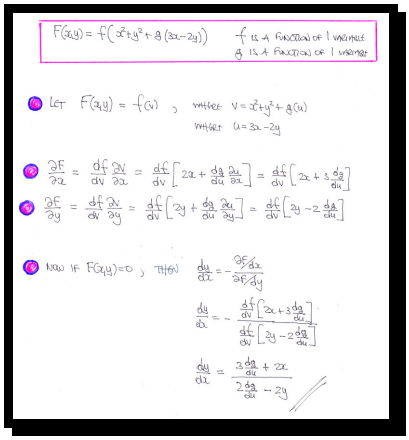
$$F(x,y)=f\left[x^2+y^2+g(3x-2y)\right].$$

If the function $y=y(x)$ is a rearrangement of $F(x,y)=0$, show that

$$\frac{dy}{dx}=\frac{3\dfrac{dg}{du}+2x}{2\dfrac{dg}{du}-2y},$$

where $u=3x-2y$

proof



Question 43 (**+)**

The surface S has Cartesian equation

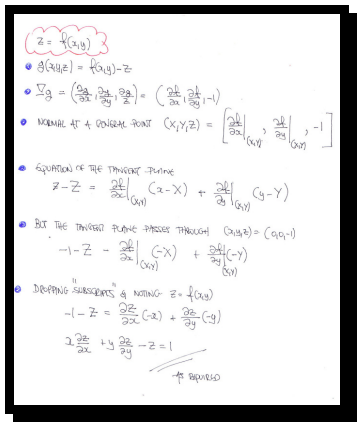
$$z = f(x, y).$$

The tangent plane at any point on S passes through the point $(0,0,-1)$.

Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 1.$$

proof



Question 44 (*****)

It is given that the function f depends on x and y, and the function g depends on u and v, so that

$$f(x,y)=g(u,v),\quad u=x^2-y^2\quad\text{and}\quad v=2xy.$$

a) Show that

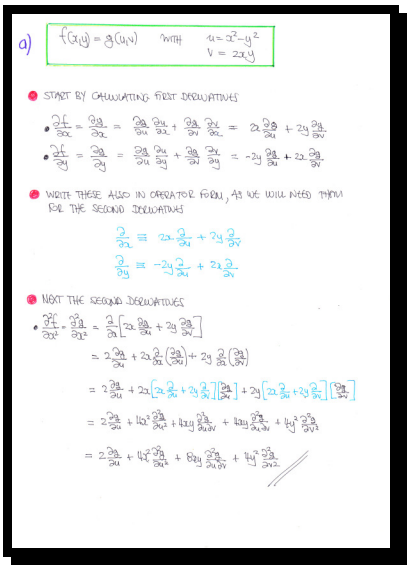
$$\frac{\partial^2 f}{\partial x^2}=2\frac{\partial g}{\partial u}+4x^2\frac{\partial^2 g}{\partial u^2}+8xy\frac{\partial^2 g}{\partial u\partial v}+4y^2\frac{\partial^2 g}{\partial v^2},$$

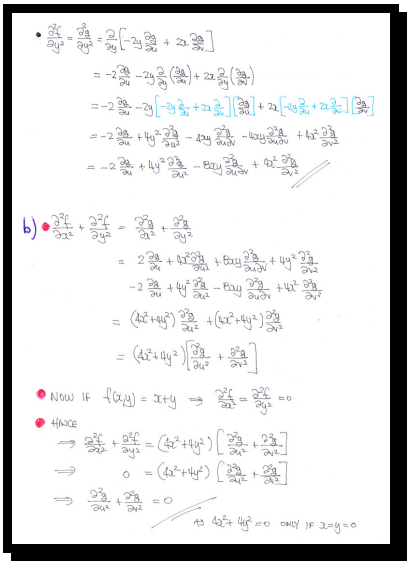
and find a similar expression for $\dfrac{\partial^2 f}{\partial y^2}$.

b) Deduce that if $f(x,y)=x+y$

$$\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}=0.$$

proof





Question 45 (*****)

The function z depends on x and y so that

$$z=f(u,v),\quad u=x-2\sqrt{y}\quad\text{and}\quad v=x+2\sqrt{y}\,.$$

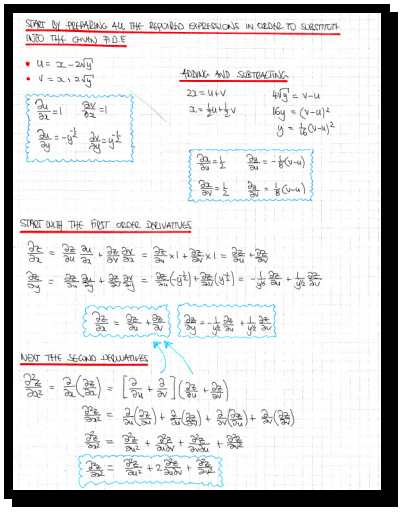
Show that the partial differential equation

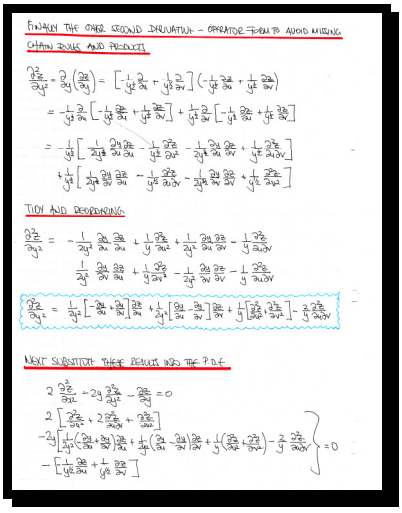
$$2\frac{\partial^2 z}{\partial x^2}-2y\frac{\partial^2 z}{\partial y^2}-\frac{\partial z}{\partial y}=0\,,$$

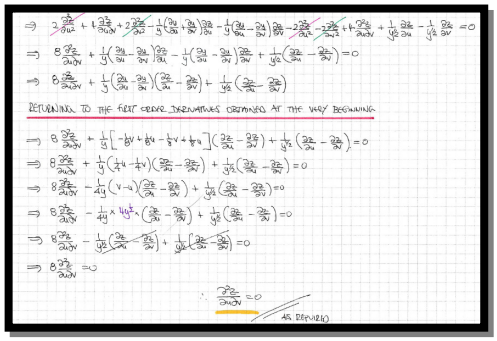
can be simplified to

$$\frac{\partial^2 z}{\partial u\partial v}=0\,.$$

, proof







Question 46 (*****)

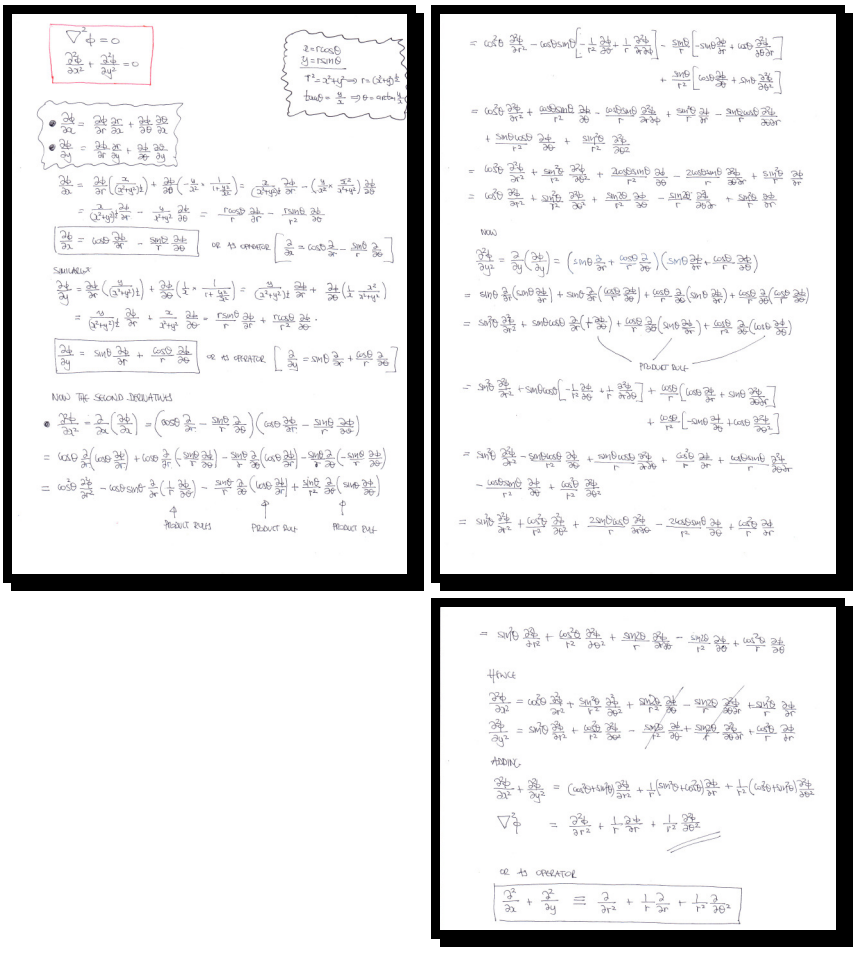
$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0\,.$$

The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2}+\frac{1}{r}\frac{\partial \varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}=0\,.$$

proof



Question 47 **(*****)**

It is given that for $\varphi=\varphi(x,y)$ and $\psi=\psi(x,y)$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y}\qquad\text{and}\qquad\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}.$$

Show that the above pair of coupled partial differential equations transform in plane polar coordinates to

$$\frac{\partial\varphi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\qquad\text{and}\qquad\frac{\partial\psi}{\partial r}=-\frac{1}{r}\frac{\partial\varphi}{\partial\theta}.$$

proof

