PARTIAL DIFFERENTIATION APPLICATIONS

STATIONARY POINTS

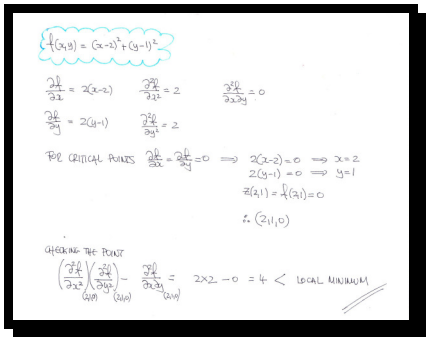
Question 1 (**)

A surface has Cartesian equation $z = f(x, y)$, given by

$$f(x, y) = (x-2)^2 + (y-1)^2 .$$

Investigate the critical points of f.

local minimum at $(2,1,0)$

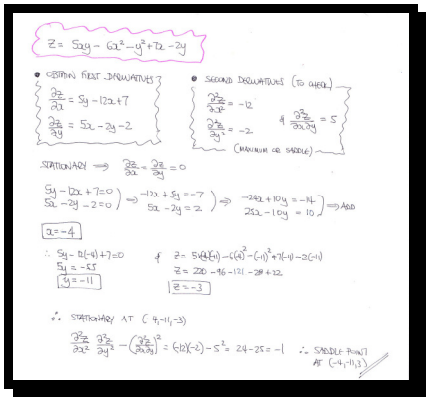


Question 2 (**)

$$z = 5xy - 6x^2 - y^2 + 7x - 2y .$$

Investigate the critical points of z.

saddle point at $(-4,-11,-3)$



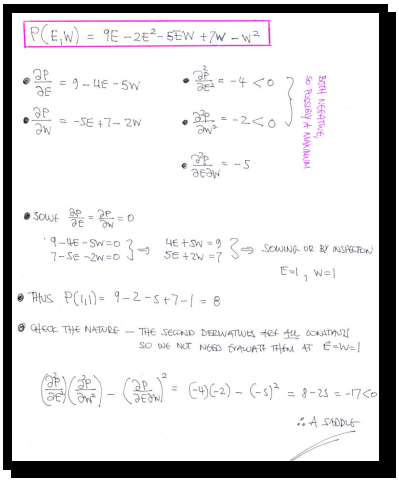
Question 3 (**)

A profit function P depends on two variables E and W, as follows.

$$P(E,W) = 9E - 2E^2 - 5EW + 7W - W^2 , \ E > 0 , \ W > 0 .$$

Investigate the critical points of P.

saddle "point" at $(E,W,P) = (1,1,8)$

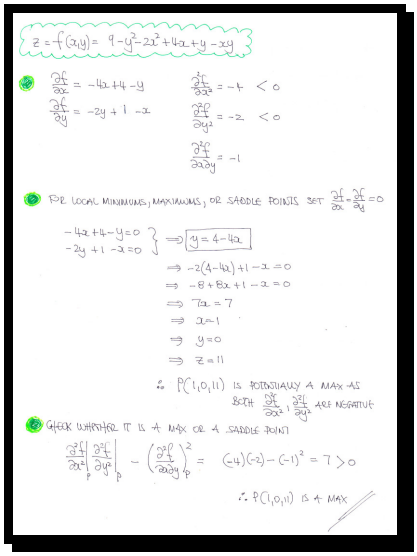


Question 4 (**)

$$f(x,y)=9-y^2-2x^2+4x+y-xy\,.$$

Investigate the critical points of f.

$\boxed{\text{maximum point at } (1,0,11)}$

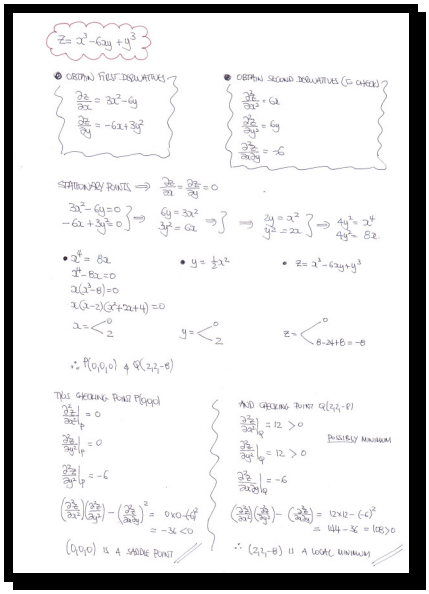


Question 5 (+)**

$$z = x^3 - 6xy + y^3 .$$

Investigate the critical points of z.

$\boxed{\text{saddle point at } (0,0,0)}$, $\boxed{\text{local minimum at } (2,2,-8)}$

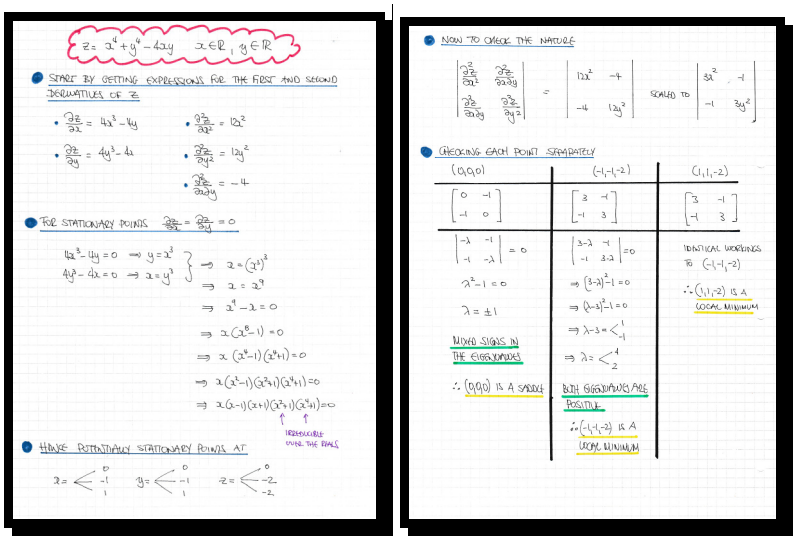


Question 6 (+)**

$$z(x,y)=x^4+y^4-4xy\,.$$

Investigate the critical points of z.

$\boxed{}$, $\boxed{\text{saddle point at } (0,0,0)}$, $\boxed{\text{local minimum at } (-1,-1,-2)}$, $\boxed{\text{local minimum at } (1,1,-2)}$

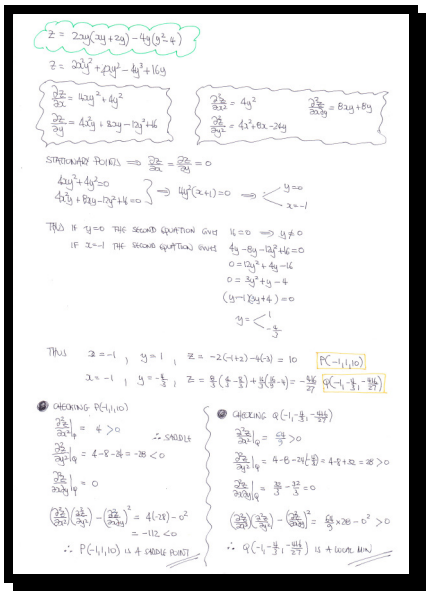


Question 7 (***)

$$z = 2xy\left(xy+2y\right)-4y\left(y^2-4\right).$$

Investigate the critical points of z.

$\boxed{\text{saddle point at } \left(-1,1,10\right)}$, $\boxed{\text{local minimum at } \left(-1,-\frac{4}{3},-\frac{416}{27}\right)}$



Question 8 (***)

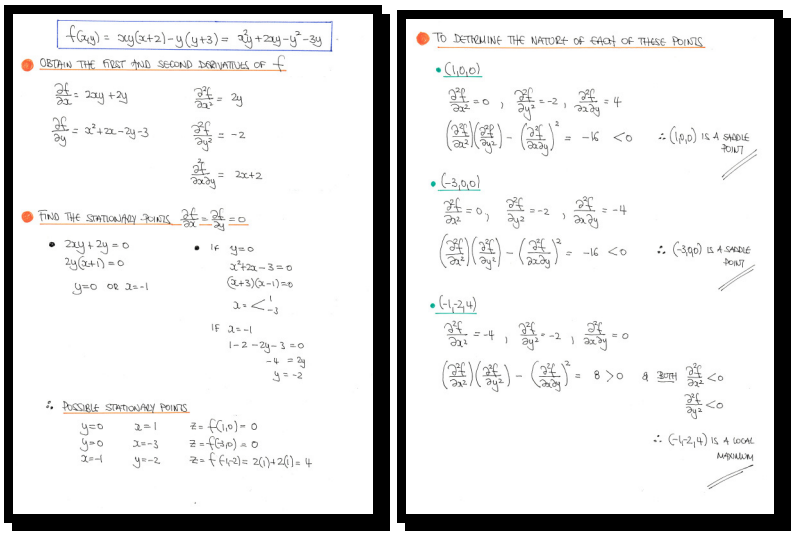
The function of two variables f is defined as

$$f(x,y)\equiv xy(x+2)-y(y+3),\quad x\in\mathbb{R},\ y\in\mathbb{R}.$$

Find the coordinates of each of the stationary points of f, where $z=f(x,y)$, and further determine their nature.

$\boxed{}$, $\boxed{\text{saddle point at } (1,0,0)}$, $\boxed{\text{saddle point at } (-3,0,0)}$,

$\boxed{\text{local maximum at } (-1,-2,4)}$



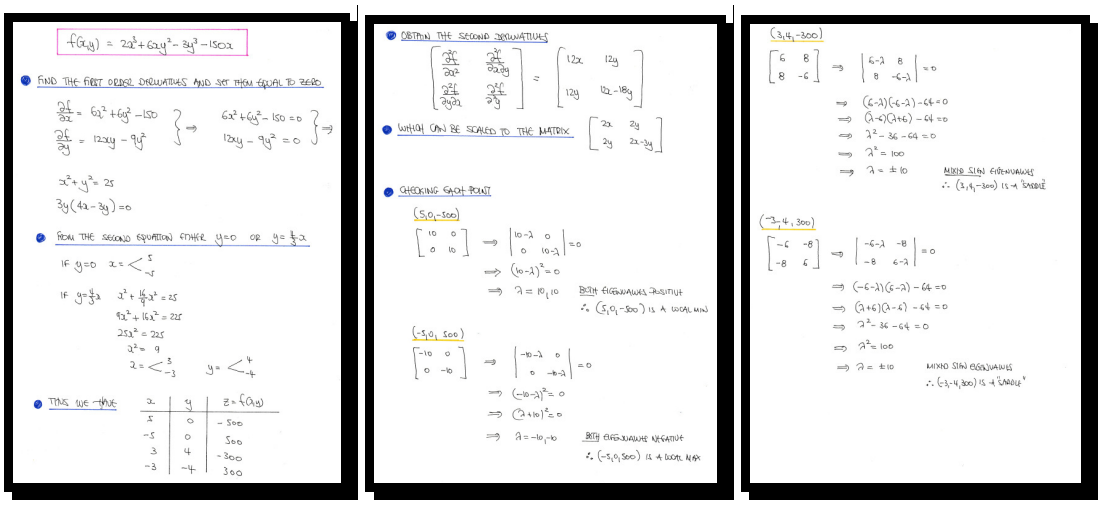
Question 9 **(***)**

The function of two variables f is defined as

$$f(x,y)\equiv 2x^3+6xy^2-3y^3-150x\,,\quad x\in\mathbb{R}\,,\ y\in\mathbb{R}\,.$$

Find the coordinates of each of the stationary points of f, where $z=f(x,y)$, and further determine their nature.

$\boxed{}$, $\boxed{\text{saddle point at } (3,4,-300)}$, $\boxed{\text{saddle point at } (-3,-4,300)}$,
$\boxed{\text{local minimum at } (5,0,-500)}$, $\boxed{\text{local maximum at } (-5,0,500)}$



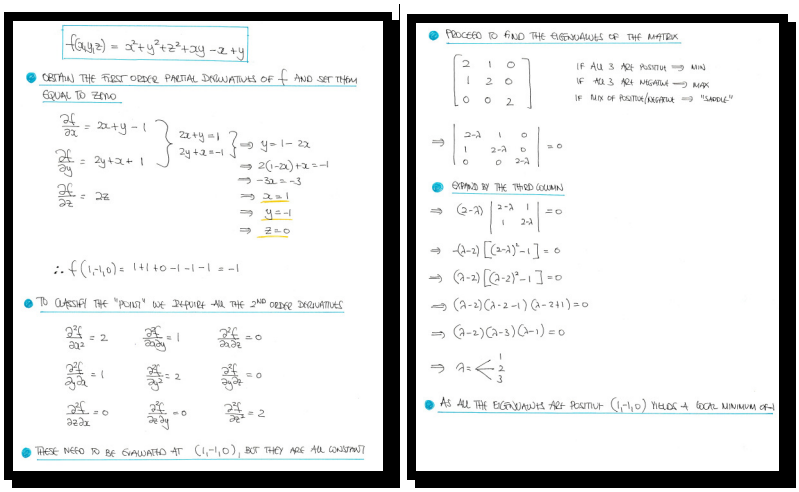
Question 10 (*+)**

The function of three variables f is defined as

$$f(x,y,z) \equiv x^2+y^2+z^2+xy-x+y\,, \quad x\in\mathbb{R}\,,\ y\in\mathbb{R}\,,\ z\in\mathbb{R}\,.$$

Find the stationary value of f, including the triple (x,y,z) which produces this value, further determining the nature of this stationary value.

$\boxed{\phantom{\text{local minimum of } -1 \text{ at } (1,-1,0)}}$, $\boxed{\text{local minimum of } -1 \text{ at } (1,-1,0)}$



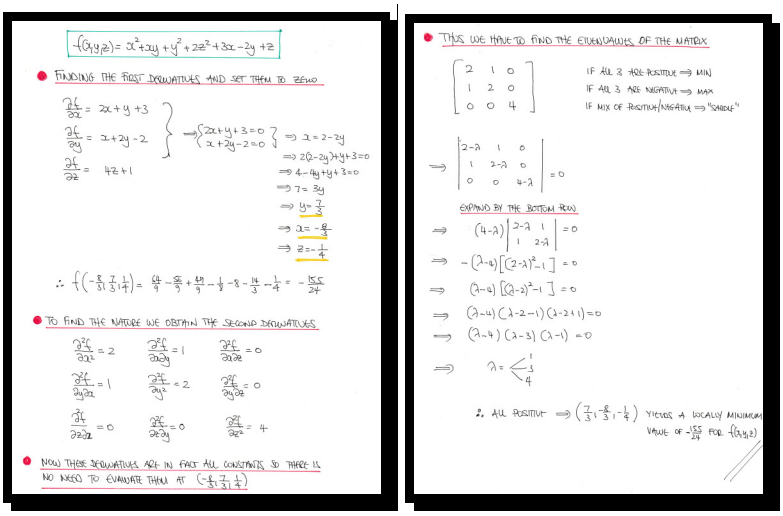
Question 11 (*+)**

The function of three variables f is defined as

$$f(x,y,z)\equiv x^2+xy+y^2+2z^2+3x-2y+z\,,\quad x\in\mathbb{R}\,,\ y\in\mathbb{R}\,,\ z\in\mathbb{R}\,.$$

Find the stationary value of f, including the triple (x,y,z) which produces this value, further determining the nature of this stationary value.

$\square$, local minimum of $-\dfrac{155}{24}$ at $\left(\dfrac{7}{3},-\dfrac{8}{3},-\dfrac{1}{4}\right)$



APPLIED OPTIMIZATION

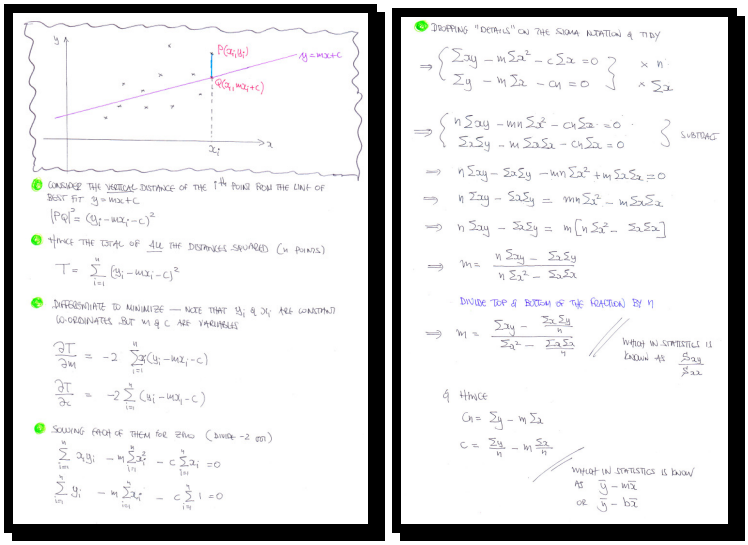
Question 1 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \leq n$, is given.

It is required to find a straight line with equation $y = mx + c$, so that the sum of the squares of the vertical distances between P_i and the straight line is least.

Find simplified expressions for each of the constants m and c, in terms of x_i and y_i.

$$m = \frac{n\sum_{i=1}^{n}(x_i y_i) - \sum_{i=1}^{n}(x_i)\sum_{i=1}^{n}(y_i)}{n\sum_{i=1}^{n}(x_i)^2 - \sum_{i=1}^{n}(x_i)\sum_{i=1}^{n}(x_i)}, \qquad c = \frac{1}{n}\sum_{i=1}^{n}(y_i) - \frac{m}{n}\sum_{i=1}^{n}(x_i)$$



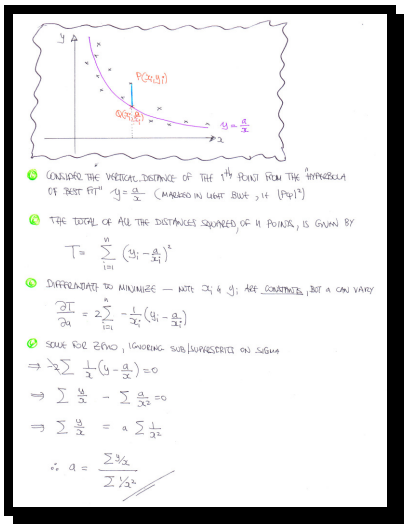
Question 2 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \leq n$, is given.

It is required to find a hyperbola with equation $y = \dfrac{a}{x}$, so that the sum of the squares of the vertical distances between P_i and the hyperbola is least.

Find a simplified expression for the constant a, in terms of x_i and y_i.

$$a = \frac{\displaystyle\sum_{i=1}^{n}\left(\frac{y_i}{x_i}\right)}{\displaystyle\sum_{i=1}^{n}\frac{1}{(x_i)^2}}$$



a) CONSIDER THE VERTICAL DISTANCE OF THE i^{th} POINT FROM THE HYPERBOLA OF BEST FIT $y = \frac{a}{x}$ (MARKED IN LIGHT BLUE, IF PRINTED)

b) THE TOTAL OF ALL THE DISTANCES SQUARED OF n POINTS, IS GIVEN BY

$$T = \sum_{i=1}^{n}\left(y_i - \frac{a}{x_i}\right)^2$$

c) DIFFERENTIATE TO MINIMIZE — NOTE x_i & y_i ARE CONSTANTS, BUT a CAN VARY

$$\frac{\partial T}{\partial a} = 2\sum_{i=1}^{n} -\frac{1}{x_i}\left(y_i - \frac{a}{x_i}\right)$$

d) SOLVE FOR ZERO, IGNORING SUBSUPERSCRIPTS ON SIGMA

$$\Rightarrow -2\sum \frac{1}{x}\left(y - \frac{a}{x}\right) = 0$$

$$\Rightarrow \sum \frac{y}{x} - \sum \frac{a}{x^2} = 0$$

$$\Rightarrow \sum \frac{y}{x} = a\sum \frac{1}{x^2}$$

$$\therefore a = \frac{\sum y/x}{\sum 1/x^2}$$

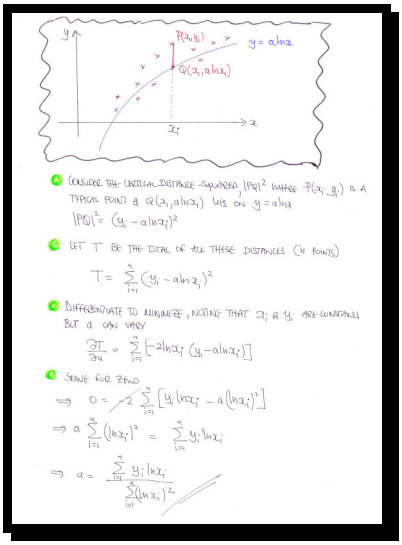
Question 3 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \leq n$, is given.

It is required to find a curve with equation $y = a \ln x$, so that the sum of the squares of the vertical distances between P_i and the curve is least.

Find a simplified expression for the constant a, in terms of x_i and y_i.

$$a = \frac{\sum_{i=1}^{n} \left[y_i \ln(x_i) \right]}{\sum_{i=1}^{n} \left[\ln(x_i) \right]^2}$$



Question 4 (****)

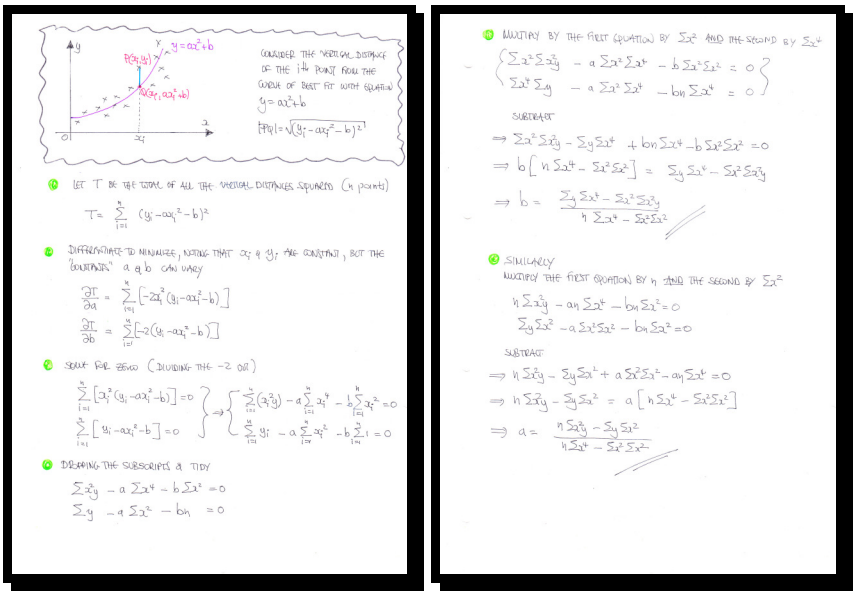
A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \leq n$, is given.

It is required to find a curve with equation $y = ax^2 + b$, so that the sum of the squares of the vertical distances between P_i and the curve is least.

Find simplified expressions for each of the constants a and b, in terms of x_i and y_i.

$$a = \frac{n\sum_{i=1}^{n}\left[x_i^2 y_i\right] - \sum_{i=1}^{n}\left[y_i\right]\sum_{i=1}^{n}\left[x_i^2\right]}{n\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2\right]},$$

$$b = \frac{\sum_{i=1}^{n}\left[y_i\right]\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2 y_i\right]}{n\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2\right]}$$



Question 5 (****)

The table below shows experimental data connecting two variables x and y.

t	5	10	15	30	70
P	181	158	145	127	107

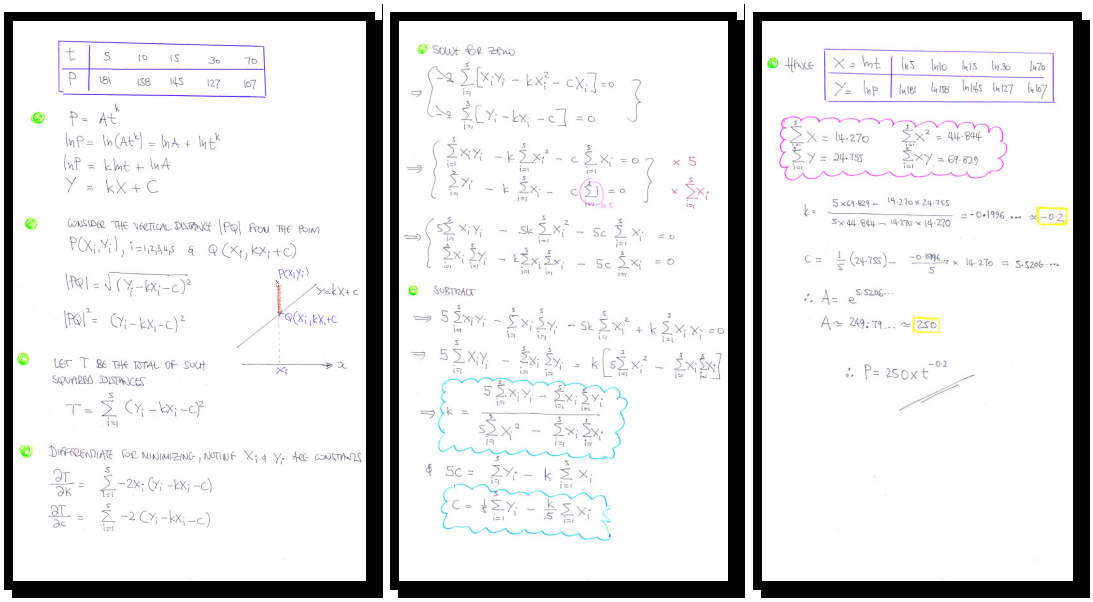
It is assumed that t and P are related by an equation of the form

$$P = A \times t^k ,$$

where A and k are non zero constants.

By linearizing the above equation, and using partial differentiation to obtain a line of least squares, determine the value of A and the value of k.

$\boxed{A \approx 250}$, $\boxed{k \approx -0.2}$



CONSTRAINED OPTIMIZATION

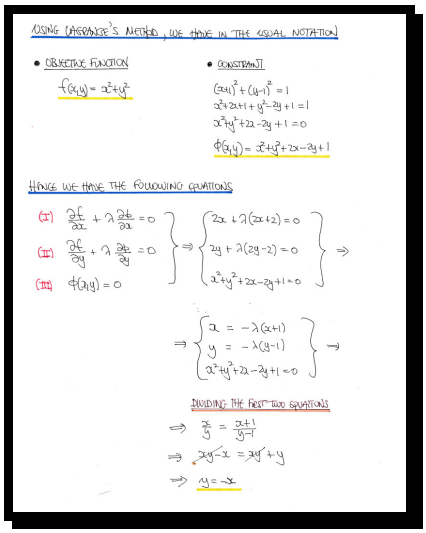
Question 1 (***)

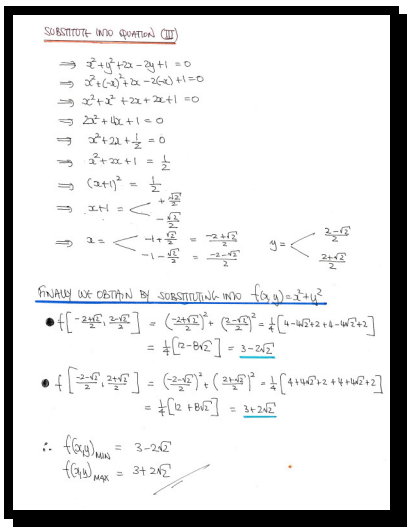
$$f(x,y)=x^2+y^2,\ x\in\mathbb{R},\ y\in\mathbb{R}.$$

The region R in the x-y plane is a circle centred at $(-1,1)$ and of radius 1.

Use partial differentiation to determine the maximum and the minimum value of f, whose projection onto the x-y plane is the region R.

$\boxed{}$, $\boxed{f_{\max}=3+2\sqrt{2}}$, $\boxed{f_{\min}=3-2\sqrt{2}}$



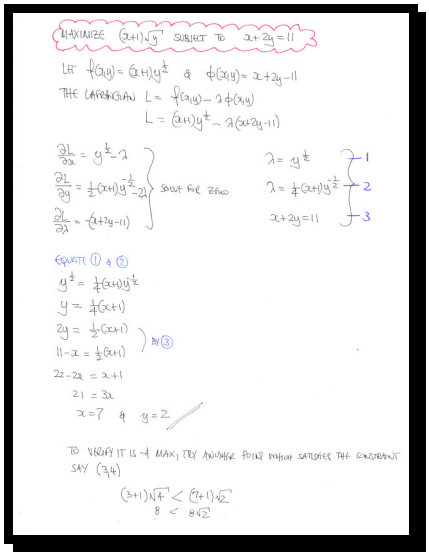


Question 2 (***)

$$f(x,y)=(x+1)\sqrt{y}\,,\ x\in\mathbb{R}\,,\ y\in\mathbb{R}\,,\ y>0\,.$$

Find the value of x and the value of y which maximizes value of f, subject to the constraint $x+2y=11$.

$$\boxed{(7,2)}$$



Question 3 (***)

The region R in the x-y plane is the ellipse with equation

$$2x^2+xy=2y^2=15\,.$$

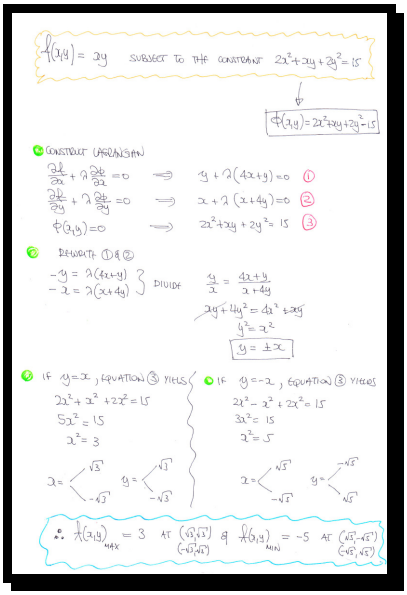
The surface with equation $z=f\left(x,y\right)$ is given by

$$f\left(x,y\right)=xy\,,\ x\in\mathbb{R}\,,\ y\in\mathbb{R}\,.$$

Determine the maximum value of f and the minimum value of f, whose projection onto the x-y plane is the region R.

Give the corresponding x and y coordinates in each case

$$\boxed{f_{\max}=3 \text{ at } \left(\sqrt{3},\sqrt{3}\right) \text{ or } \left(-\sqrt{3},-\sqrt{3}\right)}\,,\quad \boxed{f_{\max}=-5 \text{ at } \left(\sqrt{5},-\sqrt{5}\right) \text{ or } \left(-\sqrt{5},\sqrt{5}\right)}$$



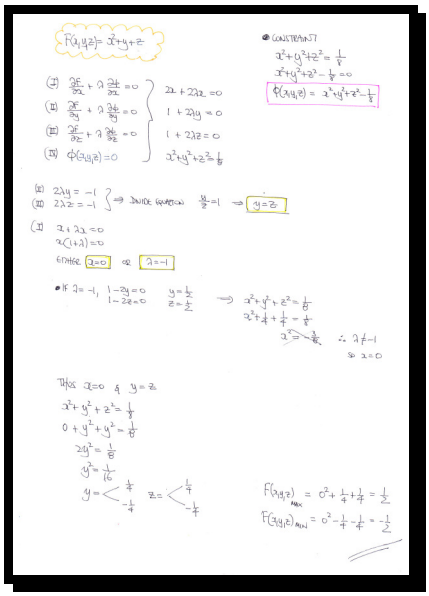
Question 4 (***)

A scalar field F exists on the surface of the sphere with Cartesian equation

$$x^2+y^2+z^2=\frac{1}{8}.$$

Given further that $F=x^2+y+z$, determine the maximum value of F and the minimum value of F.

$$\boxed{F_{\max}=\frac{1}{2}}\,,\quad \boxed{F_{\min}=-\frac{1}{2}}$$

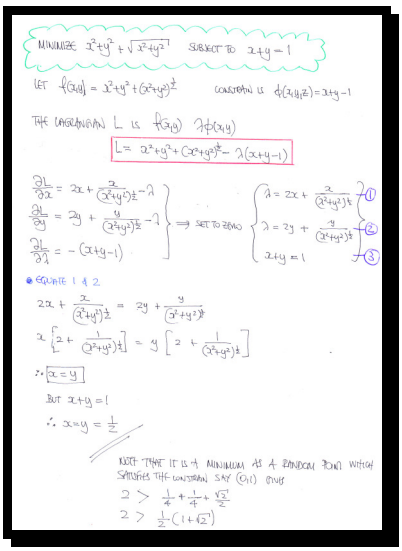


Question 5 (***)

$$f(x,y)=x^2+y^2+\sqrt{x^2+y^2}\ ,\ x\in\mathbb{R},\ y\in\mathbb{R}.$$

Find the value of x and the value of y which minimizes value of f, subject to the constraint $x+y=1$.

$$\boxed{\left(\frac{1}{2},\frac{1}{2}\right)}$$

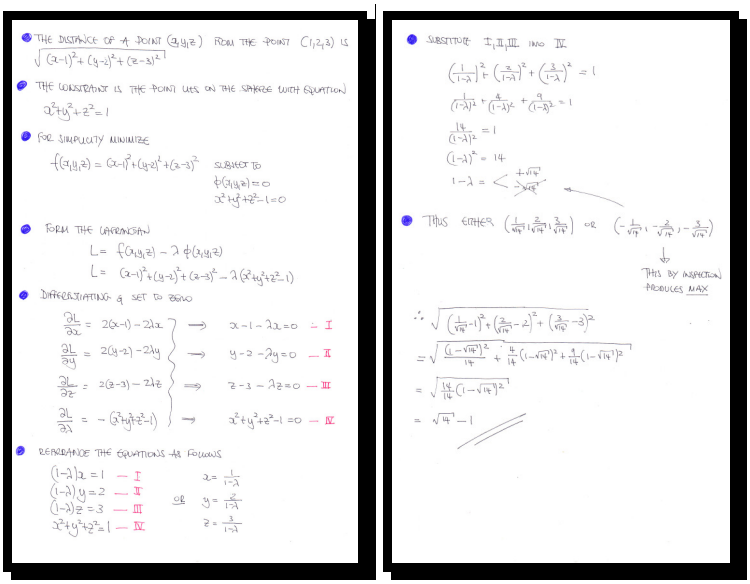


Question 6 (*+)**

Determine in exact form the shortest distance of the point $(1,2,3)$ from the sphere with equation

$$x^2+y^2+z^2=1.$$

$$\boxed{d_{\min}=\sqrt{14}-1}$$



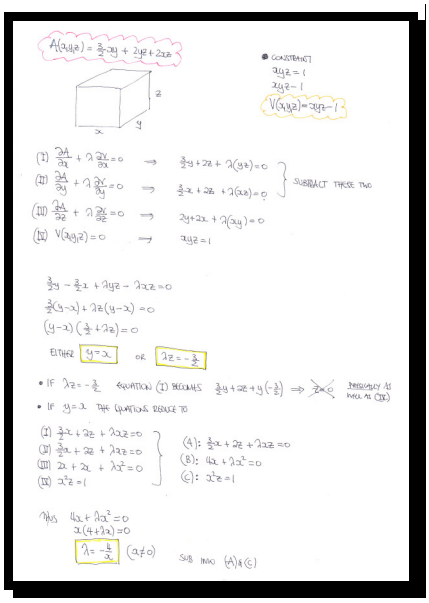
Question 7 (*+)**

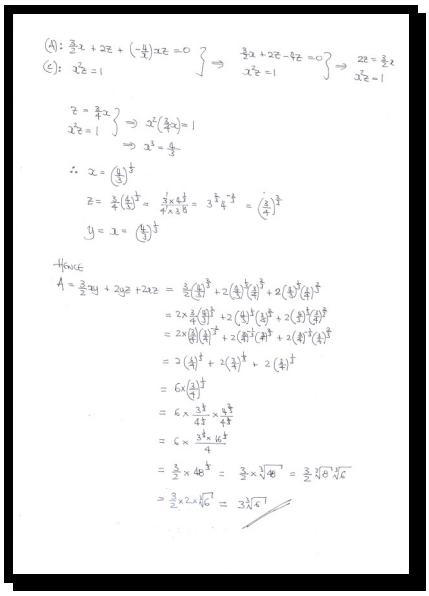
A water tank, in the shape of a cuboid, is to have a capacity of 1 m^3.

Sheet metal is used for the construction of the tank. The sheets are of uniform thickness but the density of the metal used for the lid is half the density of the metal used for the rest of the tank.

Use Lagrange's constrained optimization method to show that the minimum total sheet metal to be used is exactly $3\sqrt[3]{6}\text{ m}^2$.

proof





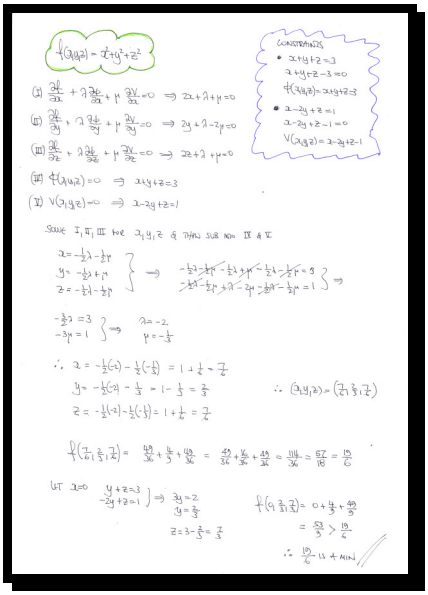
Question 8 (*+)**

$$f(x,y,z)=x^2+y^2+z^2\,.$$

Determine the minimum value of F, subject to the constraints

$$x+y+z=3 \quad \text{and} \quad x-2y+z=1\,.$$

$$\boxed{f_{\min}=\frac{19}{6}}$$

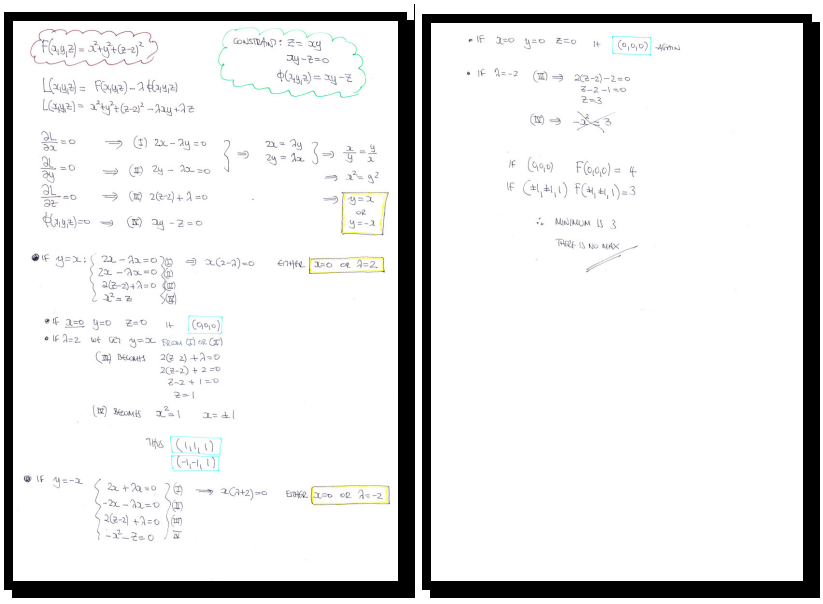


Question 9 (*+)**

$$F(x,y,z)=x^2+y^2+(z-2)^2 .$$

Determine the minimum value of F, subject to the constraint $z=xy$.

$$\boxed{F_{\min}=3}$$



Question 10 (*+)**

The points P and Q lie on the intersection of the sphere and cylinder with respective Cartesian equations

$$x^2+y^2+z^2=9 \quad \text{and} \quad x^2+y^2=8$$

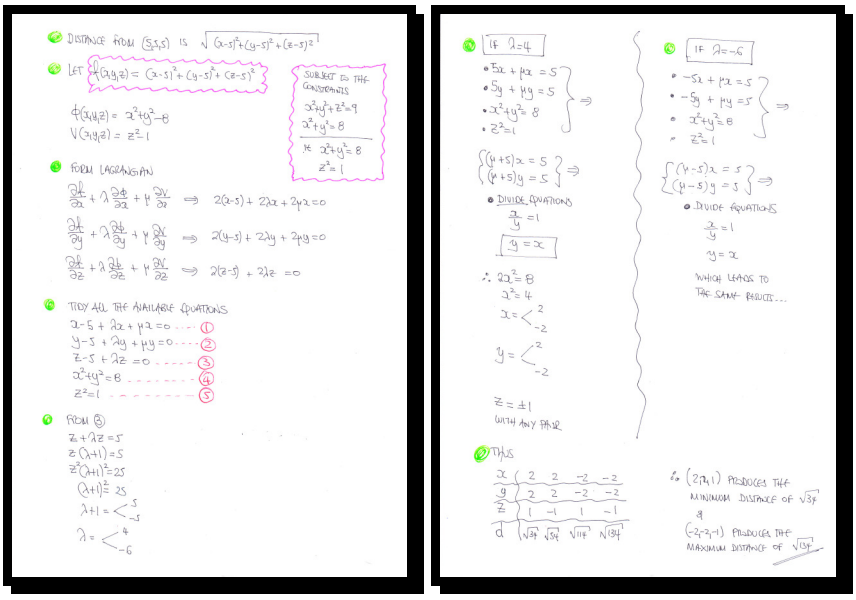
The position of P is such so that the distance of P from the point $(5,5,5)$ is least.

The position of Q is such so that the distance of Q from the point $(5,5,5)$ is greatest.

Determine the coordinates of P and the coordinates of Q.

Give the corresponding distance from $(5,5,5)$ in each case.

$$\boxed{d_{\min}=\sqrt{34},\quad P(2,2,1)}\ ,\ \boxed{d_{\max}=\sqrt{134},\quad P(-2,-2,1)}$$



Question 11 (****)

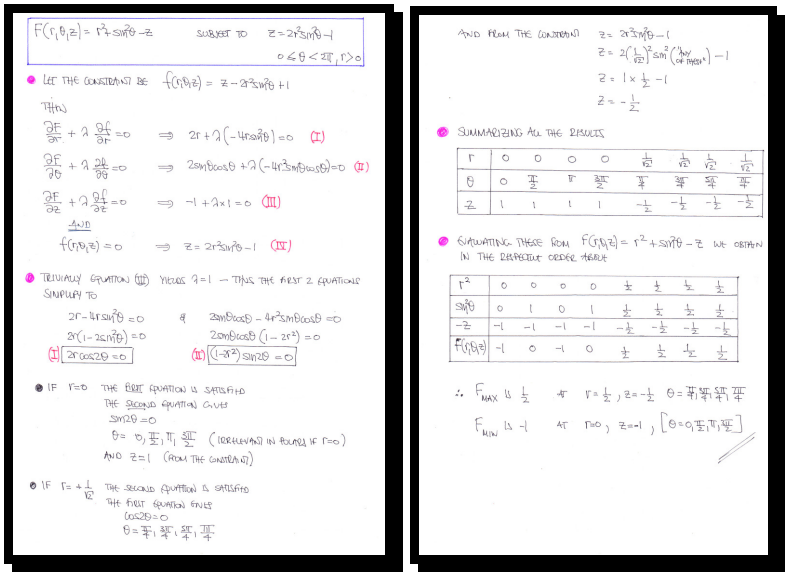
The function F is defined in cylindrical polar coordinates (r,θ,z) as

$$F(r,\theta,z)=r^2+\sin^2\theta-z\,,\quad r\geq 0\,,\ 0\leq\theta<2\pi$$

Determine the minimum value and the maximum value of F, subject to the constraint

$$z=2r^2\sin\theta-1\,.$$

$$\boxed{F_{\min}=-1}\,,\quad \boxed{F_{\max}=\frac{1}{2}}$$



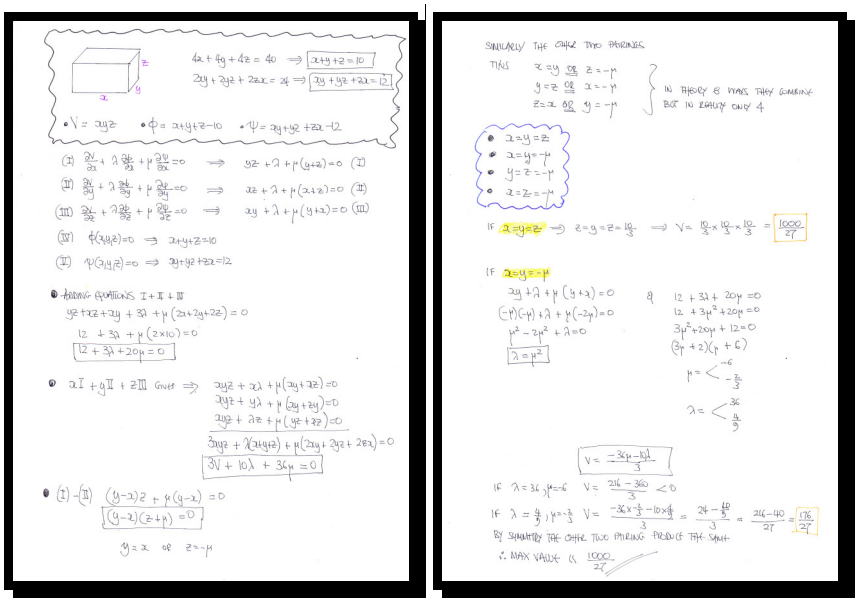
Question 12 (**+)**

A brick is made so that

- the sum of the lengths of all of its edges is 40
- the sum of the area of all of its faces is 24

Use differentiation to find the maximum volume of the brick.

$$\frac{1000}{27}$$



APPLICATIONS TO O.D.E.s

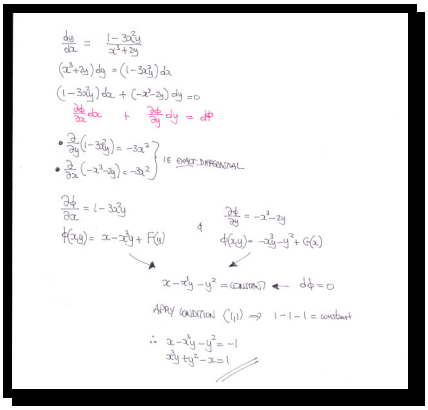
Question 1 (***)

Find the solution of the following differential equation

$$\frac{dy}{dx}=\frac{1-3x^2y}{x^3+2y},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^3y+y^2-x=1}$$



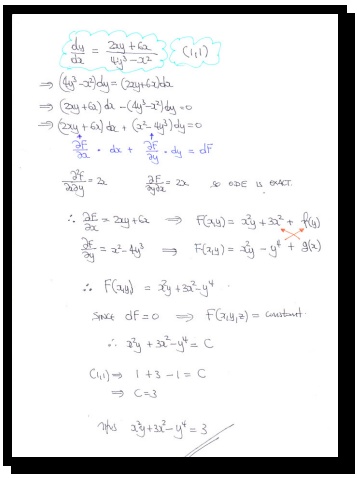
Question 2 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{2xy+6x}{4y^3-x^2},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^2y+3x^2-y^4=3}$$

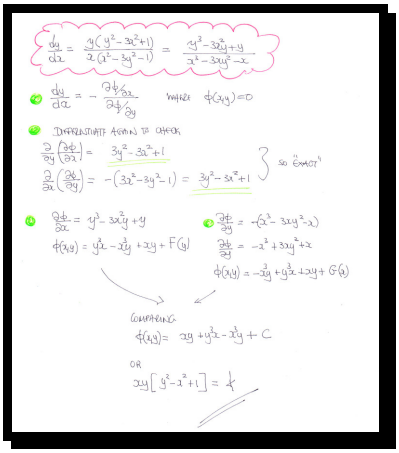


Question 3 (***)

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{y\left(y^2-3x^2+1\right)}{x\left(x^2-3y^2-1\right)}.$$

$$\boxed{xy\left(x^2-y^2-1\right)=\text{constant}}$$



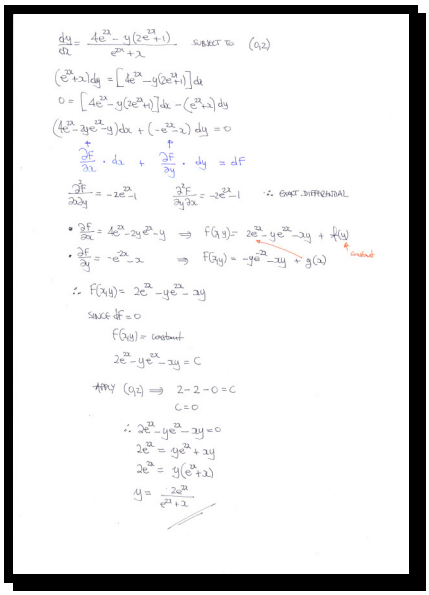
Question 4 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{4\mathrm{e}^{2x}-y\left(2\mathrm{e}^{2x}+1\right)}{\mathrm{e}^{2x}+x},$$

subject to the boundary condition $y=2$ at $x=0$.

$$\boxed{y=\frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x}+x}}$$



Question 5 (*+)**

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{\cos x\cos y+\sin^2 x}{\sin x\sin y+\cos^2 y}.$$

$$\boxed{\sin x\cos y-\frac{1}{4}\left(\sin 2x+\sin 2y\right)+\frac{1}{2}\left(x-y\right)=\text{constant}}$$

